CENTRE FOR DISTANCE AND ONLINE EDUCATION

MANONMANIAM SUNDARANAR UNIVERSITY

TIRUNELVELI- 627 012

TAMILNADU

BBA Course Material

OPERATION RESEARCH

Prepared by

Mr. P. Alex Antonyraj

Assistant Professor

Department of Business Administration

Manonmaniam Sundaranar University College

Govindaperi, Cheranmahadevi - 627414

SYLLABUS

OPERATION RESEARCH

UNIT	DETAILS
I	Introduction - Overview of Operation research - Nature - Scope and characteristics of OR - Features of OR - Stages in OR - Limitations of Operational research.
II	Linear Programming Problem - Concept and Scope of OR - General Mathematical problem of LPP - Steps of LPP model Formulation - Graphical Method of the solution of the LPP - Simple problems
III	Vogel's Approximation method to find the optimal Solution.
IV	Network Models - PERT and CPM - Difference between PERT and CPM - Constructing Network - Critical Path - Various Floats - Three times estimates for PERT
V	Game theory - Maximini Minimax Criterion - Saddle Point - Dominance Property - Graphical method for solving 2×n and m×2 game . Decision Theory - Statement of Nate's theorem application - decision trees.

	References Books
1	P.R. Vittal & V. Malini, Operation Research , Margham Publications ,Chennai, 17
2	P.K.Gupta & Man Mohan , Problems in Operation Research , Sultan Chand & Sons ,New Delhi.
3	V.K. Kapoor , Introduction to Operation Research , Sultan Chand & Sons ,New Delhi.
4	Handy A Taha , Operation Research, An Introduction Prentice Hall of India – New Delhi.
5	P. Gupta, N. Aruna Rani , M.Haritha (2018) Operation Research and Quantitative Techniques, First Edition, Himalaya publishing House.

CONTENTS

Unit I

Structure :

1.1 Introduction of OR

Operation Research (OR) is a multidisciplinary field that deals with the application of advanced analytical methods to help make better decisions in complex systems.

Definition:

Operation Research is a scientific approach to analyzing and optimizing business processes, management systems, and organizational structures. It involves using mathematical and analytical techniques to identify, analyze, and solve complex problems.

History:

OR originated during World War II, when scientists and mathematicians were tasked with optimizing military operations. After the war, OR expanded into various fields, including business, healthcare, and finance.

Key Characteristics:

1. Interdisciplinary: OR combines concepts from mathematics, statistics, computer science, engineering, and economics.

2. Analytical: OR uses mathematical models, algorithms, and statistical analysis to analyze and solve problems.

3. Problem-solving: OR focuses on identifying and solving complex problems in a systematic and scientific way.

4. Decision-making: OR provides insights and recommendations to support informed decision-making.

Methodologies:

1. Linear Programming: Optimizes linear objective functions subject to linear constraints.
2. Dynamic Programming: Breaks down complex problems into smaller sub-problems and solves them recursively.
3. Integer Programming: Deals with optimization problems involving integer variables.
4. Simulation: Uses statistical models to mimic real-world systems and analyze their behavior.
5. Queueing Theory: Studies the behavior of waiting lines and queues.

Applications:

1. Supply Chain Optimization: OR helps manage inventory, logistics, and distribution.
2. Resource Allocation: OR optimizes resource allocation in various industries, such as healthcare and finance.
3. Scheduling: OR helps schedule tasks, jobs, and personnel.
4. Risk Analysis: OR assesses and mitigates risks in various domains.
5. Data Analytics: OR uses data analytics to gain insights and inform decision-making.

Tools and Software:

1. Excel: A popular spreadsheet software used for OR applications.
2. Python: A programming language widely used for OR and data science.
3. R: A programming language and environment for statistical computing and graphics.
4. CPLEX: A commercial software for linear and integer programming.
5. Arena: A simulation software for modeling and analyzing complex systems.

Career Opportunities:

1. Operations Research Analyst: Works on optimizing business processes and solving complex problems.

2. Data Scientist: Applies OR techniques to analyze and interpret complex data.

3. Management Consultant: Uses OR methods to improve organizational performance.

4. Supply Chain Manager: Optimizes supply chain operations using OR techniques.

5. Risk Management Specialist: Applies OR methods to assess and mitigate risks.

Overview of Operation Research

Why Operation Research (OR) is important:

Why Operation Research?

1. Improved Decision-Making: OR provides a systematic and analytical approach to decision-making, helping organizations make informed choices.

2. Optimization: OR helps optimize business processes, resources, and systems, leading to increased efficiency and productivity.

3. Problem-Solving: OR provides a structured approach to solving complex problems, breaking them down into manageable parts.

4. Risk Management: OR helps identify and mitigate risks, reducing the likelihood of adverse outcomes.

5. Cost Savings: OR can help organizations reduce costs by streamlining processes, optimizing resources, and minimizing waste.

6. Competitive Advantage: Organizations that adopt OR techniques can gain a competitive edge by making better decisions and optimizing their operations.

7. Data-Driven Insights: OR provides a framework for analyzing data, extracting insights, and informing decision-making.

8. Resource Allocation: OR helps optimize resource allocation, ensuring that the right resources are allocated to the right tasks.

9. Supply Chain Optimization: OR can help organizations optimize their supply chains, reducing lead times and improving delivery performance.

10. Environmental Sustainability: OR can help organizations reduce their environmental impact by optimizing resource usage and reducing waste.

Why is OR important in today's world?

1. Complexity: Modern organizations face complex challenges, and OR provides a structured approach to addressing these challenges.

2. Uncertainty: OR helps organizations navigate uncertain environments by providing tools for risk analysis and scenario planning.

3. Data Overload: OR provides techniques for analyzing and interpreting large datasets, helping organizations extract insights and make informed decisions.

4. Globalization: OR can help organizations optimize their global supply chains, manage international logistics, and navigate cultural differences.

5. Digital Transformation: OR can help organizations optimize their digital transformation initiatives, ensuring that they are leveraging technology to drive business value.

6. Cybersecurity: OR can help organizations optimize their cybersecurity measures, reducing the risk of cyber attacks and data breaches.

7. Healthcare: OR can help healthcare organizations optimize their operations, improving patient outcomes and reducing costs.

8. Finance: OR can help financial institutions optimize their operations, reducing risk and improving returns.

Benefits of OR:

1. Improved Efficiency: OR can help organizations improve their efficiency, reducing waste and improving productivity.

2. Better Decision-Making: OR provides a systematic and analytical approach to decision-making, helping organizations make informed choices.

3. Increased Competitiveness: Organizations that adopt OR techniques can gain a competitive edge by making better decisions and optimizing their operations.

4. Cost Savings: OR can help organizations reduce costs by streamlining processes, optimizing resources, and minimizing waste.

5. Improved Customer Satisfaction: OR can help organizations improve their customer satisfaction, by optimizing their operations and improving their ability to deliver high-quality products and services.

Challenges of OR:

1. Complexity: OR can be complex, requiring specialized skills and expertise.

2. Data Quality: OR requires high-quality data, which can be a challenge in some organizations.

3. Change Management: OR often requires changes to business processes and operations, which can be challenging to implement.

4. Communication: OR requires effective communication between stakeholders, which can be a challenge in some organizations.

5. Cultural Barriers: OR may require changes to an organization's culture, which can be challenging to implement.

Future of OR:

1. Artificial Intelligence: OR will increasingly leverage artificial intelligence (AI) and machine learning (ML) to analyze data and make decisions.
2. Internet of Things: OR will increasingly leverage the Internet of Things (IoT) to collect data and optimize operations.
3. Cloud Computing: OR will increasingly leverage cloud computing to analyze data and optimize operations.
4. Big Data Analytics: OR will increasingly leverage big data analytics to analyze large datasets and make informed decisions.
5. Sustainability: OR will increasingly focus on sustainability, helping organizations reduce their environmental impact and improve their social responsibility.

1.2 Nature and Scope of Operation Research

Nature of Operation Research:

1. Interdisciplinary: OR combines concepts from mathematics, statistics, computer science, engineering, and economics.
2. Analytical: OR uses mathematical and analytical techniques to analyze and solve complex problems.
3. Problem-Solving: OR provides a structured approach to solving complex problems, breaking them down into manageable parts.
4. Decision-Making: OR provides insights and recommendations to support informed decision-making.
5. Quantitative: OR relies heavily on quantitative methods and techniques to analyze and solve problems.

Scope of Operation Research:

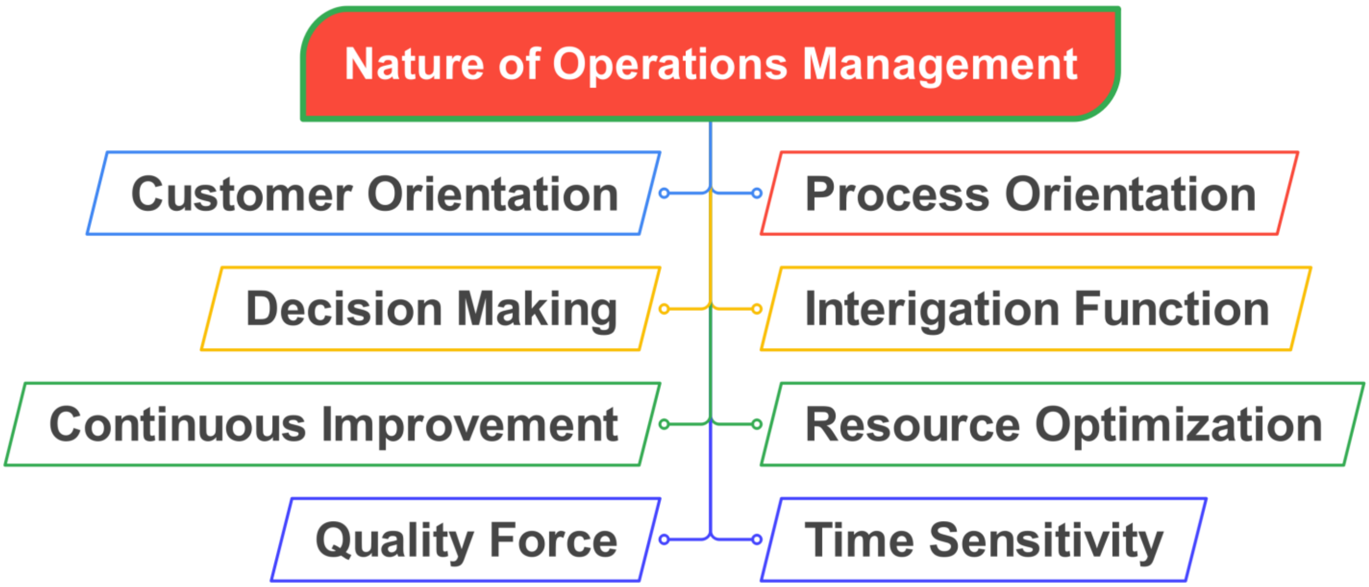
1. Strategic Planning: OR helps organizations develop strategic plans, setting goals and objectives.
2. Operations Management: OR optimizes business processes, managing resources, and improving efficiency.
3. Supply Chain Management: OR optimizes supply chains, managing inventory, logistics, and distribution.

4. Risk Management: OR identifies and mitigates risks, reducing the likelihood of adverse outcomes.

5. Financial Management: OR optimizes financial decisions, managing investments, and reducing costs.

6. Human Resources Management: OR optimizes HR decisions, managing workforce planning, and employee development.

7. Marketing Management: OR optimizes marketing decisions, managing customer relationships, and improving market share.



Phases of Operation Research:

1. Problem Definition: Define the problem or opportunity, identifying key stakeholders and objectives.
2. Data Collection: Gather relevant data, using techniques such as surveys, interviews, and observations.
3. Model Building: Develop a mathematical model, representing the problem or system.
4. Model Solution: Solve the model, using techniques such as linear programming or simulation.
5. Model Validation: Validate the model, ensuring it accurately represents the problem or system.
6. Implementation: Implement the solution, making changes to the organization or system.
7. Monitoring and Evaluation: Monitor and evaluate the solution, ensuring it meets objectives and makes necessary adjustments.

Tools and Techniques of Operation Research:

1. Linear Programming: Optimizes linear objective functions subject to linear constraints.

2. Integer Programming: Deals with optimization problems involving integer variables.

3. Dynamic Programming: Breaks down complex problems into smaller sub-problems and solves them recursively.

4. Simulation: Uses statistical models to mimic real-world systems and analyze their behavior.

5. Queueing Theory: Studies the behavior of waiting lines and queues.

6. Game Theory: Analyzes strategic decision-making in competitive environments.

7. Decision Analysis: Provides a framework for evaluating and optimizing decision-making processes.

1.3 Characteristics of Operation Research (OR)

1. Interdisciplinary

Combines multiple disciplines: OR combines concepts from mathematics, statistics, computer science, engineering, and economics.

Integrates different fields: OR integrates different fields, such as management science, decision science, and systems engineering.

2. Analytical

Uses mathematical models: OR uses mathematical models to analyze and solve complex problems.

Employs statistical techniques: OR employs statistical techniques, such as regression analysis and hypothesis testing.

Utilizes optimization techniques: OR utilizes optimization techniques, such as linear programming and dynamic programming.

3. Problem-Solving

Focuses on problem-solving: OR focuses on solving complex problems, breaking them down into manageable parts.

Uses a systematic approach: OR uses a systematic approach, involving problem definition, data collection, model building, and solution implementation.

Employs creative thinking: OR employs creative thinking, using techniques such as brainstorming and mind mapping.

4. Decision-Making

Supports informed decision-making: OR provides insights and recommendations to support informed decision-making. Uses decision analysis: OR uses decision analysis, involving techniques such as decision trees and sensitivity analysis.

Considers multiple criteria: OR considers multiple criteria, such as cost, time, and quality.

5. Quantitative

Relies on quantitative data: OR relies heavily on quantitative data, using techniques such as data mining and statistical analysis.

Uses mathematical models: OR uses mathematical models, such as linear programming and simulation models.

Employs optimization techniques: OR employs optimization techniques, such as linear programming and dynamic programming.

6. Systematic

Follows a structured approach: OR follows a structured approach, involving problem definition, data collection, model building, and solution implementation.

Uses a phased approach: OR uses a phased approach, involving phases such as problem definition, analysis, and implementation.

Employs checklists and templates: OR employs checklists and templates, such as decision trees and fishbone diagrams.

7. Objective

Aims to optimize performance: OR aims to optimize performance, whether it's maximizing profit, minimizing cost, or improving quality.

Uses objective criteria: OR uses objective criteria, such as cost, time, and quality, to evaluate alternatives.

Avoids subjective bias: OR avoids subjective bias, using techniques such as sensitivity analysis and scenario planning.

8. Adaptive

Responds to changing conditions: OR responds to changing conditions, such as changes in demand, supply, or technology.

Uses feedback mechanisms: OR uses feedback mechanisms, such as monitoring and evaluation, to adjust solutions.

Employs flexible models: OR employs flexible models, such as simulation models, to adapt to changing conditions.

9. Collaborative

Involves stakeholders: OR involves stakeholders, such as managers, employees, and customers, in the problem-solving process.

Uses teamwork: OR uses teamwork, involving cross-functional teams, to solve complex problems.

Employs communication techniques: OR employs communication techniques, such as presentations and reports, to share results.

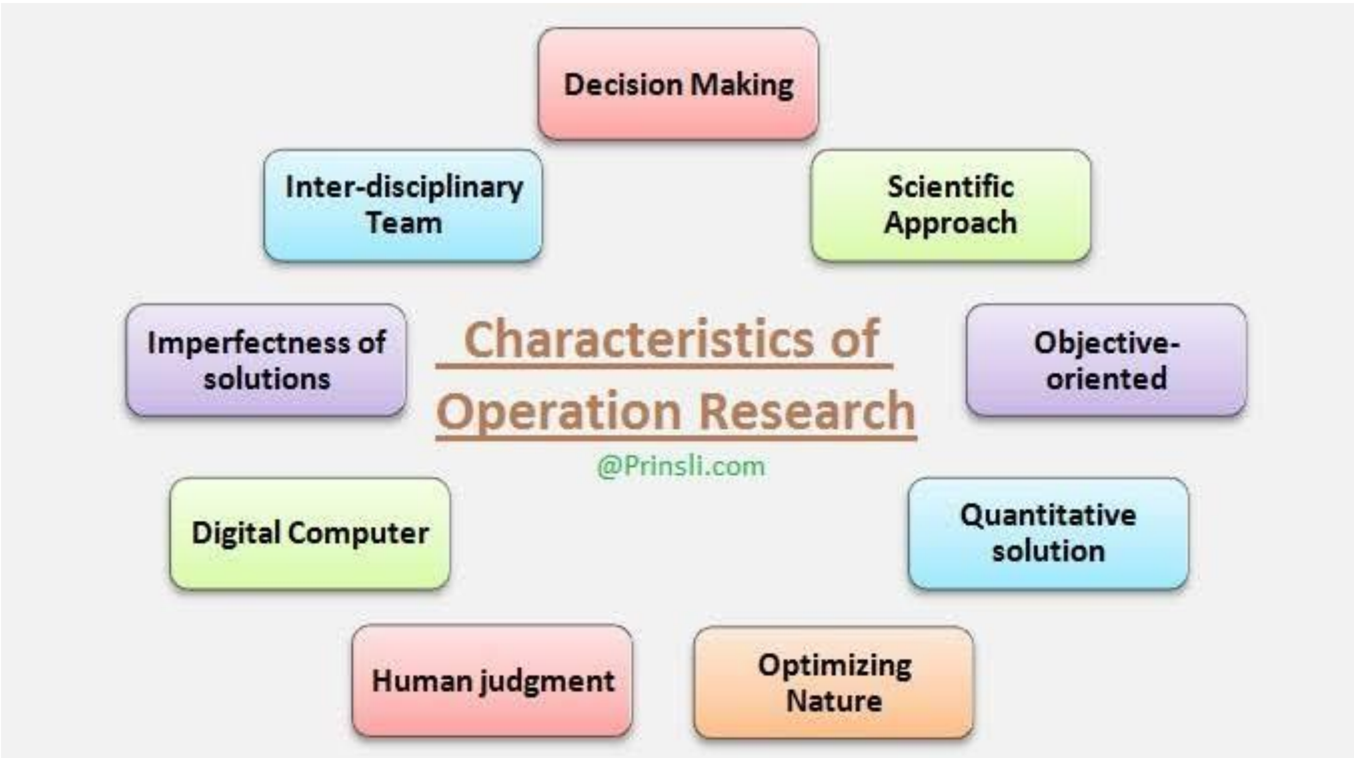
10. Ethical

Considers ethical implications: OR considers ethical implications, such as fairness, transparency, and accountability.

Uses ethical frameworks: OR uses ethical frameworks, such as utilitarianism and deontology, to evaluate alternatives.

Avoids harm: OR avoids harm, using techniques such as risk analysis and mitigation.

Operation Research is characterized by its interdisciplinary approach, analytical techniques, problem-solving focus, decision-making support, quantitative methods, systematic approach, objective criteria, adaptive nature, collaborative approach, and ethical considerations.



1.4 Features in Operation Research :

Operations Research (OR) is a multidisciplinary field that deals with the application of advanced analytical methods to help make better decisions. Here are some key features of Operations Research:

1. Interdisciplinary Approach:

1. Combines multiple disciplines: OR combines concepts from mathematics, statistics, computer science, engineering, and economics.

2. Integrated approach: OR uses an integrated approach to analyze complex problems.

2. Analytical Methods:

1. Mathematical modeling: OR uses mathematical models to represent complex systems and problems.

2. Statistical analysis: OR employs statistical techniques to analyze data and make informed decisions.

3. Optimization techniques: OR uses optimization techniques, such as linear programming and dynamic programming, to find the best solution.

3. Problem-Solving Focus:

1. Problem definition: OR involves defining and structuring complex problems.

2. Alternative solutions: OR generates and evaluates alternative solutions to the problem.

3. Optimal solution: OR aims to find the optimal solution that meets the decision-maker's objectives.

4. Decision-Making Support:

1. Decision analysis: OR provides decision-makers with a structured approach to decision-making.

2. Risk analysis: OR helps decision-makers evaluate and manage risks associated with different alternatives.

3. Sensitivity analysis: OR performs sensitivity analysis to test the robustness of the solution.

5. Quantitative Approach:

1. Quantitative models: OR uses quantitative models to analyze complex systems and problems.

2. Data-driven decision-making: OR relies on data to support decision-making.

6. Iterative Process:

1. Iterative approach: OR involves an iterative approach, where the problem is refined and reanalyzed until a satisfactory solution is found.

2. Feedback loop: OR includes a feedback loop, where the results of the analysis are fed back into the problem-solving process.

7. Focus on Efficiency and Effectiveness:

1. Optimization: OR aims to optimize resources, processes, and systems.

2. Efficiency and effectiveness: OR seeks to improve the efficiency and effectiveness of decision-making.


FEATURES OF
OPERATIONAL RESEARCH
❖Decision-Making
❖Scientific Approach
❖Inter-Disciplinary Team Approach
❖System Approach
❖Use of Computers
❖Objectives
❖Human Factors

1.5 Stages in Operation Research

Stage 1: Problem Definition

1. Identify the problem: Recognize the problem or opportunity for improvement.
2. Define the problem: Clearly articulate the problem and its objectives.
3. Establish the problem's scope: Determine the boundaries of the problem.
4. Identify the stakeholders: Determine who will be impacted by the problem and its solution.

Stage 2: Data Collection

1. Gather relevant data: Collect data relevant to the problem.

2. Analyze the data: Examine the data to understand the problem better.

3. Identify data gaps: Determine if there are any gaps in the data.

4. Develop a data collection plan: Create a plan to collect additional data if needed.

Stage 3: Model Formulation

1. Develop a mathematical model: Create a mathematical representation of the problem.

2. Validate the model: Ensure the model accurately represents the problem.

3. Test the model: Test the model to ensure it is working correctly.

4. Refine the model: Refine the model based on the results of the testing.

Stage 4: Solution Methodology

1. Choose a solution approach: Select a suitable methodology to solve the problem.

2. Apply the solution approach: Use the chosen methodology to obtain a solution.

3. Evaluate the solution: Assess the solution's feasibility and effectiveness.

4. Compare alternatives: Compare the solution with alternative solutions.

Stage 5: Solution Evaluation

1. Evaluate the solution: Assess the solution's feasibility and effectiveness.

2. Compare alternatives: Compare the solution with alternative solutions.

3. Sensitivity analysis: Perform sensitivity analysis to test the robustness of the solution.

4. Risk analysis: Perform risk analysis to identify potential risks associated with the solution.

Stage 6: Implementation

1. Implement the solution: Put the chosen solution into practice.

2. Monitor and adjust: Monitor the solution's performance and make adjustments as needed.

3. Train personnel: Train personnel on the new solution.

4. Evaluate the implementation: Evaluate the effectiveness of the implementation.

Stage 7: Review and Revision

1. Review the outcome: Evaluate the effectiveness of the solution.

2. Revise the solution: Refine the solution based on lessons learned and new information.

3. Document the results: Document the results of the solution.

4. Share the knowledge: Share the knowledge gained from the solution with others.

Stage 8: Maintenance and Update

1. Maintain the solution: Ensure the solution continues to work effectively.

3. Continuously monitor: Continuously monitor the solution's performance.

4. Make adjustments: Make adjustments as needed to ensure the solution remains effective.

By following these stages, Operations Research provides a structured approach to problem-solving, enabling organizations to make informed decisions and drive improvement.

1.6 Limitations of Operation Research

Operations Research (OR) is a powerful tool for decision-making, it has several limitations. Some of the key limitations of OR:

1. Assumptions and Simplifications:

1. Model assumptions: OR models rely on assumptions that may not always hold true in reality.

2. Simplifications: Complex problems are often simplified, which can lead to loss of important details.

2. Data Quality and Availability:

1. Data accuracy: OR models are only as good as the data they are based on. Poor data quality can lead to inaccurate results.

2. Data availability: OR models often require large amounts of data, which may not always be available.

3. Complexity and Interconnectedness:

1. Complex systems: OR models can struggle to capture the complexity of real-world systems.

2. Interconnectedness: OR models may not fully account for the interconnectedness of different components within a system.

4. Uncertainty and Risk:

1. Uncertainty: OR models often rely on probabilistic estimates, which can be uncertain.

2. Risk: OR models may not fully capture the risks associated with different courses of action.

5. Limited Scope:

1. Narrow focus: OR models often focus on a specific aspect of a problem, neglecting other important factors.

2. Short-term focus: OR models may prioritize short-term gains over long-term sustainability.

6. Overreliance on Quantitative Methods:

1. Quantitative bias: OR models often prioritize quantitative data over qualitative insights.

2. Neglect of soft factors: OR models may neglect important soft factors, such as organizational culture or stakeholder engagement.

7. Limited Consideration of Human Factors:

1. Human behaviour: OR models often assume rational human behavior, neglecting the complexities of human decision-making.

2. Social and cultural factors: OR models may neglect important social and cultural factors that influence decision-making.

8. Dependence on Technology:

1. Software limitations: OR models are often dependent on software, which can have limitations and biases.

2. Data processing: OR models require significant data processing, which can be time-consuming and prone to errors.

9. Limited Transparency and Explainability:

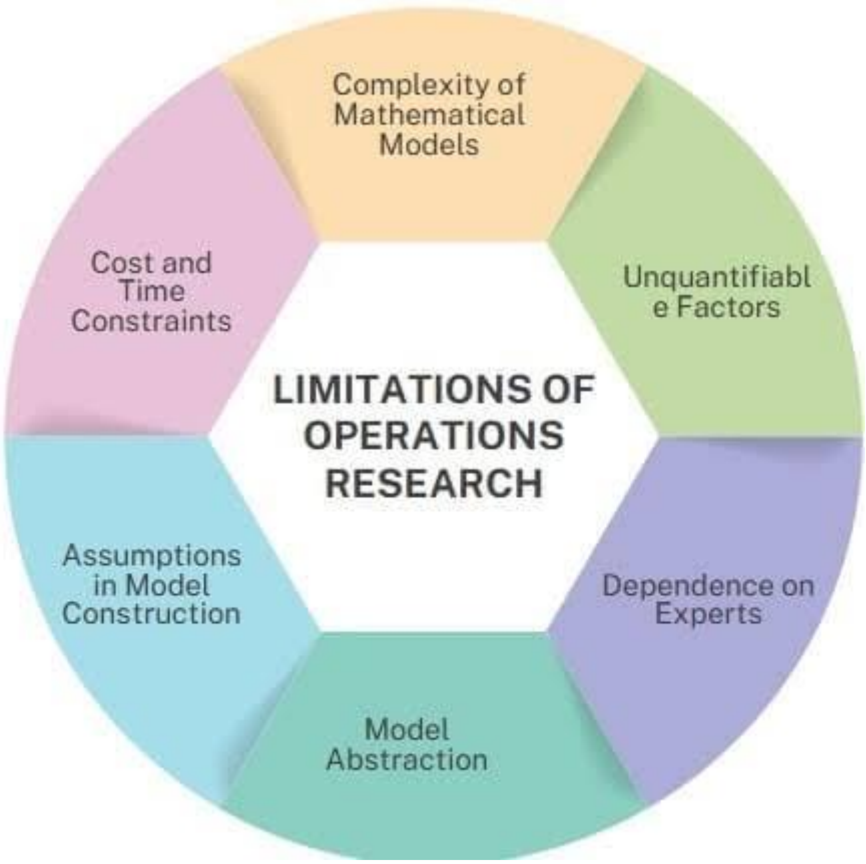
1. Black box models: Some OR models, such as machine learning algorithms, can be difficult to interpret and explain.

2. Lack of transparency: OR models may not provide clear insights into the decision-making process.

10. Ethical Considerations:

1. Bias and fairness: OR models can perpetuate biases and unfairness if they are not designed with ethics in mind.

2. Transparency and accountability: OR models should be designed to provide transparency and accountability in decision-making processes.



By acknowledging these limitations, OR practitioners can take steps to mitigate them and develop more effective and responsible OR models.

1.7 Self Assessment Questions

Part - A

1. What is the primary goal of Operations Research (OR)?

A. To replace human decision-making

B. To find optimal or near-optimal solutions to complex problems

C. To conduct laboratory experiments

D. To design new computer hardware

Answer: B

2. Which of the following best describes the nature of Operations Research?

A. Purely theoretical

B. Art based on intuition

C. Interdisciplinary and applied science

D. A branch of psychology

Answer: C

3. Operations Research helps in decision-making by:

A. Guesswork

B. Random choice

C. Scientific and mathematical modeling

D. Human intuition only

Answer: C

4. Operations Research originated during:

A. World War I

B. The Industrial Revolution

C. World War II

D. The Cold War

Answer: C

5. The main components of an OR problem are:

A. Resources, constraints, objectives

B. Time, money, emotions

C. Feelings, ethics, opinions

D. Data, intuition, assumption

Answer: A

6. Which one of the following is not a typical application area of OR?

A. Production planning

B. Inventory management

C. Consumer behavior psychology

D. Transportation and logistics

Answer: C

7. Operations Research models are primarily:

A. Legal documents

B. Physical structures

C. Mathematical representations of real-world problems

D. Literary theories

Answer: C

8. Linear programming is used in OR to:

A. Create random strategies

B. Optimize a linear objective function with linear constraints

C. Measure customer satisfaction

D. Perform quality assurance

Answer: B

9. Which of the following is a limitation of OR?

A. Too simple to handle complex problems

B. Cannot be used with computers

C. May oversimplify real-world situations

D. Only useful for academic research

Answer: C

10. Simulation in OR is used when:

A. The problem is too simple

B. The problem can’t be modeled mathematically

C. A physical prototype is needed

D. A trial-and-error method is preferred

Answer: B

Part - B

1. What is the nature of Operations Research and how does it contribute to decision-making in organizations?

2. Discuss the interdisciplinary nature of Operations Research and explain how it integrates various fields like mathematics, economics, and engineering.

3. How does Operations Research assist in solving complex real-world problems through scientific methods?

4. Describe the scope of Operations Research in modern industries like transportation, healthcare, and military.

5. How does Operations Research support optimization and resource allocation in large-scale operations?

6. Explain the key characteristics that define the methodology of Operations Research.

7. What are the main steps involved in the Operations Research process, and how do they reflect its systematic nature?

8. How does OR use mathematical modeling and simulation to analyze decision problems?

9. What are the main limitations of Operations Research in practical implementation, especially in dynamic and uncertain environments?

10. Discuss how data availability and computational complexity can affect the effectiveness of Operations Research models.

Unit II

Structure :

2.1 Concept and Scope of OR

2.2 General Mathematical problem of LPP

2.3 Steps of LPP model Formulation Generation

2.4 Graphical Method of the solution of the LPP

2.5 Simple problems

2.6 Self Assessment Questions

Linear Programming Problem:

The application of specific operations research techniques to determine the choice among several courses of action, so as to get an optimal value of the measures of effectiveness (objective or goal), requires to formulate (or construct) a mathematical model. Such a model helps to represent the essence of a system that is required for decision-analysis. The term formulation refers to the process of converting the verbal description and numerical data into mathematical expressions, which represents the relationship among relevant decision variables (or factors), objective and restrictions (constraints) on the use of scarce resources (such as labour, material, machine, time, warehouse space, capital, energy, etc.) to several competing activities (such as products, services, jobs, new equipment, projects, etc.) on the basis of a given criterion of optimality. The term scarce resources refers to resources that are not available in infinite quantity during the planning period. The criterion of optimality is generally either performance, return on investment, profit, cost, utility, time, distance and the like.

2.1 Concept and Scope of OR

Linear Programming (LP) is a powerful optimization technique used to make decisions in various fields, including business, economics, and engineering. The scope of Linear Programming is vast and can be applied to a wide range of problems.

Definition:

Linear Programming is a method used to optimize a linear objective function, subject to a set of linear constraints. It involves finding the best outcome (maximum or minimum) of a linear function, given certain limitations.

Scope:

The scope of Linear Programming includes:

1. Resource Allocation: LP is used to allocate scarce resources, such as labour, materials, and equipment, to maximize efficiency.
2. Cost Minimization: LP helps minimize costs, such as production costs, transportation costs, and inventory costs.
3. Profit Maximization: LP is used to maximize profits, revenue, or returns on investment.
4. Optimization of Business Processes: LP can optimize business processes, such as supply chain management, logistics, and production planning.
5. Portfolio Optimization: LP is used in finance to optimize investment portfolios, managing risk and return.
6. Energy and Environment: LP can optimize energy production, consumption, and distribution, as well as environmental management.
7. Transportation and Logistics: LP is used to optimize transportation routes, schedules, and inventory management.
8. Manufacturing and Production: LP can optimize production planning, scheduling, and inventory control.

Real-World Applications:

Some real-world applications of Linear Programming include:

1. Airline Scheduling: LP is used to optimize flight schedules, crew assignments, and aircraft routing.
2. Supply Chain Optimization: LP helps companies like Walmart and Amazon optimize their supply chains, reducing costs and improving efficiency.
3. Portfolio Management: LP is used by investment firms to optimize investment portfolios, managing risk and return.
4. Energy Production: LP can optimize energy production, reducing costs and environmental impact.

The scope of Linear Programming is vast and can be applied to various fields, including business, economics, engineering, and finance. Its applications are diverse, ranging from resource allocation and cost minimization to profit maximization and portfolio optimization.

2.2 General Mathematical problem of LPP

The general linear programming problem (or model) with n decision variables and m constraints can be stated in the following form:

Optimize (Max. or Min.) Z = c1x1 + c2x2 + . . . + cn xn

subject to the linear constraints,

a11x1 + a12 x2 + . . . + a1n xn (≤, =, ≥) b1

a21x1 + a22 x2 + . . . + a2 n xn (≤, =, ≥) b2

am1x1 + am2 x2 + . . . + amn xn (≤, =, ≥) bm

and x x xn

1 , 2 , . . ., ≥ 0

The above formulation can also express the compact form as follows

$$\text{Optimize (Max.orMin.)} Z = \sum_{j=1}^{n} c_j x_j \qquad \text{(Objective function)} \qquad (1)$$

Subject to the linear constraints

$$\sum_{j=1}^{n} a_{ij}x_j (\le, =, \ge) b_i;$$

where, the cj s are coefficients representing the per unit profit (or cost) of decision variable xj to the value of objective function. The aij’s are referred as technological coefficients (or input-output coefficients).

These represent the amount of resource, say i consumed per unit of variable (activity) xj. These coefficients can be positive, negative or zero. The bi represents the total availability of the ith resource.
The term resource is used in a very general sense to include any numerical value associated with the right-hand side of a constraint. It is assumed that bi ≥ 0 for all i. However, if any bi < 0, then both sides of constraint i is multiplied by –1 to make bi > 0 and reverse the inequality of the constraint.
In the general LP problem, the expression (≤, =, ≥) means that in any specific problem each constraint may take only one of the three possible forms:

(i) less than or equal to (≤)

(ii) equal to (=)

(iii) greater than or equal to (≥)

2.3 Guidelines On Linear Programming Model Formulation :

The effective use and application requires, as a first step, the mathematical formulation of an LP model.

Steps of LP model formulation are summarized as follows:

Step 1: Identify the decision variables

(a) Express each constraint in words. For this you should first see whether the constraint is of the form ≥ (at least as large as), of the form ≤ (no larger than) or of the form = (exactly equal to).

(b) Express verbally the objective function.

(c) Verbally identify the decision variables with the help of Step (a) and (b). For this you need to ask yourself the question – What decisions must be made in order to optimize the objective function?

Having followed Step 1(a) to (c) decide the symbolic notation for the decision variables and specify their units of measurement. Such specification of units of measurement would help in interpreting the final solution of the LP problem.

Step 2: Identify the problem data

To formulate an LP model, identify the problem data in terms of constants, and parameters associated with decision variables. It may be noted that the decision-maker can control values of the variables but cannot control values in the data set.

Step 3: Formulate the constraints

Convert the verbal expression of the constraints in terms of resource requirement and availability of each resource. Then express each of them as linear equality or inequality, in terms of the decision variables defined in Step 1.

Values of these decision variables in the optimal LP problem solution must satisfy these constraints in order to constitute an acceptable (feasible) solution. Wrong formulation can either lead to a solution that is not feasible or to the exclusion of a solution that is actually feasible and possibly optimal.

Step 4: Formulate the objective function

Identify whether the objective function is to be maximized or minimized. Then express it in the form of linear mathematical expression in terms of decision variables along with profit (cost) contributions associated with them.

After gaining enough experience in model building, readers may skip verbal description. The following are certain examples of LP model formulation that may be used to strengthen the ability to translate a real life problem into a mathematical model.

LP Model Formulation :

In this section a number of illustrations have been presented on LP model formulation with the hope that readers may gain enough experience in model building.

Example 1

A manufacturing company is engaged in producing three types of products: A, B and C. The Production department produces, each day, components sufficient to make 50 units of A, 25 units of B and 30 units of C. The management is confronted with the problem of optimizing the daily production of the products in the assembly department, where only 100 man-hours are available daily for assembling the products. The following additional information is available:

Type of Product	Profit Contribution per Unit of Product (Rs)	Assembly time Per Product (hrs)
A	12	0.8
B	20	1.7
C	45	2.5

The company has a daily order commitment for 20 units of products A and a total of 15 units of products B and C. Formulate this problem as an LP model so as to maximize the total profit.

LP model formulation The data of the problem is summarized as follows:

Resources / Constraints	Product A	Product B	Product C	Total
Production Capacity	50	25	30	
Man hours Per Units	0.8	1.7	2.5	100
Order Commitment Units	20	15	15	
Profit Contribution Rs/Unit	12	20	45	

Decision variables Let $x1$, $x2$ and $x3$ = number of units of products A, B and C to be produced, respectively.

The LP model

Maximize (total profit) $Z = 12x1 + 20x2 + 45x3$

subject to the constraints

(i) Labour and materials

(a) $0.8x1 + 1.7x2 + 2.5x3 \leq 100$, (b) $x\ 1 \leq 50$, (c) $x2 \leq 25$, (d) $x3 \leq 30$

(ii) Order commitment

(a) $x1 \geq 20$; (b) $x2 + x3 \geq 15$

and $x1, x2, x3 \geq 0$.

Example 2

A company has two plants, each of which produces and supplies two products: A and B. The plants can each work up to 16 hours a day. In plant 1, it takes three hours to prepare and pack 1,000 gallons of A and one hour to prepare and pack one quintal of B. In plant 2, it takes two hours to prepare and pack 1,000 gallons of A and 1.5 hours to prepare and pack a quintal of B. In plant 1, it costs Rs 15,000 to prepare and pack 1,000 gallons of A and Rs 28,000 to prepare and pack a quintal of B, whereas in plant 2 these costs are Rs 18,000 and Rs 26,000, respectively. The company is obliged to produce daily at least 10 thousand gallons of A and 8 quintals of B. Formulate this problem as an LP model to find out as to how the company should organize its production so that the required amounts of the two products be obtained at the minimum cost.

LP model formulation The data of the problem is summarized as follows:

Resources / Constraints	Product A	Product B	Total Availability Hrs
Preparation time hrs	Plant 1 : 3hrs Plant 2 :2 hrs	1hr 1.5hr	16 16
Minimum Daily Production	10 thousand	8 quintals	
Cost of Production	15000 18000	28000 26000	

Decision variables Let

x1, x2 = quantity of product A (in ’000 gallons) to be produced in plant 1 and 2, respectively.

x3, x4 = quantity of product B (in quintals) to be produced in plant 1 and 2, respectively.

The LP model

Minimize (total cost) Z = 15,000x1 + 18,000x2 + 28,000x3 + 26,000x4

subject to the constraints

(i) Preparation time

(a) 3x1 + 2x2 δ 16, (b) x3 + 1.5x4 δ 16

(ii) Minimum daily production requirement

(a) x1 + x2 ε 10, (b) x3 + x4 ε 8

and x1, x2, x3, x4 ε 0.

Example : 3

An electronic company is engaged in the production of two components *C*1 and *C*2 that are used in radio sets. Each unit of *C*1 costs the company Rs 5 in wages and Rs 5 in material, while each of *C*2 costs the company Rs 25 in wages and Rs 15 in material. The company sells both products on one period credit terms, but the company's labour and material expenses must be paid in cash. The selling price of *C*1 is Rs 30 per unit and of *C*2 it is Rs 70 per unit. Because of the company's strong monopoly in these components, it is assumed that the company can sell, at the prevailing prices, as many units as it produces.

The company's production capacity is, however, limited by two considerations. First, at the beginning of period 1, the company has an initial balance of Rs 4,000 (cash plus bank credit plus collections from past credit sales). Second, the company has, in each period, 2,000 hours of machine time and 1,400 hours of assembly time. The production of each *C*1 requires 3 hours of machine time and 2 hours of assembly time, whereas the production of each *C*2 requires 2 hours of machine time and 3 hours of assembly time. Formulate this problem as an LP model so as to maximize the total profit to the company.

LP model formulation The data of the problem is summarized as follows:

Resources / Constraints	C1 Components	C2 Components	Total Availability
Budger (Rs)	10/Unit	40/Unit	Rs 4000
Machine Time	3hrs Unit	2hrs/ Unit	2000Hrs
Assembly Time	2hrs /Unit	3Hrs /Unit	1400Hrs
Selling Price	Rs 30	Rs 70	
Cost Price (Wages + Material)	Rs 10	Rs 40	

Decision variables Let *x*1 and *x*2 = number of units of components *C*1 and *C*2 to be produced, respectively.

The LP model

Maximize (total profit) Z = Selling price – Cost price

$= (30 – 10)\, x1 + (70 – 40)\, x2 = 20x1 + 30x2$

subject to the constraints

(i) The total budget available

$10x1 + 40x2 \leq 4{,}000$

(ii) Production time

(a) $3x1 + 2x2 \leq 2{,}000$; (b) $2x1 + 3x2 \leq 1{,}400$

and $x1, x2 \geq 0$.

Example 4

A company has two grades of inspectors 1 and 2, the members of which are to be assigned for a quality control inspection. It is required that at least 2,000 pieces be inspected per 8-hour day. Grade 1 inspectors can check pieces at the rate of 40 per hour, with an accuracy of 97 per cent. Grade 2 inspectors check at the rate of 30 pieces per hour with an accuracy of 95 per cent. The wage rate of a Grade 1 inspector is Rs 5 per hour while that of a Grade 2 inspector is Rs 4 per hour. An error made by an inspector costs Rs 3 to the company. There are only nine Grade 1 inspectors and eleven Grade 2 inspectors available to the company. The company wishes to assign work to the available inspectors so as to minimize the total cost of the inspection. Formulate this problem as an LP model so as to minimize the daily inspection cost.

LP model formulation The data of the problem is summarized as follows:

Particulars	Inspector Grade1	InspectorGrade2
Number of Inspectors	9	11
Rate of Checking	40 Pieces / hr	30pieces/hr
Inaccuracy in Checking	1-0.97=0.03	1-0.95=0.05
Cost of accuracy in checking	Rs 3/ Piece	Rs 3 / Piece
Wage rate / hr	Rs 5	Rs 4
Duration of Inspection 8 hr per day Total Piece Inspected 2000		

Decision variables Let $x1$ and $x2$ = number of Grade 1 and 2 inspectors to be assigned for inspection, respectively.

The LP model

Hourly cost of each inspector of Grade 1 and 2 can be computed as follows:

Inspector Grade 1 : Rs $(5 + 3 \times 40 \times 0.03)$ = Rs 8.60

Inspector Grade 2 : Rs $(4 + 3 \times 30 \times 0.05)$ = Rs 8.50

Based on the given data, the LP model can be formulated as follows:

Minimize (daily inspection cost) $Z = 8\,(8.60x1 + 8.50x2) = 68.80x1 + 68.00x2$

subject to the constraints

(i) Total number of pieces that must be inspected in an 8-hour day

$8 \times 40x1 + 8 \times 30x2 \ \varepsilon\ 2000$

(ii) Number of inspectors of Grade 1 and 2 available

(a) $x1 \ \delta\ 9$, (b) $x2 \ \delta\ 11$

and $x1, x2 \ \varepsilon\ 0$.

Example 5

An electronic company produces three types of parts for automatic washing machines. It purchases casting of the parts from a local foundry and then finishes the part on drilling, shaping and polishing machines.

The selling prices of parts A, B and C are Rs 8, Rs 10 and Rs 14 respectively. All parts made can be sold. Castings for parts A, B and C, respectively cost Rs 5, Rs 6 and Rs 10.

The shop possesses only one of each type of casting machine. Costs per hour to run each of the three machines are Rs 20 for drilling, Rs 30 for shaping and Rs 30 for polishing. The capacities (parts per hour) for each part on each machine are shown in the table:

Machine	Part A	Part B	Part C
Drilling	25	40	25
Shaping	25	20	20
Polishing	40	30	40

The management of the shop wants to know how many parts of each type it should produce per hour in order to maximize profit for an hour's run. Formulate this problem as an LP model so as to maximize total profit to the company. [*Delhi Univ., MBA, 2001, 2004, 2007*]

LP model formulation Let $x1$, $x2$ and $x3$ = numbers of type A, B and C parts to be produced per hour, respectively.

Since 25 type A parts per hour can be run on the drilling machine at a cost of Rs 20, then

Rs 20/25 = Re 0.80 is the drilling cost per type A part. Similar reasoning for shaping and polishing gives

Profit per type A part = 0.25

Profit per type B part = 1

Profit per type C part = 0.95

On the drilling machine, one type A part consumes 1/25th of the available hour, a type B part consumes 1/40th, and a type C part consumes 1/25th of an hour. Thus, the drilling machine constraint is

$x1$ $x2$ $x3$

25 40 25

+ + δ 1

Similarly, other constraints can be established.

The LP model

Maximize (total profit) $Z = 0.25\, x1 + 1.00\, x2 + 0.95\, x3$

subject to the constraints

(i) Drilling machine: 1 2 3 1,

25 40 25

$x + x + x$ δ (ii) Shaping machine: 1 2 3 1,

25 20 20

$x + x + x$ δ

(iii) Polishing machine: 1 2 3 1,

40 30 40

$x + x + x$ δ+

and $x1$, $x2$, $x3$ ε 0.

2.4 Graphical Method of the solution of the LPP

An optimal as well as a feasible solution to an LP problem is obtained by choosing one set of values from several possible values of decision variables $x_1, x_2, \ldots, x_n$, that satisfies the given constraints simultaneously and also provides an optimal (maximum or minimum) value of the given objective function.

For LP problems that have only two variables, it is possible that the entire set of feasible solutions can be displayed graphically by plotting linear constraints on a graph paper in order to locate the best (optimal) solution. The technique used to identify the optimal solution is called the graphical solution method (approach or technique) for an LP problem with two variables.

Since most real-world problems have more than two decision variables, such problems cannot be solved graphically. However, graphical approach provides understanding of solving an LP problem algebraically, involving more than two variables.

In this chapter, we shall discuss the following two graphical solution methods (or approaches):

(i) Extreme point solution method

(ii) Iso-profit (cost) function line method

to find the optimal solution to an LP problem.

IMPORTANT DEFINITIONS

Solution The set of values of decision variables x_j ($j = 1, 2, \ldots, n$) that satisfy the constraints of an LP problem is said to constitute the solution to that LP problem.

Feasible solution The set of values of decision variables x_j ($j = 1, 2, \ldots, n$) that satisfy all the constraints and non-negativity conditions of an LP problem simultaneously is said to constitute the feasible solution to that LP problem.

Infeasible solution The set of values of decision variables x_j ($j = 1, 2, \ldots, n$) that do not satisfy all the constraints and non-negativity conditions of an LP problem simultaneously is said to constitute the infeasible solution to that LP problem.

Basic solution For a set of m simultaneous equations in n variables ($n > m$) in an LP problem, a solution obtained by setting ($n - m$) variables equal to zero and solving for remaining m equations in m variables is called a basic solution of that LP problem.

The ($n - m$) variables whose value did not appear in basic solution are called non-basic variables and the remaining m variables are called basic variables.

Basic feasible solution A feasible solution to an LP problem which is also the basic solution is called the basic feasible solution. That is, all basic variables assume non-negative values. Basic feasible solution is of two types:

(a) Degenerate A basic feasible solution is called degenerate if the value of at least one basic variable is zero.

(b) Non-degenerate A basic feasible solution is called non-degenerate if value of all m basic variables is non-zero and positive.

Optimum basic feasible solution A basic feasible solution that optimizes (maximizes or minimizes) the objective function value of the given LP problem is called an optimum basic feasible solution.

Unbounded solution A solution that can increase or decrease infinitely the value of the objective function of the LP problem is called an unbounded solution.

GRAPHICAL SOLUTION METHODS OF LP PROBLEM

While obtaining the optimal solution to the LP problem by the graphical method, the statement of the following theorems of linear programming is used

- The collection of all feasible solutions to an LP problem constitutes a convex set whose extreme points correspond to the basic feasible solutions.
- There are a finite number of basic feasible solutions within the feasible solution space.
- If the convex set of the feasible solutions of the system of simultaneous equations: **$Ax = b$, $x \geq 0$,** is a convex polyhedron, then at least one of the extreme points gives an optimal solution.
- If the optimal solution occurs at more than one extreme point, the value of the objective function will be the same for all convex combinations of these extreme points.

2.5 Simple Problems - Maximization LP Problem

Example : 1

Use the graphical method to solve the following LP problem.

Maximize $Z = 15x1 + 10x2$

subject to the constraints

(i) $4x1 + 6x2 \leq 360$, (ii) $3x1 + 0x2 \leq 180$, (iii) $0x1 + 5x2 \leq 200$

and $x1, x2 \geq 0$.

It given LP problem is already in mathematical form.

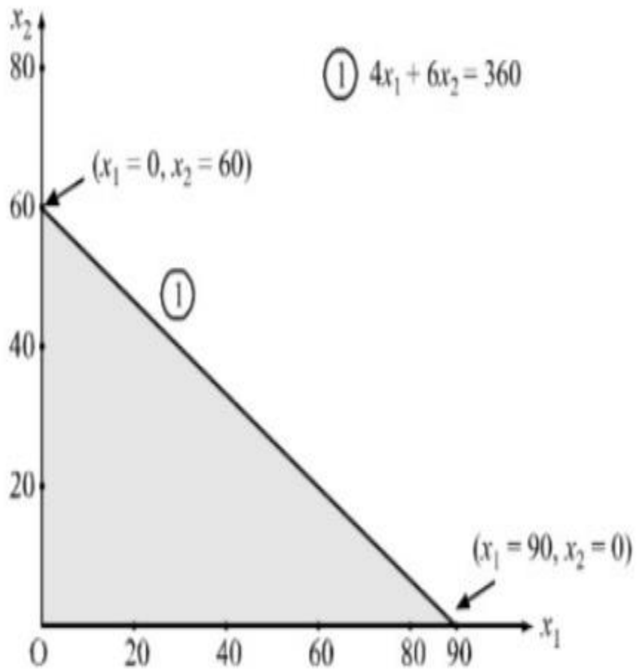
2. Treat x1 as the horizontal axis and x2 as the vertical axis. Plot each constraint on the graph by treating it as a linear equation and it is then that the appropriate inequality conditions will be used to mark the area of feasible solutions.

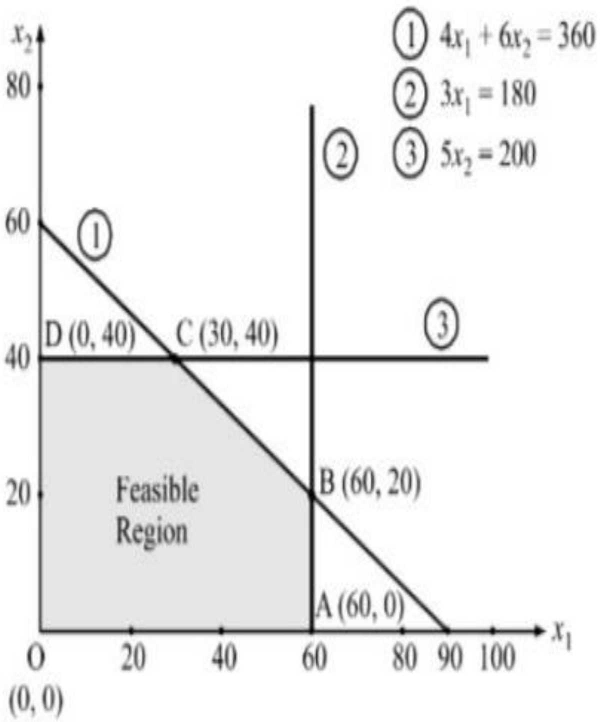
Consider the first constraint 4x1 + 6x2 ≤ 360. Treat this as the equation 4x1 + 6x2 = 360. For this find any two points that satisfy the equation and then draw a straight line through them. The two points are generally the points at which the line intersects the x1 and x2 axes.

For example, when

x1 = 0 we get 6x2 = 360 or x2 = 60. Similarly when x2 = 0, 4x1 = 360, x1 = 90.

These two points are then connected by a straight line as shown in Fig. 3.1(a). But the question is: Where are these points satisfying 4x1 + 6x2 ≤ 360. Any point above the constraint line violates the inequality condition. But any point below the line does not violate the constraint. Thus, the inequality and non-negativity condition can only be satisfied by the shaded area (feasible region) as shown below





Similarly, the constraints 3x1 ≤ 180 and 5x2 ≤ 200 are also plotted on the graph and are indicated by the shaded area

Since all constraints have been graphed, the area which is bounded by all the constraints lines including all the boundary points is called the feasible region (or solution space). The feasible region is by the shaded area OABCD.

3. (i) Since the optimal value of the objective function occurs at one of the extreme points of the feasible region, it is necessary to determine their coordinates. The coordinates of extreme points of the feasible region are: O = (0, 0), A = (60, 0), B = (60, 20), C = (30, 40), D = (0, 40).

(ii) Evaluate objective function value at each extreme point of the feasible region as shown

Extreme Point	Coordinates	Objective Function Value Z=15x1+10x2
O	(0.0)	15(0)+10(0)=0
A	(60,0)	900
B	(60,20)	1100
C	(30,40)	850
D	(0,40	400

Since objective function Z is to be maximized, from Table 3.1 we conclude that maximum value of Z = 1,100 is achieved at the point extreme B (60, 20). Hence the optimal solution to the given LP problem is: x1 = 60, x2 = 20 and Max Z = 1,100.

To determine which side of a constraint equation is in the feasible region, examine whether the origin (0, 0) satisfies the constraints. If it does, then all points on and below the constraint equation towards the origin are feasible points. If it does not, then all points on and above the constraint equation away from the origin are feasible points

Example 2

Use the graphical method to solve the following LP problem.

Maximize $Z = 2\,x1 + x2$

subject to the constraints

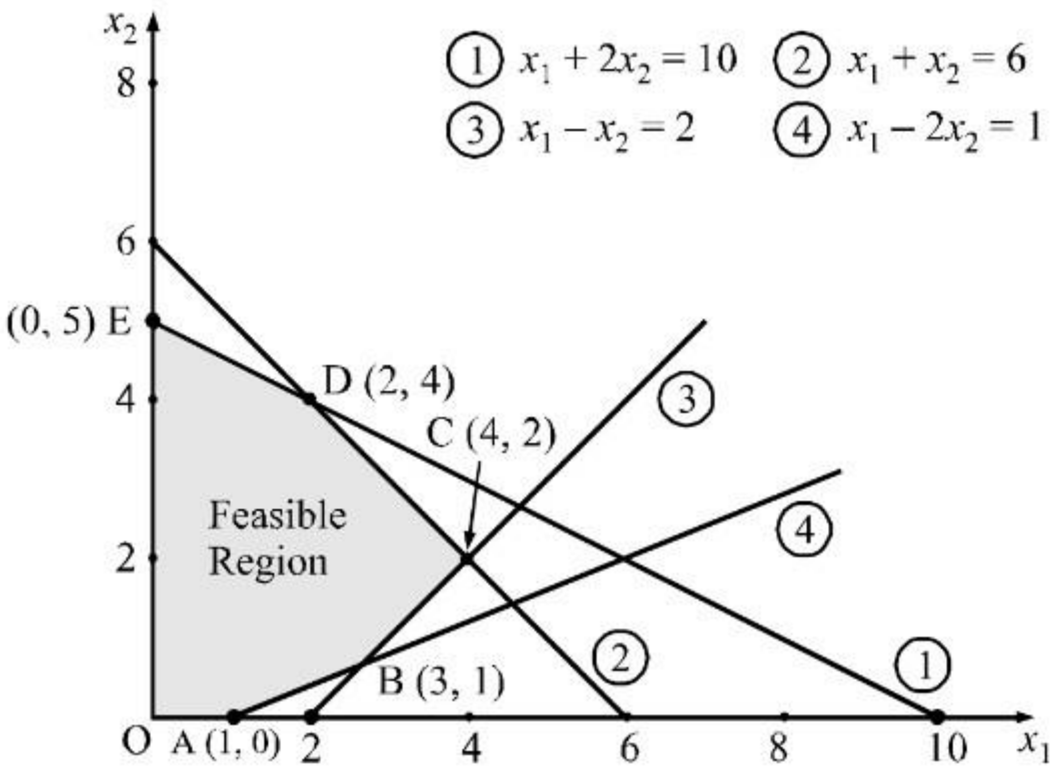
(i) $x1 + 2x2 \leq 10$, (ii) $x1 + x2 \leq 6$,

(iii) $x1 - x2 \leq 2$, (iv) $x1 - 2x2 \leq 1$

and $x1, x2 \geq 0$

Solution

Plot on a graph each constraint by first treating it as a linear equation. Then use inequality condition of each constraint to mark the feasible region by shaded area as shown in Fig. 3.2. It may be noted that we have not considered the area below the lines $x1 - x2 = 2$ and $x1 - 2x2 = 1$ for the negative values of $x2$. This is because of the non-negativity condition, $x2 \geq 0$.



The coordinates of extreme points of the feasible region are: O = (0, 0), A = (1, 0), B = (3, 1), C = (4, 2), D = (2, 4), and E = (0, 5). The value of objective function at each of these extreme points is shown

Extreme Point	Coordinates *(*X1, X2)	Objective FunctionValue (Z= 2X1+X2
0	*(0,0)*	*0*
A	*(1,0)*	*2*
B	*(3,1)*	*7*
C	*(4,2)*	*10*
D	*(2,4)*	*8*
E	*(0,5)*	*5*

The maximum value of the objective function $Z = 10$ occurs at the extreme point (4, 2).
Hence, the optimal solution to the given LP problem is: $x1 = 4$, $x2 = 2$ and Max $Z = 10$.

Example 3

Solve the following LP problem graphically:

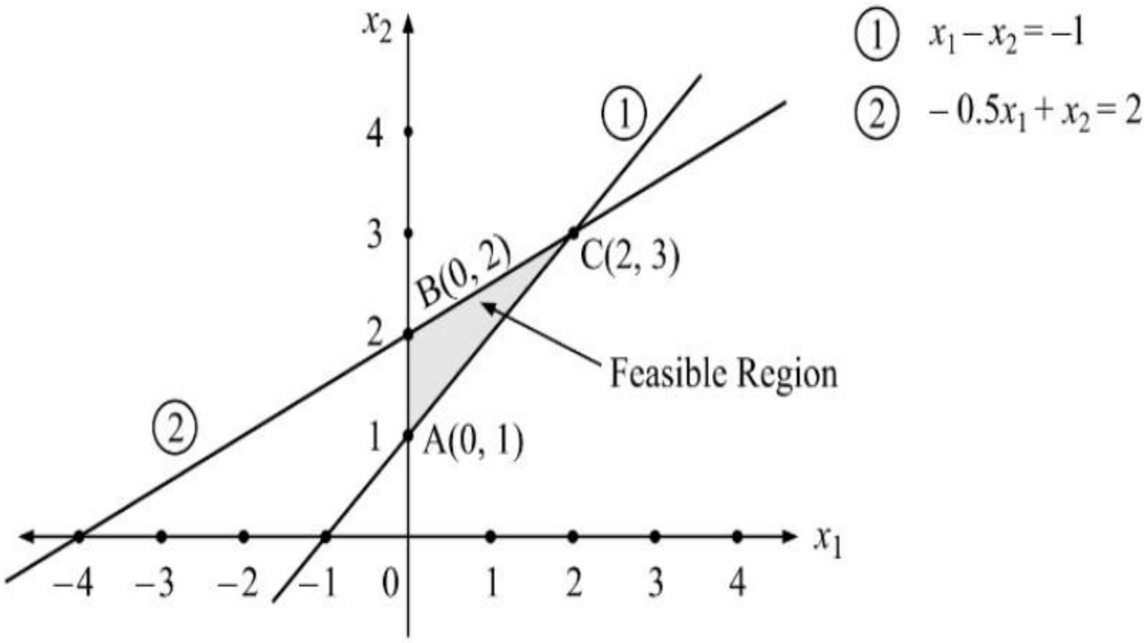
Maximize $Z = – x1 + 2x2$

subject to the constraints

(i) $x1 – x2 \leq –1$; (ii) $– 0.5x1 + x2 \leq 2$

and $x1, x2 \geq 0$

Solution Since resource value (RHS) of the first constraint is negative, multiplying both sides of this constraint by –1, the constraint becomes: $– x1 + x2 \geq 1$. Plot on a graph each constraint by first treating them as a linear equation and mark the feasible region



the value of the objective function at each of the extreme points A(0, 1), B(0, 2) and C(2, 3)

Extreme Point	Co ordinates (X1,X2)	Objective Function Value Z=- X1+2X2
A	(0,1)	2
B	(0,2)	4
C	(2,3)	4

The maximum value of objective function Z = 4 occurs at extreme points B and C. This implies that every point between B and C on the line BC also gives the same value of Z. Hence, problem has multiple optimal.

solutions: $x1 = 0$, $x2 = 2$ and $x1 = 2$, $x2 = 3$ and Max Z = 4.

Example 4

The ABC Company has been a producer of picture tubes for television sets and certain printed circuits for radios. The company has just expanded into full scale production and marketing of AM and AM-FM radios. It has built a new plant that can operate 48 hours per week. Production of an AM radio in the new plant will require 2 hours and production of an AM-FM radio will require 3 hours. Each AM radio will contribute Rs 40 to profits while an AM-FM radio will contribute Rs 80 to profits. The marketing department, after extensive research has determined that a maximum of 15AM radios and 10 AM-FM radios can be sold each week.

(a) Formulate a linear programming model to determine the optimum production mix of AM and FM radios that will maximize profits.

(b) Solve this problem using the graphical method. [*Delhi Univ., MBA, 2002, 2008*]

Solution Let us define the following decision variables

$x1$ and $x2$ = number of units of AM radio and AM-FM radio to be produced, respectively.

Then LP model of the given problem is:

Maximize (total profit) $Z = 40x1 + 80x2$

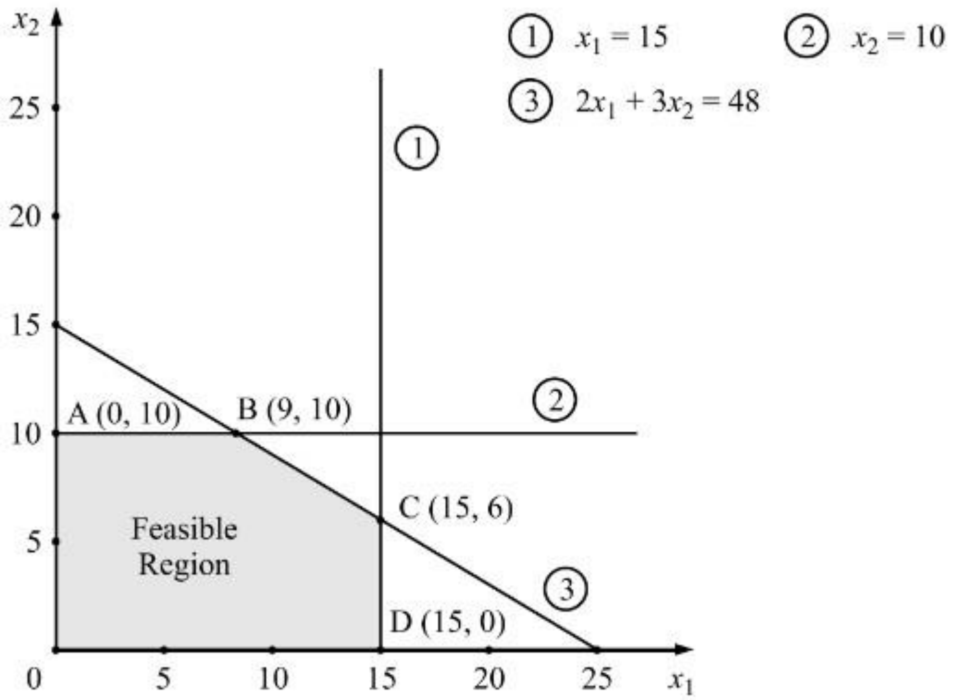
subject to the constraints

(i) Plant : $2x1 + 3x2 \leq 48$, (ii) AM radio : $x1 \leq 15$, (iii) AM-FM radio : $x2 \leq 10$

and $x1, x2 \geq 0$.

Plot on a graph each constraint by first treating it as a linear equation. Then use inequality condition of each

constraint to mark the feasible region. The feasible solution space (or region) is shaded area.



The coordinates of extreme points of the feasible region are: O (0, 0), A (0, 10), B (9, 10), C (15, 6) and D (15, 0). The value of objective function at each of corner (or extreme) points.

Extreme Point	Coordinates((X1,X2)	Objective Value Function Z=40X1+80X2
O	(0,0)	0
A	(0,10)	800
B	(9,10)	1160
C	(15,6)	1080
D	(15,0)	600

Since the maximum value of the objective function Z = 1,160 occurs at the extreme point (9, 10), the optimum solution to the given LP problem is: x1 = 9, x2 = 10 and Max. Z = Rs 1,160.

Example 5

Anita Electric Company produces two products P1 and P2. Products are produced and sold on a weekly basis. The weekly production cannot exceed 25 for product P1 and 35 for product P2 because of limited available facilities. The company employs total of 60 workers. Product P1 requires 2 man-weeks of labour, while P2 requires one man-week of labour. Profit margin on P1 is Rs. 60 and on P2 is Rs. 40. Formulate this problem as an LP problem and solve that using graphical method.

Solution Let us define the following decision variables:

$x1$ and $x2$ = number of units of product P1 and P2, to be produced, respectively.

Then LP model of the given problem is:

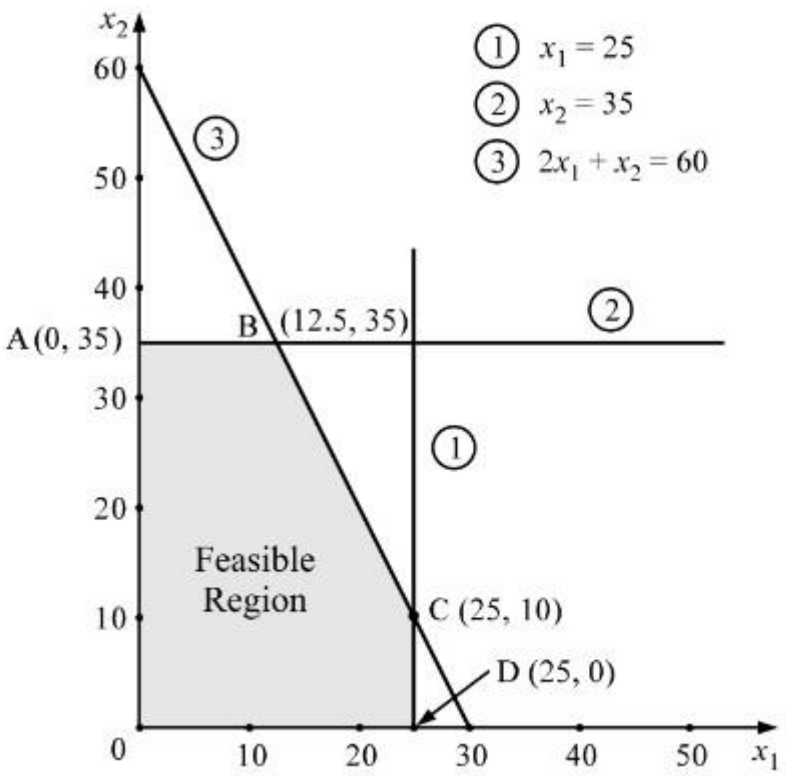
Maximize Z = $60x1 + 40x2$

subject to the constraints

(i) Weekly productions for $P1 : x1 \leq 25$, (ii) Weekly production for $P2 : x2 \leq 35$,

(iii) Workers: $2x1 + x2 = 60$

and $x1, x2 \geq 0$.



Plot on a graph each constraint by first treating it as linear equation. Then use inequality condition of each constraint to mark the feasible region

The coordinates of extreme points of the feasible solution space (or region) are: A(0, 35), B(12.5, 35), C(25, 10) and D(25, 0). The value of the objective function at each of these extreme points

Extreme point	Coordinates (X1,X2)	Objective Function Value Z= 60X1+40X2
A	(0,35)	1400
B	(12.5,35)	2150
C	(25,10)	1900
D	(25,0)	1500

The optimal (maximum) value, Z = 2,150 is obtained at the point B (12.5, 35). Hence, $x1$ = 12.5 units of product P1 and $x1$ = 35 units of product P2 should be produced in order to have the maximum profit, Z = Rs. 2,150.

2.6 Self Assessment Questions

Part A

1. Which of the following is a requirement for a linear programming problem?

A. The decision variables must be integers

B. The objective function and constraints must be linear

C. There must be at least three constraints

D. The problem must be solved graphically

Answer: B. The objective function and constraints must be linear

2. In a standard LPP, the objective function is usually meant to:

A. Minimize resources only

B. Maximize profit or minimize cost

C. Find non-linear solutions

D. Eliminate variables

Answer: B. Maximize profit or minimize cost

3. Which of the following represents a constraint in LPP?

A. $2x + 3y$

B. $x = y$

C. $2x + y \leq 20$

D. x or y

Answer: C. $2x + y \leq 20$

4. In graphical method of solving LPP, the feasible region is:

A. Always a triangle

B. A region that satisfies only the objective function

C. The region that satisfies all constraints including non-negativity

D. A single point only

Answer: C. The region that satisfies all constraints including non-negativity

5. What is the first step in solving an LPP graphically?

A. Maximizing the objective function

B. Drawing the coordinate axes

C. Plotting the constraints as lines

D. Guessing the optimal solution

Answer: C. Plotting the constraints as lines

6. The corner points of the feasible region are used to:

A. Estimate profit

B. Solve for maximum or minimum values of the objective function

C. Plot the objective function

D. Eliminate unbounded solutions

Answer: B. Solve for maximum or minimum values of the objective function

7. In a graphical solution, if the feasible region is unbounded, the LPP may have:

A. No feasible solution

B. Infinite feasible solutions but no optimal solution

C. An optimal solution only if the objective function is bounded

D. A unique solution only

Answer: C. An optimal solution only if the objective function is bounded

8. Which of the following is a valid objective function for LPP?

A. Maximize $x^2 + y^2$

B. Minimize $|x - y|$

C. Maximize $5x + 3y$

D. Minimize $x^3 - y$

Answer: C. Maximize $5x + 3y$

9. Non-negativity constraints in LPP mean:

A. Variables can only be zero

B. Variables must be positive or zero

C. Variables must be negative

D. Variables can be any real number

Answer: B. Variables must be positive or zero

10. If the feasible region of an LPP is bounded and the constraints are linear, the optimal value occurs at:

A. Midpoint of the region

B. Any random point in the region

C. A corner point (vertex) of the feasible region

D. On the constraint line only

Answer: C. A corner point (vertex) of the feasible region

Part - B

1. Define Linear Programming. What are the main assumptions and limitations of Linear Programming?

2. Explain the characteristics of a Linear Programming Problem. What makes a problem suitable for LPP?

3. Describe the significance of constraints and the objective function in a Linear Programming model.

4. Formulate the general mathematical model of a Linear Programming Problem and explain each component with an example.

5. What are the types of constraints in LPP? Explain how equality and inequality constraints affect the feasible region.

6. A company sells two different products A and B, making a profit of Rs 40 and Rs 30 per unit, respectively. They are both produced with the help of a common production process and are sold in two different markets. The production process has a total capacity of 30,000 man-hours. It takes three hours to produce a unit of A and one hour to produce a unit of B. The market has been surveyed and company officials feel that the maximum number of units of A that can be sold is 8,000 units and that of B is 12,000 units. Subject to these limitations, products can be sold in any combination. Formulate this problem as an LP model to maximize profit.

7. Consider a small plant which makes two types of automobile parts, say A and B. It buys castings that are machined, bored and polished. The capacity of machining is 25 per hour for A and 24 per hour for B, capacity of boring is 28 per hour for A and 35 per hour for B, and the capacity of polishing is 35 per hour for A and 25 per hour for B. Castings for part A cost Rs 2 and sell for Rs 5 each and those for part B cost Rs 3 and sell for Rs 6 each. The three machines have running costs of Rs 20, Rs 14 and Rs 17.50 per hour. Assuming that any combination of parts A and B can be sold, formulate this problem as an LP model to determine the product mix which would maximizes profit.

8. A manufacturer produces three models (I, II and III) of a certain product. He uses two types of raw materials (A and B) of which 4,000 and 6,000 units, respectively, are available. The raw material requirements per unit of the three models are as follows: The labour time of each unit of model I is twice that of model II and three times that of model III. The entire labour force of the factory can produce equivalent of 2,500 units of model I. A market survey indicates that the minimum demand of the three models is: 500, 500 and 375 units, respectively. However, the ratios of the number of units produced must be equal to 3 : 2 : 5. Assume that the profit per unit of models I, II and III are Rs 60, 40 and 100, respectively. Formulate this problem as an LP model to determine the number of units of each product which will maximize profit.

9. The manager of an oil refinery must decide on the optimal mix of two possible blending processes of which the input and output per production run are given as follows:

Process (Units)	Inputs(Units)		Output	
	Crude A	Crude B	Gasoline X	GasolineY
1	4	3	5	8
2	5	5	4	4

The maximum available amount of crude A and B are 200 units and 150 units, respectively. Market requirements show that at least 100 units of gasoline X and 80 units of gasoline Y must be produced. The profit per production run from process 1 and process 2 are Rs 300 and Rs 400, respectively. Formulate this problem as an LP model to maximize profit.

10. Comment on the solution of the following LP problems.

(i) Max $Z = 4x1 + 4x2$

subject to $x1 + 2x2 \leq 10$

$6x1 + 6x2 \leq 36$

$x1 \leq 6$

and $x1, x2 \geq 0$

(ii) Max $Z = 5x1 + 3x2$

subject to $4x1 + 2x2 \leq 8$

$x1 \geq 3$

$x2 \geq 7$

and $x1, x2 \geq 0$

(iii) Max $Z = 4x1 + 2x2$

subject to $-x1 + 2x2 \leq 6$

$-x1 + x2 \leq 2$

and $x1, x2 \geq 0$

Unit III

3.1 Vogel's Approximation method to find the optimal Solution

3.2 North West Corner Method

3.3 Least Cost Method

3.4 Self Assessment Questions

INTRODUCTION

One important application of linear programming is in the area of physical distribution (transportation) of goods and services from several supply centres to several demand centres. A transportation problem when expressed in terms of an LP model can also be solved by the simplex method. However a transportation problem involves a large number of variables and constraints, solving it using simplex methods takes a long time. Two transportation algorithms, namely Stepping Stone Method and the MODI (modified distribution) Method have been developed for solving a transportation problem.

The structure of transportation problem involves a large number of shipping routes from several supply centres to several demand centres. Thus, objective is to determine shipping routes between supply centres and demand centres in order to satisfy the required quantity of goods or services at each destination centre, with available quantity of goods or services at each supply centre at the minimum transportation cost and/ or time.

The transportation algorithms help to minimize the total cost of transporting a homogeneous commodity (product) from supply centres to demand centres. However, it can also be applied to the maximization of total value or utility.

There are various types of transportation models and the simplest of them was first presented by F L Hitchcock (1941). It was further developed by T C Koopmans (1949) and G B Dantzig (1951). Several extensions of transportation models and methods have been subsequently developed.

Mathematical Model of Transportation Problem

Illustrate the mathematical model formulation of transportation problem of transporting a single commodity from three sources of supply to four demand destinations.

The sources of supply are production facilities, warehouses, or supply centres, each having certain amount of commodity to supply. The destinations are consumption facilities, warehouses or demand centres each having certain amount of requirement (or demand) of the commodity.

Example 1

A company has three production facilities S1, S2 and S3 with production capacity of 7, 9 and 18 units (in 100s) per week of a product, respectively. These units are to be shipped to four warehouses D1, D2, D3 and D4 with requirement of 5, 6, 7 and 14 units (in 100s) per week, respectively.

The transportation costs (in rupees) per unit between factories to warehouses are given in the table below:

	D1	D2	D3	D4	Supply
S1	19	30	50	10	7
S2	70	30	40	60	9
S3	40	8	70	20	18
Demand	5	8	7	14	34

Formulate this transportation problem as an LP model to minimize the total transportation cost.

Model formulation Let x_{ij} = number of units of the product to be transported from a production facility

i (i = 1, 2, 3) to a warehouse j (j = 1, 2, 3, 4)

The transportation problem is stated as an LP model as follows:

Minimize (total transportation cost) $Z = 19x_{11} + 30x_{12} + 50x_{13} + 10x_{14} + 70x_{21} + 30x_{22} + 40x_{23} + 60x_{24} + 40x_{31} + 8x_{32} + 70x_{33} + 20x_{34}$

subject to the constraints

$x_{11} + x_{12} + x_{13} + x_{14} = 7$

$x_{21} + x_{22} + x_{23} + x_{24} = 9$ (Supply)

$x_{31} + x_{32} + x_{33} + x_{34} = 18$

$x_{11} + x_{21} + x_{31} = 5$

$x_{12} + x_{22} + x_{32} = 8$

$x_{13} + x_{23} + x_{33} = 7$ (Demand)

$x_{14} + x_{24} + x_{34} = 14$

and $x_{ij} \geq 0$ for i = 1, 2, 3 and j = 1, 2, 3, and 4.

In the above LP model, there are m × n = 3 × 4 = 12 decision variables, x_{ij} and m + n = 7 constraints, where m are the number of rows and n are the number of columns in a general transportation table.

The Transportation Algorithm

The algorithm for solving a transportation problem may be summarized into the following steps:

Step 1: Formulate the problem and arrange the data in the matrix form

The formulation of the transportation problem is similar to the LP problem formulation. In transportation problem, the objective function is the total transportation cost and the constraints are the amount of supply and demand available at each source and destination, respectively.

Step 2: Obtain an initial basic feasible solution

In this chapter, following three different methods are discussed to obtain an initial solution:

- North-West Corner Method,
- Least Cost Method, and
- Vogel's Approximation (or Penalty) Method.

The initial solution obtained by any of the three methods must satisfy the following conditions:

(i) The solution must be feasible, i.e. it must satisfy all the supply and demand constraints (also called rim conditions).

(ii) The number of positive allocations must be equal to m + n – 1, where m is the number of rows and n is the number of columns.

Any solution that satisfies the above conditions is called non-degenerate basic feasible solution, otherwise, degenerate solution.

Step 3: Test the initial solution for optimality

In this chapter, the Modified Distribution (MODI) method is discussed to test the optimality of the solution obtained in Step 2.

If the current solution is optimal, then stop. Otherwise, determine a new improved solution.

Step 4: Updating the solution

Repeat Step 3 until an optimal solution is reached.

METHODS OF FINDING INITIAL SOLUTION

There are several methods available to obtain an initial basic feasible solution. In this chapter, we shall discuss only following three methods:

3.2 North-West Corner Method (NWCM)

This method does not take into account the cost of transportation on any route of transportation. The method can be summarized as follows:

Step 1: Start with the cell at the upper left (north-west) corner of the transportation table (or matrix) and allocate commodity equal to the minimum of the rim values for the first row and first column, i.e. min (a1, b1)

Step 2: (a) If allocation made in Step 1 is equal to the supply available at first source (a1, in first row), then move vertically down to the cell (2, 1), i.e., second row and first column. Apply Step 1 again, for next allocation.

(b) If allocation made in Step 1 is equal to the demand of the first destination (b1 in first column), then move horizontally to the cell (1, 2), i.e., first row and second column. Apply Step 1 again for next

allocation.

(c) If a1 = b1, allocate x11 = a1 or b1 and move diagonally to the cell (2, 2).

Step 3: Continue the procedure step by step till an allocation is made in the south-east corner cell of the transportation table.

Remark If during the process of making allocation at a particular cell, the supply equals demand, then the next allocation of magnitude zero can be made in a cell either in the next row or column. This condition is known as degeneracy.

Example 1

Use North-West Corner Method (NWCM) to find an initial basic feasible solution to the transportation problem using data

Solution The cell (S1, D1) is the north-west corner cell in the given transportation table. The rim values for row S1 and column D1 are compared. The smaller of the two, i.e. 5, is assigned as the first allocation; otherwise it will violate the feasibility condition. This means that 5 units of a commodity are to be transported from source S1 to destination D1. However, this allocation leaves a supply of 7 – 5 = 2 units of commodity at S1.

Move horizontally and allocate as much as possible to cell (S1, D2). The rim value for row S1 is 2 and for column D2 is 8. The smaller of the two, i.e. 2, is placed in the cell. Proceeding to row S2, since the demand of D1 is fullfilled. The unfulfilled demand of D2 is now 8 – 2 = 6 units. This can be fulfilled by S2 with capacity of 9 units. So 6 units are allocated to cell (S2, D2). The demand of D2 is now satisfied and a balance of 9 – 6 = 3 units remains with S2.

	D1	D2	D3	D4	SUPPLY
S1	19 **5**	30 **2**	50	10	7
S2	70	30 **6**	40 3	60	9
S3	40	8	70 4	20 14	18
DEMAND	5	8	7	14	34

Continue to move horizontally and vertically in the same manner to make desired allocations. Once the procedure is over, count the number of positive allocations. These allocations (occupied cells) should be equal to m + n – l = 3 + 4 – l = 6. If yes, then solution is non-degenerate feasible solution. Otherwise degenerate solution.

The total transportation cost of the initial solution is obtained by multiplying the quantity xij in the occupied cells with the corresponding unit cost cij and adding all the values together. Thus, the total transportation cost of this solution is

Total cost = 5 × 19 + 2 × 30 + 6 × 30 + 3 × 40 + 4 × 70 + 14 × 20 = Rs 1,015

3.3 Least Cost Method (LCM)

Since the main objective is to minimize the total transportation cost, transport as much as possible through those routes (cells) where the unit transportation cost is lowest. This method takes into account the minimum unit cost of transportation for obtaining the initial solution and can be summarized
as follows:

Step 1: Select the cell with the lowest unit cost in the entire transportation table and allocate as much as possible to this cell. Then eliminate (line out) that row or column in which either the supply or demand is fulfilled. If a row and a column are both satisfied simultaneously, then crossed off either a row or a column.

In case the smallest unit cost cell is not unique, then select the cell where the maximum allocation can be made.

Step 2: After adjusting the supply and demand for all uncrossed rows and columns repeat the procedure to select a cell with the next lowest unit cost among the remaining rows and columns of the transportation table and allocate as much as possible to this cell. Then crossed off that row and column in which either supply or demand is exhausted.

Step 3: Repeat the procedure until the available supply at various sources and demand at various destinations is satisfied. The solution so obtained need not be non-degenerate.

Example 2

Use Least Cost Method (LCM) to find initial basic feasible solution to the transportation problem using data

Solution The cell with lowest unit cost (i.e., 8) is (S3, D2). The maximum units which can be allocated to this cell is 8. This meets the complete demand of D2 and leave l0 units with S3, .

In the reduced table without column D2, the next smallest unit transportation cost, is 10 in cell (S1, D4). The maximum which can be allocated to this cell is 7. This exhausts the capacity of S1 and leaves 7 units with D4 as unsatisfied demand.

	D1	D2	D3	D4	SUPPLY
S1	19	30	50	10 **7**	**7**
S2	70	30	40	60	9
S3	40	8 **8**	70	20	18
DEMAND	5	8	7	14	34

the next smallest cost is 20 in cell (S3, D4). The maximum units that can be allocated to this cell is 7 units. This satisfies the entire demand of D4 and leaves 3 units with S3, as the remaining supply,

the next smallest unit cost cell is not unique. That is, there are two cells – (S2, D3) and (S3, D1) – that have the same unit transportation cost of 40. Allocate 7 units in cell (S2, D3) first because it can accommodate more units as compared to cell (S3, D1). Then allocate 3 units (only supply left with S3) to cell (S3, D1). The remaining demand of 2 units of D1 is fulfilled from S2. Since supply and demand at each supply centre and demand centre is exhausted, the initial solution is arrived .

	D1	D2	D3	D4	SUPPLY
S1	19	30	50	10 **7**	7
S2	70 **2**	30	40 **7**	60	9
S3	40 **3**	**8** **8**	70	20 **7**	18
DEMAND	5	8	7	14	34

The total transportation cost of the initial solution by LCM is calculated as given below:
Total cost = 7 × 10 + 2 × 70 + 7 × 40 + 3 × 40 + 8 × 8 + 7 × 20 = Rs 814
The total transportation cost obtained by LCM is less than the cost obtained by NWC

3.1 Vogal's Approximation Method To Optimize Feasible Solution

Vogal's Approximation Method (VAM) is a heuristic algorithm used to find an initial feasible solution to the transportation problem. The transportation problem involves determining the most efficient way to transport goods from several suppliers (sources) to several consumers (destinations) while minimizing the total cost of transportation. The key to this problem is to determine how to allocate shipments in a way that respects supply and demand constraints and minimizes the transportation cost.

VAM is particularly used because it generally provides a good approximation of the optimal solution, though it does not guarantee the absolute optimal result. However, it often leads to solutions that are close to optimal and can be refined using methods like the stepping-stone method or MODI method.

Explanation of VAM

The main idea of VAM is to minimize the transportation cost by first considering the "penalties" for each row and column and then making allocations based on these penalties. The steps are as follows:

1. Construct the Transportation Table

Begin by constructing a transportation table, where:

Rows represent suppliers (sources), each with a supply value.

Columns represent consumers (destinations), each with a demand value.

Each cell in the table contains the transportation cost from the corresponding supplier to the corresponding consumer.

2. Calculate Row and Column Penalties

For each row and each column, calculate the penalty, which is the difference between the smallest and second-smallest costs in that row or column.

Row penalty: For each row, identify the two lowest costs in that row. Subtract the second-lowest cost from the lowest cost. This difference is the penalty for that row.

Column penalty: For each column, identify the two lowest costs in that column. Subtract the second-lowest cost from the lowest cost. This difference is the penalty for that column.

3. Select the Highest Penalty

Next, look at all the row and column penalties and select the highest penalty. The highest penalty indicates the row or column where allocating the goods will have the largest impact on the total cost (as it has the largest cost difference).

4. Make the Allocation

Once the row or column with the highest penalty is identified, do the following:

In the corresponding row or column, choose the cell with the lowest cost, as this represents the least expensive route for allocation.

Allocate as much as possible to this cell, i.e., the minimum of the supply in the corresponding row and the demand in the corresponding column.

Update the supply and demand values: subtract the allocated amount from the supply of the supplier and from the demand of the consumer. If a supply or demand becomes zero, that row or column is "closed" and no further allocations are made to that row or column.

5. Repeat the Process

After making the first allocation, repeat the process:

Recalculate the row and column penalties (since the supply and demand values have changed).

Again, select the row or column with the highest penalty and make the allocation.

Continue this process until all supply and demand constraints are satisfied (i.e., all supplies are exhausted and all demands are met).

6. Final Solution

Once all the allocations have been made, the resulting transportation plan is a feasible solution. However, this is just the initial solution. It may not be optimal, and further optimization methods like the MODI Method or Stepping-Stone Method can be applied to improve the solution.

Cost matrix (transportation cost per unit):

	C1	C2	C3
S1	10	15	20
S2	25	10	30
S3	35	20	10

Step-by-Step Process:

Calculate Row Penalties:

For S1: The two smallest values are 10 and 15. Penalty = 15 - 10 = 5.

For S2: The two smallest values are 10 and 25. Penalty = 25 - 10 = 15.

For S3: The two smallest values are 10 and 20. Penalty = 20 - 10 = 10.

Row penalties: [5, 15, 10]

Calculate Column Penalties:

For C1: The two smallest values are 10 and 25. Penalty = 25 - 10 = 15.

For C2: The two smallest values are 10 and 15. Penalty = 15 - 10 = 5.

For C3: The two smallest values are 10 and 20. Penalty = 20 - 10 = 10.

Column penalties: [15, 5, 10]

Select the highest penalty:

The highest penalty is 15 (from row S2 and column C1).

Make the allocation:

The lowest cost for S2 to C1 is 25, and the minimum between the supply (70) and demand (40) is 40.

Allocate 40 units from S2 to C1.

Updated supply:

S1 = 30, S2 = 30 (70 - 40), S3 = 50

Updated demand:

C1 = 0 (40 - 40), C2 = 50, C3 = 60

The cell C1 is now closed.

Repeat:

Recalculate the penalties, select the next highest penalty, and continue until all allocations are made.

VAM is an effective method to obtain a reasonably good solution for the transportation problem in a relatively short amount of time. The quality of the solution can often be improved with further optimization techniques, but VAM serves as a strong starting point for solving transportation problems.

Vogel's approximation (penalty or regret) is preferred over NWCR and LCM methods. In this method, anMallocation is made on the basis of the opportunity (or penalty or extra) cost that would have been incurred if the allocation in certain cells with minimum unit transportation cost were missed.

Hence, allocations are made in such a way that the penalty cost is minimized. An initial solution obtained by using this method is nearer to an optimal solution or is the optimal solution itself. The steps of VAM are as follows:

Step 1: Calculate the penalties for each row (column) by taking the difference between the smallest and next smallest unit transportation cost in the same row (column). This difference indicates the penalty or extra cost that has to be paid if decision-maker fails to allocate to the cell with the minimum unit transportation cost.

Step 2: Select the row or column with the largest penalty and allocate as much as possible in the cell that has the least cost in the selected row or column and satisfies the rim conditions. If there is a tie in the values of penalties, it can be broken by selecting the cell where the maximum allocation can be made.

Step 3: Adjust the supply and demand and cross out the satisfied row or column. If a row and a column are satisfied simultaneously, only one of them is crossed out and the remaining row (column) is assigned a zero supply (demand). Any row or column with zero supply or demand should not be used in computing future penalties.

Step 4: Repeat Steps 1 to 3 until the available supply at various sources and demand at various destinations is satisfied.

Example 1

Use Vagel's Approximation Method (VAM) to find the initial basic feasible solution to
the transportation problem using the data .

Solution The differences (penalty costs) for each row and column have been calculated as shown . In the first round, the maximum penalty, 22 occurs in column D2. Thus the cell (S3, D2) having the least transportation cost is chosen for allocation. The maximum possible allocation in this cell is 8 units and it satisfies demand in column D2. Adjust the supply of S3 from 18 to 10 (18 – 8 = 10).

	D1	D2	D3	D4	SUPPLY	Row Difference			
S1	19 **5**	30	50	10 **2**	7	9	9	40	40
S2	70	30	40 **7**	60 **2**	9	10	20	20	20
S3	40	8 **8**	70	20 **10**	18	12	20	50	-
DEMAND	5	8	7	14	34	-	-	-	-
Coloumn differnece	21	22	10	10					
	21	-	10	10					
	-	-	10	50					

The new row and column penalties are calculated except column D2 because D2's demand has been satisfied. In the second round, the largest penalty, 21 appears at column D1. Thus the cell (S1, D1) having the least transportation cost is chosen for allocating 5 units.

After adjusting the supply and demand in the table, we move to the third round of penalty calculations. In the third round, the maximum penalty 50 appears at row S3. The maximum possible allocation of 10 units is made in cell (S3, D4) that has the least transportation cost of 20 as shown in Table 9.5. The process is continued with new allocations till a complete solution is obtained. The initial solution using VAM . The total transportation cost associated with this method is:

Total cost = 5 × 19 + 2 × 10 + 7 × 40 + 2 × 60 + 8 × 8 + 10 × 20 = Rs 779

Example 2

A dairy firm has three plants located in a state. The daily milk production at each plant

is as follows:

Plant 1 : 6 million litres, Plant 2 : 1 million litres, and Plant 3 : 10 million litres

Each day, the firm must fulfil the needs of its four distribution centres. The minimum requirement of each centre is as follows:

Distribution centre 1 : 7 million litres, Distribution centre 2 : 5 million litres, Distribution centre 3 : 3 million litres, and Distribution centre 4 : 2 million litres

Cost (in hundreds of rupees) of shipping one million litre from each plant to each distribution centre is given in the following table:

Distribution Centre

	D1	D2	D3	D4
S1	2	3	11	7
S2	1	0	6	1
S3	5	8	15	9

Find the initial basic feasible solution for given problem by using following methods:

(a) North-west corner rule

(b) Least cost method

(c) Vogel's approximation method

Solution

(a) **North-West Corner**

	D1	D2	D3	D4	SUPPLY
P1	2 **6**	3	11	7	6=a1
P2	1 **1**	0	6	1	1= a2
P3	5	8 **5**	15 **3**	9 **2**	10= a3
DEMAND	7= b1	5 = b2	3= b3	2=b4	

(i) Comparing a1 and b1, since a1 < b1; allocate x11 = 6. This exhausts the supply at P1 and leaves 1 unit as unsatisfied demand at D1.

(ii) Move to cell (P2, D1). Compare a2 and b1 (i.e. 1 and 1). Since a2 = b1, allocate x21 = 1.

(iii) Move to cell (P3, D2). Since supply at P3, is equal to the demand at D2, D3 and D4, therefore, allocate x32 = 5, x33 = 3 and x34 = 2.

It may be noted that the number of allocated cells (also called basic cells) are 5 which is one less than the required number m + n – 1 (3 + 4 – 1 = 6). Thus, this solution is the degenerate solution. The transportation cost associated with this solution is:

Total cost = Rs (2 × 6 + l × l + 8 × 5 + 15 × 3 + 9 × 2) × 100 = Rs 11,600

(b) **Least Cost Method**

	D1	D2	D3	D4	SUPPLY
P1	2 **6**	3	11	7	6
P2	1	0 **1**	6	1	1
P3	5 **1**	8 **4**	15 **3**	9 **2**	10
DEMAND	7	5	3	2	

(i) The lowest unit cost in Table 9.7 is 0 in cell (P2, D2), therefore the maximum possible allocation that can be made is 1 unit. Since this allocation exhausts the supply at plant P2, therefore row 2 is crossed off.

(ii) The next lowest unit cost is 2 in cell (P1, D1). The maximum possible allocation that can be made is 6 units. This exhausts the supply at plant P1, therefore, row P1 is crossed off.

(iii) Since the total supply at plant P3 is now equal to the unsatisfied demand at all the four distribution centres, therefore, the maximum possible allocations satisfying the supply and demand conditions, are made in cells (P3, D1), (P3, D2), (P3, D3) and (P3, D4).

The number of allocated cells in this case are six, which is equal to the required number m + n – 1 (3 + 4 – 1 = 6). Thus, this solution is non-degenerate. The transportation cost associated with this solution is

Total cost = Rs (2 × 6 + 5 × l + 8 × 4 + 15 × 3 + 9 × 2) × 100 = Rs 11,200

(c) **Vogel's Approximation Method:** First calculating penalties as per rules and then allocations are made in accordance of penalties

	D1	D2	D3	D4	SUPPLY	ROW PENALTY		
P1	2 **1**	3 **5**	11	7	6	1	1	5
P2	1	0	6	1 **1**	1	0	-	-
P3	5 **6**	8	15 **3**	9 **1**	10	3	3	4
DEMAND	7	5	3	2				
COLUMN PENALTY	1	3	5	6				
	3	5	4	2				
	3	-	4	2				

The number of allocated cells are six, which is equal to the required number $m + n - 1$ ($3 + 4 - 1 = 6$), therefore, this solution is non-degenerate. The transportation cost associated with this solution is:

Total cost = Rs $(2 \times 1 + 3 \times 5 + 1 \times 1 + 5 \times 6 + 15 \times 3 + 9 \times 1) \times 100$ = Rs 10,200

3.4 Self Assessment Questions :

Part - A

1. What is the primary objective of Vogel's Approximation Method?

A) To find the optimal solution

B) To maximize profit

C) To find an initial feasible solution

D) To eliminate transportation cost

Answer: C

2. VAM is applied in which type of problems?

A) Assignment problems

B) Transportation problems

C) Game theory

D) Linear programming

Answer: B

3. In VAM, what is calculated to select the row or column?

A) Minimum cost

B) Maximum cost

C) Penalty cost

D) Average cost

Answer: C

4. The penalty in VAM is calculated as:

A) Sum of two smallest costs in a row/column

B) Difference between two lowest costs in a row/column

C) Maximum value in the row

D) Average of all values

Answer: B

5. What is the next step after assigning the maximum possible allocation in VAM?

A) Stop the process

B) Recalculate penalties for remaining cells

C) Delete the entire row

D) Subtract allocated value from the cost

Answer: B

6. Which method generally gives a better initial solution than the North-West Corner Rule?

A) Least Cost Method

B) MODI Method

C) VAM

D) Dual Simplex Method

Answer: C

7. VAM always provides:

A) Optimal solution

B) Infeasible solution

C) Initial feasible solution

D) Final solution

Answer: C

8. VAM is considered better than which of the following in terms of initial cost?

A) MODI Method

B) North-West Corner Rule

C) Big M Method

D) Simplex Method

Answer: B

9. In a balanced transportation problem using VAM, what must be true?

A) Supply = Demand

B) Supply > Demand

C) Demand > Supply

D) Cost = Supply

Answer: A

10. Which of the following is the last step in VAM?

A) Recalculate penalties

B) Allocate zero

C) All supplies and demands are satisfied

D) Choose the highest penalty

Answer: C

Part - B

1. Explain in brief three, methods of initial feasible solution for transportation problem.

2. Explain the various steps involved in solving transportation problem using (i) Least cost method, and (ii) Vogel's
approximation method.

3. Explain the (i) North-West Corner method, (ii) Least-Cost method, and (iii) Vogel's Approximation method, for obtaining an initial basic feasible solution of a transportation problem.

4. State the transportation problem. Describe clearly the steps involved in solving it.

5. Is the transportation model an example of decision-making under certainty or under uncertainty? Why?

6. Determine an initial basic feasible solution to the following transportation problem by using (a) NWCR, (b) LCM and (c) VAM.

Destination

Source

	D1	D2	D3	D4	Supply
S1	21	16	15	3	11
S2	17	18	14	23	13
S3	32	27	18	41	19
Demand	6	6	8	23	

7. Determine an initial basic feasible solution to the following transportation problem by using (a) the least cost method, and

(b) Vogel's approximation method.

Source **Destination**

	D1	**D2**	**D3**	**D4**	**Supply**
S1	**1**	**2**	**1**	**4**	**30**
S2	**3**	**3**	**2**	**1**	**30**
S3	**4**	**2**	**5**	**9**	**40**
Demand	**20**	**40**	**30**	**10**	

8. Why does Vogel's approximation method provide a good initialfeasible solution? Can the North-West Corner method ever be able to provide an initial solution with a cost as low as this?

9. Explain in brief three, methods of initial feasible solution for transportation problem.

10. With reference to a transportation problem define the following terms:

(i) Feasible solution (ii) Basic feasible solution

(iii) Optimal solution

(iv) Non-degenerate basic feasible solution

UNIT IV

Structure :

4.1 Network Models

4.2 PERT and CPM

4.3 Difference between PERT and CPM

4.4 Constructing Network

4.5 Critical Path,

4.6 Various Floats.

4.7 Three times estimates for PERT

4.8 Self Assessment Questions

4.1 Network Models

In Operations Research (OR), network models are used to solve problems that involve the flow of materials, information, or resources through a system of interconnected nodes and arcs (edges). These models are essential for optimizing processes like transportation, supply chain management, project planning, and communication systems. Here are the most common types of network models:

1. Shortest Path Problem

Objective: To find the shortest path between two nodes in a network.

Application: Used in transportation, navigation, and routing systems.

Example: Finding the quickest route in a road network.

2. Maximum Flow Problem

Objective: To determine the maximum amount of flow (e.g., goods, information, etc.) that can be sent from a source node to a sink node in a network without violating capacity constraints on the arcs.

Application: Used in telecommunications, transportation, and supply chain networks.

Example: Maximizing the number of goods that can be shipped from a warehouse to multiple retail locations.

3. Minimum Cost Flow Problem

Objective: To determine the cheapest way to send flow through a network while satisfying supply and demand constraints at nodes, subject to capacity constraints on the arcs.

Application: Often used in logistics, transportation, and supply chain management to minimize transportation costs.

Example: Finding the least costly way to transport goods from multiple sources to multiple destinations while respecting the constraints.

4. Transportation Problem

Objective: A special case of the minimum cost flow problem where the goal is to transport goods from several suppliers to several consumers at minimum cost.

Application: Used in logistics and supply chain optimization.

Example: Minimizing transportation costs for delivering goods from factories to distribution centers.

5. Assignment Problem

Objective: To assign resources to tasks in such a way that the total cost is minimized (or profit maximized). This can be represented as a bipartite graph where one set of nodes represents tasks, and the other represents resources.

Application: Used in workforce planning, scheduling, and resource allocation.

Example: Assigning employees to jobs at minimum cost or maximum efficiency.

6. Project Scheduling (PERT/CPM)

Objective: Involves determining the optimal schedule for a set of activities (or tasks) in a project, where the tasks are represented as nodes, and the dependencies between tasks are represented as arcs. The goal is to minimize the project completion time (makespan).

Application: Used in project management for scheduling and managing project timelines.

Example: Determining the critical path and scheduling of activities in a construction project.

Key Characteristics of Network Models:

Nodes (Vertices): Represent points, entities, or stages in the system (e.g., locations in transportation problems, tasks in project scheduling).

Arcs (Edges): Represent connections or paths between nodes that carry flow (e.g., roads in transportation models, tasks dependencies in project scheduling).

Flow: Represents the movement of goods, resources, or information through the network.

Capacity: The maximum allowable flow along an arc or edge.

Supply/Demand: Represents the amount of goods or resources available or required at the nodes.

Network models in Operations Research are essential tools for optimizing a variety of systems involving complex interconnections, helping organizations make data-driven, efficient decisions.3

Network Models : Introduction

A project involves a large number of interrelated activities (or tasks) that must be completed on or before a specified time limit, in a specified sequence (or order) with specified quality and minimum cost of using resources such as personnel, money, materials, facilities and/or space. Examples of projects include, construction of a bridge, highway, power plant, repair and maintenance of an oil refinery or an air plane; design, development and marketing of a new product, research and development work, etc. Since a project involves large number of interrelated activities, therefore it is necessary to prepare a plan for scheduling and controlling these activities (or tasks). This approach will help in identifying bottlenecks and even discovering alternate work-plan for the project.

Network Analysis, Network Planning or Network Planning and Scheduling Techniques are used for planning, scheduling and controlling large and complex projects. These techniques are based on the representation of the project as a network of activities. A network is a graphical presentation of arrows and nodes for showing the logical sequence of various activities to be performed to achieve project objectives. In this chapter, we shall discuss two of these well-known techniques – PERT and CPM.

PERT (Programme Evaluation and Review Technique) was developed in 1956–58 by a research team to help in the planning and scheduling of the US Navy's Polaris Nuclear Submarine Missile project involving thousands of activities. The objective of the team was to efficiently plan and develop the Polaris missile system. This technique has proved to be useful for projects that have an element of uncertainty in the estimation of activity duration, as is the case with new types of projects which have never been taken up before.

CPM (Critical Path Method) was developed by E.I. DuPont company along with Remington Rand Corporation almost at the same time, 1956-58. The objective of the company was to develop a technique to monitor the maintenance of its chemical plants. This technique has proved to be useful for developing time-cost trade-off for projects that involve activities of repetitive nature

4.2 PERT vs CPM

PERT (Program Evaluation and Review Technique) and CPM (Critical Path Method) are both project management tools used to plan, schedule, and control complex projects. They help in determining the longest sequence of activities (critical path) to complete the project on time. However, they differ in their approach and application. Here's a comparison of the two:

1. Purpose

PERT is primarily used for planning and coordinating large and complex projects, where the time required to complete various tasks is uncertain. It focuses on time and event-oriented scheduling.

CPM is used for projects where the activities and time required are well-defined. It focuses on the cost and time optimization for project scheduling.

2. Focus

PERT focuses on uncertainty and variability in project timelines. It uses probabilistic time estimates, accounting for uncertainty in task durations.

CPM focuses on predictable tasks with defined durations and aims to optimize both time and cost.

3. Time Estimates

PERT uses three time estimates for each task: optimistic, pessimistic, and most likely. These estimates help calculate the expected time for each activity.

CPM uses a single time estimate for each task, assuming the duration is known and fixed.

4. Nature of Tasks

PERT is more suited for research and development projects, where tasks may have uncertain durations.

CPM is better for projects with well-defined tasks that have predictable timelines, such as construction or manufacturing.

5. Type of Network

PERT uses a event-oriented network, focusing on milestones and the events that occur at the end of tasks.

CPM uses an activity-oriented network, focusing on the tasks or activities that must be completed.

6. Risk

PERT helps in analyzing the risks and uncertainties in the scheduling of a project by considering multiple possible outcomes.

CPM assumes minimal risk and uncertainty, as the time estimates are fixed.

7. Usage of Resources

PERT doesn't directly consider resource allocation or optimization.

CPM is often used in conjunction with resource allocation, focusing on optimizing resources along with time.

Aspect PERT CPM

Focus Time & Uncertainty Time & Cost Optimization

Time Estimates 3 estimates (Optimistic, Pessimistic, Most Likely) 1 estimate per activity

Nature of Tasks Uncertain, Research-oriented Defined, Predictable, Construction-oriented

Network Type Event-oriented (Milestones) Activity-oriented (Tasks)

Risk Management High, due to uncertainty Low, assuming fixed durations

Resource Focus Does not focus on resource allocation Often involves resource optimization

Use PERT for projects with high uncertainty in task durations (e.g., research, development).

Use CPM for projects where the tasks are well-defined with predictable durations (e.g., construction, manufacturing).

4.3 Difference between PERT vs CPM

PERT (Program Evaluation and Review Technique) and CPM (Critical Path Method) are both project management techniques used for planning, scheduling, and managing projects. While they share some similarities, they differ in several key aspects:

1. Purpose:

PERT: Primarily used for projects where the duration of activities is uncertain. It helps in estimating project timelines by considering various time estimates (optimistic, pessimistic, and most likely).

CPM: Focuses on determining the longest path (critical path) in a project, where the duration of activities is fixed and known. It helps in scheduling and controlling project activities.

2. Time Estimation:

PERT: Uses three time estimates for each activity — Optimistic (O), Pessimistic (P), and Most Likely (M). These estimates are used to calculate the expected time for each activity, typically using the formula:

CPM: Assumes fixed, deterministic activity durations, so it doesn’t account for uncertainty in activity durations.

3. Type of Projects:

PERT: Suitable for projects with a high level of uncertainty and research or development work (e.g., scientific research, new product development).

CPM: Suitable for construction or manufacturing projects where activity durations are well-defined.

4. Focus:

PERT: Emphasizes time estimation and the probability of completing a project on time. It focuses more on time-related uncertainties.

CPM: Focuses on the critical path, the sequence of activities that dictates the minimum project duration. It identifies which activities should be prioritized to avoid project delays.

5. Calculation of Critical Path:

PERT: The concept of the critical path is not central to PERT, as it's more concerned with time estimation and probabilistic analysis.

CPM: The critical path is a key feature in CPM. It's the longest path through the project network, and any delay in the critical path will delay the entire project.

6. Flexibility:

PERT: More flexible as it considers uncertainty and the potential variability in activity durations.

CPM: Less flexible as it assumes a fixed duration for each activity.

7. Visual Representation:

Both PERT and CPM use network diagrams (also known as flow charts or activity-on-node diagrams) to represent activities and their relationships. The primary difference is the emphasis on uncertainty (PERT) vs. fixed scheduling (CPM).

In summary, PERT is used for projects with uncertain durations, focusing on probability and time estimates, while CPM is used for projects with known durations and emphasizes the critical path to control the project schedule.

BASIC DIFFERENCE BETWEEN PERT AND CPM

Both PERT and CPM share in common the determination of a critical path and are based on the network representation of activities and their scheduling that determines the most critical activities to be controlled so as to meet the completion date of a project. However, the following are some of their major differences.

PERT

1. In PERT analysis, a weighted average of the expected completion time of each activity is calculated given three time estimates of its completion. These time estimates are derived from probability distribution of completion times of an activity.

2. In PERT analysis emphasis is given on the completion of a task rather than the activities required to be performed to complete a task. Thus, PERT is also called an event-oriented technique.
3. PERT is used for one time projects that involve activities of non-repetitive nature (i.e. activities that may never have been performed before), where completion times are uncertain.
4. PERT helps in identifying critical areas in a project so that necessary adjustments can be made to meet the scheduled completion date of the project.

CPM

1. In CPM, the completion time of each activity is known with certainty that too unique.
2. CPM analysis explicitly estimate the cost of the project in addition to the completion time. Thus, this technique is suitable for establishing a trade-off for optimum balancing between schedule time and cost of the project.
3. CPM is used for completing of projects that involve activities of repetitive nature.

Significance of Using PERT/CPM

1. A network diagram helps to translate complex project into a set of simple and logical arranged activities and therefore, helps in the clarity of thoughts and actions. It helps in clear and unambiguous communication developing from top to bottom and vice versa , among the people responsible for executing the project.
2. Detailed analysis of a network helps project incharge to peep into the future because difficulties and problems that can be reasonably expected to crop up during the course of execution, can be foreseen well ahead of its actual execution. delays and holdups during course of execution are minimized. Corrective action can also be taken well in time.
3. Isolates activities that control the project completion and therefore, results in expeditious completion of the project.
4. Helps in the division of responsibilities and therefore, enhance effective coordination among different departments/agencies involved.
5. Helps in timely allocation of resources to various activities in order to achieve optimal utilization of resources.

PHASES OF PROJECT MANAGEMENT

In general, project management consists of three phases: Planning, Scheduling and Control.

1. Project planning phase In order to understand the sequencing or precedence relationship among activities in a project, it is essential to draw a network diagram. The steps involved during this phase are listed below:

(i) Identify various activities (tasks or work packages/elements) to be performed in the project, that is, develop a breakdown structure (WBS).

(ii) Determine the requirement of resources such as men, materials, machines, money, etc., for carrying out activities listed above.

(iii) Assign responsibility for each work package. The work packages corresponds to the smallest work efforts defined in a project and forms the set of tasks that are the basis for planning, scheduling and controlling the project.

(iv) Allocate resources to work packages.

(v) Estimate cost and time at various levels of project completion.

(vi) Develop work performance criteria.

(vii) Establish control channels for project personnel.

2. Scheduling phase Once all activities have been identified and given unique codes, the project scheduling (when each of the activities is required to be performed) is taken up. Prepare an estimate of the likelihood of the project to be completed on or before the specified time. The steps involved during this phase are listed below:

(i) Identify all people who will be responsible for each task.

(ii) Estimate the expected duration(s) of each activity, taking into consideration the resources required for their execution in the most economic manner.

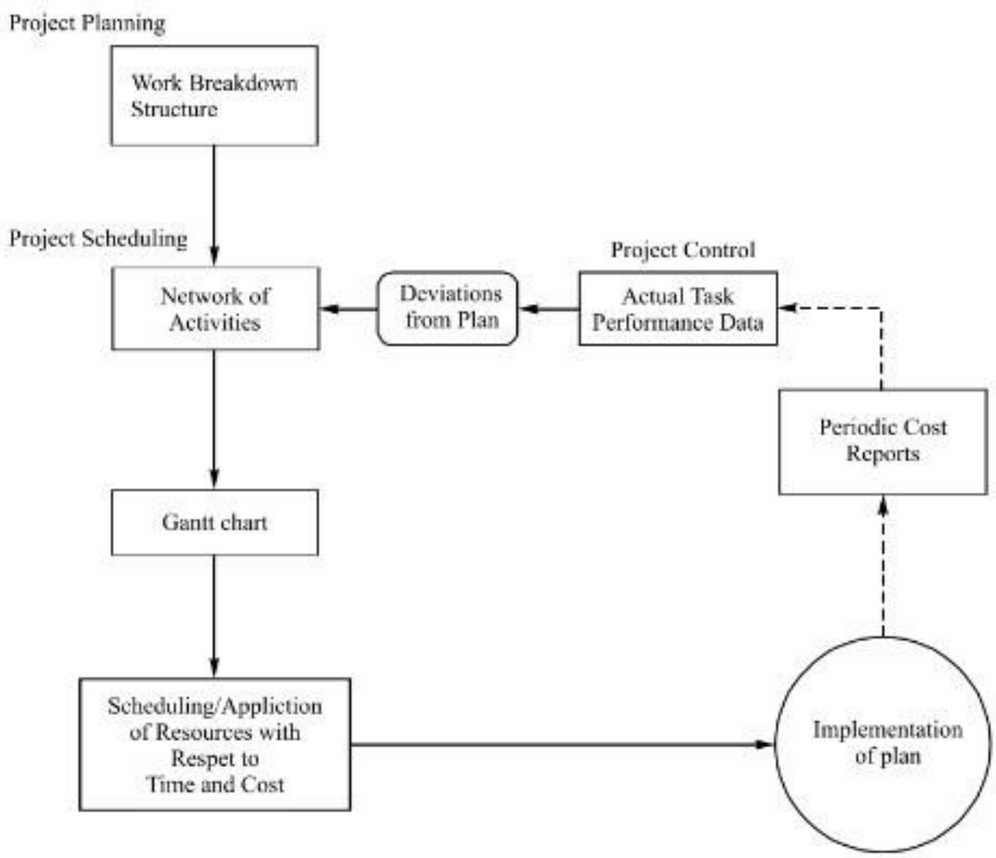
(iii) Specify the interrelationship (i.e. precedence relationship) among various activities.

(iv) Develop a network diagram, showing the sequential interrelationship between various activities.

For this, tips such as; what is required to be done; why it must be done, can it be dispensed with; how to carry out the job; what must precede it; what has to follow; what can be done concurrently, may be followed.

(v) Based on these time estimates, calculate the total project duration, identify critical path; calculate floats; carry out resources smoothing (or levelling) exercise for critical (or scare) resources, taking into account the resource constraints (if any).

3. Project control phase Project control refers to the evaluation of the actual progress (status) against the plan. If significant differences are observed, then remedial (modifying planning) or reallocation of resources measures are adopted in order to update and revise the uncompleted part of the project.



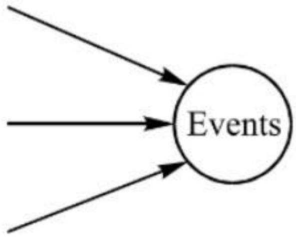
PERT/CPM network components and precedence relationships

PERT/CPM network consists of two major components. These are discussed below:

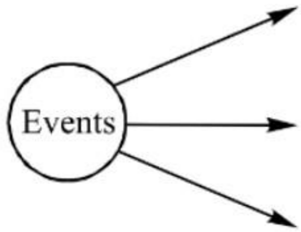
Events Events in the network diagram represent project milestones, such as the start or the completion of an activity (task) or activities, and occur at a particular instant of time at which some specific part of the project has been or is to be achieved. Events are commonly represented by circles (nodes) in the network diagram.

The events can be further classified into the following two categories:

(i) Merge Event : An event which represents the joint completion of more than one activity is known as a merge event.



(a) Merge event



(b) Burst event

(ii) *Burst Event* : An event that represents the initiation (beginning) of more than one activity is known as burst event. This is shown above .

Events in the network diagram are identified by numbers. Each event should be identified by a number higher than that the one allotted to its immediately preceding event to indicate progress of work. The numbering of events in the network diagram must start from left (start of the project) to the right (completion of the project) and top to the bottom. Care should be taken that there is no duplication in the numbering.

Activities Activities in the network diagram represent project operations (or tasks) to be conducted. As such each activity except dummy activity requires resources and takes a certain amount of time for completion. An arrow is commonly used to represent an activity with its head indicating the direction of progress in the project.

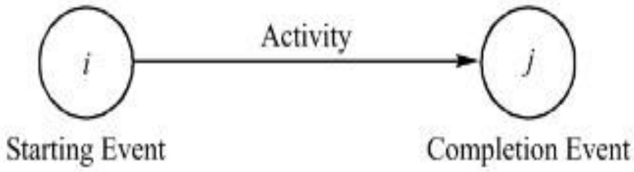
Activities are identified by the numbers of their starting (tail or initial) event and ending (hea terminal) event, for example, an arrow (i, j) between two events; the tail event i represents the start of the activity and the head event j represents the completion of the activity. The activities can be further classified into the following three categories:

(i) Predecessor Activity: An activity which must be completed before one or more other activities start is known as predecessor activity.

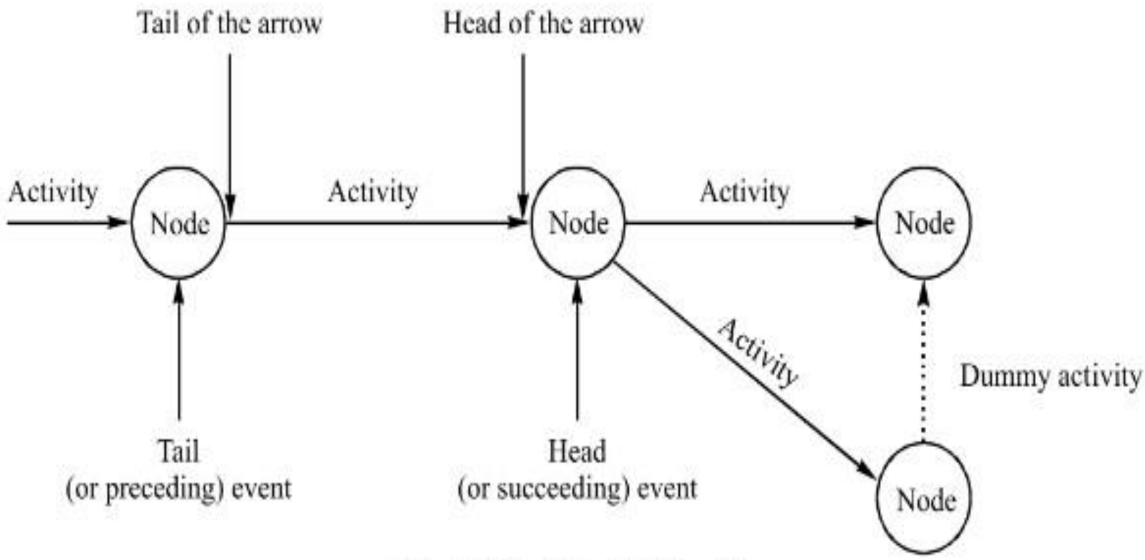
(ii) Successor Activity: An activity which starts immediately after one or more of other activities are completed is known as successor activity.

(iii) Dummy Activity: An activity which does not consume either any resource and/or time is known as dummy activity.

A dummy activity in the network is added only to establish the given precedence relationship among activities of the project. It is needed when (a) two or more parallel activities in a project have same head and tail events, or (b) two or more activities have some (but not all) of their immediate predecessor activities in common. A dummy activity is shown by a dotted line in the network diagram



(a) Activity-Node Relationship



(b) Activity-Node Relationship

Network models use the following two types of precedence network to show precedence requirements of the activities in the project.

Activity-on-Node (AON) network In this type of precedence network each node (or circle) represents a specific task while the arcs represent the ordering between tasks. AON network diagrams place the activities within the nodes, and the arrows are used to indicate sequencing requirements. Generally, these diagrams have no particular starting and ending nodes for the whole project. The lack of dummy activities in these diagrams always make them easier to draw and to interpret.

Activity-on-Arrow (AOA) network In this type of precedence network at each end of the activity arrow is a node (or circle). These nodes represent points in time or instants, when an activity is starting or ending.

The arrow itself represents the passage of time required for that activity to be performed.

These diagrams have a single beginning node from which all activities with no predecessors may start. The diagram then works its way from left to right, ending with a single ending node, where all activities with no followers come together. Three important advantages of using AOA are as follows:

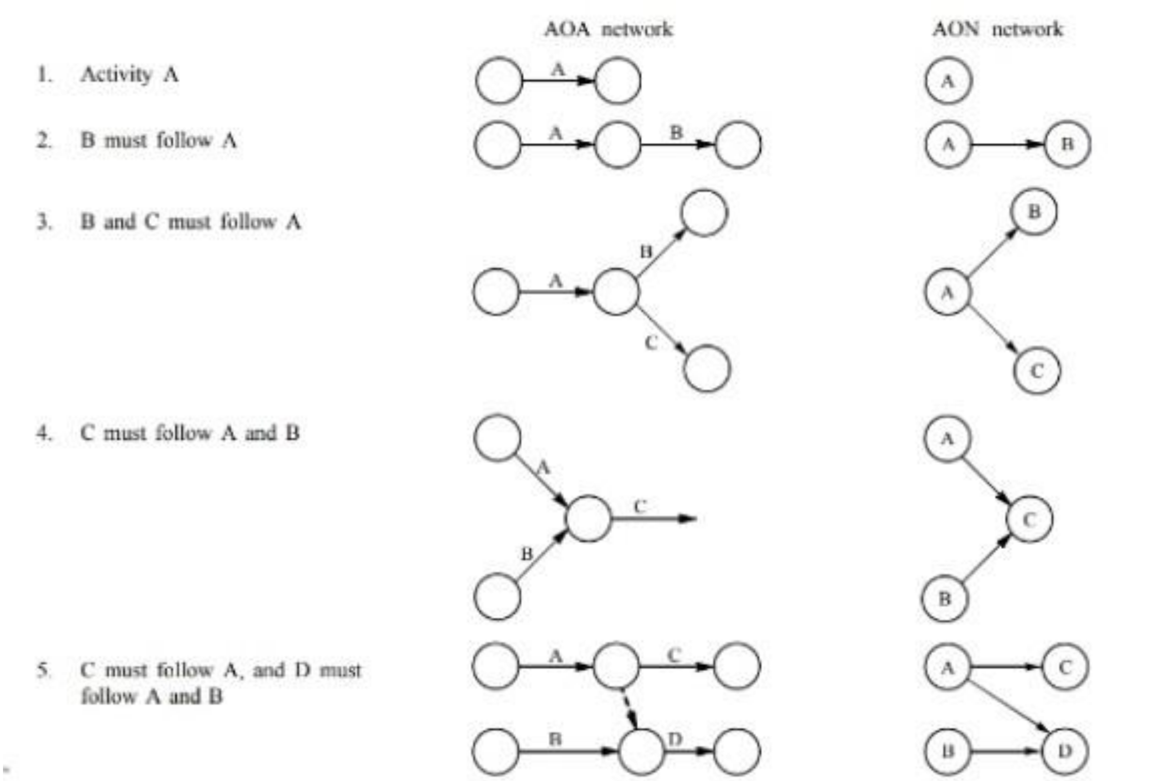
(i) Many computer programs are based on AOA network.

(ii) AOA diagrams can be superimposed on a time scale with the arrows drawn, the correct length to indicate the time requirement.

(iii) AOA diagrams give a better sense of the flow of time throughout a project. In this chapter, only AOA network diagrams will be used

4.4 Rules for AOA Network Construction

Following are some of the rules that have to be followed while constructing a network:



1. In network diagram, arrows represent activities and circles the events. The length of an arrow is of no significance.

2. Each activity should be represented only by one arrow and must start and end in a circle called *event*. The tail of an activity represents the start, and head the completion of work.

3. The event numbered 1 denotes the start of the project and is called *initial event*. All activities emerging (or taking off) from event 1 should not be preceded by any other activity or activities. An event carrying the highest number denotes the completion event. A network should have only one initial event and only one terminal event.

4. The general rule for numbering the event is that the head event should always be numbered larger than the number at its tail. That is, events should be numbered such that for each activity (i, j), $i < j$.

5. An activity must be uniquely identified by its starting and completion event, which implies that:

(a) An event number should not get repeated or duplicated.

(b) Two activities should not be identified by the same completion event.

(c) Activities must be represented either by their symbols or by the corresponding ordered pair of starting-completion events.

6. The logical sequence (or interrelationship) between activities must follow following rules:

(a) An event cannot occur until all its incoming activities have been completed.

(b) An activity cannot start unless all the preceding activities, on which it depends, have been completed.

(c) Though a dummy activity does not consume either any resource of time, even then it has to follow

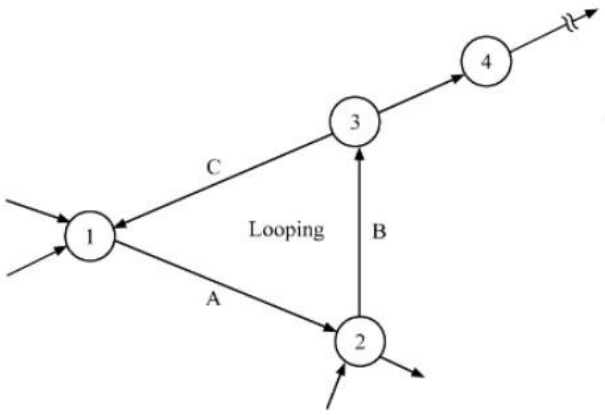
the rules 6(a) and (b).

Errors and Dummies in Network

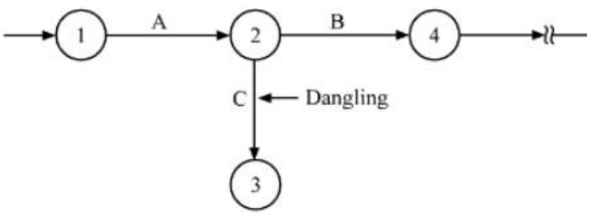
Looping and Dangling Looping (cycling) and dangling are considered as faults in a network. Therefore, these must be avoided.

(i) A case of endless loop in a network diagram, which is also known as *looping,*where activities *A*, *B* and *C* form a cycle.

Due to precedence relationships, it appears that every activity in looping (or cycle) is a predecessor of itself. In this case it is difficult to number three events associated with activity *A*, *B* and *C* so as to satisfy rule 6 of constructing the network.



(a) Looping

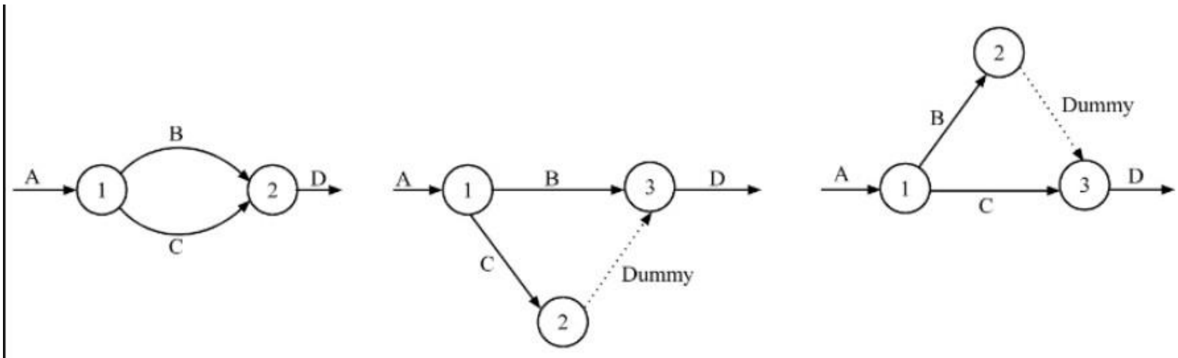


(b) Dangling

(ii) A case of disconnect activity before the completion of all activities, which is also known as dangling, .In this case, activity C does not give any result as per the rules of the network. The dangling may be avoided by adopting rule 5 of constructing the network.

Dummy (**or Redundant**) **Activity :** The following are the two cases in which the use of dummy activity may help in drawing the network correctly, as per the various rules.

(i) When two or more parallel activities in a project have the same head and tail events, i.e. two events are connected with more than one arrow.

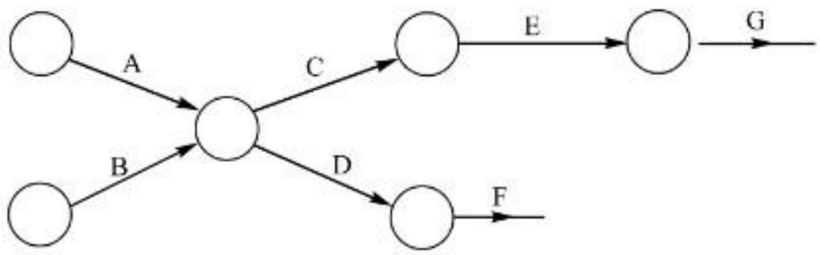


(a) Parallel Activities (b) Dummy Activity (c) Dummy Activity

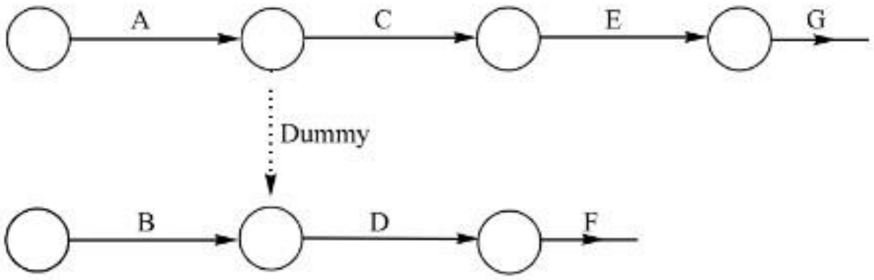
In activities B and C have a common predecessor – activity A. At the same time, they have activity D as a common successor. To arrive correct network, a dummy activity for the ending event B to show that D may not start before B and C, is completed. .

(ii) When two chains of activities have a common event, yet are completely or partly independent of each other, . A dummy which is used in such a case, to establish proper logical relationships, is also known as logic dummy activity.

if head event of C and D do not depend on the completion of activities A and B, then the network can be redrawn,. Otherwise, the pattern must be followed:



(a) Dependent Events



(b) Independent Events

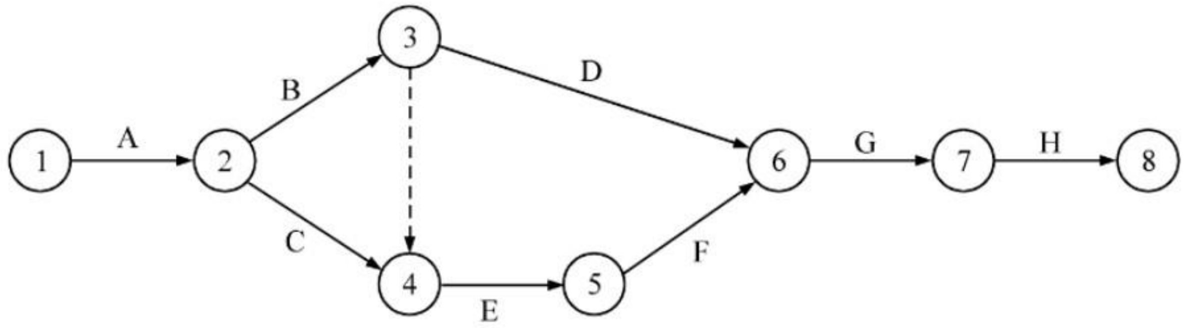
Example: 1

An assembly is to be made from two parts X and Y. Both parts must be turned on a lathe. Y must be polished whereas X need not be polished. The sequence of activities, together with their predecessors, is given below.

Activity	Description Predecessor	Activity
A	Open work order	–
B	Get material for X	A
C	Get material for Y	A
D	Turn X on lathe	B
E	Turn Y on lathe	B, C
F	Polish Y	E
G	Assemble X and Y	D, F
H	Pack	G

Draw a network diagram of activities for the project.

Solution : The network diagram for the project



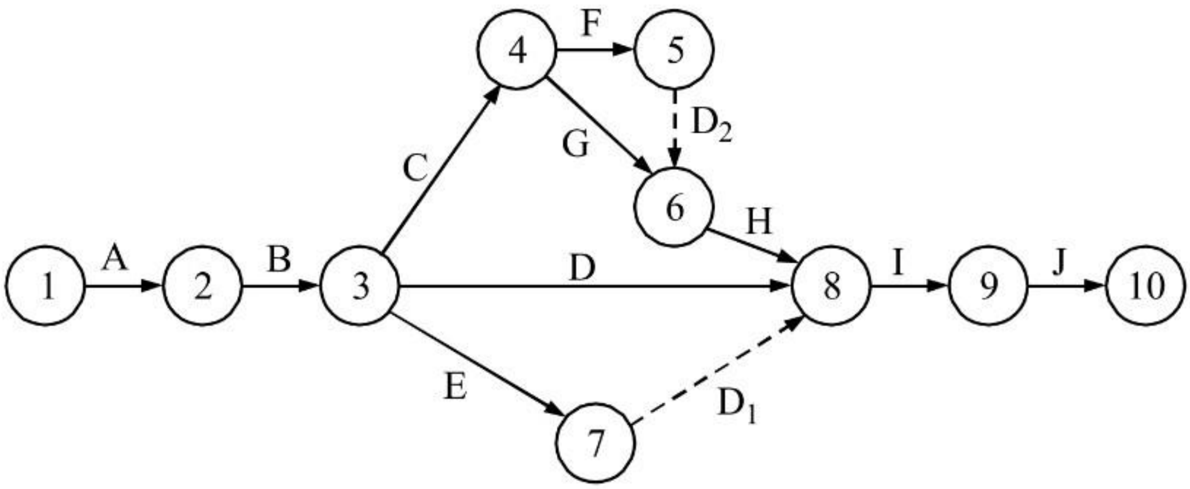
Example 2

Listed in the table are the activities and sequencing necessary for a maintenance job on the heat exchangers in a refinery.

Activity	Description Predecessor	Activity
A	Dismantle pipe connections	—
B	Dismantle heater, closure, and floating front	A
C	Remove tube bundle	B
D	Clean bolts	B
E	Clean heater and floating head front	B
F	Clean tube bundle	C
G	Clean shell	C
H	Replace tube bundle	F, G
I	Prepare shell pressure test	D, E, H
J	Prepare tube pressure test and reassemble	I

Draw a network diagram of activities for the project.

Solution The network diagram for the project



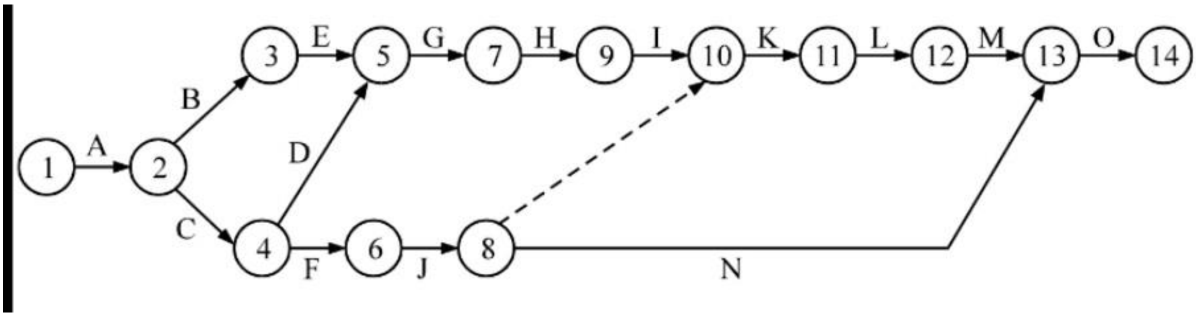
Example 3

Listed in the table are the activities and sequencing necessary for the completion of a recruitment procedure for management trainees (MT) in an organization.

Activity	Description Predecessor	Activity
A	Asking units for requirements	–
B	Ascertaining management trainees (MTs) requirements for commercial functions	A
C	Ascertaining MTs requirement for Accounts/Finance functions	A
D	Formulating advertisement for MT(A/C)	C
E	Formulating advertisement for MT (Commercial)	B
F	Calling applications from the successful candidates passing through the Institute of Chartered Accountants (ICA)	C
G	Releasing the advertisement	D, E
H	Completing applications received	G
I	Screening of applications against advertisement	H
J	Screening of applications received from ICA	F
K	Sending of personal forms	I, J
	Issuing interview/regret letters	K
M	Preliminary interviews	L
N	Preliminary interviews of outstanding candidates from ICA	J
O	Final interview	M, N

Draw a network diagram of activities for the project.

Solution The network diagram for the project



4.5 CRITICAL PATH ANALYSIS

The objective of critical path analysis is to estimate the total project duration and to assign starting and finishing times to all activities involved in the project. This helps to check the actual progress against the scheduled duration of the project.

The duration of individual activities may be uniquely determined (in case of CPM) or may involve the three time estimates (in case of PERT), out of which the expected duration of an activity is computed.

Having done this, the following factors should be known in order to prepare the project scheduling.

(i) Total completion time of the project.

(ii) Earlier and latest start time of each activity.

(iii) Critical activities and critical path.

(iv) Float for each activity, i.e. the amount of time by which the completion of a non-critical activity can be delayed, without delaying the total project completion time.

Consider the following notations for the purpose of calculating various times of events and activities.

Ei = Earliest occurrence time of an event, i. This is the earliest time for an event to occur when all the preceding activities have been completed, without delaying the entire project.

Li = Latest allowable time of an event, i. This is the latest time at which an event can occur without causing a delay in project's completion time.

ES_{ij} = Early starting time of an activity (i, j). This is the earliest time an activity should start without affecting the project completion.

LS_{ij} = Late starting time of an activity (i, j). This is the latest time an activity should start without delaying the project completion.

EF_{ij} = Early finishing time of an activity (i, j). This is the earliest time an activity should finish without affecting the project completion.

LF_{ij} = Late finishing time of an activity (i, j). This is the latest time an activity should finish without delaying the project completion.

t_{ij} = Duration of an activity (i, j).

As mentioned earlier, a network diagram should have only one initial event and one end event. The other events are numbered consecutively with integer 1, 2, . . ., n, such that $i < j$ for any two events i and j connected by an activity, which starts at i and finishes at j.

For calculating the earliest occurrence and latest allowable times for events, following two methods:

Forward Pass method and Backward Pass method are used:

Forward Pass Method (For Earliest Event Time)

In this method, calculations begin from the initial event 1, proceed through the events in an increasing order of event numbers and end at the final event, say *N.* At each event, its *earliest occurrence time* (*E*) and earliest start and finish time for each activity that begins at that event is calculated. When calculations end at the final event *N*, its earliest occurrence time gives the earliest possible completion time of the project.

The method may be summarized as follows:

1. Set the earliest occurrence time of initial event 1 to zero. That is, $E_1 = 0$, for $i = 1$.

2. Calculate the earliest start time for each activity that begins at event *i* (= 1). This is equal to the earliest occurrence time of event, *i* (tail event). That is:

$ES_{ij} = E_i$, for all activities (*i, j*) starting at event *i.*

3. Calculate the earliest finish time of each activity that begins at event *i.* This is equal to the earliest start time of the activity plus the duration of the activity. That is:

$EF_{ij} = ES_{ij} + t_{ij} = E_i + t_{ij}$, for all activities (*i, j*) beginning at event *i*.

4. Proceed to the next event, say *j*; $j > i$.

5. Calculate the earliest occurrence time for the event *j*. This is the maximum of the earliest finish times of all activities ending into that event, that is,

Ej = Max $\{EFij\}$ = Max $\{Ei + tij\}$, for all immediate predecessor activities.

6. If $j = N$ (final event), then earliest finish time for the project, that is, the earliest occurrence time EN for the final event is given by

EN = Max $\{EFij\}$ = Max $\{EN - 1 + tij\}$, for all terminal activities

Backward Pass Method (For Latest Allowable Event Time)

In this method, calculations begin from the final event N. Proceed through the events in the decreasing order of event numbers and end at the initial event 1. At each event, *latest occurrence time* (L) and latest finish and start time for each activity that is terminating at that event is calculated. The procedure continues till the initial event. The method may be summarized as follows:

1. Set the latest occurrence time of last event, N equal to its earliest occurrence time (known from forward pass method). That is, $LN = EN$, $j = N$.
2. Calculate the latest finish time of each activity which ends at event j. This is equal to latest occurrence time of final event. That is: $LFij = Li$, for all activities (i, j) ending at event j.
3. Calculate the latest start times of all activities ending at j. This is obtained by subtracting the duration of the activity from the latest finish time of the activity. That is:

$LFij = Lj$ and $LSij = LFij - tij = Lj - tij$, for all activity (i, j) ending at event j.

4. Proceed backward to the event in the sequence, that decreases j by 1.
5. Calculate the latest occurrence time of event i ($i < j$). This is the minimum of the latest start times of all activities from the event. That is:

Li = Min $\{LSij\}$ = Min $\{Lj - tij\}$, for all immediate successor activities.

6. If $j = 1$ (initial event), then the latest finish time for project, i.e. latest occurrence time $L1$ for the initial event is given by:

$L1$ = Min $\{LSij\}$ = Min$\{Lj - 1 - tij\}$, for all immediate successor activities.

Float (Slack) of an Activity and Event

The float (slack) or free time is the length of time in which a non-critical activity and/or an event can be delayed or extended without delaying the total project completion time.

Slack of an Event

The slack (or float) of an event is the difference between its latest occurrence time (Li) and its earliest occurrence time (Ei). That is: Event float = Li – Ei

It is a measure of how long an event can be delayed without increasing the project completion time.

(a) If L = E for certain events, then such events are called critical events.

(b) If $L \neq E$ for certain events, then the float (slack) on these events can be negative ($L < E$) or positive ($L > E$).

Slack of an Activity

It is the amount of activity time that can increased or delayed without delaying project completion time. This float is calculated as the difference between the latest finish time and the earliest finish time for the activity.

There are three types of floats for each non-critical activity in a project.

(a) Total float: This is the length of time by which an activity can be delayed until all preceding activities are completed at their earliest possible time and all successor activities can be delayed until their latest permissible time.

For each non-critical activity (i, j) the total float is equal to the latest allowable time for the event at the end of the activity minus the earliest time for an event at the begining of the activity minus the activity duration. That is:

Total float (TF_{ij}) = $(L_j - E_i) - t_{ij} = LS_{ij} - ES_{ij} = LF_{ij} - EF_{ij}$

(b) Free float: This is the length of time by which the completion time of any non-critical activity can be delayed without causing any delay in its immediate successor activities. The amount of free float time for a non-critical activity (i, j) is computed as follows:

Free float (FF_{ij}) = $(E_j - E_i) - t_{ij}$ = Min {ES_{ij} , for all immediate successors of activity (i, j)} – EF_{ij}

(c) Independent float: This is the length of time by which completion time of any non-critical activity (i, j) can be delayed without causing any delay in its predecessor or the successor activities. Independent float time for each non-critical activity is computed as follows:

Independent float (IF_{ij}) = $(E_j - L_i) - t_{ij} = \{ES_{ij} - LS_{ij}\} - t_{ij}$

The negative value of independent float is considered to be zero.

Remarks 1. Latest occurrence time of an event is always greater than or equal to its earliest occurrence time (i.e. $L_i \geq E_i$), $TF_{ij} \geq (L_j - E_i) - t_{ij}$

This implies that the value of free float may range from zero to total float but will not exceed total float value. That is, Independent float $\leq$ Free float $\leq$ Total float.

2. The calculation of various floats can help the decision-maker in identifying the underutilized resources, flexibility in the total schedule and possibilities of redeployment of resources.

3. Total float for a non-critical activity may be viewed as follows:

(a) **Negative (i.e. L – E < 0):** Project completion is behind the schedules date, i.e., resources are not adequate and activities may not finish in time. This needs extra resources or certain activities need crashing in order to reduce negative float value.

(b) **Positive (i.e. L – E > 0):** Project completion is ahead of the schedule date, i.e., resources are surplus. These resources can be deployed elsewhere or execution of the activities can be delayed.

(c) **Zero (i.e. L = E):** Resources are just sufficient for the completion of activities in a project. Any delay in activities execution will necessarily increase the project cost and time.

4.6 Critical Path – Various Floats

Certain activities in any project are called critical activities because delay in their execution will cause further delay in the project completion time. All activities having zero total float value are identified as critical activities, i.e., L = E

The critical path is the sequence of critical activities between the start event and end event of a project. This is critical in the sense that if execution of any activity of this sequence is delayed, then completion of the project will be delayed. A critical path is shown by a thick line or double lines in the network diagram.

The length of the critical path is the sum of the individual completion times of all the critical activities and defines the longest time to complete the project. The critical path in a network diagram can be identified as:

(i) If Ei-value and Lj-value for any tail and head events is equal, then activity (i, j) between such events is referred as critical, That is, Ej = Lj and Ei = Li.

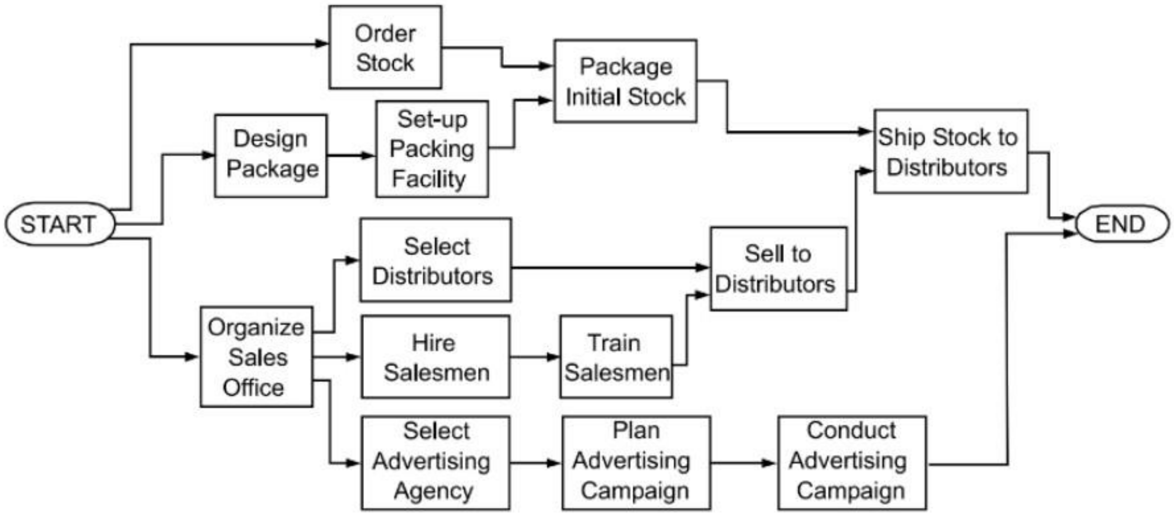
(ii) On critical path Ej – Ei = Lj – Li = tij.

Example 1

An established company has decided to add a new product to its line. It will buy the product from a manufacturing concern, package it, and sell it to a number of distributors that have been selected on a geographical basis. Market research has already indicated the volume expected and the size of sales force required. The steps shown in the following table are to be planned.

Activity	Description	Predecessors	Duration (days)
A	Organize sales office	–	6
B	Hire salesmen	A	4
C	Train salesmen	B	7
D	Select advertising agency	A	2
E	Plan advertising campaign	D	4
F	Conduct advertising campaign	E	10
G	Design package	–	2
H	Setup packaging facilities	G	10
I	Package initial stocks	J, H	6
J	Order stock from manufacturer	–	13
K	Select distributors	A	9
L	Sell to distributors	C, K	3
M	Ship stocks to distributors	I, L	5

The precedence relationship among these activities are shown in the following figure.



As the figure shows, the company can begin to organize the sales office, design the package, and order the stock immediately. Also the stock must be ordered and the packing facility must be set up before the initial stocks are packaged.

(a) Draw an arrow diagram for this project.

(b) Indicate the critical path.

(c) For each non-critical activity, find the total and free float.

Solution (a) The arrow diagram for the given project, along with E-values and L-values, is shown, Determine the earliest start time – Ei and the latest finish time – Lj for each event by proceeding as follows:

Forward Pass Method

$E1 = 0$ $E2 = E1 + t1, 2 = 0 + 6 = 6$

$E3 = E1 + t1, 3 = 0 + 2 = 2$ $E4 = Max \{Ei + ti, 4\}$

$E5 = E2 + t2, 5 = 6 + 4 = 10 = Max \{E1 + t14; E3 + t34\}$

$i = 1, 3= Max \{0 + 13, 2 + 10\} = 13$

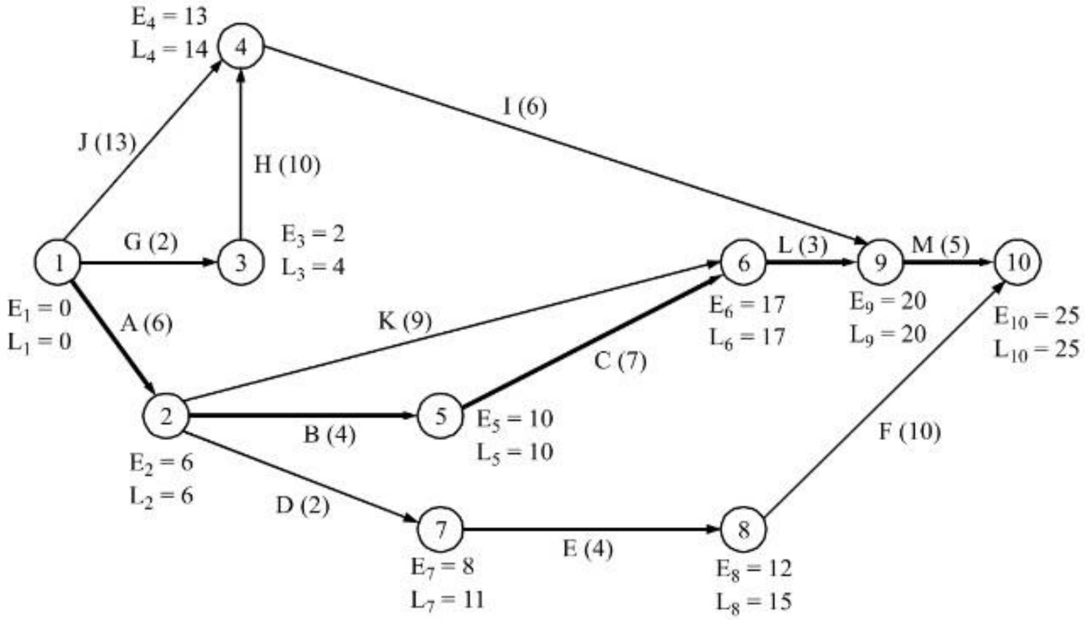
$E6 = Max \{Ei + ti, 6\} = Max \{E2 + t2, 6 ; E5 + t5, 6\}$ $E8 = E7 + t7, 8 = 8 + 4 = 12$

$i = 2, 5$ $E10 = Max \{Ei + ti,10\} = Max \{6 + 9; 10 + 7\} = 17$ $i = 8, 9$

$E7 = E2 + t2, 7 = 6 + 2 = 8 = Max \{E8 + t8, 10 ; E9 + t9, 10\}$

$E9 = Max \{Ei + ti, 9\} = Max \{E4 + t4, 9; E6 + t6, 9\} = Max \{12 + 10; 20 + 5\} = 25.$

$i = 4, 6 = Max \{13 + 6; 17 + 3\} = 20$



Backward Pass Method

L10 = E10 = 25 L9 = L10 – t9, 10 = 25 – 5 = 20

L8 = L10 – t8, 10 = 25 – 10 = 15 L7 = L8 – t7, 8 = 15 – 4 = 11

L6 = L9 – t6, 9 = 20 – 3 = 17 L5 = L6 – t5, 6 = 17 – 7 = 10

L4 = L9 – t4, 9 = 20 – 6 = 14 L3 = L4 – t3, 4 = 14 – 10 = 4

L2 = Min {Lj – t2, j} L1 = Min {Lj – t1, j}

j = 5, 6, 7 j = 2, 3, 4

= Min { L5 – t2, 5; L6 – t2, 6; L7 – t2, 7} = Min {L2 – t1, 2; L3 – t1, 3; L4 – t1, 4}

= Min {10 – 4; 17 – 9; 11 – 2} = 6 = Min {6 – 6; 4 – 2; 14 – 13} = 0

(b) The critical path in the network diagram has been shown. This has been done by double lines by joining all those events where E-values and L-values are equal.

The critical path of the project is: 1 – 2 – 5 – 6 – 9 – 10 and critical activities are A, B, C, L and M. The total project completion time is 25 weeks.

(c) For each non-critical activity, the total float and free float calculations are shown in table

Activity (i, j)	Duration (t_{ij})	Earliest Time Start (E_i)	Earliest Time Finish $(E_i + t_{ij})$	Latest Time Start $(L_j - t_{ij})$	Latest Time Finish L_j	Float Total $(L_j - t_{ij}) - E_i$	Float Free $(E_j - E_i) - t_{ij}$
1 – 3	2	0	2	2	4	2	0
1 – 4	13	0	13	1	14	1	0
2 – 6	9	6	15	8	17	2	2
2 – 7	2	6	8	9	11	3	0
3 – 4	10	2	12	4	14	2	1
4 – 9	6	13	19	14	20	1	1
7 – 8	4	8	12	11	15	3	0
8 – 10	10	12	22	15	25	3	3

Example 2

An insurance company has decided to modernize and refit one of its branch offices. Someof the existing office equipments will be disposed of but the remaining will be returned to the branch after the completion of the renovation work. Tenders are invited from a number of selected contractors.

The contractors would be responsible for all the activities in connection with the renovation work excepting the prior removal of the old equipment and its subsequent replacement.

The major elements of the project have been identified, as follows, along with their durations and immediately preceding elements.

Activity	Description	Duration	Immediate Predecessors
A	Design new premises	14	–
B	Obtain tenders from the contractors	4	A
C	Select the contractor	2	B
D	Arrange details with selected contractor	1	C
E	Decide which equipment is to be used	2	A
F	Arrange storage of equipment	3	E
G	Arrange disposal of other equipment	2	E
H	Order new equipment	4	E
I	Take delivery of new equipment	3	H, L
J	Renovations take place	12	K
K	Remove old equipment for storage or disposal	4	D, F, G
L	Cleaning after the contractor has finished	2	J
M	Return old equipment for storage	2	H, L

(a) Draw the network diagram showing the interrelations between the various activities of the project.

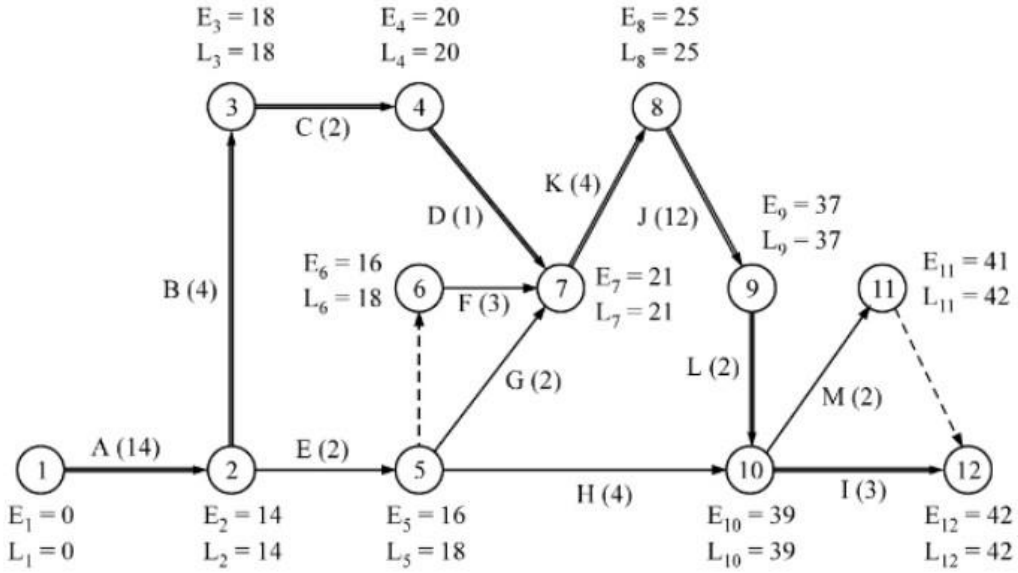
(b) Calculate the minimum time that the renovation can take from the design stage.

(c) Find the effect on the overall duration of the project if the estimates or tenders can be obtained in two weeks from the contractors by reducing their numbers.

(d) Calculate the 'independent float' that is associated with the non-critical activities in the network diagram.

Solution (a) The network diagram for the given project, along with *E*-values and *L*-values, is shown

5.12.



The critical path in the network diagram has been shown by double lines joining all those events where E-values and L-values are equal.

(b) The critical path of the project is: 1 – 2 – 3 – 4 – 7 – 8 – 9 – 10 – 12 and critical activities are A, B, C, D, K, J, L and I. The total project completion time is 42 weeks.

For non-critical activities, the total float, free float and independent float calculations are shown

Activity (i, j)	Duration (t_{ij})	Earliest Time		Latest Time		Float		
		Start (E_i)	Finish $(E_i + t_{ij})$	Start $(L_j - t_{ij})$	Finish L_j	Total $(L_j - t_{ij}) - E_i$	Free $(E_j - E_i) - t_{ij}$	Independe $(E_j - L_i) -$
2 – 5	2	14	16	16	18	2	0	0
6 – 7	3	16	19	18	21	2	2	0
5 – 7	2	16	18	19	21	3	3	1
5 – 10	4	16	20	35	39	19	19	17
10 – 11	2	39	41	40	42	1	0	0

(c) The effect on the overall project duration, if the time of activity B is reduced to 2 weeks instead of 4 weeks, is shown

	Path	Duration	
(i)	A – E – H – I	23	
(ii)	A – E – H – M	22	
(iii)	A – B – C – D – K – J – L – I (Critical path 42 weeks)	40	(New critical path)
(iv)	A – B – C – D – K – J – L – M (Critical path 41 weeks)	39	
(v)	A – E – G – K – J – L – I	39	
(vi)	A – E – G – K – J – L – M	39	
(vii)	A – E – F – K – J – L – I	40	(New critical path)
(viii)	A – E – F – K – J – L – M	39	

4.7 THREE TIMES ESTIMATES FOR PERT

Project Scheduling With Uncertain Activity Times

PERT was developed to handle projects where the time duration for each activity is not known with certainty but is a random variable that is characterized by □□(beta)-distribution. To estimate the parameters: mean and variance, of the □-distribution three time estimates for each activity are required to calculate its expected completion time. The three-time estimates that are required are as under.

(i) **Optimistic time** (**to or a**) The shortest possible time (duration) in which an activity can be performed assuming that everything goes well.

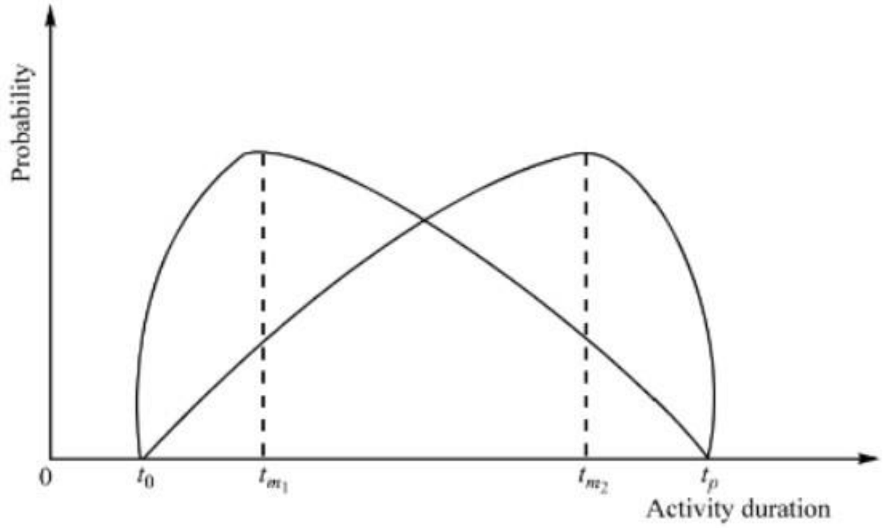
(ii) **Pessimistic time** (**tp or b**) The longest possible time required to perform an activity under extremely bad conditions. However, such conditions do not include natural calamities like earthquakes, flood, etc.

(iii) **Most likely time** (**tm or m**) The time that would occur most often to complete an activity, if the activity was repeated under exactly the same conditions many times. Obviously, it is the completion time that would occur most frequently (i.e. model value).

The □-distribution is not necessarily symmetric, the degree of skewness depends on the location of tm to to and tp. The range of optimistic time (to) and pessimistic time (tp) is assumed to enclose every possible duration of the activity. The most likely completion time (tm) for an activity may not be equal to the midpoint (to + tp)/2 and may occur to its left or to its right as shown

In Beta-distribution the midpoint (to + tp)/2 is given half weightage than that of most likely point (tm).

Thus, the expected or mean (te or □) time of an activity, that is also the weighted average of three time estimates, is computed as the arithmetic mean of (to + tp)/2 and 2 tm. That is:



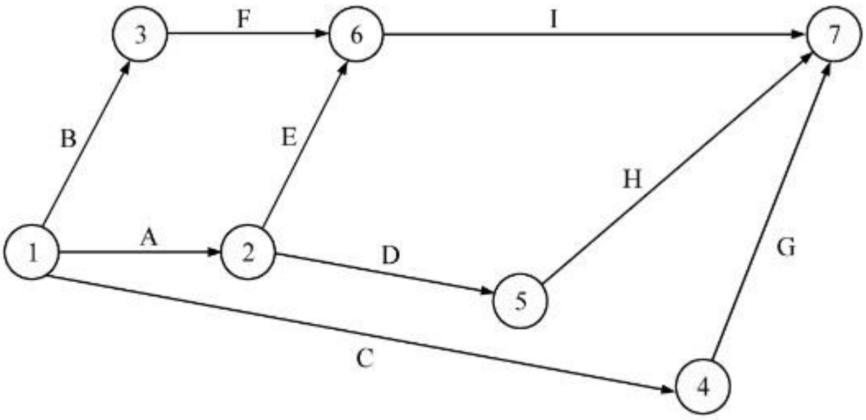
$$\text{Expected time of an activity } (t_e) = \frac{(t_o + t_p)/2 + 2\,t_m}{3} = \frac{t_o + 4\,t_m + t_p}{6}$$

If duration of activities associated with the project is uncertain, then variance describes the dispersion (variation) in the activity time values. The calculations are based on the concept of normal distribution where 99 per cent of the area under normal curve falls within from the mean or fall within the range approximately 6 standard deviation in length. Therefore, the interval (to, tp) or range (tp – to) is assumed to enclose about 6 standard deviations of a symmetric distribution. Thus, is the standard deviation of the duration of activity.

Example 1

The following network diagram represents activities associated with a project:

Activities	:	A	B	C	D	E	F	G	H	I
Optimistic time, t_0	:	5	18	26	16	15	6	7	7	3
Pessimistic time, t_p	:	10	22	40	20	25	12	12	9	5
Most likely time, t_m	:	8	20	33	18	20	9	10	8	4



Determine the following:

(a) Expected completion time and variance of each activity

(b) The earliest and latest expected completion times of each event.

(c) The critical path.

(d) The probability of expected completion time of the project if the original scheduled time of completing the project is 41.5 weeks.

(e) The duration of the project that will have 95 per cent chance of being completed

Solution (a) Calculations for expected completion time (*te*) of an activity and variance (□2), using following formulae

$$t_e = \frac{1}{6}(t_o + 4t_m + t_p) \quad \text{and} \quad \sigma_i^2 = \left\{\frac{1}{6}(t_p - t_o)\right\}^2$$

The earliest and latest expected completion time for all events considering the expected completion time of each activity

Activity	t_o	t_p	t_m	$t_e = \frac{1}{6}(t_o + 4t_m + t_p)$	$\sigma^2 = [\frac{1}{6}(t_p - t_o)]^2$
1 – 2	5	10	8	7.8	0.696
1 – 3	18	22	20	20.0	0.444
1 – 4	26	40	33	33.0	5.429
2 – 5	16	20	18	18.0	0.443
2 – 6	15	25	20	20.0	2.780
3 – 6	6	12	9	9.0	1.000
4 – 7	7	12	10	9.8	0.694
5 – 7	7	9	8	8.0	0.111
6 – 7	3	5	4	4.0	0.111

Forward Pass Method

E1 = 0 E2 = E1 + t1, 2 = 0 + 7.8 = 7.8

E3 = E1 + t1, 3 = 0 + 20 = 20 E4 = E1 + t1, 4 = 0 + 33 = 33

E5 = E2 + t2, 5 = 7.8 + 18 = 25.8 E6 = Max {Ei + ti, 6} = Max {E2 + t2, 6 ; E3 + t3, 6}

E7 = Max {Ei + ti,7} = Max {7.8 + 20; 20 + 9} = 29 = Max {E4 + t4, 7 ; E5 + t5, 7 ; E6 + t6, 7} = Max {33 + 9.8 ; 25.8 + 8; 29 + 4} = 42.8

Backward Pass Method

L7 = E7 = 42.8 L6 = L7 – t6, 7 = 42.8 – 4 = 38.8

L5 = L7 – t5, 7 = 42.8 – 8 = 34.8 L4 = L7 – t4, 7 = 42.8 – 9.8 = 33

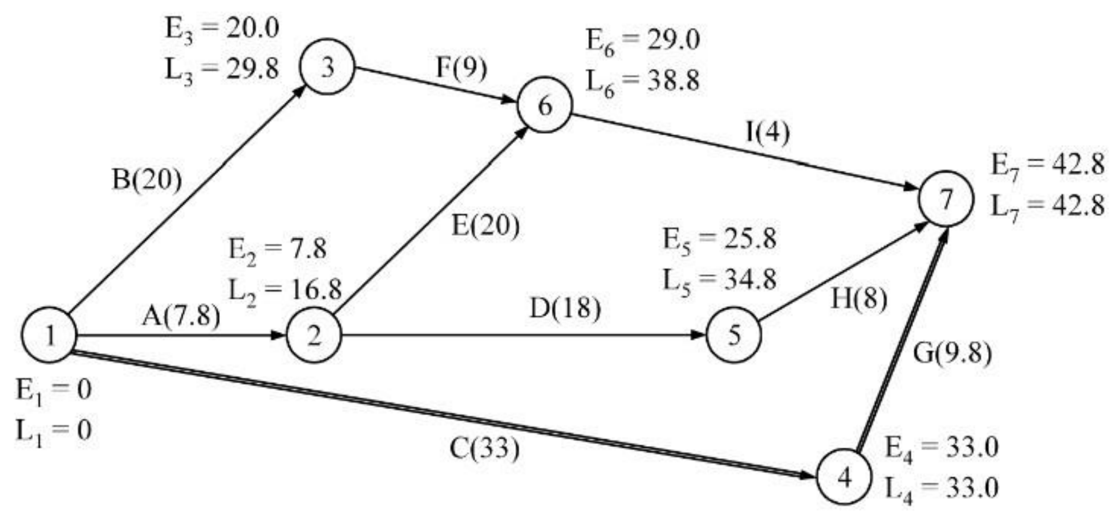
L3 = L6 – t3, 6 = 38.8 – 9 = 29.8 L1 = Min {Lj – t1j}

L2 = Min {Lj – t2, j} = Min {L2 – t1, 2; L3 – t1, 3 ; L4 – t1, 4}

= Min {L5 – t2, 5 ; L6 – t2, 6 } = Min {16.8 – 7.8 ; 29.8 – 20 ; 33 – 33} = 0

= Min {34.8 – 18 ; 38.8 – 20} = 16.8

The E-value and L-values are



(c) The critical path is shown by thick line in Fig. 13.14 where E-values and L-values are the same. The critical path is: 1 – 4 – 7 and the expected completion time for the project is 42.8 weeks.

(d) Expected length of critical path, Te = tC + tG = 33 + 9.8 = 42.8 weeks (Project duration).

Variance of critical path length, 2 = C , 2 + G

2 = 5.429 + 0.694 = 6.123 weeks.

Example 2

A small project involves 7 activities, and their time estimates are listed in the following table. Activities are identified by their beginning (i) and ending (j) node numbers.

Activity Estimated Duration (weeks)

(i – j)	Optimistic	Most Likely	Pessimistic
1 – 2	1	1	7
1 – 3	1	4	7
1 – 4	2	2	8
2 – 5	1	1	1
3 – 5	2	5	14
4 – 6	2	5	8
5 – 6	3	6	15

(a) Draw the network diagram of the activities in the project.

(b) Find the expected duration and variance for each activity. What is the expected project length?

(c) Calculate the variance and standard deviation of the project length. What is probability that the project

will be completed:

(i) at least 4 weeks earlier than expected time.

(ii) no more than 4 weeks later than expected time.

(d) If the project due date is 19 weeks, what is the probability of not meeting the due date.

Given: Z : 0.50 0.67 1.00 1.33 2.00

Prob. : 0.3085 0.2514 0.1587 0.0918 0.0228

Solution The network diagram of activities in the project is shown . The earliest and latest expected time for each event is calculated by considering the expected time of each activity

Activity	t_o	t_m	t_p	$t_e = \frac{1}{6}(t_o + 4t_m + t_p)$	$\sigma^2 = [\frac{1}{6}(t_p - t_o)]^2$
1 – 2	1	1	7	2	1
1 – 3	1	4	7	4	1
1 – 4	2	2	8	3	1
2 – 5	1	1	1	1	0
3 – 5	2	5	14	6	4
4 – 6	2	5	8	5	1
5 – 6	3	6	15	7	4

The E-values and L-values based on expected time (te) of each activity

(a) Critical path is: 1 – 3 – 5 – 6.

(b) The expected duration and variance for each activity is shown . The expected project length is the sum of the duration of each critical activity:

Expected project length = 1 – 3 – 5 – 6 = 4 + 6 + 7 = 17 weeks

(c) Variance of the project length is the sum of the variances of each critical activity:

Variance of project length = 1 – 3 – 5 – 6 = 1 + 4 + 4 = 9 weeks

(i) Probability that the project will be completed at least 4 weeks earlier (i.e. 13 weeks) than the expected project duration of 17 weeks is given by

Prob. {Z ≤: – 1.33} = 0.5 – 0.4082 = 0.0918

Thus the probability of completing the project in less than 13 days is 9.18 per cent.

(ii) Probability that the project will be completed in 4 weeks later (i.e. 21 weeks) than expected project duration of 17 weeks is given by\

P(Z ≥ ; 1.33) = 0.5 + 0.4082 = 0.9082

4.8 Self Assessment Questions : **Part - A**

1. In PERT, the expected time (TE) is calculated using which formula?

A) (O + P) / 2

B) (O + 4M + P) / 6

C) (O + M + P) / 3

D) (M + P) / 2

Answer: B) (O + 4M + P) / 6

2. CPM is most suitable for which type of projects?

A) Research-based projects

B) Projects with uncertain activities

C) Well-defined and repetitive projects

D) Creative and innovative tasks

Answer: C) Well-defined and repetitive projects

3. Which of the following is a characteristic of PERT?

A) Deterministic activity times

B) Focuses on cost optimization

C) Probabilistic activity times

D) No estimation of time

Answer: C) Probabilistic activity times

4. The critical path in a network diagram is the path that:

A) Has the least total float

B) Has the maximum slack

C) Is the longest duration path

D) Is the shortest duration path

Answer: C) Is the longest duration path

5. Float or slack in project management refers to:

A) The time taken to float resources

B) The delay that will not affect the project completion time

C) Time gained by speeding up tasks

D) Time lost due to delays

Answer: B) The delay that will not affect the project completion time

6. In CPM, activities with zero total float are:

A) Least important

B) On the critical path

C) Independent tasks

D) Dummy activities

Answer: B) On the critical path

7. Which of the following is NOT an assumption of PERT?

A) Time estimates are uncertain

B) There is only one time estimate per activity

C) Events follow a logical sequence

D) Activity time estimates are based on beta distribution

Answer: B) There is only one time estimate per activity

8. Total float is defined as:

A) Time by which an activity can be delayed without delaying the project

B) Time by which an activity can be delayed without delaying subsequent activities

C) Time to complete an activity

D) Extra resources required for an activity

Answer: A) Time by which an activity can be delayed without delaying the project

9. Free float is the time:

A) An activity can be delayed without delaying the early start of any successor

B) An activity can be delayed without delaying the project

C) By which the project can be preponed

D) Between the earliest and latest start times

Answer: A) An activity can be delayed without delaying the early start of any successor

10. Which technique is used when activity durations are uncertain?

A) CPM

B) Gantt Chart

C) PERT

D) Line of Balance

Answer: C) PERT

Part - B

1.Briefly mention the areas of application of network techniques.

2. Compare and contrast CPM and PERT. Under what conditions would you recommend the scheduling by PERT? Justify your
answer with reasons.

3. Discuss the various steps involved in the applications of PERT and CPM.

4. Highlight the difficulties encountered in using network techniques.

5. Explain the basic logic of arrow networks.

6. Explain the reasons for incorporating dummy activities in a network diagram. In what way do these differ from the normal activities?

7. Draw a network diagrams from the following list of activities:

Activity			*Predecessor Activity*	
	Set 1	*Set 2*	*Set 3*	*Set 4*
A	–	–	–	–
B	–	–	–	*A*
C	–	–	–	*A*
D	*A*	*A*	*A*	*B*
E	*B*	*A,B*	*A,B*	*B*
F	*B, C*	*A, B, C*	*B, C*	*D, E*
G	*D, E, F*	*D, E, F*	*C*	*D*
H	*E, F*	*F*	*D, E,F*	*C, F, G*

8. A new type of water pump is to be designed for an automobile. Its major specifications are given in the table below. Draw the network diagram of activates involved in the project.

Activity	*Description*	*Predecessor Activity*
A	Drawing prepared and approved	–
B	Cost analysis	*A*
C	Tool feasibility (economics)	*A*
D	Tool manufactured	*C*
E	Favourable cost	*B, C*
F	Raw materials procured	*D, E*
G	Subassemblies ordered	*E*

H	Subassemblies received	*G*
I	Parts manufactured	*D*, *F*
J	Final assembly	*I*, *H*
K	Testing and shipment	*J*

9. Explain the following terms in PERT/CPM.

(i) Earliest time, (ii) Latest time, (iii) Total activity time, (iv) Even slack, and (v) Critical path.

10.. (a) What is float? What are the different types of floats?

(b) Discuss in brief: (i) total float, and (ii) free float. Also explain their uses in network.

UNIT V

Structure :

5.1 Game Theory - Introduction

In general, the term 'game' refers to a situation of conflict and competition in which two or more competitors (or participants) are involved in the decision-making process in anticipation of certain outcomes over a period of time. The competitors are referred to as players. A player may be an individual, individuals, or an organization. A few examples of competitive and conflicting decision environment, that involve the interaction between two or more competitors are: Pricing of products, where sale of any product is determined not only by its price but also by the price set by competitors for a similar product

The success of any TV channel programme largely depends on what the competitors presence in the same time slot and the programme they are telecasting.

The success of a business strategy depends on the policy of internal revenue service regarding the expenses that may be disallowed,

The success of an advertising/marketing campaign depends on various types of services offered to the customers.

For academic interest, theory of games provides a series of mathematical models that may be useful in explaining interactive decision-making concepts, where two or more competitors are involved under conditions of conflict and competition. However, such models provide an opportunity to a competitor to evaluate not only his personal decision alternatives (courses of action), but also the evaluation of the competitor's possible choices in order to win the game.

Game theory came into existence in 20th Century. However, in 1944 John Von Neumann and Oscar Morgenstern published a book named Theory of Games and Economic Behavior, in which they discussed how businesses of all types may use this technique to determine the best strategies given a competitive business environment. The author's approach was based on the principle of 'best out of the worst'. The models in the theory of games can be classified based on the following factors:

Number of players If a game involves only two players (competitors), then it is called a two-person game. However, if the number of players are more, the game is referred to as n-person game.

Sum of gains and losses If, in a game, the sum of the gains to one player is exactly equal to the sum of losses to another player, so that, the sum of the gains and losses equals zero, then the game is said to be a zero-sum game. Otherwise it is said to be non-zero sum game.

Strategy The strategy for a player is the list of all possible actions (moves, decision alternatives or courses of action) that are likely to be adopted by him for every payoff (outcome). It is assumed that the players are aware of the rules of the game governing their decision alternatives (or strategies). The outcome resulting from a particular strategy is also known to the players in advance and is expressed in terms of numerical values (e.g. money, per cent of market share or utility). The particular strategy that optimizes a player's gains or losses, without knowing the competitor's strategies, is called optimal strategy. The expected outcome, when players use their optimal strategy, is called value of the game.

Generally, the following two types of strategies are followed by players in a game:

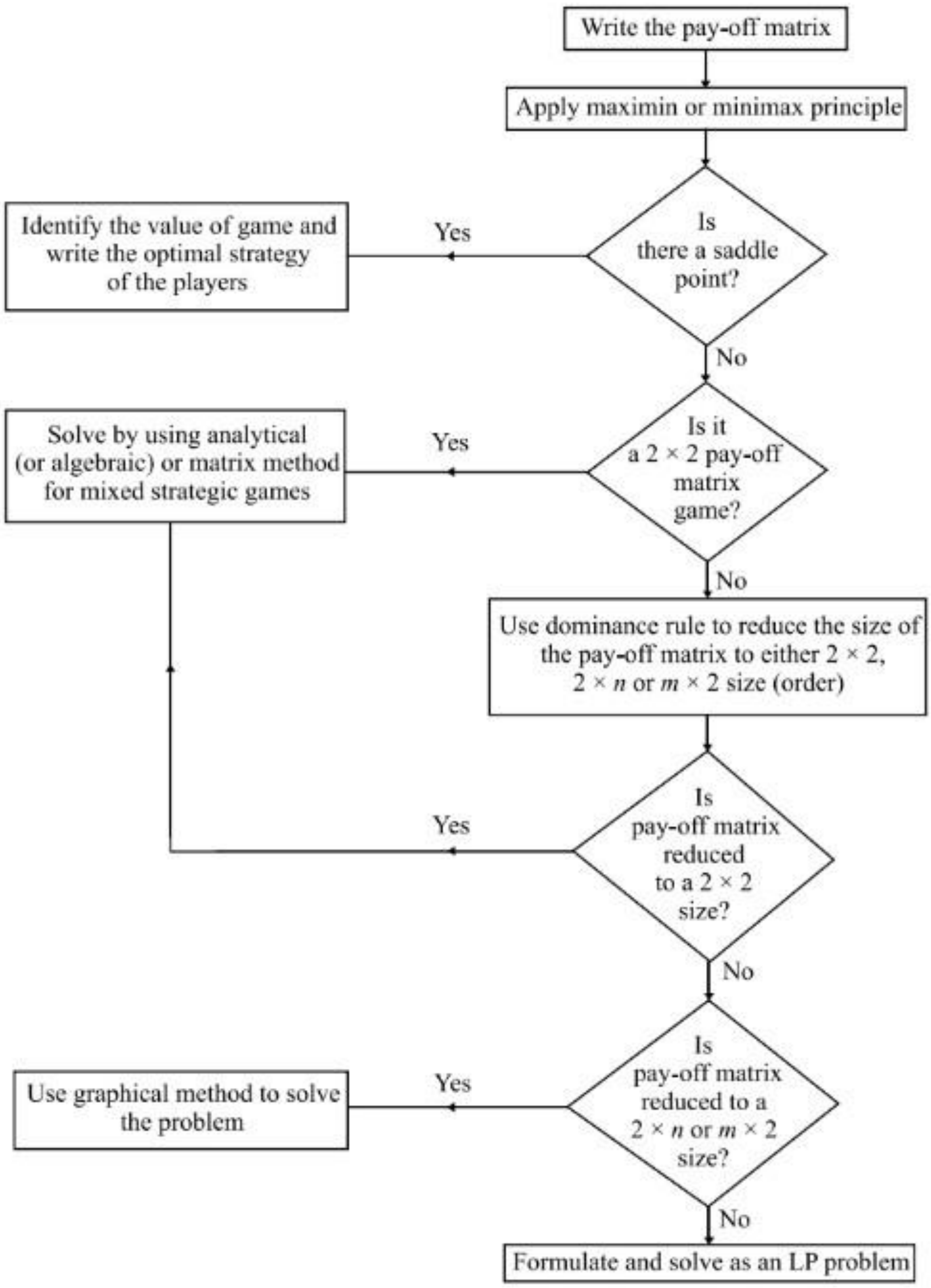
(a) Pure Strategy A particular strategy that a player chooses to play again and again regardless of other player's strategy, is referred as pure strategy. The objective of the players is to maximize their gains or minimize their losses.

(b) Mixed Strategy A set of strategies that a player chooses on a particular move of the game with some fixed probability are called mixed strategies. Thus, there is a probabilistic situation and objective of the each player is to maximize expected gain or to minimize expected loss by making the choice among pure strategies with fixed probabilities.

Mathematically, if p_j ($j = 1, 2, \ldots, n$) is the probability associated with a pure strategy j to be chosen by a player at any point in time during the game, then the set S of n non-negative real numbers (probabilities) whose sum is unity associated with pure strategies of the player is written as: S = { p1, p2,

. . ., pn} where p1 + p2 + . . . + pn = 1 and pj ≥□0 of all j.

Remark If a particular pj = 1 (j = 1, 2, . . ., n) and all others are zero, the player is said to select pure strategy j. A flow chart of using game theory approach to solve a problem



TWO-PERSON ZERO-SUM GAMES

A game with only two players, say A and B, is called a two-person zero-sum game, only if one player's gain is equal to the loss of other player, so that total sum is zero.

Payoff matrix The payoffs (a quantitative measure of satisfaction that a player gets at the end of the play) in terms of gains or losses, when players select their particular strategies (courses of action), can be represented in the form of a matrix, called the payoff matrix. Since the game is zero-sum, the gain of one player is equal to the loss of other and vice versa. In other words, one player's payoff table would contain the same amounts in payoff table of other player, with the sign changed. Thus, it is sufficient to construct a payoff table only for one of the players.

Player A's Strategies	Player B's Strategies			
	B_1	B_2	...	B_n
A_1	a_{11}	a_{12}	...	a_{1n}
A_2	a_{21}	a_{22}	...	a_{2n}
$\vdots$	$\vdots$	$\vdots$		$\vdots$
A_m	a_{m1}	a_{m2}	...	a_{mn}

Since player A is assumed to be the gainer, therefore he wishes to gain as large a payoff aij as possible, player B on the other hand would do his best to reach as small a value of aij as possible. Of course, the gain to player B and loss to A must be – aij.

Various methods discussed in this chapter to find value of the game under decision-making environment of certainty are as follows:

Assumptions of the game

1. Each player has available to him a finite number of possible strategies (courses of action). The list may not be the same for each player.
2. Players act rationally and intelligently.
3. List of strategies of each player and the amount of gain or loss on an individual's choice of strategy is known to each player in advance.
4. One player attempts to maximize gains and the other attempts to minimize losses.
5. Both players make their decisions individually, prior to the play, without direct communication between them.
6. Both players select and announce their strategies simultaneously so that neither player has an advantage resulting from direct knowledge of the other player's decision.
7. The payoff is fixed and determined in advance.

5.2 PURE STRATEGIES (MINIMAX AND MAXIMIN PRINCIPLES: GAMES WITH SADDLE POINT

The selection of an optimal strategy by each player, without the knowledge of the competitor's strategy, is the basic problem of playing games. Since the payoffs for either player provides all the essential information, therefore, only one player's payoff table is required to evaluate the decisions. By convention, the payoff table for the player whose strategies are represented by rows (say player A) is constructed. The objective of the study is to know how these players must select their respective strategies so that they are able to optimize their payoff. Such a decision-making criterion is referred to as the minimax-maximin principle. Such principle in pure strategies game always leads to the best possible selection of a strategy for both players.

Maximin principle For player A the minimum value in each row represents the least gain (payoff) to him, if he chooses his particular strategy. These are written in the matrix by row minima. He will then select the strategy that gives the largest gain among the row minimum values. This choice of player A is called the maximin principle, and the corresponding gain is called the maximin value of the game.

Minimax principle For player B, who is assumed to be the looser, the maximum value in each column represents the maximum loss to him, if he chooses his particular strategy. These are written in the payoff matrix by column maxima. He will then select the strategy that gives the minimum loss among the column maximum values. This choice of player B is called the minimax principle, and the corresponding loss is the minimax value of the game.

Optimal strategy A course of action that puts any player in the most preferred position, irrespective of the course of action his competitor(s) adopt, is called as optimal strategy. In other words, if the maximin value equals the minimax value, then the game is said to have a saddle (equilibrium) point and the corresponding strategies are called optimal strategies.

Value of the game This is the expected payoff at the end of the game, when each player uses his optimal strategy, i.e. the amount of payoff, V, at an equilibrium point. A game may have more than one saddle points. A game with no saddle point is solved by choosing strategies with fixed probabilities.

Remarks 1. The value of the game, in general, satisfies the equation: maximin value ≤ V ≤ minimax value.

2. A game is said to be a fair game if the lower (maximin) and upper (minimax) values of the game are equal and both equals zero.

3. A game is said to be strictly determinable if the lower (maximin) and upper (minimax) values of the game are equal and both equal the value of the game.

Rules to Determine Saddle Point

The reader is advised to follow the following three steps, in this order, to determine the saddle point in the payoff matrix.

1. Select the minimum (lowest) element in each row of the payoff matrix and write them under 'row minima' heading. Then, select the largest element among these elements and enclose it in a rectangle, .

2. Select the maximum (largest) element in each column of the payoff matrix and write them under 'column maxima' heading. Then select the lowest element among these elements and enclose it in a circle, .

3. Find out the element(s) that is same in the circle as the well as rectangle and mark the position of such element(s) in the matrix. This element represents the value of the game and is called the saddle (or equilibrium) point.

Example 1

For the game with payoff matrix:

	Player B	
Player B	B1	B2
A1	-1	2
A2	6	4

Determine the optimal strategies for players A and B. Also determine the value of game. Is this game (i) fair? (ii) strictly determinable?

Solution In this example, gains to player A or losses to player B are represented by the positive quantities, whereas, losses to A and gains to B are represented by negative quantities. It is assumed that A wants to maximize his minimum gains from B. Since the payoffs given in the matrix are what A receives, therefore, he is concerned with the quantities that represent the row minimums. Now A can do no worse than receive one of these values. The best of these values occurs when he chooses strategy A1. This choice provides a payoff of – 2 to A when B chooses strategy B3. This refers to A's choice of A1 as his maximum payoff strategy because this row contains the maximum of A's minimum possible payoffs from his competitor B.

Player A	Player B B_1	B_2	B_3	Row minimum	
A_1	–1	2	– 2	– 2	← *Maximin*
A_2	6	4	– 6	– 6	
Column maximum	6	4	– 2	← *Minimax*	

Similarly, it is assumed that B wants to minimize his losses and wishes that his losses to A be as small as possible. The column maximums also represent the greatest payments B might have to make to A. The smallest of these losses is – 2, which occurs when A chooses his course of action, A1 and B chooses his course of action, B3. This choice of B3 by B is his minimax loss strategy because the amount of this column is the minimum of the maximum possible losses. The quantity – 2 in row A1 and column B3 is enclosed both in the box and the circle. That is, it is both the minimum of the column maxima and the maximum of the row minima. This value is referred to as saddle point.

The payoff amount in the saddle-point position is also called value of the game. For this game, value of the game is, V = – 2, for player A. The value of game is always expressed from the point of view of the player whose strategies are listed in the rows.

The game is strictly determinable. Also since the value of the game is not zero, the game is not fair.

Example 2

A company management and the labour union are negotiating a new three year settlement. Each of these has 4 strategies:

I : Hard and aggressive bargaining II : Reasoning and logical approach

III : Legalistic strategy IV : Conciliatory approach

The costs to the company are given for every pair of strategy choice.

Company Strategies

Union Strategies	I	II	III	IV
I	20	15	12	35
II	25	14	8	10
III	40	2	10	5
IV	– 5	4	11	0

What strategy will the two sides adopt? Also determine the value of the game.

Solution Applying the rule of finding out the saddle point, we obtain the saddle point that is enclosed both in a circle and a rectangle,

	Company Strategies				
Union Strategies	*I*	*II*	*III*	*IV*	Row minimum
I	20	15	12	35	12 ← *Maximin*
II	25	14	8	10	8
III	40	2	10	5	2
IV	– 5	4	11	0	– 5
Column maximum	40	15	12	35	
			↑ *Minimax*		

Since Maximin = Minimax = Value of game = 12, therefore the company will
always adopt strategy III – Legalistic strategy and union will always adopt strategy I – Hard and aggressive bargaining.

5.3 MIXED STRATEGIES: GAME WITHOUT SADDLE POINT

In certain cases, no saddle point exists, i.e. maximin value ≠ minimax value. In all such cases, players must choose the mixture of strategies to find the value of game and an optimal strategy.

The value of game obtained by the use of mixed strategies represents the least payoff, which player A can expect to win and the least which player B can expect to lose. The expected payoff to a player in a game with payoff matrix [aij] of order m × n is defined as:

where **P** = (p1, p2, . . ., pm) and **Q** = (q1, q2, . . ., qn) denote probabilities (or relative frequency with which a strategy is chosen from the list of strategies) associated with m strategies of player A and n strategies of player, B respectively, where p1 + p2 + . . . + pm = 1 and q1 + q2 + . . . + qn = 1.

A mixed strategy game can be solved by using following methods:

- Algebraic method
- Analytical or calculus method
- Matrix method
- Graphical method, and
- Linear programming method.

Remark For solving a 2 × 2 game, without a saddle point, the following formula is also used. If payoff matrix for player *A* is given by:

$$\text{Player } A \overset{\text{Player } B}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}$$

then the following formulae are used to find the value of game and optimal strategies:

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - (a_{12} + a_{21})}; \quad q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$$

where $p_2 = 1 - p_1; \quad q_2 = 1 - q_1$

and $V = \dfrac{a_{11}a_{22} - a_{21}a_{12}}{a_{11} + a_{22} - (a_{12} + a_{21})}$

5.4 THE RULES (PRINCIPLES) OF DOMINANCE

The rules of dominance are used to reduce the size of the payoff matrix. These rules help in deleting certain rows and/or columns of the payoff matrix that are inferior (less attractive) to at least one of the remaining rows and/or columns (strategies), in terms of payoffs to both the players. Rows and/or columns once deleted can never be used for determining the optimum strategy for both the players.

The rules of dominance are especially used for the evaluation of two-person zero-sum games without a saddle (equilibrium) point. Certain dominance principles are stated as follows:

1. For player B, who is assumed to be the loser, if each element in a column, say Cr is greater than or equal to the corresponding element in another column, say Cs in the payoff matrix, then the column Cr is said to be dominated by column Cs and therefore, column Cr can be deleted from the payoff matrix.

In other words, player B will never use the strategy that corresponds to column Cr because he will loose more by choosing such strategy.

2. For player A, who is assumed to be the gainer, if each element in a row, say Rr, is less than or equal to the corresponding element in another row, say Rs, in the payoff matrix, then the row Rr is said to be dominated by row Rs and therefore, row Rr can be deleted from the payoff matrix. In other words, player A will never use the strategy corresponding to row Rr, because he will gain less by choosing such a strategy.

3. A strategy say, k can also be dominated if it is inferior (less attractive) to an average of two or more other pure strategies. In this case, if the domination is strict, then strategy k can be deleted. If strategy k dominates the convex linear combination of some other pure strategies, then one of the pure strategies involved in the combination may be deleted. The domination would be decided as per rules 1 and 2 above.

Remark Rules (principles) of dominance discussed are used when the payoff matrix is a profit matrix for the player A and a loss matrix for player B. Otherwise the principle gets reversed.

Example 1

Players A and B play a game in which each has three coins, a 5p, 10p and a 20p. Each selects a coin without the knowledge of the other's choice. If the sum of the coins is an odd amount, then A wins B's coin. But, if the sum is even, then B wins A's coin. Find the best strategy for each player and the values of the game.

Solution The payoff matrix for player A is

	Player B		
Player A	5p : B1	10p : B2	20p :B3
5p:A1	-5	10	20
10p :A2	5	-10	-10
20p: A3	5	-20	-20

It is clear that this game has no saddle point. Therefore, further we must try to reduce the size of the given payoff matrix as further as possible. Note that every element of column B3 (strategy B3 for player B) is more than or equal to every corresponding element of row B2 (strategy B2 for player B). Evidently, the choice of strategy B3, by the player B, will always result in more losses as compared to that of selecting the strategy B2. Thus, strategy B3 is inferior to B2. Hence, delete the B3 strategy from the payoff matrix. The
reduced payoff matrix is shown below:

	Player B		
Player A	B1	B2	B3
A1	-5	10	20
A2	5	-10	-10
A3	5	-20	-20

After column B3 is deleted, it may be noted that strategy A2 of player A is dominated by his A3 strategy, since the profit due to strategy A2 is greater than or equal to the profit due to strategy A3, regardless of which strategy player B selects. Hence, strategy A3 (row 3) can be deleted from further consideration. Thus, the reduced payoff matrix becomes:

	Player B		
Player A	B1	B2	RowMinimum
A1	**-5**	10	**-5 maximum**
A2	**5**	-10	-10
ColumnMaximum	**5 Minimum**	10	

As shown in the reduced 2 × 2 matrix, the maximin value is not equal to the minimax value. Hence, there is no saddle point and one cannot determine the point of equilibrium. For this type of game situation, it is possible to obtain a solution by applying the concept of mixed strategies. The solution to this game can now be obtained by applying any of the methods used for mixed-strategy games .

SOLUTION METHODS FOR GAMES WITHOUT SADDLE POINT

Algebraic Method

This method is used to determine the probability of using different strategies by players A and B. This method becomes quite lengthy when a number of strategies for both the players are more than two. Consider a game where the payoff matrix is: $[a_{ij}]_{m \times n}$. Let ($p_1, p_2, \ldots, p_m$) and ($q_1, q_2, \ldots, q_n$) be the probabilities with which players A and B select their strategies ($A_1, A_2, \ldots, A_m$) and ($B_1, B_2, \ldots, B_n$), respectively. If V is the value of game, then the expected gain to player A, when player B selects strategies
$B_1, B_2, \ldots, B_n$, one by one, is given by left-hand side of the following simultaneous equations, respectively. Since player A is the gainer player and expects at least V, therefore, we must have

	Player B				
Player A	B_1	B_2	. . .	B_n	Probability
A_1	a_{11}	a_{12}	. . .	a_{1n}	p_1
A_2	a_{21}	a_{22}	. . .	a_{2n}	p_2
A_m	a_{m1}	a_{m2}	. . .	a_{mn}	p_m
Probability q_1 q_2 . . . q_n					

$a_{11} p_1 + a_{21} p_2 + \ldots + a_{m1} p_m \geq V$

$a_{12} p_1 + a_{22} p_2 + \ldots + a_{m2} p_m \geq V$ (1)

$a_{1n} p_1 + a_{2n} p_2 + \ldots + a_{mn} p_m \geq V$

where $p_1 + p_2 + \ldots + p_m = 1$ and $p_i \geq 0$ for all i

Similarly, the expected loss to player B, when player A selects strategies $A_1, A_2, \ldots, A_m$, one by one, can also be determined. Since player B is the loser player, therefore, he must have:

$a_{11} q_1 + a_{12} q_2 + \ldots + a_{1n} q_n \leq V$

$a_{21} q_1 + a_{22} q_2 + \ldots + a_{2n} q_n \leq V$ (2)

$a_{m1} q_1 + a_{m2} q_2 + \ldots + a_{mn} q_n \leq V$

where q1 + q2 + . . . + qn = 1 and qj ≥:0 for all j

To get the values of pi's and qj's, the above inequalities are considered as equations and are then solved for given unknowns. However, if the system of equations, so obtained, is inconsistent, then at least one of the inequalities must hold as a strict inequality. The solution can now be obtained only by applying the trial and error method.

Example 1

A company is currently involved in negotiations with its union on the upcoming wage contract. Positive signs in table represent wage increase while negative sign represents wage reduction. What
are the optimal strategies for the company as well as the union? What is the game value?
Conditional costs to the company (Rs. in lakhs)
\

Union Strategies

		U1	U2	U3	U4
	C1	0.25	0.27	0.35	– 0.02
Company	C2	0.20	0.06	0.08	0.08
Strategies	C3	0.14	0.12	0.05	0.03
	C4	0.30	0.14	0.19	0.00

Solution Suppose, Company is the gainer player and Union is the looser player. Transposing payoff matrix because company's interest is to minimize the wage increase while union's interest is to get the maximum wage increase.

Company Strategies

		C1	C2	C3	C4
	U1	0.25	0.20	0.14	0.30
Union	U2	0.27	0.16	0.12	0.14
Strategies	U3	0.35	0.08	0.15	0.19
	U4	– 0.02	0.08	0.13	0.00

In this payoff matrix strategy U4 is dominated by strategy U1 as well as U3. After deleting this strategy, we get

Company Strategies

		C1	C2	C3	C4
	U1	0.25	0.20	0.14	0.30
Union	U2	0.27	0.16	0.12	0.14
Strategies	U3	0.35	0.08	0.15	0.19

Company's point of view, strategy C1 is dominated by C2 as well as C3, while C4 is dominated C3. Deleting strategies C1 and C4 we get

Company Strategies

		C2	C3
	U1	0.20	0.14
Union	U2	0.16	0.12
Strategies	U3	0.08	0.15

Again strategy U2 is dominated by U1 and is, therefore, deleted to give

Company Strategies

C2 C3 Probability

Union U1 0.20 0.14 0.07/0.13 = 0.538

Strategies U3 0.08 0.15 0.06/0.13 = 0.461

Probability 0.01/0.13 0.12/0.13

= 0.076 = 0.923

Optimal strategy for the company : (0, 0.076, 0.923, 0)

Optimal strategy for the union : (0.538, 0, 0.461, 0)

Value of the game, V : 0.538 × 0.20 + 0.461 × 0.08 = Rs. 14360

5.5 Graphical Method

The graphical method is useful for the game where the payoff matrix is of the size 2 × n or m × 2, i.e. the game with mixed strategies that has only two undominated pure strategies for one of the players in the
two-person zero-sum game.

Optimal strategies for both the players assign non-zero probabilities to the same number of pure strategies. Therefore, if one player has only two strategies, the other will also use the same number of strategies. Hence, this method is useful in finding out which of the two strategies can be used.

Consider the following 2 × n payoff matrix of a game, without saddle point.

Player B

Player A B1 B2 . . . Bn Probability

A1 a11 a12 . . . a1n p1

A2 a21 a22 . . . a2n p2

Probability q1 q2 . . . qn

Player A has two strategies A1 and A2 with probability of their selection p1 and p2, respectively, such

that p1 + p2 = 1 and p1, p2 ≥ 0. Now for each of the pure strategies available to player B, the expected

pay off for player A would be as follows:

B's Pure Strategies A's Expected Payoff

B1 a11p1 + a21 p2

B2 a12p1 + a22 p2

Bn a1np1 + a2n p2

According to the maximin criterion for mixed strategy games, player A should select the value of probability p1 and p2 so as to maximize his minimum expected payoffs. This may be done by plotting the straight lines representing player A's expected payoff values.

The highest point on the lower boundary of these lines will give the maximum expected payoff among the minimum expected payoffs and the optimum value of probability p1 and p2.

Now, the two strategies of player B corresponding to those lines which pass through the maximin point can be determined. This helps in reducing the size of the game to (2 × 2), which can be easily solved by any of the methods discussed earlier.

The (m × 2) games are also treated in the same way except that the upper boundary of the straight lines corresponding to B's expected payoff will give the maximum expected payoff to player B and the lowest point on this boundary will then give the minimum expected payoff (minimax value) and the optimum value of probability q1 and q2.

Example 1

Use the graphical method for solving the following game and find the value of the game.

Player B

Player A	B1	B2	B3	B4
A1	2	2	3	–2
A2	4	3	2	6

Solution The game does not have a saddle point. If the probability of player A's playing A1 and A2 in the strategy mixture is denoted by p1 and p2, respectively, where p2 = 1 – p1, then the expected payoff (gain) to player A will be B's Pure Strategies A's Expected Payoff

B1 2 p1 + 4 p2

B2 2 p1 + 3 p2

B3 3 p1 + 2 p2

B4 – 2 p1 + 6 p2

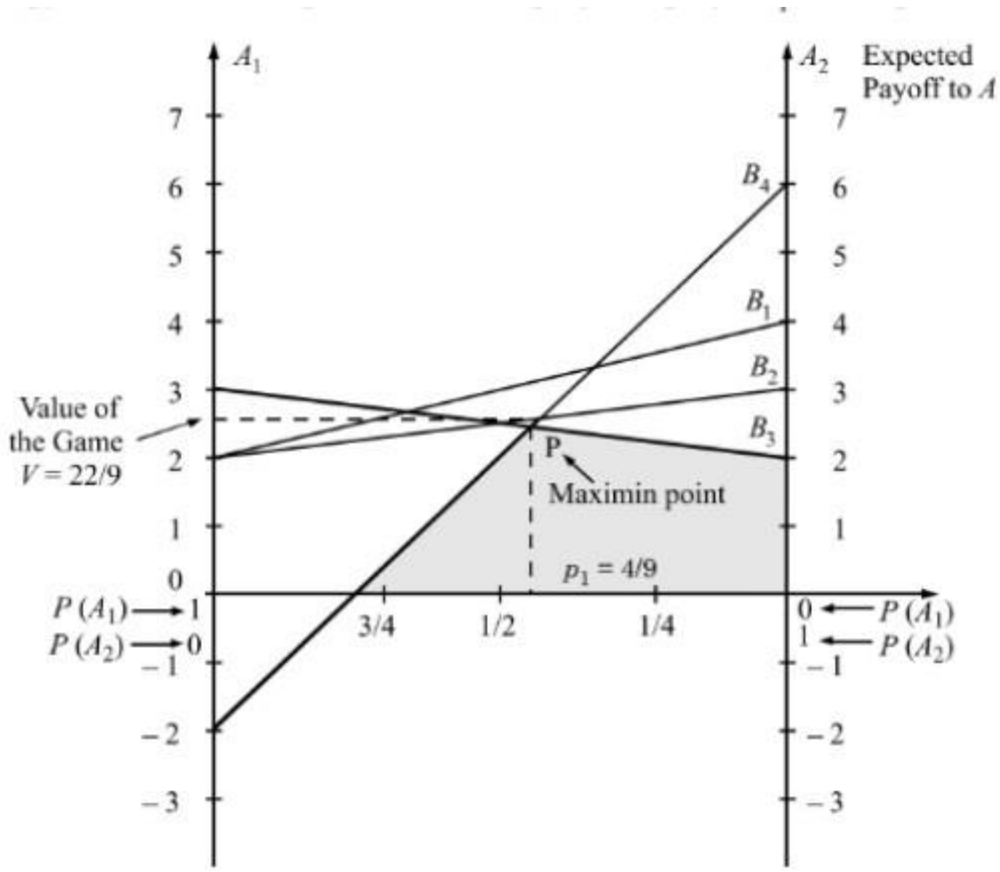
These four expected payoff lines can be plotted on a graph to solve the game.

The graph for player A graphic solution is shown in Fig. 12.2. Here, the probability of player A's playing A1, i.e. p1 is measured on the x-axis. Since p1 cannot exceed 1, the x-axis is cutoff at p1 = 1. The expected payoff of player A is measured along y-axis. From the game matrix, if player B plays B1, the expected payoff of player A is 2 when A plays A1 with p1 = 1 and 4 when A plays A2 with p1 = 0. These two extreme points are connected by a straight line, which shows the expected payoff of A when B plays B1. Three other straight lines are

similarly drawn for B2, B3 and B4. It is assumed that player B will always play his best possible strategies yielding the worst result to player A. Thus, the payoffs (gains) to A are represented by the lower boundary when he is faced with the most unfavourable situation in the game. Since player A must choose his best possible strategies in order to realize a maximum expected gain, the highest expected gain is found at point P, where the two straight lines

E3 = 3p1 + 2p2 = 3p1 + 2(1 – p1)

E4 = – 2p1 + 6p2 = – 2p1 + 6(1 – p1)



meet. In this manner the solution to the original (2 × 4) game reduces to that of the game with payoff matrix of size (2 × 2) as given below:

		Player B	
Player A		B3	B4
	A1	3	–2
	A2	2	6

The optimum payoff to player A can now be obtained by setting E3 and E4 equal and solving for p1, i.e.3p1 + 2(1 – p1) = – 2p1 + 6(1 – p1) or p1 = 4/9; p2 = 1 – p1 = 5/9

Substituting the value of p1 and p2 in the equation for E3 (or E4) we have:

Value of the game, V = 3 × 4/9 + 2 × 5/9 = 22/9

The optimal strategy mix of player B can also be found in the same manner as for player A. If the probabilities of B's selecting strategy B3 and B4 are denoted by q3 and q4, respectively, then the expected loss to B will be:

L3 = 3q3 – 2q4 = 3q3 – 2(1 – q3) (if A selects A1)

L4 = 2q3 + 6q4 = 2q3 + 6(1 – q3) (if A selects A2)

To solve for q3, equate the two equations:

3q3 – 2(1 – q3) = 2q3 + 6(1 – q3) or q3 = 8/9; q4 = 1 – q3 = 1/9

Substituting the value of q3 and q4 in the equation for L3 (or L4), we have

Value of the game, V = 3 × 8/9 – 2 × 1/9 = 22/9

Example 2

Two firms A and B make colour and black & white television sets. Firm A can make either 150 colour sets in a week or an equal number of black & white sets, and make a profit of Rs 400 per colour set, or 150 colour and 150 black & white sets, or 300 black & white sets per week. It also has the same profit margin on the two sets as A. Each week there is a market of 150 colour sets and 300 black & white sets and the manufacturers would share market in the proportion in which they manufacture a particular type of set. Write the pay-off matrix of A per week. Obtain graphically A's and B's optimum strategies and value of the game.

Solution For firm A, the strategies are:

A1 : make 150 colour sets, A2 : make 150 black & white sets. For firm B, the strategies are:

B1 : make 300 colour sets, B2 : make 150 colour and 150 black & white sets. B3 : make 300 black and white sets.

For the combination A1B1, the profit to firm A would be: {150/(150 + 300)} × 150 × 400 = Rs 20,000 wherein 150/(150 + 300) represents share of market for A, 150 is the total market for colour television sets and 400 is the profit per set. In a similar manner, other profit figures may be obtained as shown in the following pay-off matrix:

		B's Strategy		
A's Strategy		B1	B2	B3
	A1	20,000	30,000	60,000
	A2	45,000	45,000	30,000

This pay-off table has no saddle point. Thus to determine optimum mixed strategy, the data are plotted on graph Lines joining the pay-offs on axis A1 with the pay-offs on axis A2 represents each of B's strategies. Since firm A wishes to maximize his minimum expected pay-off, we consider the highest point of intersection, P on the lower envelope of A's expected payoff equation. This point P represents the maximum expected value of the game.

The lines B1 and B3 passing through P, define the strategies which firm B needs to adopt. The solution to the original 2 × 3 game, therefore, reduces to that of the simpler game with 2 × 2 pay-off matrix as follows:

		B's Strategy	
A's Strategy	B1	B3	Probability
A1	20,000	60,000	p1
A2	45,000	30,000	p2

Probability q1 q2

The optimal mixed strategies of player A are: A1 = 3/11, A2 = 8/11. Similarly, the optimal mixed strategies for B are: B1 = 6/11, B2 = 0, B3 = 5/11. The value of the game is V = 38,182.

Example 3

Obtain the optimal strategies for both persons and the value of the game for two-person zero-sum game whose payoff matrix is as follows:

	Player B	
Player A	B1	B2
A1	1	–3
A2	3	5
A3	– 1	6
A4	4	1
A5	2	2
A6	– 5	0

Solution The game does not have any saddle point. If the probability of player B's playing strategies B1 and B2 in the strategy mix is denoted by q1 and q2 such that q1 + q2 = 1, then the expected payoff to player B will be:

A's Pure Strategies B's Expected Payoff

A1 q1 – 3q2

A2 3q1 + 5q2

A3 – q1 + 6q2

A4 4q1 + q2

A5 2q1 + 2q2

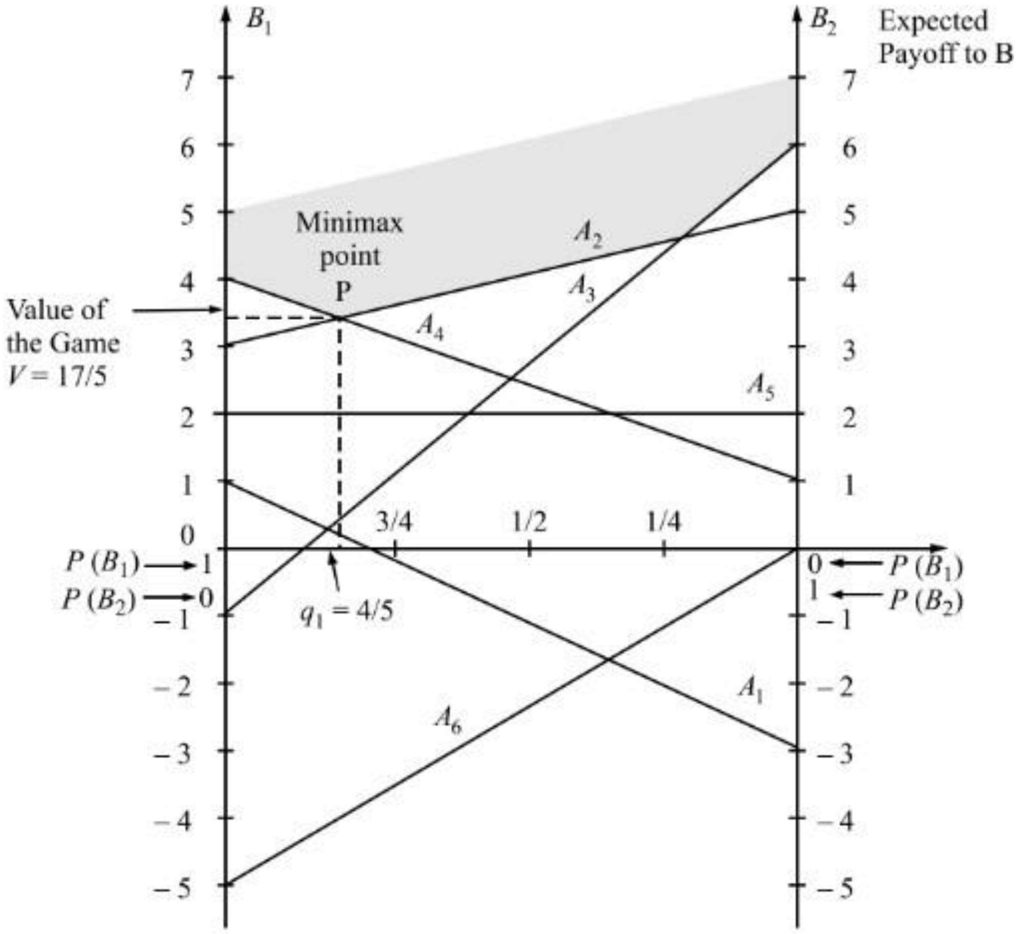
A6 – 5q1 + 0q2

The six expected payoff lines can be plotted on the graph to solve the game.

The graph for player B A graphic solution is shown in Fig. 12.3 where the probability of player B's playing B1, i.e. q1 is measured on the x-axis. Since q1 cannot exceed 1, therefore the x-axis is cutoff at q1 = 1. The expected payoff of player B is measured along y-axis. From the game matrix, if player A plays

A1, the expected payoff of player B is 1 when he plays B1 with q1 = 1 and –3 when he plays B2 with q1 = 0. These two extreme points are connected by a straight line, which shows the expected payoff to B when A plays A1. Five other straight lines are similarly drawn for A2 to A6.

It is assumed that player A will always play his best possible strategies, yielding the worst result to player B. Thus, payoffs (losses) to B are represented by the upper boundary when he is faced with the most unfavourable situation in the game. According to the minimax criterion, player B will always select a combination of strategies B1 and B2, so that he minimizes the losses. Even in this case the optimum solution occurs at the intersection of the two payoff lines.



$E_3 = 3q_1 + 5q_2 = 3q_1 + 5(1 - q_1)$

$E_4 = 4q_1 + q_2 = 4q_1 + (1 - q_1)$

The solution to the original (6 × 2) game reduces to that of the game with payoff matrix of size (2 × 2) as shown below:

	Player B	
Player A	B1	B2
A2	3	5
A4	4	1

Now using the usual method of solution for a (2 × 2) game, the optimum strategies can be obtained as given below:

Player A: (0, 3/5, 0, 2/5, 0, 0) ; Player B: (4/5, 1/5) and, Value of the game, V = 17/5.

5.6 Decision Theory and Decision Trees - Introduction

The success or failure that an individual or organization experiences, depends to a large extent, on the ability of making acceptable decisions on time. To arrive at such a decision, a decision-maker needs to enumerate feasible and viable courses of action (alternatives or strategies), the projection of consequences associated with each course of action, and a measure of effectiveness (or an objective) to identify the best course of action.

Decision theory is both descriptive and prescriptive business modeling approach to classify the degree of knowledge and compare expected outcomes due to several courses of action. The degree of knowledge is divided into four categories: complete knowledge (i.e. certainty), ignorance, risk and uncertainty.

Ignorance	Uncertainty	Risk	Certainty

Increasing Knowledge

Irrespective of the type of decision model, following essential components are common to all:

Decision alternatives There is a finite number of decision alternatives available to the decision-maker at each point in time when a decision is made. The number and type of such alternatives may depend on the previous decisions made and their outcomes. Decision alternatives may be described numerically, such as stocking 100 units of a particular item, or non-numerically, such as conducting a market survey to know the likely demand of an item.

States of nature A state of nature is an event or scenario that is not under the control of decision makers.

For instance, it may be the state of economy (e.g. inflation), a weather condition, a political development, etc.

The states of nature may be identified through Scenario Analysis where a section of people are interviewed – stakeholders, long-time managers, etc., to understand states of nature that may have serious impact on a decision.

The states of nature are mutually exclusive and collectively exhaustive with respect to any decision problem. The states of nature may be described numerically such as, demand of 100 units of an item or non-numerically such as, employees strike, etc.

Payoff It is a numerical value (outcome) obtained due to the application of each possible combination of decision alternatives and states of nature. The payoff values are always conditional values because of unknown states of nature.

The payoff values are measured within a specified period (e.g. within one year, month, etc.) called the decision horizon. The payoffs in most decisions are monetary. Payoffs resulting from each possible combination of decision alternatives and states of natures are displayed in a matrix (also called payoff matrix) form

		Courses of Action (Alternatives)		
States of Nature	Probability	S1	S2	S3
N1	P1	P11	P12	P1n
N2	P2	P21	P22	P2n
Nm	Pm	Pm1	Pm2	Pmn

STEPS OF DECISION-MAKING PROCESS

The decision-making process involves the following steps:

1. Identify and define the problem.
2. List all possible future events (not under the control of decision-maker) that are likely to occur.
3. Identify all the courses of action available to the decision-maker.
4. Express the payoffs (pij) resulting from each combination of course of action and state of nature.
5. Apply an appropriate decision theory model to select the best course of action from the given list on the basis of a criterion (measure of effectiveness) to get optimal (desired) payoff.

A firm manufactures three types of products. The fixed and variable costs are given below:

	Fixed Cost (Rs)	Variable Cost per Unit (Rs)
Product A	25,000	12
Product B	35,000	9
Product C	53,000	7

The likely demand (units) of the products is given below:

Poor demand : 3,000

Moderate demand : 7,000

High demand : 11,000

If the sale price of each type of product is Rs 25, then prepare the payoff matrix.

Solution Let D1, D2 and D3 be the poor, moderate and high demand, respectively. The payoff is determined as:

Payoff = Sales revenue – Cost

The calculations for payoff (in '000 Rs) for each pair of alternative demand (course of action) and the types of product (state of nature) are shown below:

D1 A = 3 × 25 – 25 – 3 × 12 = 14 D2 A = 7 × 25 – 25 – 7 × 12 = 66

D1 B = 3 × 25 – 35 – 3 × 9 = 13 D2 B = 7 × 25 – 35 – 7 × 9 = 77

D1 C = 3 × 25 – 53 – 3 × 7 = 1 D2 C = 7 × 25 – 53 – 7 × 7 = 73

D3 A = 11 × 25 – 25 – 11 × 12 = 118

D3 B = 11 × 25 – 35 – 11 × 9 = 141

D3 C = 11 × 25 – 53 – 11 × 7 = 145

The payoff values are shown

Product Type	Alternative Demand in (000 Rs)		
	D1	D2	D3
A	14	66	118
B	13	77	141
C	1	73	145

TYPES OF DECISION-MAKING ENVIRONMENTS

To arrive at an optimal decision it is essential to have an exhaustive list of decision-alternatives, knowledge of decision environment, and use of appropriate quantitative approach for decision-making. In this section three types of decision-making environments: certainty, uncertainty, and risk, have been discussed. The knowledge of these environments helps in choosing the quantitative approach for decision-making.

Type 1 Decision-Making under Certainty

In this decision-making environment, decision-maker has complete knowledge (perfect information) of outcome due to each decision-alternative (course of action). In such a case he would select a decision alternative that yields the maximum return (payoff) under known state of nature. For example, the decision to invest in National Saving Certificate, Indira Vikas Patra, Public Provident Fund, etc., is where complete information about the future return due and the principal at maturity is know.

Type 2 Decision-Making under Risk

In this decision-environment, decision-maker does not have perfect knowledge about possible outcome of every decision alternative. It may be due to more than one states of nature. In a such a case he makes an assumption of the probability for occurrence of particular state of nature.

Type 3 Decision-Making under Uncertainty

In this decision environment, decision-maker is unable to specify the probability for occurrence of particular state of nature. However, this is not the case of decision-making under ignorance, because the possible states of nature are known. Thus, decisions under uncertainty are taken even with less information than decisions under risk. For example, the probability that Mr X will be the prime minister of the country 15 years from now is not known.

5.7 DECISION-MAKING UNDER UNCERTAINTY

When probability of any outcome can not be quantified, the decision-maker must arrive at a decision only on the actual conditional payoff values, keeping in view the criterion of effectiveness (policy). The following criteria of decision-making under uncertainty have been discussed in this section.

(i) Optimism (Maximax or Minimin) criterion

(ii) Pessimism (Maximin or Minimax) criterion

(iii) Equal probabilities (Laplace) criterion

(iv) Coefficient of optimism (Hurwiez) criterion

(v) Regret (salvage) criterion

Optimism (Maximax or Minimin) Criterion

In this criterion the decision-maker ensures that he should not miss the opportunity to achieve the largest possible profit (maximax) or the lowest possible cost (minimin). Thus, he selects the decision alternative that represents the maximum of the maxima (or minimum of the minima) payoffs (consequences or outcomes).

The working method is summarized as follows:

(a) Locate the maximum (or minimum) payoff values corresponding to each decision alternative.

(b) Select a decision alternative with best payoff value (maximum for profit and minimum for cost).

Since in this criterion the decision-maker selects an decision-alternative with largest (or lowest) possible payoff value, it is also called an optimistic decision criterion.

Pessimism (Maximin or Minimax) Criterion

In this criterion the decision-maker ensures that he would earn no less (or pay no more) than some specified amount. Thus, he selects the decision alternative that represents the maximum of the minima (or minimum of the minima in case of loss) payoff in case of profits. The working method is summarized as follows:

(a) Locate the minimum (or maximum in case of profit) payoff value in case of loss (or cost) data corresponding to each decision alternative.

(b) Select a decision alternative with the best payoff value (maximum for profit and mimimum for loss or cost).

Since in this criterion the decision-maker is conservative about the future and always anticipates the worst possible outcome (minimum for profit and maximum for cost or loss), it is called a pessimistic decisioncriterion. This criterion is also known as Wald's criterion.

Equal Probabilities (Laplace) Criterion

Since the probabilities of states of nature are not known, it is assumed that all states of nature will occur with equal probability, i.e. each state of nature is assigned an equal probability. As states of nature are mutually exclusive and collectively exhaustive, so the probability of each of these must be: 1/(number of states of nature). The working method is summarized as follows:

(a) Assign equal probability value to each state of nature by using the formula:

1 (number of states of nature).

(b) Compute the expected (or average) payoff for each alternative (course of action) by adding all the payoffs and dividing by the number of possible states of nature, or by applying the formula:

(Probability of state of nature j) × (Payoff value for the combination of alternative i

and state of nature j.)

(c) Select the best expected payoff value (maximum for profit and minimum for cost).

This criterion is also known as the criterion of insufficient reason. This is because except in a few cases,

(a) some information of the likelihood of occurrence of states of nature is available.
(b) **Coefficient of Optimism (Hurwicz) Criterion**
(c) This criterion suggests that a decision-maker should be neither completely optimistic nor pessimistic and,
(d) therefore, must display a mixture of both. Hurwicz, who suggested this criterion, introduced the idea of a
(e) coefficient of optimism (denoted by) to measure the decision-maker’s degree of optimism. This coefficient
(f) lies between 0 and 1, where 0 represents a complete pessimistic attitude about the future and 1 a complete
(g) optimistic attitude about the future. Thus, if is the coefficient of optimism, then (1 – □) will represent
(h) the coefficient of pessimism.
(i) The Hurwicz approach suggests that the decision-maker must select an alternative that maximizes
(j) H (Criterion of realism) = (Maximum in column) + (1 –) (Minimum in column)
(k) The working method is summarized as follows:
(l) (a) Decide the coefficient of optimism (alpha) and then coefficient of pessimism (1 –).
(m) (b) For each decision alternative select the largest and lowest payoff value and multiply these with

(n) and (1 –) values, respectively. Then calculate the weighted average, H by using above formula.

(o) (c) Select an alternative with best weighted average payoff value.

Regret (Savage) Criterion

This criterion is also known as opportunity loss decision criterion or minimax regret decision criterion because decision-maker regrets for choosing wrong decision alternative resulting in an opportunity loss of payoff. Thus, he always intends to minimize this regret. The working method is summarized as follows:

(a) From the given payoff matrix, develop an opportunity-loss (or regret) matrix as follows:

(i) Find the best payoff corresponding to each state of nature

(ii) Subtract all other payoff values in that row from this value.

(b) For each decision alternative identify the worst (or maximum regret) payoff value. Record this value in the new row.

(a) (c) Select a decision alternative resulting in a smallest anticipated opportunity-loss value.

Example 1

A food products' company is contemplating the introduction of a revolutionary new product with new packaging or replacing the existing product at much higher price (S1). It may even make a moderate change in the composition of the existing product, with a new packaging at a small increase in price (S2), or may mall a small change in the composition of the existing product, backing it with the word 'New' and a negligible increase in price (S3). The three possible states of nature or events are: (i) high increase in sales (N1), (ii) no change in sales (N2) and (iii) decrease in sales (N3). The marketing department of the company worked out the payoffs in terms of yearly net profits for each of the strategies of three.

(a) events (expected sales). This is represented in the following table:

Strategies	States of Nature		
	N1	N2	N3
S1	700000	300000	150000
S2	500000	450000	0
S3	300000	300000	300000

Which strategy should the concerned executive choose on the basis of

(a) Maximin criterion (b) Maximax criterion

(c) Minimax regret criterion (d) Laplace criterion?

Solution The payoff matrix is rewritten as follows:

(a) Maximin Criterion

States of nature	Strategies		
	S1	S2	S3
N1	700000	500000	
N2	300000	450000	300000
N3	150000	0	300000
Column Minimum	150000		300000 maximumPayoff

The maximum of column minima is 3,00,000. Hence, the company should adopt strategy S3

(b) Maximax Criterion

States of nature	Strategies		
	S1	S2	S3
N1	700000	500000	300000
N2	300000	450000	300000
N3	150000	0	300000
Column Maximum	700000 Maximum Payoff	500000	300000

The maximum of column maxima is 7,00,000. Hence, the company should adopt strategy S1.

(c) Minimax Regret Criterion Opportunity loss table is shown below:

States of nature	Strategies		
	S1	S2	S3
N1	700000 – 700000 =0	700000-500000= 200000	700000-300000=400000
N2	450000-300000=150000	450000-450000	450000-300000=150000
N3	300000-150000=150000	300000-0=300000	300000-300000=0
Column Maximum	150000 Minimax Regret	300000	400000

(d)Laplace Criterion Assuming that each state of nature has a probability 1/3 of occurrence. Thus,

Strategy	Expected Return (Rs)
S1	(700000+300000+150000)/3=383333.33Largest Payoff
S2	(500000+450000+0)/3=316666.66
S3	(300000+300000+30000)//3=300000

Since the largest expected return is from strategy S1, the executive must select strategy S1.

5.8 DECISION TREE ANALYSIS

Decision-making problems discussed earlier were limited to arrive at a decision over a fixed period of time.

That is, payoffs, states of nature, courses of action and probabilities associated with the occurrence of states of nature were not subject to change.

However, situations may arise when a decision-maker needs to revise his previous decisions due to availability of additional information. Thus he intends to make a sequence of interrelated decisions over several future periods. Such a situation is called a sequential or multiperiod decision process.

For example, in the process of marketing a new product, a company usually first go for 'Test Marketing' and other alternative courses of action might be either 'Intensive Testing' or 'Gradual Testing'. Given the various possible consequences – good, fair, or poor, the company may be required to decide between redesigning the product, an aggressive advertising campaign or complete withdrawal of product, etc. Based on this decision there might be an outcome that leads to another decision and so on. A decision tree analysis involves the construction of a diagram that shows, at a glance, when decisions are expected to be made – in what sequence, their possible outcomes, and the corresponding payoffs. A decision tree consists of nodes, branches, probability estimates, and payoffs. There are two types of nodes:

Decision (or act) node: A decision node is represented by a square and represents a point of time where a decision-maker must select one alternative course of action among the available. The courses of action are shown as branches or arcs emerging out of decision node.

Chance (or event) node: Each course of action may result in a chance node. The chance node is represented by a circle and indicates a point of time where the decision-maker will discover the response to his decision.

Branches emerge from and connect various nodes and represent either decisions or states of nature. There are two types of branches:

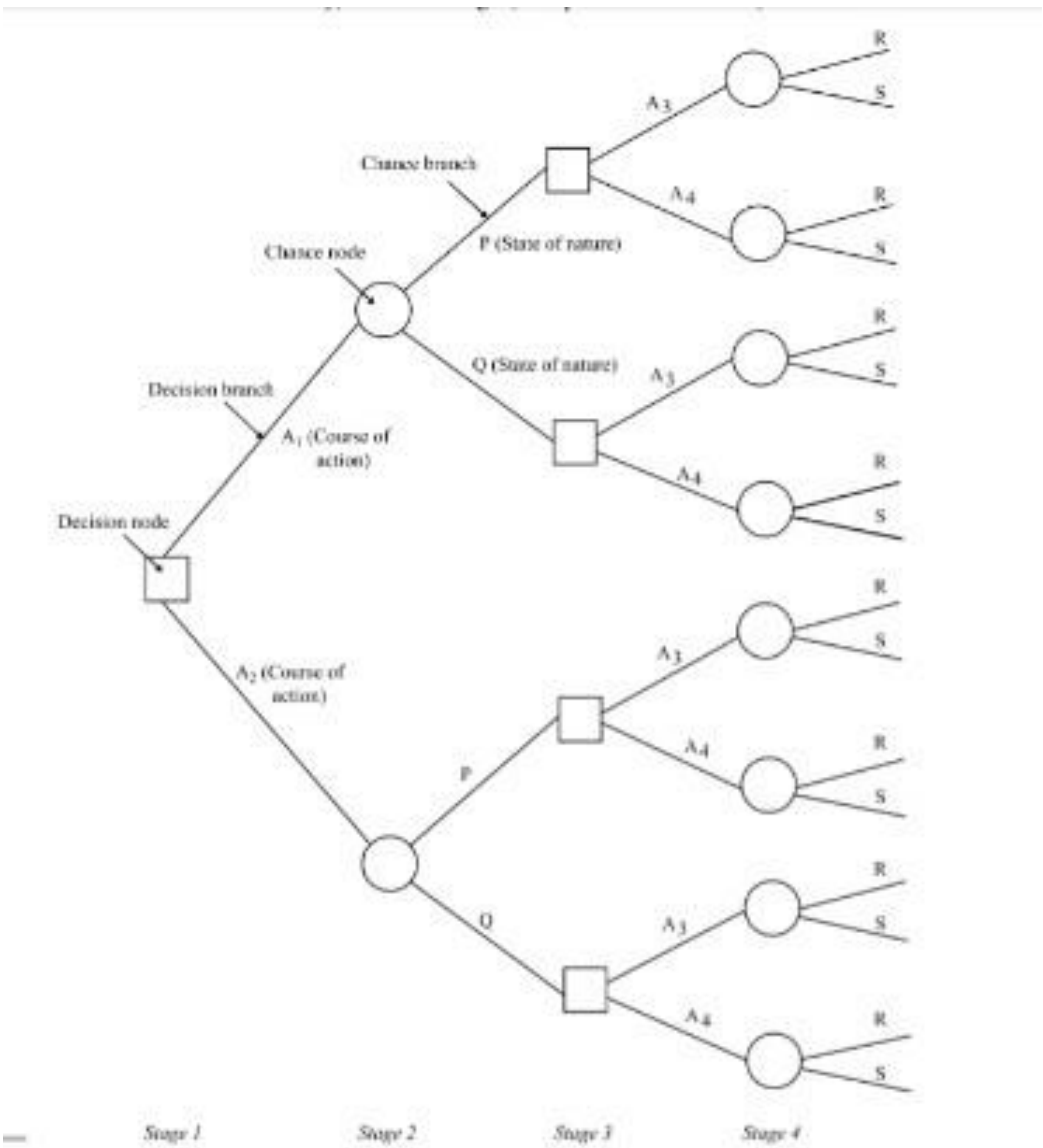
Decision branch: It is the branch leading away from a decision node and represents a course of action that can be chosen at a decision point.

Chance branch: It is the branch leading away from a chance node and represents the state of nature of a set of chance events. The assumed probabilities of the states of nature are written alongside their respective chance branch.

Terminal branch: Any branch that makes the end of the decision tree (not followed by either a decision or chance node), is called a terminal branch. A terminal branch can represent either a course of action. The terminal points of a decision tree are supposed to be mutually exclusive points so that exactly one course of action will be chosen.

The payoff can be positive (i.e. revenue or sales) or negative (i.e. expenditure or cost) and it can be associated either with decision or chance branches.

An illustration of a decision tree is shown . It is possible for a decision tree to be deterministic or probabilistic. It can also further be divided in terms of stages – into single stage (a decision under condition of certainty) and multistage (a sequence of decisions).



The optimal sequence of decisions in a tree is found by starting at the right-hand side and rolling backwards. At each node, an expected return is calculated (called position value). If the node is a chance node, then the position value is calculated as the sum of the products of the probabilities or the branches emanating from the chance node and their respective position values. If the node is a decision node, then the expected return is calculated for each of its branches and the highest return is selected. This procedure continues until the initial node is reached. The position values for this node corresponds to the maximum expected return obtainable from the decision sequence

Remark Decision trees versus probability trees Decision trees are basically an extension of probability trees. However, there are several basic differences:

(i) The decision tree utilizes the concept of 'rollback' to solve a problem. This means that it starts at the right-hand terminus with the highest expected value of the tree and works back to the current or beginning decision point in order to determine the decision or decisions that should be made.

It is the multiplicity of decision points that make the rollback process necessary.

(ii) The probability tree is primarily concerned with calculating the probabilities, whereas the decision

tree utilizes probability factors as a means of arriving at a final answer.

(iii) The most important feature of the decision tree, is that it takes time differences of future earnings into account. At any stage of the decision tree, it may be necessary to weigh differences in immediate cost or revenue against differences in value at the next stage.

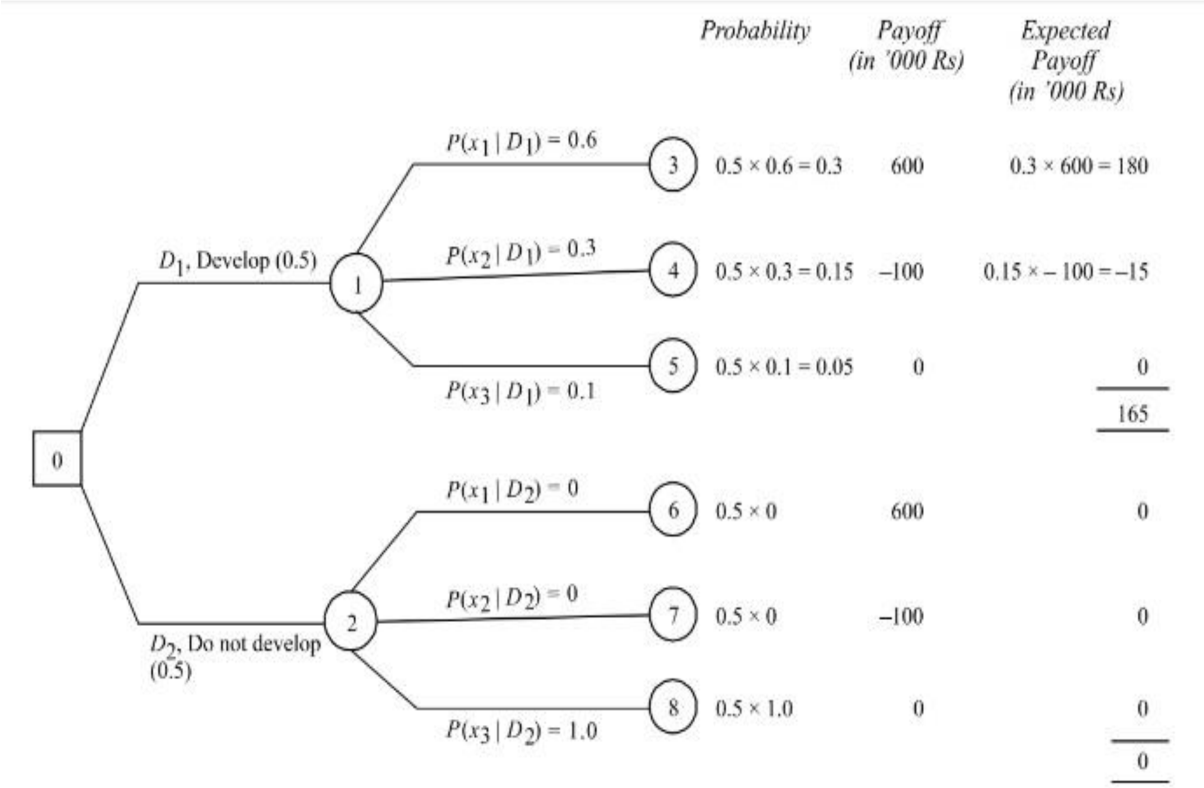
Example 1

You are given the following estimates concerning a Research and Development programme:

Decision D_i	*Probability of Decision* D_i *Given Research R* $P(D_i \mid R)$	*Outcome Number*	*Probability of Outcome* x_i *Given* D_i $P(x_i \mid D_i)$	*Payoff Value of Outcome,* x_i *(Rs '000)*
Develop	0.5	1	0.6	600
		2	0.3	– 100
		3	0.1	0
Do not develop	0.5	1	0.0	600
		2	0.0	– 100
		3	1.0	0

Construct and evaluate the decision tree diagram for the above data. Show your workings for evaluation.

Solution The decision tree of the given problem along with necessary calculations is



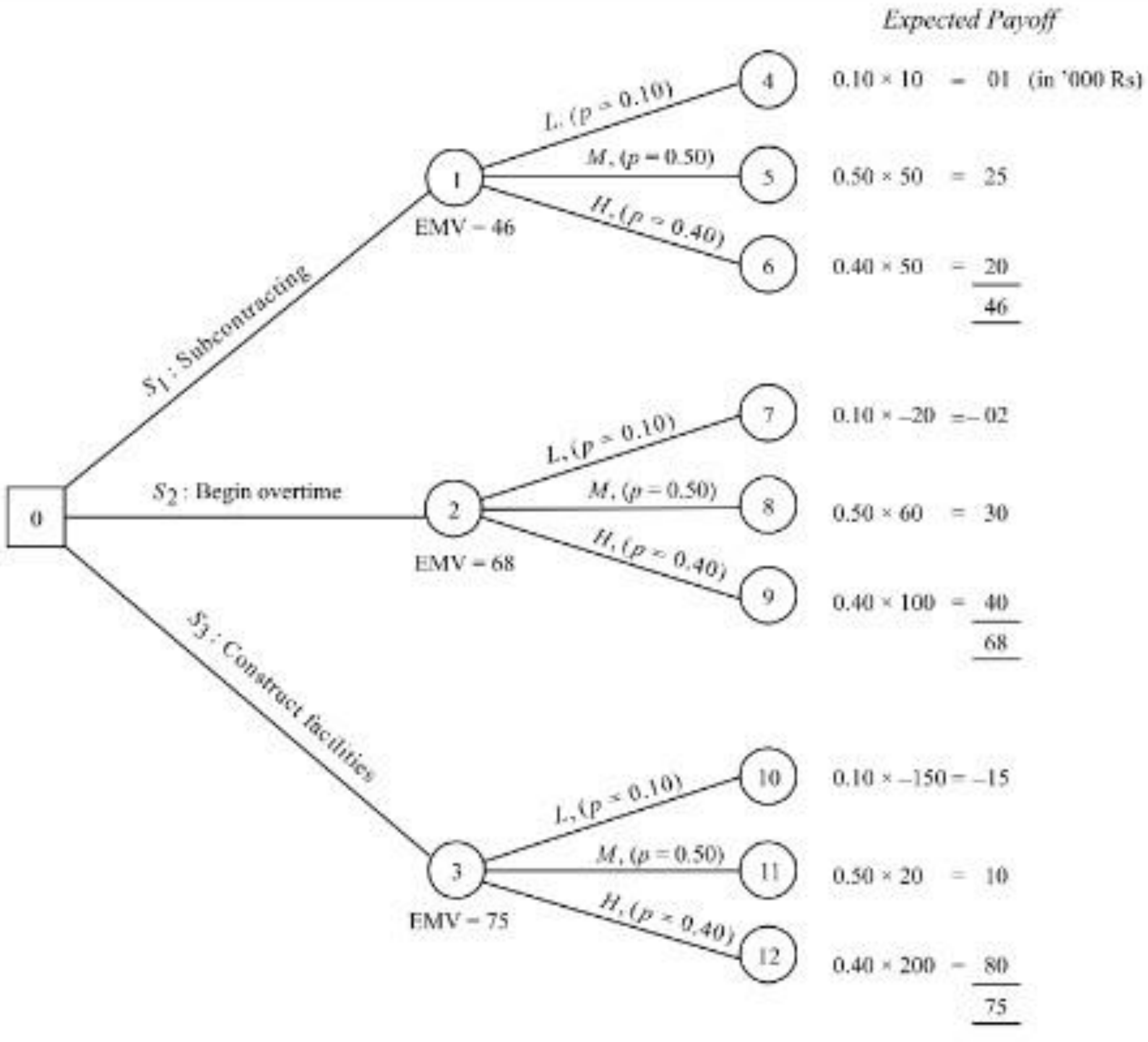
Example 2

A glass factory that specializes in crystal is developing a substantial backlog and for this the firm's management is considering three courses of action: To arrange for subcontracting (*S*1), to begin overtime production (*S*2), and to construct new facilities (*S*3). The correct choice depends largely upon the future demand, which may be low, medium, or high. By consensus, management ranks the respective probabilities as 0.10, 0.50 and 0.40. A cost analysis reveals the effect upon the profits. This is shown

Demand	*Probability*	*Course of Action*		
		S_1 (*Subcontracting*)	S_2 (*Begin Overtime*)	S_3 (*Construct Facilities*)
Low (L)	0.10	10	– 20	– 150
Medium (M)	0.50	50	60	20
High (H)	0.40	50	100	200

Show this decision situation in the form of a decision tree and indicate the most preferred decision and its corresponding expected value.

Solution A decision tree that represents possible courses of action and states of nature is shown . In order to analyze the tree, we start working backwards from the end branches.
The most preferred decision at the decision node 0 is found by calculating the expected value of each decision branch and selecting the path (course of action) that has the highest value.



Since node 3 has the highest EMV, therefore, the decision at node 0 will be to choose the course of action *S*3, i.e. construct new facilities.

5.9 Self Assessment Questions

Part _ A

1. In a two-player zero-sum game, the Saddle Point represents:

A) The average payoff

B) The maximum loss

C) The equilibrium point where optimal strategies intersect

D) A dominated strategy

Answer: C

2. If a game has a saddle point, the value of the game is:

A) Not defined

B) Equal to the payoff at the saddle point

C) Always zero

D) Infinite

Answer: B

3. The Dominance Property is used to:

A) Increase the size of a payoff matrix

B) Identify mixed strategies

C) Eliminate inferior strategies

D) Calculate Nash equilibrium

Answer: C

4. Which of the following is true about a strictly dominated strategy?

A) It always forms part of the optimal strategy

B) It can be removed without affecting the solution

C) It increases the complexity of the game

D) It gives the highest payoff

Answer: B

5. In a 2x2 zero-sum game, if there is no saddle point, then:

A) One player has a dominant strategy

B) The game has no solution

C) The players use mixed strategies

D) The game becomes non-zero sum

Answer: C

6. What is the minimum number of strategies a player should have in a graphical method?

A) 1

B) 2

C) 3

D) 4

Answer: B

(Graphical method is applicable for 2 strategies of one player and 3 or more of the other)

7. Which of the following games is suitable for solving using the Graphical Method?

A) 3x3

B) 2x5

C) 5x5

D) 4x2

Answer: B

8. In game theory, if a strategy yields better or equal payoff in all conditions compared to another, it is said to:

A) Be a pure strategy

B) Dominate the other

C) Be a mixed strategy

D) Be an equilibrium

Answer: B

9. In dominance property, row dominance means:

A) One strategy is better than another for Player B

B) One row gives higher payoffs than another for all column choices

C) One column dominates the row

D) A Nash equilibrium is found

Answer: B

10. In Graphical Method, the best strategy is found by:

A) Identifying the point where two lines cross and gives the highest payoff

B) Selecting the highest point on the graph

C) Taking the midpoint of all payoffs

D) Choosing any random point

Answer: A

Part - B

1. Define: (i) competitive game, (ii) payoff matrix, (iii) pure and mixed strategies, (iv) saddle point, (v) optimal strategies, and
(vi) rectangular (or two-person zero-sum) game.

2. Explain: Minimax and Maximin principle used in the theory of games.

3. What is a game in game theory? What are the properties of game? Explain the 'best strategy' on the basis of minimax
criterion of optimality.

4. Explain the two-person zero-sum game, giving a suitable example.

5. Define 'saddle point'. Is it necessary that a game should always possess a saddle point?

6. Consider the game with the following payoff table:

		Player *B*	
Player *A*		*B*1	*B*2
	*A*1	2	6
	*A*2	– 2	λ

(a) Show that the game is strictly determinable, whatever λ may be.

(b) Determine the value of the game.

7. Determine which of the following two-person zero-sum games are strictly determinable and fair. Give the optimum strategies for each player in the case of it being strictly determinable.

(a)

Player *A*	Player *B*	
	*B*1	*B*2
*A*1	1	2
*A*2	4	–3

(b)

Player *A*	Player *B*	
	*B*1	*B*2
*A*1	– 5	2
*A*2	– 7	– 4

8. Solve the following games by using maximin (minimax) principle, whose payoff matrix are given below: Include in your answer:

(i) strategy selection for each player, (ii) the value of the game to each player. Does the game have a saddle point?

(a)

Player *A*	Player *B*			
	*B*1	*B*2	*B*3	*B*4
*A*1	1	7	3	4
*A*2	5	6	4	5
*A*3	7	2	0	3

(b)

Player *A*	Player *B*				
	*B*1	*B*2	*B*3	*B*4	*B*5
*A*1	– 2	0	0	5	3
*A*2	3	2	1	2	2
*A*3	– 4	– 3	0	– 2	6
*A*4	5	3	–4	2	6

(c)

Player *A*	Player *B*			
	*B*1	*B*2	*B*3	*B*4
*A*1	3	–5	0	6
*A*2	– 4	– 2	1	2
*A*3	5	4	2	3

9. Define: (i) Competitive game; (ii) Pure strategies; (iii) Mixed strategies (iv)Two-person zero-sum (or rectangular) game, (v) Payoff matrix.

10. State the major limitations of the game theory.
