UNIT I
OPERATION RESEARCH INTRODUCTION

Definitions and Characteristics of Operations Research

Some of the definitions of O R. by leaders and pioneers are as follows:

1. Churchman, Ackoff and Arnoff. "Operations Research is the application of scientific methods, techniques and tools to problems involving the operations of systems, so as to provide those in control of operations with optimum solution to the problems".

2. Morse and Kimbel. "Operations Research is the scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control".

3. Ellis Johnson. "Operations Research is the prediction and comparison of values, effectiveness and cost of a set of alternative courses of action involving man-machine systems".

4. The Operational Research Society of India defines "Operations Research" as the attack of modern science on complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence. The distinctive approach is to develop a scientific model of system incorporating measurement of factors, such as chance and risk, with which to predict and compare the outcomes of alternative decisions, strategies or controls. The purpose is to help management determine its policy and actions scientifically".

Considering the above definitions we can define operations research as follows: "Operations Research is a scientific approach to find optimal solutions to problems of complex man-machine systems".

Characteristics of Operations Research

1. Operations Research has a team approach – O.R. is a research carried out by a team of scientists whose individual members have been drawn from different scientific and engineering disciplines.

2. Operations Research uses scientific methods, techniques and tools to analyse executive type problems.

The terms tools, techniques and methods are frequently used interchangeably but in fact they differ from each other. Scientific tools can be compared with the cutting tools used in a workshop Scientific techniques indicate how these tools can be used to obtain results. Scientific methods indicate a plan of using these techniques to achieve the desired objectives. For example, calculus is a scientific tool. Application of calculus to find the optimal value of a variable in a mathematical model is a scientific technique and the plan of using mathematical model to optimize the objective is scientific method.

3. Operations Research aims to help executives to make optimal decisions.

4. Operations Research relies mainly on mathematical model.

5. Operations Research depends largely on electronic computers for the analysis of mathematical model.

Limitations of Operations Research

The limitations of Operations Research are generally the same as for any applied scientific discipline. Some of the drawbacks crop up because many a time, it has to deal with people who know very little about its tools and language.

Some important limitations of Operations Research

1. The research nature of undertaking Operations Research is actually a research of complex industrial and business systems. Hence it is time consuming and the results are difficult to control and evaluate.

2. **The inadequate industrial background of practitioners**. Generally the Operations Research experts have to deal with problems in fields wherein they have no previous industrial experience. In such situations Operations Research experts have to study thoroughly the background of the problems. The application of Operations Research without sufficient background information can only result in failures. The strength of this limitation will weaken slowly with time as Operations Research experience develops.

3. **Poor communication of results.** One of the strongest limitations of Operations Research is the inability to transmit and communicate results to business executives who have to make decisions.

Generally the solution obtained by an Operations Research team is in the form of mathematical terminology unknown to managers and executives. This poses a problem of communication. The Operations Research team or expert has to translate the solution in the form of procedures, methods and schedules, so that the executives do not face any difficulties in understanding and executing the same. The Operations Research team generally deals with problems of complex man-machine systems. The solutions to such problems are equally complicated. Even if the solution obtained is precise and logical, it may not be feasible. Feasibility depends on interpretability and workability of the solution. An Industrial engineer having sufficient knowledge of production processes and procedures can help the implementation solution to a great extent.

Linear Programming

Introduction

Linear Programming is a formidable Quantitative technique of Decision making. Such technique is very much useful in the fields of uncertainty, viz., Business and commerce where decisions are taken on every matter. This technique was introduced for the first time in 1947 by a Russian Mathematician George B. Dantzig.

The name 'Linear Programming' consists of the two important terms viz., Linear and Programming. The term Linear refers to the relationship of the interrelated variables which is of the form of $y = a + bx$ where x and y are the variables of power one and 'a' and 'b' are constants.

The term programming means planning a way of action in a systematic manner with a view to achieving some desired optimal results, viz., the minimization, of cost, maximization of profit etc.

Thus Linear Programming is a Mathematical technique which is applied in the form of a linear formula for arriving at a rational proportion of the variables to be used as inputs to get the optimum result from a course of action to be planned accordingly.

Definitions

The term Linear Programming has been defined variously by various authors. Some such definitions are quoted here as under:

According to Dantzig Linear Programming is defined as "a programming of interdependent activities in a linear structure".

According to Galton, "Linear Programming is a mathematical technique for determining the optimal solution of resources and obtaining a particular objective where there are alternative uses of resources, viz., man material, machinery and money etc".

From the above definitions it will be clear that linear programming is a mathematical device of ascertaining the optimal allocation of resources for obtaining the desired objective, viz., maximization of profit, or minimization of cost where various resources can be used alternatively.

Types of Problems (where L.P. can be applied)

There are three important types of Problems concerning various fields where linear programming technique can be applied advantageously. They are:

(1) Problems of allocation
(2) Problems of assignment, and
(3) Problems of transportation.

TYPES OF LINEAR PROGRAMMING PROBLEMS

Linear Programming problems are classified into two types. They are

(1) General or Primal linear programming problem
(2) Duality linear programming problem.

The procedures for determining the desired results under the above two types of problems are contradicting to each other. These procedures are annotated here as under.

1. General or Primal Linear Programming Problem

Determination of the desired results under this type of problem will involve the following steps:

Step No. 1. Formulation of the Given Problem

Step No. 2. Solution of the Formulated Problem

Each of the above steps will again require the following sub steps.

Formulation of L.P. Problem

Under this step

(i) Objective function
(ii) Constraint functions, and
(iii) Non – negative functions.

(i) Objective Function. The objective of a problem may be either to maximize or minimize some result. If it is a case of profit or income or output the objective must be maximization. But if it is a case of loss, cost or input, the objective will be minimization. For this the rate of profit or cost per variable in issue must be assessed first and then the number of each variable will be represented in the function through some letters viz. x, y to be ascertained through the process of solution. As such the objective function will be presented in the following form:

$$Z_{(p)} = P_1X_1 + P_2X_2 + ...+P_nX_n$$

where, $Z_{(p)}$ = Maximum amount of profit

$P_1, P_2, ..., P_n$ = Rate of profit per different variables to be produced, viz., goods or services

$X_1, X_2, ..., X_n$ = The number of different variables to be produced under decision.

In case of variables involving cost or loss the objective will be minimization and in that case the objective function will be formulated as under:

$$Z_{(c)} = C_1X_1 + C_2X_2 + ...+C_nX_n$$

where, $Z_{(c)}$ = Minimum amount of cost

$C_1, C_2, ..., C_n$ = Cost per unit of the variable.

$X_1, X_2, ..., X_n$ = Different number of the different variable..

(ii) Constraint Function. To accomplish the desirable objective it is necessary to put some resources, viz., manpower, material, machine or money in the process of production or performance. But such resources may not be available in unlimited quantity in all the cases. There are some resources which may be available to a limited extent and thus create constraints or bottlenecks in the process of performance. There are also some resources whose availability cannot be obtained below a certain extent and thus compels the management to procure them in certain larger quantities. However, there might be some resources, which may be available to the extent just required for the purpose. These resources, therefore, do not create any obstacle. But the resources which are available up to or beyond certain limit create constraints in achieving the objective. Thus the objective function will be adjusted in the light of the given constraints relating to the availability of the various resources. Thus, the objective function will be flollws by the constraint functions in the following manner.

Constraints of the

Process – I : $a_{11}X_1 + a_{12}X_2 + ... + a_{1n}X_n \leq b_1$

Process – II : $a_{21}X_1 + a_{22}X_2 + ... + a_{2n}X_n \geq b_2$

Process – III : $a_{31}X_1 + a_{32}X_2 + ... + a_{3n}X_n = b_3$

Here a represents the quantity of a particular resource required in particular process, and b represents the total of quantity of the resources available for a process.

(iii) Non – Negative Function. This function implies that the production or performance of the variables in issue will never be negative. It will be either zero or greater than zero but never but never less than zero. This function is therefore represented as under:

$X_1 \geq 0, X_2 \geq 0, ..., X_n$

Thus the formulation phase of a primal linear programming problem will be constituted as below:

Formulation of L.P.P. (Primal):

Maximize or Minimize $Z = C_1X_1 + C_2X_2 + ..., C_nX_n$

Subject to constraints:

$a_{11}X_1 + a_{12}X_2 + ... + a_{1n}X_n \leq b_1$
$a_{21}X_1 + a_{22}X_2 + ... + a_{2n}X_n \geq b_2$
$a_{31}X_1 + a_{32}X_2 + ... + a_{3n}X_n = b_3$
Subject to non-negativity condition that
$X_1 \geq 0, X_2 \geq 0, ..., X_n \geq 0$
Alternatively:
Determine the real numbers X_1, X_2, ... and X_n such that
$a_{11}X_1 + a_{12}X_2 + ... + a_{1n}X_n \leq b_1$
$a_{21}X_1 + a_{22}X_2 + ... + a_{2n}X_n \geq b_2$
$a_{31}X_1 + a_{32}X_2 + ... + a_{3n}X_n = b_3$
$X_1, X_2, ..., X_n \geq 0$
and for which expression (Objective Function)
$Z = C_1X_1 + C_2X_2 + ...+C_nX_n$
may be a maximum or minimum.

Optimal Product Line Problem: An optimal product line problem is one which needs decision as to how much of 'n' different products should a firm produce or sell when each of the products require a particular proportion of various resources, viz., material, labour, machine hour etc. the supply of which are limited to a certain extent.

Illustration: A firm produces two types of products P and Q through two processes, viz., Foundry and Machine shop. The number of manhours required for each unit of P and Q in each of the processes and the number of manhours that can be availed at best in the two processes are given as follows.

	Foundry process	**Machine process**
Product P	10 units	5 units
Product Q	6 units	4 units
Available at best	1000 units	600 units

Net profits expected from each unit of the product are: A – Rs. 50 and B – Rs. 40. Formulate the problem for solution to arrive at the optimal number of the two products P and Q to be produced.

Solution: Here the problem obviously involves the maximization of profits. Thus the formulation of the problem will be made in the following order:

1st Step: Notation

Let	Z =	Total of maximum possible Net profit
	X_1 =	No. of product P to be produced
	X_2 =	No. of product Q to be produced
	F =	Foundry process
	M =	Machine shop process

2nd Step: Decision Table

In this step the given data will be arranged in a table in the following order:

Product	Decision variable	F (process) units of man hour	M (process) units of man hour	Net profit per unit (Rs.)
P Q	X_1 X_2	10 6	5 4	50 40
	Maximum Labour hours available	1000	600 units	

3rd Step: Constitution of the different linear functions

(i) Objective function

Maximize profit $Z = 50X_1 + 40X_2$

(ii) Constraint functions

(a) Foundry constraints $10X_1 + 6X_2 \leq 1000$

(b) Machine shop constraints $5X_1 + 4X_2 \leq 600$

(iii) Non – negative functions: $X_1 \geq 0, X_2 \geq 0$

4th Step: Formulation of L.P.P.

Determine the real numbers X_1 and X_2 such that

$10X_1 + 6X_2 \leq 1000$

$5X_1 + 4X_2 \leq 600$

$X_1 X_2 \geq 0$

and for which the objective function $Z = 50X_1 + 40X_2$ may be a – maximum.

Diet Problem: A diet problem is one in which decision is taken as to how much of 'n' different foods to be included in a diet given the cost of each good and the particular combination of nutrient each food contains. Here, the objective is to minimize the cost of diet such that it contains a certain minimum amount of each nutrient.

Illustration: A poultry firm contemplates to procure four special feeds in a combination which would provide the required vitamin contents and minimize the cost as well. From the following data formulate the linear programming problem.

Feed	Units of vitamins A, B, C in each feed			Feed Cost (Rs.)
	A	B	C	
P	4	1	0	2
Q	6	1	2	5
R	1	7	1	6
S	2	5	3	8

Minimum vitamin contents needed per feed mix in units

A – 12

B – 14

C – 8

Solution: 1st Step: Notations

Let $Z =$ Total of minimum possible costs

$X_1 =$ Decision variable for the feed P

X_2 = Decision variable for the feed Q

X_3 = Decision variable for the feed R

X_4 Decision variable for the feed S

2nd Step: Decision Table

Feed	**Decision variable**	**Vitamins constraints**			**Feed Cost (Rs.)**
		A	**B**	**C**	
p	X_1	4	1	0	2
Q	X_2	6	1	2	5
R	X_3	1	7	1	6
S	X_4	2	5	3	8
Minimum needed		12	14	18	

3rd Step: Constitution of the different Linear Functions.

(i) Objective function

Minimize cost

$$Z = 2X_1 + 5X_2 + 6X_3 + 8X_4$$

(ii) Constraint functions

Vitamin A constraints

$$4X_1 + 6X_2 + X_3 + 2X_4 \geq 12$$

Vitamin B constraints

$$X_1 + X_2 + 7X_3 + 5X_4 \geq 14$$

Vitamin C constraints

$$0X_1 + 2X_2 + X_3 + 3X_4 \geq 8$$

(iii) Non – negative functions:

$$X_1, X_2, X_3, X_4 \geq 0$$

4th Step: Formulation (Table) of the L.P.P.

(Minimize cost) $Z = 2X_1 + 5X_2 + 6X_3 + 8X_4$

Subject to the constraints

$$4X_1 + 6X_2 + X_3 + 2X_4 \geq 12$$

$$X_1 + X_2 + 7X_3 + 5X_4 \geq 14$$

$$0X_1 + 2X_2 + X_3 + 3X_4 \geq 8$$

and subject to the non – negativity condition that

$$X_1, X_2, X_3, X_4 \geq 0$$

Transportation Problem: A transportation problem is one in which decision is taken over a shipping schedule for particular goods, viz., Pig iron, Petroleum, Cotton, Jute etc. from each of a number of production centres at different location to each of a number of marketing centres at different location in such a manner that it would minimize the total shipping cost subject to the constraints.

(i) that demand at each market will be satisfied and

(ii) that supply at the production centre will not be ex**ceeded.**

2nd Step: Decision Table.

Production Centre	Decision variables	Marketing Centre constraints				No. of units that can be supplied
		M	N	O	P	
		X_1	X_2	X_3	X_4	
D	X_1	5	15	7	6	50
E	X_2	8	7	9	1	60
F	X_3	15	25	30	40	90
No. of units that can be demanded		30	40	60	70	200 Total of supply and demand

3rd Step: Constitution of the different Linear Functions.

(i) Objective function. The objective here is to minimize the cost which can be represented as thus

$$\text{Minimize cost } Z = 5X_{11} + 15X_{12} + 7X_{13} + 6X_{14} + 8X_{21} + 7X_{22} + 9X_{23} + X_{24} + 15X_{31} + 25X_{32} + 30X_{33} + 40X_{34}$$

(ii) Constraint functions

Supply constraints

$D = X_{11} + X_{12} + X_{13} + X_{14} = 50$

$E = X_{21} + X_{22} + X_{23} + X_{24} = 60$

$F = X_{31} + X_{32} + X_{33} + X_{34} = 90$

Demand constraints

$M = X_{11} + X_{21} + X_{31} = 30$

$N = X_{12} + X_{22} + X_{32} = 40$

$O = X_{31} + X_{23} + X_{33} = 60$

$P = X_{14} + X_{24} + X_{34} = 70$

(iii) Non – negative functions:

$X_{11}, X_{12}, X_{13}, \ldots$ etc. ≥ 0

4th Step: Formulation of the L.P.P.

$$\text{Minimize cost } (Z) = 5X_{11} + 15X_{12} + 7X_{13} + 6X_{14} + 8X_{21} + 7X_{22} + 9X_{23} + X_{24} + 15X_{31} + 25X_{32} + 30X_{33} + 40X_{34}$$

Subject to the constraints

$X_{11} + X_{12} + X_{13} + X_{14} = 50$

$X_{21} + X_{22} + X_{23} + X_{24} = 60$

$X_{31} + X_{32} + X_{33} + X_{34} = 90$

$X_{11} + X_{21} + X_{31} = 30$

$X_{12} + X_{22} + X_{32} = 40$

Solution of the Formulated Problems

After a linear programming problem has been properly formulated, the next step is to attempt at its solution to determine the values of the different decisive variables, viz., X_1, X_2, X_3 etc. depicted in the formulation of the said linear programming problem.

Solution of the L.P.P. may be of three types, viz. :

(1) Feasible solution,

(2) Non – Feasible solution, and

(3) Optimal solution.

(1) Feasible Solution: A solution which satisfies the non – negativity conditions of a general L.P.P. is called feasible solution.

(2) Non – Feasible Solution: A process of solution which does not satisfy the non – negativity conditions of a general L.P.P. is called Non – feasible solution.

(3) Optimal Solution: A feasible solution which optimizes (minimizes or maximizes) the objective function of a general L.P.P. is called an optimum solution of a general L.P.P.

Methods of Solution of L.P.P.

There are two methods of solving a linear programming problem involving allocations of resources viz.:

(1) Graphic Method, and

(2) Simplex Method.

(1) Graphic Method. A linear programming problem which involves only two decisive variables, viz, X_1 and X_2 can be easily solved by Graphic method. But a problem which involves more than two decisive variables cannot be solved by the Graphic Method. This is because a graph ordinarily has two axis only, i.e., horizontal and vertical and thus more than two decisive variables cannot be represented and solved through a graph.

Procedure for Graphic Solution: For solution of a L.P.P. through a graph the following steps are to be taken up one after another.

1st Step: Formulation of the L.P.P.

2nd Step: Conversion of the constraints functions into the equations and determination of the values of each of the variables under each equation by assuming the other variable to be zero.

3rd Step: Drawal of the 1st quadrants of the graph in which only positive values of both the variables and plotted on the basis of the non – negativity condition, i.e., $X_1, X_2 \geq 0$.

4th Step: Plotting of each set of points on the graph for the pair of values obtained under each of the equations and joining them differently by straight lines.

5th Step: Identification of the feasible region through shaded area which satisfies all the constraints. For "less than or equal to constraints" such region will the below all the constraint lines but for "greater than or equal to constraints" the said region will lie above all the constraint lines.

6th Step: Location of the corner points or the extreme points of the feasible region.

7th Step: Evaluation of the objective function at each of the corner points through the following table.

Evaluation Table

Corner Points	Values of the Variables		Objective Function	Total of Values of
	X_1	X_2	$Z = RX_1 + RX_2$	Z

If the objective function relates to maximization, the corner point showing the maximum value in the above table will give the optimal solution for the values of X_1 and X_2. On the other hand, if the objectives function relates to minimization, the corner point showing the minimum value in the above table will give the optimal solution for the value of X_1 and X_2.

Alternative Method of Evaluation

(i) Plot a line for the objectives function assuming any value for it so that it falls within the shade area of the graph. Such line is known as iso–profit line or ISO – Cost line.

(ii) Move this iso–profit line parallel to itself and farther (closer) from (to) the origin till it goes completely outside the feasible region.

(iii) Identify the optimal solution as the co-ordinates of that pair on the feasible region touched by the highest possible profit line (lower possible cost line).

(iv) Read the optimal co-ordinates of X_1 and X_2 from the graph and compute the profit or the cost.

Illustration (On Maximization): A firm proposes to purchase some fans and sewing machines. It has only Rs. 5760 to invest and space for at most 20 items. A fan costs Rs. 360 and a sewing machine Rs. 240. Profit expected from a fan is Rs. 22 and a sewing machine is Rs. 18. Using graphic method of solution determines the number of fans and sewing machines he dhows purchase to maximize his profit. Also ascertain the maximum possible profit he can ear.

Solution: 1st Step: Formulation of the Problem.

Formulation of the problem can be made directly by the following two steps.

(i) Decision Table

Articles	Decision Variables	Constraints		Profit per unit
		Investment (Rs.)	Space	Rs.
Fan	X_1	360	1	22
Sewing Machine	X_2	240	1	18
	Maximum Capacity	≤ 5760	≤ 20	

(ii) Formulation of L.P.P.

Maximize (Profit) $Z = 22X_1 + 18X_2$

Subject to Constraints

(1) $360X_1 + 240X_2 \leq 5760$

(2) $X_1 + X_2 \leq 20$ and

Subject to non – negativity condition

$X_1, X_2 \geq 0$

2nd Step: Conversion of constraints into equations and determination of the values of the ordinates.

(i) $360X_1 + 240X_2 = 5760$

Here, when $X_1 = 0$ $X_2 = 24$

when $X_2 = 0$ $X_1 = 16$

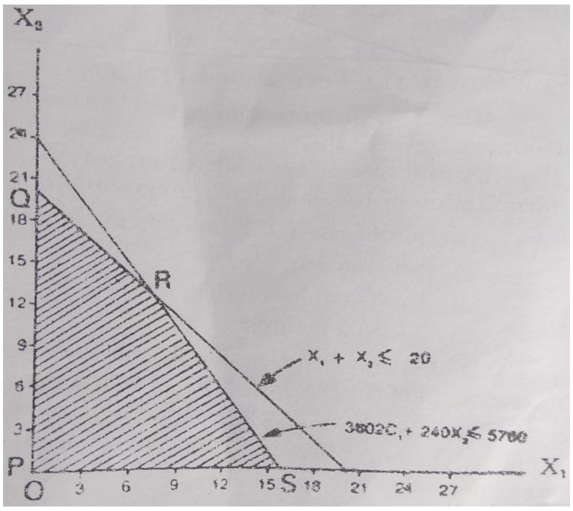
(ii) $X_1 + X_2 = 20$

Here, when $X_1 = 0$ $X_2 = 20$

when $X_2 = 0$ $X_1 = 20$

3rd Step to 6th Step

Graphic Representation of the Constraints lines of the L.P.P.



7th Step: Evaluation of the Objective Function

Corner Points	Values of the Co-ordinates		Objective Function	Total of Values of
	X_1	X_2	$Z = 22X_1 + 18X_2$	Z
P	0	0	$22 \times 0 + 18 \times 0 =$	0
Q	0	20	$22 \times 0 + 18 \times 20 =$	360
R	0	12	$22 \times 8 + 18 \times 12 =$	392 (Max)
s	16	0	$22 \times 16 + 18 \times 0 =$	352

Hence the company should purchase 8 units of X_1 i.e. Fan and 12 units of and X_2 i.e., sewing machine to make the maximum profit of Rs. 392.

Illustration: (On minimization): A firm produces three different products, viz., R, S and T through two different plants, viz., P_1 and P_2 the capacities of which in number of products per day are as follows:

	Product R	Product S	Product T
P_1	3000	1000	2000
P_2	1000	1000	6000

The operating cost per day of running the plants P_1 and P_2 are Rs. 600 and Rs. 400 respectively. The expected minimum demands during any month for the products R, S and T are 24000 units, 16000 units and 4800 units respectively. Show, by Graphic method how many days should the firm run each plant during a month so that the production cost is minimized while still meeting the market demand.

Solution

1. Notation

Let Z = objective function (Minimization of cost)

X_1 X_2 = number of working days of the plants P_1 and P_2 respectively.

2. **Decision Table**.

Name of Plants	**Decision variables**	**Constraints of the Products**			**Cost per day (Rs.)**
		R units	**S units**	**T units**	
P_1	X_1	3000	1000	2000	600
P_2	X_2	1000	1000	6000	400
Minimum demands		24000	16000	48000	

(iii) Formulation of L.P.P.

Objective Function:

$$Z(\text{Minimize}) = 600X_1 + 400X_2$$

Subject to

$3000\ X_1 + 1000\ X_2 \geq 24000$

$1000\ X_1 + 1000\ X_2 \geq 16000$

$2000\ X_1 + 6000\ X_2 \geq 48000$

and $X_1 + X_2 \geq 0$

4. Conversion of the constraints into equations and determination of the values of the different sets of ordinates.

(i) $3000\ X_1 + 1000\ X_2 = 24000$

Let, $X_1 = 0$ then $X_2 = 24$

$X_2 = 0$ then $X_1 = 8$

(ii) $1000\ X_1 + 1000\ X_2 = 16000$

Let, $X_1 = 0$ then $X_2 = 16$

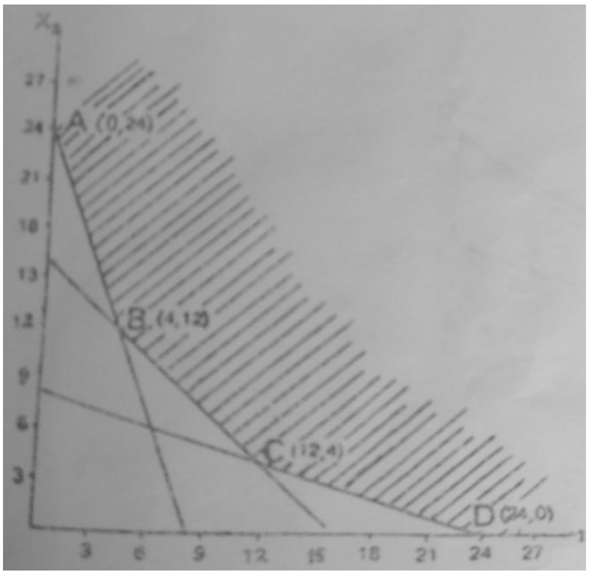
$X_2 = 0$ then $X_1 = 16$

(iii) $2000\ X_1 + 6000\ X_2 = 48000$

Let, $X_1 = 0$ then $X_2 = 8$

$X_2 = 0$ then $X_1 = 24$

5. **Graphic Representation of the Constraints lines of the L.P.P.**



6. Evaluation of the Objective Function by corner points.

Corner Points	Co-ordinates		Objective Function	Total of
	X_1	X_2	$Z = 600\ X_1 + 400\ X_2$	Z
A	0	24	600 × 0 + 400 × 24	9600
B	4	12	600 × 4 + 400 × 12	7200 (Min.)
C	12	4	600 × 12 + 400 × 4	8800
D	24	0	600 × 24 + 600 × 0	14400

From the above evaluation table it comes out that the optimal solution lies at the corner point B, where the total cost is minimum, i.e., 7200. Hence the firm should run the plant I, for 4 days and plant II for 12 days to minimize its cost and to meet with the expected demand of the market as well.

UNIT II
SIMPLEX METHOD

L.P.P. involving three or more variables cannot be solved by graphical method. Such problem can be solved by a method mown as Simplex Method. To use this method a L.P.P. given in canonical form must be converted into a form called standard form in which all constraints are of equality type.

Inequality constraints are changed into equations by adding or subtracting a non-negative variable. If the constraint is ≤, a new variable called slack variable is added and if the constraints is ≥ a new variable, called surplus is subtracted

Ex: If $a_1x_1 + a_2x_2 \leq b$

$a_1x_1 + a_2x_2 + a_3 = b, x_3 \leq b_0$ is the slack variable

If $a_1x_1 + a_2x_2 \geq b$ then

$a_1x_1 + a_2x_2 - a_4 = b,\ x_4 \geq b\ 0$ is the surplus variables.

Used slack or surplus variables, a general L.P.P. can be written in the standard form as follows”

Maximize or Minimize

$z = c_1x_1 + c_2x_2 + ...+ c_n x_n$ ... (1)

Subject to the constraints

$a_{11}x_1 + a_{12} x_2 +......+ a_{1n} x = b_1$

$a_{21}x_1 + a_{22} x_3 +......+ a_{2n} x = b_2$ ... (2)

....

$a_{m1}x_1 + a_{m2} x_2 +......+ a_{mn} x = b_m$

$x_j = 0, j = 1,2, ... n$ ... (3)

Here $b_1, b_2, ... b_m \geq 0$

There are m constraints and n variables (n > m)

A Basic Solution to (2) is obtained by putting (n – m) variables equal to zero and solving for the remaining m variable. Then variables are called basic variables. The (n – m) variables which are made to zero are called non basic variables.

If all the basic variables are non-negative them it is called a basic feasible solution. If all the basic variables are positive, then the solution is called a non-degenerate basic feasible solution. If one or more of the m basic variable are zero, the solution is called a degenerate basic feasible.

For a n variable, m constraints, (n > m) L.P.P. we can find n < m basic solutions. By a well known theory is it can be proved that that one of the basic solution which is feasible will optimize the given objective function. It will be very difficult to find the n < m different solution and than identify the optimum solution among them. It will be time consuming. Simplex method gives us an elegant way of converging to the optimum, solution, starting from a basic feasible solution we check, whether this solution is optimum if not we identify a non-basic variable which may improve the solution and introduce that as a new basic variable acid drop one of the present basic variables. We repeat the process till we get the optimum solution.

The simplex algorithm is given as follows:

Step 1: Write the L.P.P. in the standard form

Step 2: Obtain an initial basic feasible solution, by making (n-m) of the variables equal to zero let $x_{B_1}, x_{B_2} \ldots\ldots x_{B_m}$ be the m basic variables and let $c_{B_1}, c_{B_2} \ldots\ldots c_{B_m}$ be their cost coefficients in the objective function.

Step 3: Compute the net evaluations $z_j - c_j = 1,2,...,1,2,......m_j$ the n variables, by using the relation.

$Z_j - C_j = \sum_{i=1}^{m} C_{Bi} Y_{ij} - C_j$ where y_{ij} are the coefficient of the variable x_j in the m constraints.

Step 4: (i) For a maximization problem, if all $Z_j - c_j \geq 0$ then the initial basic feasible solution is an optimum basic feasible solution.

(ii) For a minimisation problem, if all $Z_j - c_j \geq 0$ then the initial basic feasible solution is an optimum basic feasible solution.

(iii) It atleast one $Z_j - c_j < 0$ for a maximization problem or $Z_j - c_j > 0$ for a minimization problem go to Step 5.

Step 5: (i) For a maximization problem if there are more than one negative $Z_j - c_j$, then choose the most negative of them. Let it be $Z_r - c_r$, for j = r. If $y_{ir} \leq 0$, (i = 1, 2, ... m) then there is an unbounded solution to the given L.P.P.

If at least one $y_{ir} > 0$, (i = 1, 2, ... m) then the non-basic variable x_r will become the new basic variable.

For minimization problem, if $z_j - c_j$, is the most positive net evaluation, the x_r will become the new basic variable.

Step 6: Compute the ratios be $\left\{\frac{x_{Bk}}{y_{ir}}, y_n > 0, i = 1,2, \ldots \ldots m\right\}$ and chose the minimum of them.

Let the minimum of these ratios be, $\frac{x_{Bk}}{y_{kr}}$. Then the variable x_k, will leave the set of basic variables.

The common element y_{kr}, which is in the k^{th} row and r^{th} column is known as the pivotal element of the simplex table.

Step 7: Convert the pivotal element to unity by dividing its raw by the leading element. Itself and all other elements in its column to zeroes by making use of the relations.

$$\hat{y}_{ij} = y_{ij} - \frac{\hat{y}_{kj}}{\hat{y}_{kj}}, i = 1,2, \ldots \ldots m$$

$$\hat{y}_{kj} = y_{kj}/u_{kr}, \quad j = 1,2, \ldots \ldots n$$

Step 8: Go to step 3 and repeat the computations procedure until either an optimum solution is obtained or there is an indication of an unbounded solution.

Example 1:

Solve the L.P.P.

Maximize $z = 5x_1 + 3x_2$

Subject to the constraints

$x_1 + x_2 + 2$

$5x_1 + 2x_2\ 10$

$3x_1 + 8x_2\ 12$

$x_1, x_2 \geq 0$

Let us first convert the problem to standard form by introducing slack variables. Since the slack variables do not contribute anything to the objective function, take the cost for all such variables as zero in the objective function.

Standard form

Maximize $z = 5x_1 + 3x_2 + \mathbf{O}\, x_3 + O\, x_4 + O\, x_5$

Subject to $x_1 + x_2 + x_3 = 2$

$5x_1 + 2x_2 + x_4 = 10$

$3x_1 + 8x_2 + x_5 = 12.\ x_j > 0\ \ j = 1, \dots 5$

An initial basic solution can be obtained by making the variables x_1 and x_2 equal to zero. Then the basic variables are $x_3 = 2.\ x_4 = 10, x_5 = 12$.

The constraint equations, the basic feasible solution and the costs of the variables can be put in the form of a table called simplex table.

x_B denotes the set of basic variables

C_B the cost corresponding to the basic variables.

		5	3	0	0	0	
C_B	X_B	x_1	x_2	x_3	x_4	x_5	$x_{Bi}/y_{11}, y_{11} > 0$
0	$x_3 = 2$	①	1	1	0	0	$\frac{2}{1} = 2$ ←
0	$x_4 = 10$	5	2	0	1	0	$\frac{10}{5} = 2$
0	$x_5 = 12$	3	8	0	0	1	$\frac{12}{5} = 2$
Net evaluations $z_j - c_j$		-5 ↑	-3	0	0	0	

Note that the next evaluation corresponding to the variables x_1 is evaluated as follows.

$$z_j - c_j = (0)(1) + (0)(5) + (0)(3) - 5 = 0 - 5 = -5$$

Similarly the next evaluations corresponding to the other variables are evaluated. x_1 and x_2 have negative net evaluations and hence the current solution is not optimum. We have to find a new basic feasible solution. Among the net evaluations -5 is the most negative.

This corresponds to the variable x_1. Hence x_1 must become a basic variable. To identify the leaving basic variable, evaluate the ratio $\frac{x_{B1}}{y_{11}}, y_{11} > 0.$

i.e., evaluate $\frac{2}{1}, \frac{10}{5}, \frac{12}{3}$. They are 2, 2, 4. The minimum of these ratios as 2. They correspond

to x_3 and x_4. Therefore we can drop either x_3 and x_4 a basic variable. We can use a tie breaker and drop x_3 as a basic variables. The $\boxed{1}$ becomes the pivotal or leading element. Since it is already unit make the other elements in that column to zero. as a basic variables. Note that the column corresponding to the leaving variables x_3 has elements 1.0, 0. The new basic variables xi should have the same elements 1, 0, 0. This can be done by the following row operations, which is the same as the formula given in Step 7 of the simplex algorithm.

In the new simplex table, the variable x_3 will be replaced by x_1. Since the pivotal element is 1 there is no change, in the first row. Row 2 of the new table can be obtained as follows".

$\hat{R}_2 = 5R_1 + R_2$

$5R_1$	= - 10	-5	-5	-5	0	0
R_2	= 10	5	2	0	1	0
$-5R_1+R_2$	= 0	0	-3	-5	1	0

Similarly the new 3rd row can be obtained by $-3R_1 + R_3$. $\hat{R}_3 = -3R_1 + R_3$.

$-3R_1$	= - 6	-3	-3	-3	0	0
R_3	= 12	3	8	0	0	1
	= 6	0	5	-3	1	0

The new simplex table then will be

C_B	X_B	5	3	0	0	0
		x_1	x_2	x_3	x_4	x_5
5	$x_1 = 2$	1	1	1	0	0
0	$x_4 = 0$	0	-3	-5	1	0
0	$x_6 = 6$	0	5	-3	0	1
$z_j - c_j$		0	2	5	0	0

Since all $z_j - c_j \geq 0$ the current solution is optimum, $x_1 = 2$ and $x_2 = 0$ (non-basic variable x_2 has zero value) is the optimum solution and maximum z = 5(2) + 310) = 10.

Example 2

Maximize $z = 4x_1 + 3x_2$

Subject to

$2x_1 + 3x_2 \leq 6$

$-3x_1 + 2x_2 \leq 3$

$2x_2 \leq 5$

$2x_1 + x_2 \leq 4$

Standard form

Maximize $z = 4x_1 + 3x_2 + 0.x_3 + 0.x_4 + 0.x_5 + 0.x_6$

Subject to $2x_1 + 2x_2 + x_3 = 6$

$-3x_1 + 2x_2 + x_4 = 3$

$2x_2 + x_5 = 5$

$2x_1 + x_2 + x_6 = 4, x_j \geq 0, j = 1, 2$ ... (6)

A basic feasible solution is $x_1 = 0, x_2 = 0, .x_3 = 6, x_4 = 5, x_6 = 4$.

Simplex table

		4	3	0	0	0	0		
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_{B1}/y_{11}	$y_{11}>0$
0	$x_3 = 6$	2	3	1	0	0	0	$\frac{6}{2} = 3$	
0	$x_4 = 3$	-3	2	0	1	0	0	-	
0	$x_5 = 5$	0	2	0	0	1	0	-	←
	$x_6 = 4$	②	1	0	0	0	1	$\frac{4}{2} = 2$	
	$z_j - c_j$	-4	-3	0	0	0	0		

When we evaluate the ratio $x_{B1}./y_{1j}$, the radio should be evaluated only if the elements y_{ij} is positive. Since $y_{21} = -3$. $y_{31} = 0$, the corresponding ratio's need not be evaluated. The minimum positive ratio is 2 and hence x_1 should become the new basic variable in the place of x_6 2 is the pivotal element.

New Simplex table is obtained by the following row operations.

$$\hat{R}_4 = \frac{1}{2} R_4$$

$$\hat{R}_1 = R_4 + R_1$$

$$\hat{R}_2 = \frac{3_R}{2} R_2$$

$$\hat{R}_3 = R_3$$

In the above row operations, R_1 should not be obtained by the tow operation $R_4 - R_1$. In this case x_3 will become negative, which is not feasible.

		4	3	0	0	0	0	
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	$x_{Bi}/y_{11}, y_{11} > 0$
0	$x_3 = 2$	0	②	1	0	0	-1	$2 \div 2 = 1$ ←
0	$x_4 = 9$	0	7/2	0	1	0	3/2	$9 \div 7/2 = \frac{18}{2}$

0	$x_5 = 5$	0	2	0	0	1	0	5 ÷ 5/2
0	$x_1 = 2$	1	½	0	0	0	½	9 ÷ 7/2 = 4
$z_j - c_j$		0	-1	0	0	0	2	

↑

-1 is the only negative net evaluation, x_2 enters as a basic variable. The minimum of the ratios x_{B1}/y_{12} is 1 × 3 is dropped as a basic variable.

② is the pivotal element. The new simplex table is obtained by the following row operations.

$$\hat{R} = \frac{1}{2}R_1; \hat{R}_2 = - + \frac{-7}{4}R_1 + R_2; \hat{R}_3 = R_1 + R_3; \hat{R}_4 = -\frac{1}{4}R_1 + R_4$$

		4	3	0	0	0	0
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6
0	$x_1 = 1$	0	1	½	0	0	½
0	$x_4 = \frac{11}{2}$	0	0	$-\frac{7}{4}$	1	0	13/2
0	$x_5 = 3$	0	0	-1	0	1	1
0	$x_5 = 3/2$	1	0	$-\frac{1}{4}$	0	0	3/4
$z_j - c_j$		0	0	½	0	0	3/2

Note that the next evaluation corresponding to the variables x_1, x_2, x_4, x_5, are all equal to zero. For the non basic variables x_3 and x_6, the net evaluations are positive. Hence the current solution is the optimum solution.

The optimum solution is $x_1 = 3/2$. $x_2=1$.

Maximize z = 4(3/2) + 3(1) = 9

Example 3

Maximize $z = x_1 + 3x_2 + 2x_3$

Subject to the constraints

$3x_1 + x_2 + 2x_3 \leq 7$

$-2x_1 + 4x_2 \leq 12$

$4x_2 + 3x_2 + 8x_3 \leq 10$

$x_2 , x_2, x_2 \leq 0$

Standard form

Minimize $z = x_1 + 3x_2 + 2x_3 + 0.4x_4 + 0.x_5 + 0.x_6$

Subject to $3x_1 - x_2 + 2x_3 + x_4 = 7$

$- 2x_1 + 4x_2 + x_5 = 12$

$-4x_1 + 3x_2 + 8x_3 + x_6 = 10$

$x_j = 0, \quad j = 1, 2$ ... (6)

A basic feasible solution is $x_1 = 0$, $x_2 = 0$, $.x_3 = 0$, $x_4 = 7$, $x_5 = 12$, $x_6 = 10$.

Simplex table

C_B	X_B	x_1 (4)	x_2 (-3)	x_3 (0)	x_4 (0)	x_5 (0)	x_6 (0)	x_{B1}/y_{11} $y_{12} > 0$
0	$x_4 = 7$	3	-1	2	0	0	0	
0	$x_5 = 12$	-2	④	0	1	0	0	$12 \div 4 = 3$ ←
0	$x_6 = 10$	-4	3	0	0	0	1	$10 \div 3 = \frac{1}{3}$
	$z_j - c_j$	-1	+3	-2	0	0	0	

↑

For a minimization problem, optimization is obtained when all $z_j - c_j \leq 0$. If we have +ive net evaluations, the variable corresponding to the most +ive net evaluation will become the new basic variable. Here 3 is the only +ive net evaluation. Hence x_2 becomes the new basic variable. Minimum of the ratios x_{B1}/y_{12}, $y_{12} > 0 = 3$. That corresponds to the variable x_5. Hence x_2 enters as the basic variable in the place of x_5.

$\boxed{4}$ is the pivotal element. The new simplex table can be obtained by the following row operations.

$$\hat{R}_2 = \frac{R_2}{4}; \hat{R}_1 = \frac{1}{4}R_2 + R_1; \hat{R}_2 = \frac{3}{4}R_2 + R_3$$

New Simplex table

C_B	X_B	x_1 (4)	x_2 (-3)	x_3 (0)	x_4 (0)	x_5 (0)	x_6 (0)	x_{B1}/y_{11} $y_{12} > 0$
0	$x_4 = 10$	$\boxed{5/2}$	0	2	1	¼	0	$10 \div 3 = \frac{1}{3}$
0	$x_2 = 3$	-½	1	0	0	¼	0	-
0	$x_6 = 1$	-5/2	0	8	0	-¾	1	–
	$z_j - c_j$	-1	0	-2	0	-¾	0	

x_1 enters as a basic variable. In the place of x_4. To get the new simplex table, use the following row operations.

$$\hat{R}_1 = \frac{2}{5}R_1; \hat{R}_2 = \frac{1}{5}R_1 + R_2; \hat{R}_3 = R_1 + R_3$$

The new simplex table is

C_B	X_B	x_1 (4)	x_2 (3)	x_3 (0)	x_4 (0)	x_5 (0)	x_6 (0)
0	$x_3 = 4$	0	0	4/5	2/5	0	0
0	$x_2 = 5$	0	1	2/5	1/5	0	0
0	$x_6 = 11$	0	0	10	1	1	1

$z_j - c_j$	0	0	-12/5	-4/5	-4/5	0

Since all $z_j - c_{j \le 0,}$

$x_1 = 4$, $x_2 = 5$, $x_3 = 0$ (non-basic variables has the value zero) and minimum z = 4(1) + 5(-3) + (0) (2) = - 11.

Example 4

A company makes products A, B and C which flows through three departments: Milling, lathe and grinder. The variable time per unit of different products are given below in hours.

Product	Milling	Lathe	Grinder
A	8	4	2
B	2	3	0
c	3	0	1

The milling, lathe and grinder machines can work for 250. 150 and 50 hours respectively per week. The unit profit would be Rs. 20, Rs. 6 and Rs. 8 respectively for products A, B and C. Find how much of each product the company should produce in order to maximize profit?

Solution

Let the company produce x_1 units of A. x_2 units of B and x_3 of C.

Maximize profit $z = 20x_1 + 6x_2 + 8x_3$ subject to the following time constraints

For milling machine

$8x_1 + 2x_2 + 3x_3 \le 250$

For lathe $4x_1 + 3x_2 \le 150$

For grinder $2x_1 + x_3 \le 50$. $x_1, x_2, x_3, \ge 0$

Standard form

Maximize $z = 20x_1 + 6x_2 + 8x_3 + 0.x_4 + 0.x_5 + 0.x_6$ Subject to

$8x_1 + 2x_2 + 2x_3 + x_4 = 250$

$4x_1 + 3x_2 + 0.x_3 + x_5 = 150$

$2x_1 + 0.x_2 + x_3 + x_6 = 50$

$x_j \ge 0$, 1,2, ... 6

A basic feasible solution is

$x_1 = 0$, $x_2 = 0$, $x_4 = 250$, $x_5 = 150$, $x_6 = 50$

Simplex table is

		20	6	8	0	0	0	↑
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_{B1}/y_{11} $y_{11} > 0$
0	$x_4 = 250$	8	2	3	1	0	0	250 ÷ 8 = 31.25
0	$x_5 = 150$	4	3	0	0	1	0	150 ÷ 4 = 37/5
0	$x_6 = 50$	[2]	0	1	0	0	1	50 ÷ 2 =25 ←
	$z_j - c_j$	-20	-6	-8	0	0	0	

x_1 enters as a basic variable in the place of x_6 [2] is the pivotal element.

The row operations to form the new table are

$$\hat{R}_1 = 4R_3 + R_1; \hat{R}_2 = -R_3 + R_2; \hat{R}_3 = \frac{R_3}{2}$$

The new simplex table is

		20	6	8	0	0	0		
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_{B1}/y_{12} $y_{12} > 0$	
0	$x_4 = 50$	0	2	-1	1	0	-4	$50 \div 2 = 25$	
0	$x_5 = 50$	0	[3]	-2	0	1	-2	$50 \div 3 = 16.67$	←
0	$x_6 = 25$	1	0	½	0	0	½	-	
	$z_j - c_j$	0	-6	2	0	0	10		
			↑						

x_2 enters as a basic variable in the place of x_5 [3] is the pivotal element.
The new row operations to get the simplex table are

$$\hat{R}_1 = \frac{2}{3}R_2 + R_1; \hat{R}_2 = \frac{R_2}{3}; \hat{R}_3 = R_3$$

		20	6	8	0	0	0		
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_{B1}/y_{13} $y_{13} > 0$	
0	$x_4 = \frac{50}{3}$	0	0	1/3	1	-2/3	-8/3	$\frac{50}{3} \div \frac{1}{3} = 50$	
0	$x_3 = \frac{50}{3}$	0	1	-2/3	0	-2/3	-2/3	-	
0	$x_1 = 25$	1	0	½	0	½	½	$50 \div \frac{1}{2} = 50$	←
	$z_j - c_j$	0	0	-2	0	2	6		
									↑

x_3 enters- as a basic variable in the place of x_4.
The row operation are

$$\hat{R}_1 = -\frac{2}{3}R_3 + R_1; \hat{R}_2 = \frac{4}{3}R_3 + R_2; \hat{R}_3 = 2R_3$$

The new table are

		20	6	8	0	0	0
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6
0	$x_4 = 0$	-2/3	0	0	1	-2/3	-3

0	$x_2 = 50$	4/3	1	0	0	+1/3	0
0	$x_3 = 50$	2	0	1	0	0	1
	$z_j - c_j$	4	0	0	0	+2	+8

Since all $z_j - c_j \geq 0$, current solution is optimum

Solution is $x_1 = 0$; $x_2 = 50$, $x_3 = 50$

Maximum z = 0 + 6(50) + 8(50) = Rs. 700

Artificial Variables

If an L.P.P. has constraints with ≥ or = sign, then the simplex method will not give us a basic feasible solution, because we will end up with negative solution, which is not feasible. In such cases, after converting the inequality to equality by subtracting a surplus variable, add another variable to the L.H.S. to get a basic feasible solution. Such a variable that is added to the L.H.S is called artificial variable. For example, if we have a constraint a, x, + a x, > b,, convert this to equality by subtracting the surplus variable x_3.

i.e., $a_1x_1 + a_2 x_2 – x_3 – b_1$

Add another variable x_4 to the LHS. Then we get $a_1x_1 + a_2 x_2 – x_3 – x_4 = b_1$. This will enable us to get a positive basic solution, x_4 is called artificial variable. For problems with ≥ or = sign, we can make use of

i) Big–M method or charmer's Penalty method

ii) Two–phase method.

Big–M Method

For problems with ≥ or = sign, we add artificial variables to the constraints. Their addition causes violation of the corresponding constraints. This difficulty is overcome by ensuring that the artificial variables will be zero in the final solution. This is achieved by assigning a very large per unit penalty to these variables in the objective function. Such a penalty will be designated by – M for maximization problems and + M for minimization problems. Then the L.P.P. is solved by simplex method as usual. But we must ensure that the artificial variables should not be present in the optimum solution at non-zero level. If an artificial variable is present at non-zero level, that indicates the L.P.P. has no solution. This technique is called Big-M method.

Example

Solve the following L.P.P. by Big-M method

Maximize $z = 8x_2 - 5x_3$

Subject to $x_1 + x_3 \geq 2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 = 0$

$x_1, x_2, x_3 \geq 0$

The constrains is standard form are

$x_1 + x_3 – x_4 = 2$

$2x_1 - x_2 + 6x_3 + x_6 = 6$

$x_1 - x_2 + 3x_3 = 0$

Introducing artificial variables x_5 and x_7 in the first and third equation, we get

$x_1 + x_3 – x_4 + x_5 = 2$

$x_1 – x_2 + 3x_3 + x_7 = 0$

The L.P.P. in standard from will be

Maximize $z = 0.x_1 + 2x_2 – 5x_3 + 0.x_4 – Mx_5 + 0x_6 – Mx_7$

Subject to

$x_1 + 0.x_2 – x_3 – x_4 + x_5 \quad = 6$

$2x_1 + x_2 + 6x_3 + x_6 \quad = 6$

$x_1 – x_2 + 3x_3 + x_7 \quad = 0$

$x_j = 0.\ j = 1,2,.....7$

A basic feasible solution is

$x_1 = 0,\ x_3 = 0, x_4 = 0, x_5 = 2, x_6 = 6\ (x_6 = 6) = x_7 = 0$

Simplex table is

			0	2	–5	0	–M	0	M	↑
x_3	C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$x_{B1}/y_{11}\ \ y_{11} > 0$
	–M	$x_5 = 2$	1	0	1	–1	1	0	0	$\frac{2}{1} = 2$
	0	$x_6 = 6$	2	1	6	0	0	1	0	6/6 =1
	–M	$x_7 = 0$	1	–1	[3]	0	0	0	1	0/3 = 0 ←
		$z_j - c_j$	–2M	M–2	–4M+5	M	0	0	0	

enters as a basic variable in the place of x_7 [3] is the pivotal element.

The row operations to form the new simplex table are

$$\hat{R}_1 = \frac{1}{3}R_3 + R_1; \hat{R}_2 = 2R_3 + R_2; \hat{R}_3 = \frac{R_3}{3}$$

			0	2	–5	0	–M	0	M	↑
x_1	C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	$x_{B1}/y_{11}\ \ y_{11} > 0$
	–M	$x_5 = 2$	2/3	1/3	0	–1	1	0	–1/3	2÷2/3 =3
	0	$x_6 = 6$	0	3	0	0	0	1	–2	–
	–5	$x_7 = 0$	[1/3]	–1/3	1	0	0	0	1/3	0 ←
		$z_j - c_j$	–3M –5/3	–M–2 +11/3	0	,	0	0	4/3 M – 5/3	

enters as a basic variable in the place of x_3 [1/3] is the pivotal element.

The row operations are

$$\hat{R}_1 = 2R_3 + R_1; \hat{R}_2; \hat{R}_3 = 3R_3$$

The simplex table is

		0	2	–5	0	–M	0	–M	
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_{B1}/y_{11} $y_{11} > 0$
–M	$x_5 = 2$	0	[1]	–2	–1	1	0	0	$2 \div 1 = 3$
0	$x_6 = 6$	0	3	0	0	0	1	–2	$6 \div 3 = 2$
0	$x_7 = 0$	1	–1	3	0	0	0	1	–
	$z_j - c_j$	0	–M–2	2M+5	M	0	0	M	

↑

↑

Since x_5 and x_6 have the same ratio 2, we can drop either x_5 or x_6 as a basic variable. But preference s

$$\hat{R}_1 = R_1; \hat{R}_2 = -3R_2 + R_2; \hat{R}_3 = R_1 = R_3$$

x_4

		0	2	–5	0	–M	0	–M	
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_{B1}/y_{11} $y_{11} > 0$
2	$x_2 = 2$	0	1	–2	–1	1	0	0	
0	$x_6 = 0$	0	0	6	[3]	–3	1	–2	⟵
0	$x_1 = 2$	1	0	1	–1	1	0	1	–
	$z_j - c_j$	0	0	1	–2	2+M	0	M	

enters as a basic variable in the place of x_6 [3] is the pivotal element.
The row operations are;

$$\hat{R}_1 = \frac{1}{3}R_2 + R_1; \hat{R}_2 = \frac{R_2}{3}; \hat{R}_3 = \frac{1}{3}R_2 + R_3$$

		0	2	–5	0	–M	0	–M
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
	$x_2 = 2$	0	1	0	0	1	1/3	–2/3
0	$x_4 = 0$	0	0	2	1	–1	1/3	–2/3
0	$x_1 = 2$	1	0	3	0	0	1/3	–2/3
	$z_j - c_j$	0	0	5	0	M	2/3	M–4.3

↑

Since all $z_j - c_j \geq 0$, current solution is optimum $x_1 = 2$, $x_2 = 2$, $x_3 = 0$ the optimum solution.

Maximum z = 2(0) + 2(2) + 0(–5) = 4

Example

Using Big-M method, solve the L.P.P.

Minimize $z = 4x_1 + x_2$

Subject to the constraints

$3x_1 + x_2 = 3$

$4x_1 + x_2 \geq 6$

$x_1 + 2x_2 + \leq 3$, $x_1, x_2 \geq 0$

The standard form is

Minimize $z = 4x_1 + x_2 + Mx_3 + 0.x_4 + Mx_5 + 0.x_6$

Subject to

$3x_1 + x_2x_3 = 3$

$4x_1 + 3x_2 – x_4 + x_5 = 6$

$x_1 + 2x_2 + x_6 = 3$

$x_j = 0, j = 1,2,.....6$

A basic feasible solution is

$x_1 = x_2 = 0, x_4 = 0, x_3 = 3, x_5 = 6, x_6 = 2$

Simplex table is

		4	1	M	0	M	0	↑
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_{B1}/y_{11} $y_{11} > 0$
M	$x_3 = 3$	[3]	1	1	0	1	0	$3 \div 3 = 1$ ←
M	$x_5 = 6$	4	3	0	–1	1	0	$3 \div 4 = 1.5$
0	$x_6 = 3$	1	2	0	0	0	1	$3 \div 1 = 3$
	$z_j - c_j$	7M–4	4M–1	0	–M	0	0	

$7M–4x_3$ is the most positive net evaluation. Hence x_1 enters as a basic variable in the place of x_3.

The row operations are

$$\hat{R}_1 = \frac{R_1}{3};\ \hat{R}_2 = \frac{-4}{3}R_1 + R_2;\ \hat{R}_3 = \frac{-1}{3}R_1 + R_3$$

		4	1	M	0	M	0	↑
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_{B1}/y_{11} $y_{11} > 0$
M	$x_1 = 1$	1	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	$1 \div \frac{1}{3} = 3$
M	$x_5 = 2$	0	[5/3]	–4/3	–1	1	0	$2 \div 5/3 = \frac{6}{5}$
0	$x_6 = 2$	0	5/3	1/3	0	0	1	$2 \div 5/3 = \frac{6}{5}$
	$z_j - c_j$	0	5/3 M +1/3	$\frac{-7}{3}M + \frac{4}{3}$	–M	0	0	

Introduce x_2 as a basic variable in the place of x_5 [5/3] is the pivotal element.

The row operations are

$$\hat{R}_1 = \frac{1}{5}\hat{R}_2 + R_1; R_2 = \frac{3}{5}\hat{R}_2;\ \hat{R}_3 =-R_2 + R_3$$

The new simplex table is

x4 as a

		4	1	M	0	M	0	↑
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_{B1}/y_{12} $y_{12} > 0$
4	$x_1 = 3/5$	1	0	3/5	1/5	–1/5	0	$\frac{3}{5} \div \frac{1}{5} = 3$
1	$x_2 = \frac{6}{5}$	0	1	$-\frac{4}{5}$	$-\frac{3}{5}$	$\frac{3}{5}$	0	–
0	$x_6 = 0$	0	0	1	$\boxed{1}$	–1	1	$\frac{6}{5} = 1$ ←
	$z_j - c_j$	0	0	$\frac{8}{3}M$	1/5	$-M-\frac{1}{5}$	0	

enters basic

variable in the place of x_6 $\boxed{1}$ is the pivotal element.

$$\hat{R}_1 = \frac{-1}{5}R_3 + R_1; \hat{R}_2 = \frac{3}{5}R_3 + R_2; \hat{R}_3; \hat{R}_3 = R_3$$

The new simplex table is

		4	1	M	0	M	0
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6
4	$x_1 = 3/5$	1	0	$\frac{2}{5}$	0	0	$-\frac{1}{5}$
1	$x_6 = 6/5$	0	1	–1/5	0	0	3/5
0	$x_4 = 0$	0	0	1	1	–1	1
	$z_j - c_j$	0	0	$\frac{7}{5}M$	0	–M	$-\frac{1}{5}$

↑

Since all $z_j - c_j \leq 0$, current solution is optimum. The optimum solution is $x_1 = 3/5$, $x_2 = 6$.

Maximum z = 4(3/5) + 1(6/5) = 18/5

Two–Phase Simplex Method

This method is an alternate method to solve a given L.P.P. in which some artificial variables are involved i.e. when the constraints are of the type ≥ or = This method involves two phases.

Phase I: Formulate a new problem by replacing the original objective function by the sum of the artificial variables. The new objective function is then minimized subject to the constraints of the original problem has a feasible solution, the minimum value of the new objective function will be zero, which indicates that all the artificial variables are zero. Now go to phase II. Otherwise, if the minimum value is > 0, the problem has no feasible solution.

Phase II: Use the optimum basic solutions of phase I as the starting solution for the original problem. In this case the original objective function must be expressed in terms of the non-basic variables, by elimination technique.

Example

Solve the following L.P.P.

Minimize $z = 4x_1 + x_2$

Subject to $3x_1 + x_2 = 3$

$4x_1 + 2x_2 \geq 6$

$x_1 + 2x_2 \leq 3,$

$x_1, x_2 \geq 0$

by two-phase method

Solution

The standard form of the L.P.P. is

Minimize $z = 4x_1 + x_2$

Subject to $3x_1 + x_2 + x_3 = 3$

$4x_1 + 3x_2 - x_4 + x_5 = 6$

$x_1 + 2x_2 + x_6 = 3,$

$x_j = 0, j = 1,2, ... 6$

Here x_3 and x_4 are artificial variables

Phase I: We have to now minimum $u = x_3 + x_5$(sum of the artificial variables)

Subject to the above conditions

A basic feasible solution is $x_1 = 0, x_2 = 0, x_4 = 0, x_3 = 3, x_5 = 6, x_6 = 3$

The starting simplex table is

		0	0	1	0	1	0	↑
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_{B1}/y_{11} $y_{11} > 0$
1	$x_3 = 3$	[3]	1	1	0	0	0	$3 \div 3 = 1$ ←
1	$x_5 = 6$	4	3	0	–1	1	0	$6 \div 4 = 1.5$
0	$x_6 = 3$	1	2	0	0	0	1	$3 \div 1 = 3$
	$z_j - c_j$	7	4	0	–1	0	0	

7 is the most positive net evaluation and hence x_1 enters as a basic variable in the place of x_3.

[3] is pivotal element

The row operations to get the new table are

$$\hat{R}_1 = \frac{R_1}{3}; \hat{R}_2 = \frac{-4}{3}R_1 + R_2; \hat{R}_3 = \frac{-1}{3}R_1 + R_3$$

The new table is

			0	0	1	0	1	0		
↑ x_2 as a	C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_{B1}/y_{12} $y_{12} > 0$	↓ enters basic
	0	$x_1 = 1$	1	$\frac{1}{3}$	$\frac{1}{3}$	0	0	0	3	
	1	$x_5 = 2$	0	$\boxed{5/3}$	–4/3	–1	1	0	$\frac{6}{5}$	←
	0	$x_6 = 2$	0	5/3	–1/3	0	0	1	$\frac{6}{5}$	
		$z_j - c_j$	0	5/3	–7/3	–1	0	0		

variable in the place of x_5 $\boxed{5/3}$ is the pivotal element.
The row operations are

$$\hat{R}_1 = \frac{1}{5} R_2 + R_1; \hat{R}_2 = \frac{3}{5}R_2\ 6;\ \hat{R}_3 = -R_2 + R_3$$

The new table is

		0	0	1	0	1	0
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6
0	$x_1 = 3/5$	1	0	$+\frac{3}{5}$	$\frac{1}{5}$	$-\frac{1}{5}$	0
0	$x_5 = 6/5$	0	1	$-\frac{4}{5}$	$-\frac{3}{5}$	$\frac{3}{5}$	0
0	$x_6 = 0$	0	0	1	1	–1	1
	$z_j - c_j$	0	0	–1	0	–1	0
			↑				

Since all $z_j - c_j \le 0$, current solution is optimum for u. The optimum solution is $x_1 = 0$, $x_5 = 0$, i.e. the artificial variables are zero.

Phase II: Now we solve the original problem

Minimize $z = 4x_1 + x_2$

Starting with basic solution $x_1 = 3/5$, $x_2 = 6/5$, $x_6 = 0$

We can drop the columns corresponding to x_3 and x_5 as they are artificial variables, which are not present in the basic feasible solution.

The new starting solution is

		0	0	1	0		
C_B	X_B	x_1	x_2	x_3	x_4	x_{B1}/y_{11} $y_{11} > 0$	
4	$x_1 = 3/5$	1	0	$\frac{1}{5}$	0	$\frac{3}{5} \div \frac{1}{5} = 3$	
1	$x_2 = \frac{6}{5}$	0	1	$-\frac{3}{5}$	0	-	
0	$x_6 = 0$	0	0	1	1	$\frac{0}{1} = 0$ $\leftarrow$	$\uparrow$
	$z_j - c_j$	0	0	$+\frac{1}{5}$	0		

x_4 enters as a basic variable in the place of x_6 is the pivotal element.

$$\hat{R}_1 = \frac{1}{5} R_3 + R_1; \hat{R}_2 = \frac{3}{5} R_3 + R_2; \hat{R}_3 = R_3$$

The new table is

		0	0	1	0
C_B	X_B	x_1	x_2	x_3	x_4
4	$x_1 = 3/5$	1	0	0	$\frac{-1}{5}$
1	$x_2 = \frac{6}{5}$	0	1	0	$\frac{3}{5}$
0	$x_4 = 0$	0	0	1	1
	$z_j - c_j$	0	0	0	$\frac{-1}{5}$

For the minimum problem, since all $z_j - c_j \leq 0$, current solution is optimum.
The optimum solution is

$x_1 = 3/5$, $x_2 = 6/5$, and min $z = 4(3/5) + \frac{6}{5} = \frac{18}{5}$

Example 2

Solve the L.P.P.

Minimize $z = 3x_1 + x_2 - x_3$

Subject to the constraints

$x_1 + 2x_2 + 3x_2 \leq 11$

$-4x_1 + x_2 + 2x_3 \geq 3$

$2x_1 + x_3 = -1$

$x_1, x_2\ x_3 \geq 0$

The standard form is

Minimize $z = 3x_1 - x_2 - x_3$

Subject to

$x_1 + 2x_2 + x_3 - x_4 = 11$

$-4x_1 + x_2 + 2x_3 \geq 3$

$2x_1 + x_3 + x_7 = 1$

$x_j \geq 0$, j = 1,2,.....7

In the given third constraint the R.H.S. is negative. It should be always positive. First multiply by a –tive sign and then add artificial variable x_7

Phase I: Minimize $u = x_6 + x_7$ subject to the above conditions

The basic feasible solution is $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_5 = 0$, $x_4 = 11$, $x_6 = 3$, $x_7 = 1$.

The initial simplex table is

		0	0	0	0	0	1	1	↑
C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_{B1}/y_{13} $y_{13} > 0$
0	$x_4 = 11$	1	–2	1	1	0	0	0	11 ÷ 1 = 11
1	$x_6 = 3$	–4	1	2	0	–1	1	0	3 ÷ 2 = 1.5
1	$x_7 = 1$	–2	0	[1]	0	0	0	1	1 ÷ 1 = 1 ←
	z_j - c_j	–6	1	3	0	–1	0	0	

For the maximization problem, 3 is the most positive net evaluation. Hence x_4 enters as a basic variable in the place of x_7 [1] is pivotal element.

The row operations to get the new table are

$$\hat{R}_1 = -R_3 + R_1; \hat{R}_2 = 2R_3 + R_2;\ \hat{R}_3 = R_3$$

			0	0	0	0	0	1	1	↑
x_2	C_B	X_B	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_{B1}/y_{13} $y_{13} > 0$
	0	$x_4 = 10$	3	2	0	1	0	0	–1	
	1	$x_6 = 1$	–2	[1]	0	0	–2	1	–2	1←
	0	$x_3 = 1$	–2	0	1	0	0	0	1	–
		z_j - c_j	–2	1	0	0	–2	0	–3	

enters as a basic variable in the place of x_6

The row operations to get the new table are;

$$\hat{R}_1 = 2R_2 + R_1; \hat{R}_2 = R_2; \hat{R}_3 = R_3$$

The new table is

C_B	X_B	x_1 (0)	x_2 (0)	x_3 (0)	x_4 (0)	x_5 (0)	x_6 (1)	x_7 (1)
0	$x_4 = 12$	–1	0	0	1	–4	3	–5
0	$x_4 = 1$	–2	1	0	0	–2	1	–2
0	$x_3 = 1$	–2	0	1	0	0	0	1
	$z_j - c_j$	0	0	0	0	0	–1	–1

Since all $z_j - c_j \leq 0$, current solution is optimum. Minimum $u = 0$. The values of the artificial variables x_6 and x_7 are equal to zero. We can now go to Phase II.

Phase II: Now we solve the original problem

Minimize $z = 3x_1 - x_2 - x_3$ taking

$x_4 = 12$, $x_2 = 1$, $x_3 = 1$ as a basic solution. We can drop the columns corresponding to x_6 and x_7.

C_B	X_B	x_1	x_2	x_3	x_4	x_5
0	$x_4 = 12$	–1	0	0	1	–4
–1	$x_2 = 1$	–2	1	0	0	–2
–1	$x_3 = 1$	–2	0	1	0	0
	$z_j - c_j$	1	0	0	0	2

Since all $z_j - c_j \geq 0$, for the maximization problem, the current solution is optimum. $x_1 = 0$, $x_2 = 1$, $x_3 = 1$ is the optimum solution

Maximum $z = 3(0) - 1(1) - 1(1) = -2$

Dual Simplex Method

This is another method to solve a L.P.P. having constraints of the hope ≥. The algorithm for this method is given below:

Step 1: Convert all the constraints into the type ≤ and introduce slack variables in the constraints and obtain an initial basic solution.

Step 2: Find the net evaluation $z_j - c_j$ in the starting table

a) For a maximization problem, if all $z_j - c_j \geq 0$ and the basic variables x'_{B_1} are also > 0, then an optimum basic feasible solution has been obtained.

b) If all z, – c . ≥ 0 and atleast one basic variable is negative then to go Step 3.

c) If atleast one, z, – c . < 0, the method is not applicable to the given problem.

Step 3: Select the most negative of the basic variables. Let x_{B_k} be the most negative basic variables. Then x_{B_k} leaves as a basic variables.

Step 4: a) If all the elements y_{kj}, j = 1, 2, ... n ≥ 0 there is no feasible solution to the problem.

b) If atleast one y_{kj} is negative, compute the ratio $\frac{z_j - c_j}{y_{kj}}$ y_{kj}, j = 1, 2, ... n and choose the maximum of these. Then the corresponding variables, say $x_1 \left(\frac{z_r - c_r}{y_{kr}} \text{ is maximum}\right)$enters as basic variable.

Step 5: Test the new iterated dual simplex table for optimality. Repeat the procedure until the optimum feasible solution is obtained.

Note: For minimizing problems, in step 2 the net evaluations must be ≤ 0. In step 4. Atleast one y_{kj} must be negative. If $\frac{z_r - c_r}{y_{kr}}$ is minimum, than x_r will become a basic variable.

Example 1

Using dual simplex method solve the L.P.P.

Minimize $z = 2x_1 + x_2$ Subject to constraints

$3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \leq 3,$

$x_1 - x_2 \geq 0$

Solution

Convert all the constraints to ≤ type

$-3x_1 - x_2 \leq -3$

$-4x_1 - 3x_2 \leq -6$

$x_1 - 2x_2 \leq 3$

Adding surplus variables, the L.P.P. becomes

Subject to $-3x_1 - x_2 + x_3 = -3$

$-4x_1 - 3x_2 + x_4 = -6$

$x_1 + 2x_2 + x_5 = 3$

A basic solution is $x_1 = 0$, $x_2 = 0$, $x_3 = 3$, $x_4 = 6$, $x_5 = 3$. Note that this is not a feasible solution. We cannot apply simplex method, but we can apply dual simplex method. The starting dual simplex table is

		2	1	0	0	0
C_B	X_B	x_1	x_2	x_3	x_4	x_5
0	$x_3 = -3$	–3	–1	1	0	0
0	$x_4 = -6$	–4	–3	0	1	0
0	$x_5 = 3$	1	2	0	0	1
	$z_j - c_j$	–2	–1	0	0	0

↑

Since all z_j - $c_j \le 0$, for the maximization problem, the current solution is optimum, but not feasible. We can apply dual simplex method since two basic variables $x_3 = -3$, $x_4 = -6$, are negative $x_4 = -6$ is the most negative variable. It will be dropped as a basic variable. To find the new basic variable evaluate the ratios $\frac{z_j - c_j}{y_{2i}} y_{2j} \le 0.$

The ratio are $\left[\frac{-2}{-4}, \frac{-1}{-3}\right] = \left[\frac{1}{2}, \frac{1}{3}\right]$ The minimum of these ratios is $\frac{1}{3}$ and that corresponds to variable x_2. Hence x_2 becomes a new variable in the place of $\boxed{-3}$ is pivotal value.

The row operations to get the new dual simplex table are

$$\hat{R}_1 = \frac{1}{3}R_2 + R_1; \hat{R}_2 = \hat{R}_3 = \frac{2}{3}R_2 + R_3$$

The new dual simplex table is

		2	1	0	0	0	
C_B	X_B	x_1	x_2	x_3	x_4	x_5	
0	$x_3 = -1$	$\boxed{-5/3}$	0	1	–1/3	0	←
1	$x_2 = -2$	4/3	1	0	–1/3	0	
0	$x_5 = -1$	–5/3	0	0	2/3	1	
	z_j - c_j	$-\frac{2}{3}$	0	0	$-\frac{1}{3}$	0	

↑

Since all z_j - $c_j \le 0$, for the minimization problem, the current solution is optimum, but not feasible. Both x_3 and x_4 are equal to –1. So we can select any one of them to leave the basic select $x_3 = -1$ leave the basis.

$$Min\left\{\frac{Z_j - C_j}{y_{ij}} y_{ij} \le 0.\right\} = Min\left\{\frac{-2/3}{-5/3}, \frac{-1/3}{-1/3}\right\} = Min\left\{\frac{2}{5}, 1\right\}\frac{2}{5}$$

The corresponds to variable x_1. Hence x_1 enters as a basic variable in the place of x_3 $\boxed{\frac{-5}{3}}$ is the pivotal element.

The row operations are

$\hat{R}_1 = -\frac{-3}{5} R_1; \hat{R}_2 = \frac{4}{5} - R_1 + R_2; \hat{R}_3 = -R_1 + R_3$

The new dual simplex table is

		2	1	0	0	0
C_B	X_B	x_1	x_2	x_3	x_4	x_5
2	$x_1 = 3/5$	1	0	–3/5	1/5	0
1	$x_2 = 6/5$	0	1	4/5	–3/5	0
0	$x_5 = 0$	0	0	–1	1	1
	$z_j - c_j$	0	0	–2/5	–1/5	0

Since all $z_j - c_j \leq 0$, current solution is optimum and also feasible. The solution is $x_1 = \frac{3}{5}, x_2 = \frac{6}{5}$ and minimum $z = 2(3/5) + 1(6/5) = \frac{12}{5}$.

UNIT III
TRANSPORTATION PROBLEM

Transportation Problem is a special case L.P.P. and occurs very frequently in practical life.

Let non-negative quantities a_1, a_2 ... an of a product be available at m places called

$$\text{Subject} \sum_{i=1}^{n} x_{ij} = a_i, \qquad 1 = 1, 2, \ldots\ldots m$$

$$\text{and} \sum_{i=1}^{n} x_{ij} = b_i, \quad 1 = 1, 2, \ldots\ldots m$$

x_{ij} + 0, i ≥ 1, 2 ,,,,,, m, j = 1,2......m

The data can be given in the form of a table, called transportation table. In this table there are mn squares, called cells. The per unit c_{ij} of transporting from the i^{th} origin to the j^{th} designation is in the lower right position of the $(i, j)^{th}$ cell. Any feasible solution to the transportation problem is displayed in the table by entering the value of x_{ij} in the small square at the upper left position of the $(i, j)^{th}$ cell. The various origin capacities and destination requirements are listed in the right most columns and bottom respectively. These are called rim requirements. If the total availability is equal to the total requirement, i.e., if $\sum_{j=1}^{m} a_i = \sum_{j=1}^{n} b_i$, the problem is called a balanced transportation problem, otherwise it is called unbalanced. The above is necessary and sufficient condition for the existence of a feasible to the transportation problem.

Transportation Table

Destinations

		D_1	D_2		D_J		D_n	Availability
	O_1	x_{11} c_{11}	x_{12} c_{13}	c_{13}	x_{1i} c_{ij}		x_{1n} c_{in}	a_1
	O_2	x_{21} c_{21}	x_{22} c_{22}	c_{13}	x_{2i} c_{2j}		x_{2n} c_{2n}	a_2
Origins								
	O_i	x_{11} c_{11}	x_{12} c_{12}	c_{13}	x_{1i} c_{ij}		x_{1n} c_{in}	a_i
	O_m	x_{m1}	x_{m2}	c_{13}	x_{mi}		x_{mn}	a_m

		c_{m1}	c_{m2}		c_{mj}		c_{mn}	
Requirement		b_1	b_2		b_J		b_n	

A set of non-negative variables $x_{ij} > 0$ which satisfies the rim requirements is called a feasible solution.

For a m×n transportation problem, if the number of a feasible is not more then (m+n–1), then the feasible solution is called basic feasible solution. If the number of feasible solution is exactly (m+n–1), then it is called non-degenerate basic feasible solution.

Methods to find basic feasible solution (B.F.S) **to a transportation problem(T.P)**

North West corner rule

Step 1: Start with the North West corner of the transportation table. This cell is (1, 1) has a_1 and b_1 as availability and requirement.

If $a_1 > b_1$, then assign $x_{11} = \min(a_1, b_1)$ in the cell (1,1). The requirement at D_1 is satisfied but the capacity of O_1 completely exhausted. Score out the first column. Move to right horizontally to the Second Column and make the second allocation of magnitude $x_{12} = \min(a_1 - x_{11}, -b_2)$ in the cell (1,2).

If $a_1 > b_1$, assign $x_{11} = a_1$ in the cell (1,1). The availability satisfied. Move vertically below to the nest row to the cell (2,1) until the demand of D_1 is satisfied and allocation $x_{21} = \min(a_2 - b_2 - x_{11,})$ Score out the first row.

If $a_1 = b_1$, the capacity of O_1 is completely exhausted as well as the requirements at D_1 is completely satisfied. There is a tie for second allocation. Allocation can be made in the horizontal or vertical cell. make the second allocation of magnitude $x_{12} = \min\{a_1, a_{13}\ b_2\}= 0$ in the cell (1,2) or $x_{21} = \min\{a_2, b_1 - b_1\}= 0$. Score out either the first column or row.

Step 2

Start from the new north – west corner of the transportation table, satisfying destination requirements and exhausting the origin capacities, one at a time move down towards the lower right corner of the transportation table until all the rim requirements are satisfied.

Example 1.

Determine an initial basic feasible solution to the following T.P. using North West corner rule.

		I	II	III	IV	V	Availability
	A	3	4	8	8	9	20
From	B	2	10	1	5	8	30
	C	7	11	20	40	3	15
	D	2	1	9	14	16	13
Requirement		40	6	8	18	6	

Solution

Total availability = Total requirement = 78

This is a balanced transportation problem

Feasible solution exists

Assign x_{11} = min [20, 40] = 20 in the cell (1, 1). Availability at A is exhausted. Score out the first row.

	I	II	III	IV	V	
A	20 / 3	4	6	8	9	~~20~~
B	2	10	1	5	8	30
C	7	11	20	40	3	15
D	2	1	9	14	16	13
	~~40~~ 20	6	8	18	6	

Next allocate x_{21} = min [30, 20] = 20 in the cell (2, 1). Requirements of I are satisfied. Score out the first column.

	I	II	III	IV	V	
B	20 / 2	10	1	5	8	~~30~~ - 10
C	7	11	20	40	3	15
D	2		9	14	16	13
	~~20~~	6	8	18	6	

Next allocate x_{22} = min [10, 6] = 6 in the cell (2, 2). Requirements of II are satisfied. Score out the second column.

	II	III	IV	V	
B	6 / 10	1	5	5	~~10~~ - 4
C	11	20	40	3	15
D	1	9	14	16	13
		8	18	6	

Next allocate x_{23} = Min [4, 8] = 4 in the cell (2, 3). Availability at B is exhausted. Score out the second column.

	III	IV	IV	
B	4 / 1	5	8	4

C	20	40	3	15
D	9	14	16	13
	~~4~~	18	6	

Next allocate x_{33} = Min [4, 15] = 20 in the cell (3, 3). Requirements of III are satisfied. Score out the third column.

	IV	V	
C	[11] 40	3	~~11~~
D	14	16	13
	7	6	

Next allocate x_{34} = Min [11, 18] = 11 in the cell (3, 4). Availability at C is exhausted. Score out the first column.

	IV	V	
D	[11] 14	[6] 16	13
	7	6	

Finally allocate $x_{44} = 7$, $x_{45} = 6$

Hence a basic solution to the T.P. is

$x_{11} = 20$, $x_{22} = 6$, $x_{23} = 4$, $x_{33} = 4$, $x_{34} = 11$, $x_{44} = 7$, $x_{45} = 6$,

Number of allocation is 8, which is equal to m + n – 1 = 4 + 5 – 1 = 8

Hence the solution the solution the solution is a non-degenerate basic feasible solution.

		II	III	IV	V	
A	[20] 3	4	6	8	9	20
B	[20] 2	[6] 10	[4] 1	5	8	30
C	7	11	[4] 20	[11] 40	3	15
D	2	1	9	[7] 14	[6] 16	13
		8	18	6		

The cells which get allocation are called cells. Total transportation cost according to this allocation = 20(3) + 20(2) + 6(10) + 4(20) + 11940+7(14) + 6(16) = Rs. 878.

Example 2:

Find an initial basic feasible solution to the following T.P. by North West corner method.

		A	B	C	D	Availability
	I	1	5	3	3	34
Origin	II	3	3	1	2	15
	III	2	2	2	3	12
	IV	2	7	7	4	19
Requirement		21	25	17	17	

Solution

Total availability = Total requirement = 80

Allocate x_1 = min [34, 21] = 21 in the cell (1, 1). Requirement of A is satisfied. Score out the first column.

	A	B	C	D	
I	[21] 1	~~4~~	6	~~8~~	~~34~~ 13
B	2	3	1	5	15
C	7	2	2	3	12
D	2	7	2	4	19
	~~21~~	25	17	17	

Allocate x_{12} = min [13, 25] = 13 in the cell (1, 2). Availability of I is exhausted. Score out the first column.

	A	B	C	
I	13 5	3	5	~~13~~
II	3	1	2	15
III	2	2	3	12
IV	7	2	4	19
	~~20~~	6	8	

Next allocate x_{22} = Min [15, 12] in the cell (2, 2). Requirements of B is satisfied. Score out the 2^{nd} column.

	B	C	D	
II	[12] 3	1	2	~~15~~ 3
C	2		3	

		3		12
D	7	2	4	19
	~~12~~	17	17	

Next allocate x_{23} = Min [3, 17] = 3 in the cell (2, 3). Availability of II is exhausted. Score out 2nd row.

	C	D	
II	[3] 2	2	~~3~~
III	2	3	12
IV	2	4	19
	~~17~~ 14	17	

Next allocate x_{33} = Min [12, 14] = 12 in the cell (3, 3). Availability of III is exhausted. Score out 3rd row.

	C	D	
IV	[2] 2	[17] 2	19
	2	17	

Next allocate $x_{43} = 2$, $x_{44} = 17$

The basic solution is

$x_{11} = 20$, $x_{21} = 20$, $x_{22} = 6$, $x_{23} = 4$, $x_{33} = 4$, $x_{34} = 11$, $x_{44} = 7$, $x_{45} = 6$,

Since there are m + n – 1 = 4 + 5 – 1 = 4 + 4 – 1 + 7 basic cells, this is a given in the following transpiration table.

	A	B	C	D	
I	[21] 1	[21] 5	3	3	34
II	3	[12] 3	[3] 1	2	15
III	0	2	[12] 2	3	12
IV	2	7	[2] 2	[17] 4	19
		8	18	6	

According to this allocation the total transportation cost.
= 21(1) + (13) (5) + (12) (3) (1) + 4(12) (2) + (2) + (17)

2. Vogel's Approximation Method (VAM)

Step 1: For each row of transportation table, identify the smallest and next to smallest costs. Determine the difference between them each row. Display them alongside the transportation table by enclosing them in parenthesis against the respective rows. Similarly, compute the difference for each column.

Step 2: Identify the row or column with the largest difference among all the rows and columns. Let the greatest cost in the i^{th} row. Allocate x_{ij} = min (a_i, b_j) in the cell (i, j) and cancel the i^{th} row or j^{th} column.

Step 3: Repeat step 1 and step 2 for the reduced transportation table, until all the rim requirements are satisfied.

Example 1:

Find on initial feasible solution to the following T.P by Vogel's approximation method.

		I	II	III	IV	V	Availability
	A	3	4	6	8	9	20
From	B	2	10	1	5	8	30
	C	7	11	20	40	3	15
	D	2	1	9	14	16	13
Requirement		40	6	8	18	6	

Solution:

In the first row minimum and next minimum cost are 3 and 4. Their difference is 1. In the 2^{nd} row the difference between minimum and next minimum is 2 – 1 = 1. In the 3^{rd} row the difference is 7 – 3 = 4. For the 4^{th} row it is 1. Repeating this for columns, the difference respective differences are 1, 3, 5, 3, 5. Denoting them in the transportation table we get.

	I	II	III	IV	V		
A	3	4	6	8	9	20	(1)
B	2	10	1	5	8	22	(1)
C	7	11	20	40	3	15	(4)
D	2	1	9	14	16	13	(1)
	40	6	8	18	6		
	(1)	(3)	(5)	(3)	(5)		

The maximum of all the difference is 5. That corresponds to the 3^{rd} and 5^{th} columns. 1 is the minimum transportation cost among all the elements of these two columns. First allocation is made to the cell (2, 3) corresponding to this cost. Allocate x_{23} = Min [30, 8] = 8. Cancel the 3^{rd} column. The reduced transportation table and the respective difference of the least and next least cost are given below.

	I	II	IV	V		
A	3	4	8	9	20	(1)
B	2	10	5	8	22	(1)
C	7	11	40	3	15	(4)
D	2	1	14	16	13	(1)
	40	6	18	6		
	(1)	(3)	(3)	(5)		
		↑				

5 is the maximum difference and the least cost in the corresponding column is 3. Allocate x_{35} = Min [15, 6] = 6. Cancel the 5th column and reduce 6 from 15.

The reduce table and the respective difference are:

	I	II	IV		
A	3	4	8	20	(1)
B	2	10	5	22	(1)
C	7	11	40	15	(4)
D	2	1	14	13	(1)
	40	6	18		
	(1)	(3)	(3)		

4 is the maximum difference and 7 is the minimum cost in the 3rd row. Allocate x_{31} = Min [9, 40] = 9 cancel the 3rd row and reduce 9 from 40.

	I	II	IV		
A	3	4	8	20	(1)
B	2	10	5	22	(1)
D	2	1	14	13	(1)
	31	6	18		
	(1)	(3)	(3)		

3 is the maximum difference and I is the minimum cost in the 2nd column. Allocate x_{42} = Min [13, 6] = 6 and cancel the 2nd column. Reduce 6 from 13.

The reduced table and the respective difference are:

	I	IV		
A	3	8	20	(5)
B	2	5	22	(3)
D	2	14	13	(12)
	31	18		
	(1)	(3)		

Allocate x_{41} = Min [7, 31] = 7 and cancel the row corresponding to D.

We now get

	I	IV		
A	3	8	20	(5)

B	2	5	22	(3)
	31	18		
	(1)	(3)		

Allocate x_{12} = Min [20, 24] = 20 and cancel the row corresponding to A.

	I	IV	
B	2	5	22
	4	18	

Allocate x_{21} = 4, x_{24} = 18

The basic feasible solution is

		II	III	IV	V	
A	(20) 3	4	6	8	9	20
B	(4) 2	10	(8) 1	(18) 5	8	30
C	(9) 7	11	20	40	(6) 3	15
D	(7) 2	(6) 1	9	14	16	13
	40	6	8	18	6	

Transportation cost according to this allocation

= (20) (3) + (4) (2) + (8) (1) + (18) (5) + (9) (7) + (6) (3) + (7) (2) + (6) (1)

= Rs. 267

Note: Of all the methods Vogel's methods is the best.

ASSIGNMENT PROBLEM

This is a special case of the transportation problem in which the objective is to assign a number of origins to the equal number of destinations at a minimum cost or a maximum profit.

Suppose there are n jobs in a factory and the factory has n machines to process the jobs. Let c_{ij} be the cost of assignment when the job i (i = 1.2...n) is processed by the machine j (j = 1,2...n). The assignment is to be made in such a way that each job can associate with one and only one machined. Let x_{ij} be the variable defined by

$$x_{ij} = \begin{cases} 0, & \text{if the } i^{th} \text{ job is not assigned to the } j^{th} \text{ machine} \\ 1, & \text{if the } i^{th} \text{ job is assigned to the } j^{th} \text{ machine} \end{cases}$$

$$\therefore \sum_{i=1}^{n} x_{ij} = x_{1j} + x_{2j} + \cdots\cdots\cdots + x_{nj} = 1$$

$$\text{Also} \quad \sum_{i=1}^{n} x_{ij} = 1$$

$$\text{The total assignment cost is } z = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}\, x_{ij}$$

Hence the assignment problem is to find $x_{ij} > 0$ so as to minimize

$z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}\, x_{ij}$ subject to the constraints.

$$\sum_{j=1}^{n} x_{ij} = 1$$

$$z = \sum_{i=1}^{n} x_{ij} = 1 \quad i,j = 1, 2, \ldots n$$

This is nothing but a special case of transportation problem.

We are not, going to solve an assignment problem by MODI method. Instead this problem can be solved by another elegant method called Hungarian method.

HUNGAIRAN METHOD TO SOLVE ASSIGNMENT PROBLEM

Step 1: If the problem is a minimization problem, subtract the minimum cost of each row of the cost matrix, from all the elements of the respective row. Then modify the resulting matrix by subtracting the minimum cost of each column from all the elements of the respective column, to get the starting matrix.

Step 2: Draw the least possible number of horizontal and vertical lines to cover all the zeros of the starting table. Let the number of these lines be m.

If m = n. the order of the cost matrix, then the optimum assignment can be made. Go to step 5. If m < n, go to step 3.

Step 3: Determine the smallest cost in the starting table, not covered by the m lines. Subtract this cost from all the surving (uncovered by lines) elements of the starting matrix and add the same to all those elements of the starting matrix which are lying at the intersection of horizontal and vertical lines, thus obtaining the second modified cost matrix.

Step 4: Repeat the steps 1, 2, and 3 until we get m = n.

Step 5: Mark the zeros of the optimum cost matrix In a separate table. Examine the rows successfully until a row with exactly one zero is found- Enclose this zero inside a circle and an assignment will be made there. Make a cross (x) in the cells of all other zeros lying in this column of the encircled zero, to show that they cannot be considered for future assignment. Continue in this manner until all the rows are taken care of.

Step 6: Examine the columns successfully until a column with exactly one unmarked zero is found. Encircle that zero. An assignment will be made there. Make a cross in the cells of all other zeros lying in the row of encircled zero. Continue in this way until all the columns have been taken care of.

Step 7: Repeat steps 5 and 6 successfully until one of the following arises.

i) no unmarked zero is left

ii) there lies more than one unmarked zero in one column or row.

In case (i). The algorithm steps.

In case (ii) encircle one of the unmarked zeros arbitrarily and mark a cross in the cell remaining zeros in its row and column. Repeat the process until no unmarked zero is left.

Step 8: We now have exactly one encircled zero in each row and each column of the cost matrix. The assignment schedule corresponding to these zeroes is the optimal assignment.

Note: In case of maximization problem, make all the elements of the cost matrix negative and convert it into minimization problem.

Example 1:

Consider for problem of assigning five jobs to five persons. The assignment costs are given below:

		Job				
		1	2	3	4	5
	A	8	4	2	6	1
	B	0	9	5	5	4
Persons	C	3	8	9	2	6
	D	4	3	1	0	3
	E	9	5	8	9	5

Determine the optimum assignment schedule.

Solution: From each row of the cost matrix the minimum cost and subtract that from all the elements of the respective row.

We get

7	3	1	5	0
0	9	5	5	4
1	7	8	0	4
4	3	1	0	4
4	0	3	4	0

Now, subtract the minimum cost of each column from all the elements of that column. We get

--7 –	3 --	0 –	5 –	0--
0	9	4	5	4

1	7	4	0	4
--4 –	3 –	0 –	0 –	3--
--4 –	0 –	2 –	4 –	0--

Now we draw the minimum possible number of horizontal and vertical lines so as to cover all the zeros.

--7 –	3 --	0 –	5 –	0--
0	9	4	5	4
1	7	4	0	4
--4 –	3 –	0 –	0 –	3--
--4 –	0 –	2 –	4 –	0--

Since the minimum of lines used 5 = order of the cost matrix, an optimum assignment can be made.

Now consider the zeros of the above matrix.

	1	2	3	4	5
A			O		O
B	O				
C				O	
D				O	
E		O			O

Now examine the row successively to find out a row with exactly one zero and mark it with a circle and cancel any other zero in the respective columns with a cross. Again examine the zeros successively to find out the row with only me unmarked zero.

	1	2	3	4	5
A			⊠		⊙
B	⊙				
C				⊙	
D			⊙	⊠	
E		⊙			⊠

In this table, each row and each column has exactly one encircled zero. Hence the optimum assignment is achieved. The optimal assignment is

A --> 5, B --> 1, C --> 4, D --> 3. E--> 2

The minimum assignment cost = 1 + 0 + 2 + 1 + 5 = 9

Example 2:

Solve the following assignment problem to minimize total cost of assignment.

6	5	8	11	16
1	13	16	1	10
16	11	8	8	8
9	14	12	10	16
10	13	11	8	16

Solution select the minimum cost from each row and subtract it from the other elements of the row we get.

1	0	3	6	11
0	12	15	0	9
8	3	0	0	0
2	5	3	0	8

Since all the columns have zero elements, we will get the same matrix for the column operation. Therefore, the above is the starting matrix. Now we cover the zeros by horizontal and vertical lines.

--1	0	3	6	11--
0	12	15	0	9
--8	3	0	0	0--
0	5	3	1	7
2	5	3	0	8

All the zero are covered by 4 lines which is less then 5, the order of the cost matrix. Optimal assignment cannot be made. Now modify this matrix by subtracting the least element among the surviving elements, and adding the same at the inter section of horizontal and vertical lines.

3 is the least element. The modified matrix is

4	0	3	9	11
0	9	13	0	6
11	3	0	3	0
0	2	0	1	4

2	2	0	0	5

Now we cover the zeros by horizontal and vertical lines.

--4 –	0 –	3 –	9 –	11--
0	9	13	0	6
--11 –	3 –	0	3 –	0--
0	2 –	0 –	1 –	4
2	2 –	0 –	0 –	5

Now we have lines covering all the zeros. Hence optimum assignment can be made. The table of zero is written now.

	I	II	III	IV	V
A		⊙			
B	O			O	
C			O		O
D	O		O		
E			O	O	

In the cell (1, 2) we have a single zero. Enclose it by a circle. All the other rows have two zeros. Now check the columns. Column 5 has a single zero. Enclose it and put a cross for the cell (3, 3).

	I	II	III	IV	V
A		⊙			
B	O			O	⊙
C			⊗		
D	O		O		
E			O	O	

Now each row or column has two zeros. We have encircle a zero arbitrarily. In such cases the problem has more then one solution. Encircle the cell (2, 4) and score out cell (2, 1) and (5, 4).

		II	III	IV	V
A		⊙			
B	⊗			⊙	
C			⊗		⊙
D	O		O		
E			O	⊗	

Now 5th row has a zero. Encircle and score out cell (4, 3) cell (4, 1) is the only zero left. Encircle it.

		II	III	IV	V
A		⊙			
B	~~O~~			⊙	
C			~~O~~		⊙
D	⊙		~~O~~		
E			⊙	~~O~~	

Now only one zero has a circle around it in every row or column.

$\therefore$ optimum is achieved. The optimal assignment is

A $\rightarrow$ III, B $\rightarrow$ IV, C $\rightarrow$ V, D $\rightarrow$ I, E $\rightarrow$ III

Minimum cost assignment is = 5 + 1 + 8 + 9 + 11 = 34.

UNIT IV
GAME THEORY

Meaning, Definition and Characteristics of Game Meaning

The name ‘Game Theory’ seems interesting as it given an impressionto layman to mean a body of knowledge relating to the conduct of the various games played by the people. Although, to some extent this sense holds good, but in reality it does not relate to the conduct of any game or play. Lexically, speaking, ‘Game’ means some play or gambling between two or more individuals either with or without some pay offs (i.e., settlement through payments). But broadly speaking, the term ‘game’ refers to any conflicting situation in which two or more individuals or organizations compete with each other to gain some thing at the cost of the other or others. The examples of such a situation are competition between the business units, military battles, election contests between some candidates, gambling, horse-race, cross word puzzles etc.

Definition

The term game theory may be defined as a conflicting situation between two or more independent competing entities individuals or organizations) who carry on certain activities according to a set of rules with a view to gaining some benefits or satisfaction at the cost of the opponents.

Characteristics

From the above definition, the essential characteristics of a game may be brought out as under:

1. There must be some interest of conflicting nature that calls for competition between some persons.
2. There must be some competitors who may be individuals or any organization and whose number is finite and countable.
3. There must be several alternative courses of action available to each of the competitors.
4. There must be a set of rules for taking up a course of action and they must be known to each of the competing participants.
5. The course of actions must be taken independently and simultaneously by each of the competitors before knowing the line of action taken by his competitors.
6. There must be some outcome of all the different courses of action taken up by the competitors and all such outcomes must be qualitatively defined and made known to each of the competitors much in advance of the game.
7. The outcome of the game must be affected by the course of action taken up the different participants and they must result in a set of payments by one to the other.
8. There is an assumption of rationality that each competitor will act so as to maximize his minimum gain or minimize his maximum loss.

Assumptions of Game Theory

As pointed out above, the game theory works or certain assumptions which are as under:

1. It is assumed that 'as if each competitor behaves rationally and intelligently to optimize his objective in the game. As such, this theory is also called as the "as if theory" which implies that as if rational decision makers behave in some well defined way such as maximizing the minimum gain and minimizing the maximum loss.
2. It is assumed that each participant in the game has available to him a finite set of possible course of actions.
3. It is assumed that each player is in know of all the information relating to the game he plays.
4. It is assumed that each player attempts to maximize the gains and minimize the losses for himself and minimize the gains and maximize the losses for his opponents.
5. It is assumed that each competitor takes his course of action simultaneously with that of his opponents.
6. It is assumed that each player takes his decision individually in a skilled manner without any direct communication with his opponents.
7. It is assumed that the pay offs (payments to be made in settlement of a game) are already fixed in relation, to the different courses of actions of the players i.e., the participants.

Definition of the Terms used in the Game Theory

For a clear understanding of the game theory it is necessary to have full knowledge of the various terms technically used in such theory. These terms are defined here as under.

(i) **Game.** It refers to any field of conflicting situation to which the competitors are attracted to optimize their objective in terms of profit, benefit, expenses or losses etc. at the cost of their opponents.

(ii) **Play.** It refers to a course of action taken up by a competitor in a game.

(iii) **Player.** It refers to a competitor or participant who takes active part in a game.

(iv) **Pay-off**. It means quantitative measure of satisfaction which a player gains or losses at the end of play. This is usually expressed in the form of rupees or numbers.

(v) **Strategy**. It refers to the skill or predetermined rules by which a competitor selects his/her course of action from a list of course of actions, during a game.

(vi) **Pure Strategy**. It refers to a decision taken in advance of all the plays to choose a particular course of action. It is usually represented by a number with which a course of action is associated.

(vii) **Mixed Strategy**. It refers to a decision taken in advance of all plays to choose of action for each play in accordance with some fixed proportions.

(viii) **Pay-off matrix**. It refers to a table of pay-offs or outcomes of different strategies of a game. The following is the specimen of a pay off matrix:

Pay-off matrix

Minimizing player Y

Maximizing Player X		Y_1	Y_2	Y_3	Y_4
	X_1	10	15	20	25
	X_2	5	6	0	8
	X_3	7	9	1	3

(ix) Optimal Strategy. It refers to the best possible strategy or a plan of action adopted by a competitor that gives him the maximum possible benefit. In terms of the pay off matrix beyond, it also refers to the most possible reduced from of the pay off matrix beyond which the strategies can not be altered.

(x) **Maximin.** It refers to the maximum of the minimum values of a row. It is usually indicated by a box □.

(xi) **Minimax.** It refers to the minimum of the maximum values of a column. It is usually indicated by a circle O.

(xii) **Value of game.** It means the sum of the pay-offs expected by the players at this level of their optimum strategy. In a game with saddle point(s), the pay off at the saddle point [o] is called the value of the game and it is obviously equal to the maximum and minimax of the game. In other cases, it is the total of the pay-offs expected by the players at the level of their optimum strategies. The value of a game for a winning player is positive whereas for a losing player it is negative.

(xiii) **Saddle Point.** It refers to the cell(s) of a pay-off matrix that contain(s) the element equal to the maximum of the minimum values of the rows and the minimum of the maximum values of the columns. For convenience sake, it is indicated by the mark [o] . In a two persons zero-sum game the element at the saddle point is the value of the game. Thus, there exists a saddle point in a pay-off matrix, where the maximum is equal to the minimax. There can be more than one saddle point in a particular game. A saddle point is also, otherwise called as the equilibrium point of a pay-off matrix.

Rules of determining a saddle point

1. Select the minimum element of each row of a pay-off matrix and put them, in a column to the right. Also put a circle O over all such minimum elements selected including the ones that appear in the pay-off matrix.
2. Select the maximum element of each column of the pay off matrix and put them in a row at the bottom of the table. Also put a box □ over all such maximum elements selected including the ones that appear in the pay-off matrix.
3. Select the cell(s) that contain(s) the element(s) marked with both a circle and a box [o] . The cells, thus selected constitutes the saddle point(s) and the element of such cell(s) is the value of the game.

Illustration 1: Determine the saddle-point(s) of the following matrix:

Pay-off Matrix

B / A	B_1	B_2	B_3	B_4
A_1	5	15	7	8
A_2	13	12	10	20
A_3	25	30	15	40
A_4	60	50	10	8

Solution: Following the rules of saddle point the given matrix is marked as under

Marked-up Matrix

B / A	B_1	B_2	B_3	B_4	Row minimum	
A_1	5	15	7	8	5	
A_2	13	12	10	20	10	
A_3	25	30	15	40	15	←Maximin value
A_4	60	50	10	8	8	
Column maximum	60	50	15	40		
			↑ Minimax Value			

From the above marked up matrix we find that the element 15 in the cell (3, 3) is covered by both the marks o and is equal to both the maximum and minimax values shown in the row minimum column and column maximum row of the matrix. Hence, this cell, (3, 3) constitutes the saddle point and the value of the game is 15. This indicates that the player A will gain Rs. 15 if he takes up the A_3 course of action. This also indicates that the player B will loose the same Rs. 15 if he adopts the B_3 course of action when A adopts the A_3 course of action.

Illustration (on multi saddle points). Find the saddle point(s) from the following pay off matrix.

Minimizing Player Y

Maximizing Player X

Y / X	Y_1	Y_2	Y_3
X_1	–2	15	–2
X_2	–5	–6	–4
X_3	–5	20	–8

Solution: Applying the saddle point rules the given matrix is marked as under:

Minimizing player Y

Maximizing Player X

Y / X	Y_1	Y_2	Y_3	Row minimum	
X_1	– 2	15	– 2	–2	Maximin value
X_2	– 5	– 6	– 4	–6	
X_3	– 5	20	– 8	– 8	
Column Max.	– 2	20	2		

Minimax Values

From the above marked matrix we find that there are two cells (1, 1) and (1, 3) that contain the element – 2 covered by both the mark o which is equal to both the maximum and minimax indicated therein. This shows that there are two saddle points in the given game viz., cell (1,1) and cell (1, 3), where

(i) the best strategy for the player X is X_1

(ii) the best strategy for the player Y is both Y_1 and Y_3.

(iii) the value of the game is – 2 for X and +2 for Y.

Game of Dominance

A game of dominance is one to which the rules of dominance can be applied to reduce the size of its pay off matrix for making it convenient for solution. In the pay off matrix of such a game each element of one or more rows (columns) is less than or equal to the corresponding elements of some other rows (columns) or their averages. A row containing all the bigger and the equal elements, and a column containing all the smaller and the equal elements is called a dominating row and column respectively. On the other hand, a row containing all the smaller and equal elements and a column containing all the all the bigger and equal elements is called a dominated row or column respectively. According to the rule of dominance all the dominated rows and columns of a pay off matrix are to be deleted to reduce its size to an easily workable form. The rules of dominance read as follows.

Rules of dominance

1. If the strategy of a player dominates over the other strategy in all conditions then the latter strategy (dominated row or column) can be ignored as it will not affect the solution in any way.
2. If all the elements in a row are less than or equal to the corresponding elements in another row or the average of the corresponding elements of some other rows, then that row is dominated and hence, is liable to be deleted.
3. If all the elements in a column are greater than or equal to the corresponding elements in another column or the average of the corresponding elements of some other columns, then that column is dominated and hence is liable to be deleted.
4. If any row (column) dominates the average of some other rows (columns) then one of the rows (columns) thus averaged may be deleted from the matrix.
5. Such rules of dominance can be applied to both the problems of pure strategy and mixed strategy to reduce the size off a pay of matrix to 2×2 as far as possible beyond which it need not be reduced.

Illustration: Indicate, which of the following pay of matrices represent (i) game of dominance and (ii) game of non-dominance. Also, reduce the game of dominance to the possible extent by applying the rules of dominance.

Pay-off Matrix

(a)

Player B

	A \ B	B_1	B_2	B_3
Player A	A_1	2	4	6
	A_2	10	3	8
	A_3	4	7	3

(b)

B / A	B_1	B_2	B_3	B_4
A_1	7	6	8	9
A_2	–4	–3	9	10
A_3	3	0	4	9
A_4	10	5	–2	0

Solution: (i) The pay off matrix given under (a) represents a game of non-dominance since it does not contain any row (column), each of the elements of which is less than or equal to the corresponding elements of other row (column) or to the average of corresponding elements of any other row (column). Hence, such a matrix cannot be reduced by the rules of dominance.

(ii) The pay off matrix given under (b) represents a game of dominance as it contains some dominated rows and columns which can be deleted to reduce its size of 2×2 matrix as under:

In the first instance: We observe that each of the elements in the row A_3 is less than or equal to its corresponding element in the row A_1. Thus, row A_3 is dominated by row A_1. Hence we can delete the row A_3 and get the matrix reduced as under:

B / A	B_1	B_2	B_3	B_4
A_1	7	6	8	9
A_2	–4	–3	9	10
A_4	10	5	–2	0

In the second instance: We observe that each of the average values of row A_2 and row A_4 is less than the corresponding value of the row A_1 as under:

	B_1	B_2	B_3	B_4
A_1	7	6	8	9
$(A_2 + A_4)/2$	3	1	3.5	5

Thus, the row A_2 and A_4 involved in the average are dominated by the row A_1. Hence, we can reduce any of these rows say A_2 and get the matrix reduced as under.

B / A	B_1	B_2	B_3	B_4
A_1	7	6	8	9
A_4	10	5	–2	0

In the third instance: Now, we observe that each of the elements in the column B1 is greater than the corresponding element in the column B_2. Thus, B_1 is dominated by B_2. Hence, we can delete the columns B_1 and get the matrix reduced as under.

B / A	B_2	B_3	B_4
A_1	6	8	9
A_4	5	–2	0

In the 4th Instance: Now, we see that there is no such column, each of the elements of which is greater than or equal to the corresponding element in another column or the corresponding average elements of some other column. Thus, the above matrix cannot be reduced any further. Hence, the possible reduced form of the given matrix (optimal strategy of the two persons A and B) is given as above.

Note: Now, we can compare the two matrices (original one and the reduced one) with reference to their maximum value of the minimum of the rows and the minimum value of the maximum of the columns and see that both of them produce the same result as under:

Original Matrix

B / A	B_1	B_2	B_3	B_4	Row minimum	
A_1	7	6	8	9	6	← Maximum
A_2	–4	–3	9	10	–4	
A_3	3	0	4	9	0	
A_4	10	5	–2	0	–2	
Column Max.	10	6	9	10		
		↑ Minimax				

Reduced Matrix

B / A	B_2	B_3	B_4	Row minimum	
A_1	6	8	9	6	← Maximum
A_4	5	–2	0	–2	
Column Max.	6	8	9		
	↑ Minimax				

Graphic Method

Under this method the given pay off matrix is reduced to 2 × 2 matrix by means of graphs and then the same is solved by any other method to arrive at the probable values of the mixed strategy of the players and the value of the game.

The applicability of this method, however, is subject to the following conditions.

(1) The pay off matrix must be of the order of 2 × n or m × 2.

(2) There must not be a saddle point in the problem.
(3) There must not be the possibility of reducing the matrix any further by the rules of dominance.

Steps: This method involves the following steps to be taken up in turn.

1. Represent the probabilities of the the player with only two strategies by p_1 and p_2 respectively the sum of which is equal to one (i.e., $p_1 + p_2 = 1$).
2. Represent the two pure strategies I and II of a player by two parallel lines drawn one unit apart on the Y-axis and mark a natural scale on each of them upto the maximum value given in the matrix.
3. Represent the probability line by P–P with a unit length drawn on the X–axis, touching the points of origin of the two parallel lines drawn as above.
4. Put the dots for each of the coordinated values of the different strategies of the other player, B at the proper points of the two respective parallel lines. This will be put as under:
   Value of e_{11} on the line 1 and value of e_{21} on the line II.
   Value of e_{12} on the line 1 and value of e_{22} on the line II.
   Value of e_{13} on the line 1 and value of e_{23} on the line II.
   Join the coordinated dots thus put by straight lines and represent them as the respective pay off lines.
5. In case of a 2 × n problem, mark the lower boundary of the pay off lines by a thick line and mark the highest point of intersection on this line by P to indicate the optimum value of P_1 (maximin).
   But in case of m × 2 problems, mark the upper boundary of the pay off lines by a thick line and mark the lowerst point of intersection on this line by P to indicate the optimum value of p1 (minimax).
6. Take up the two lines that pass through the point P to represent the two optimum strategies for the other player and thereby form a reduced matrix of 2 × 2 order.
7. Solve the reduced matrix so obtained by any method (preferably arithmetic method) and obtain the different values of the optimal solution.

Note: In case, there are more than two lines passing through maximum/minimax point P, select any two lines for forming the reduced matrix whose sum of the vertical oddments is equal to the sum of the horizontal oddments.

Illustration: Reduce the following game to 2 × 2 by graphic method and solve the same to find the relevant optimal results.

	B			
A	1	4	–2	–3
	2	1	4	5

Solution: The pay off matrix thus given is :

		B					
		B_1	B_2	B_3	B_4		
	A_1	1	4	–2	–3	–3	
A	A_2	2	1	4	5	①	←Maximin

2	4	4	5

↑ Minimax

1. **Tests**: From the above matrix it is observed
   (i) that the game is in 2 × n order
   (ii) that there is no saddle point in the problem, and
   (iii) that the matrix cannot be reduced further by any rule of dominance.

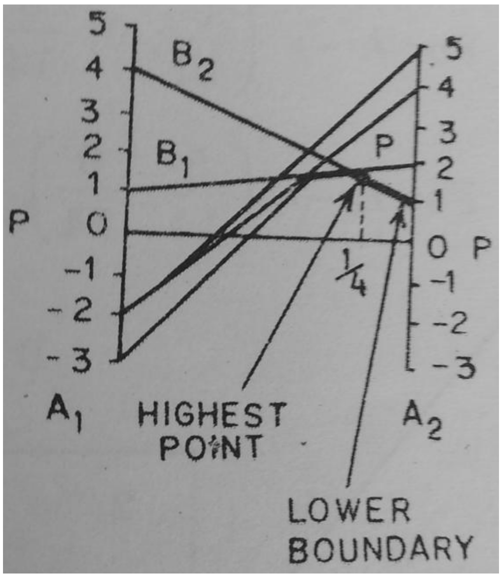
Since all the above conditions are satisfied, we can now proceed with the graphic method to reduce the game to (2 × 2) as under:

2. Here, the player A has two strategies viz., A_1 and A_2. Representing the probability of his strategies by p_1 and p_2 respectively, we get the coordinated values of the various pay off lines as under:

	A_1	A_2
For the pay off line B_1	1	2
For the pay off line B_2	4	1
For the pay off line B_3	–2	4
For the pay off line B_4	–3	5

3. Representing the strategy A_1 by the scalled line A_1, the strategy A_2 by the scalled line A_2 the probability line of a unit length by P - $P_ι$, and drawing the pay off lines with their coordinated values as above we get the graphic representation of the game as follows:

Graphic Representation of the Game



4. **Analysis:** From the above graph it is revealed that the point P indicates the point of maximum when the two pay off lines B_1 and B_2 intersect each other. These two lines represent the two optimum strategies for B. Thus, the 2 × n game is reduced to 2 × 2 game as under:

		B 1	B 2
A	1	1	4
	2	2	1

5. Solving the above 2 × 2 matrix by the arithmetic method, we get,

		B 1	B 2	Oddment	Propn.
A	1	1	4	1	¼(p_1)
	2	2	1	3	¾ (p_2)
	Oddment	3	1	4	
	Propn.	¾ (q_1)	¼ (q_2)		

6. **Interpretation:** thus, the various optimal results of the game are as under:

Optimum strategies for A : $\left(\frac{1}{4}, \frac{3}{4}\right)$

Optimum strategies for B : $\left(\frac{1}{4}, \frac{3}{4}, 0, 0\right)$

Value of the game = $\frac{1 \times 1 + 2 \times 3}{1 + 3} = \frac{7}{5}$

Maximum gain for A = $\frac{7}{4}$

and Maximum loss for B = $-\frac{7}{4}$

Illustration: Reduce the following game to 2 × 2 by graphic method and find the various optimum results.

	B	
A	–6	7
	4	–5
	–1	–2
	–2	5
	7	–6

Solution: The pay off matrix given is:

		B 1	B 2	Row min.
A	1	–6	7	–6
	2	4	–5	–5
	3	–1	–2	–2 (Maximum)
	4	–2	5	–2 (Maximum)

	5	7	–6	–6
Col.	Max.	7	7	

Minimax

1. **Test:** From the above matrix it is noticed.

(i) that the game is in m × 2 order.

(ii) that there is no saddle point in the problem

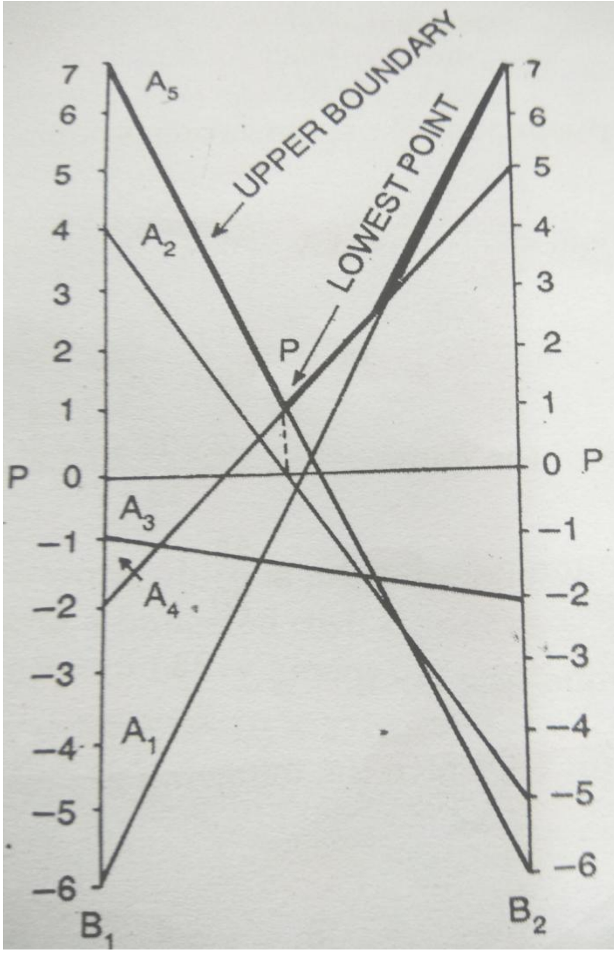
(iii) that the matrix can not be reduced further by any rule of dominance.

Since, all the above criteria are satisfied we now proceed we now proceed with the graphic method to reduce the game to (2 × 2) as under:

2. Here, the player B has two strategies viz., B_1 and B_2. Representing the probabilities of his strategies by p_1 and p_2 respectively we get the coordinated values of the various pay off lines as follows:

	B_1	B_2
For the pay off line A_1	–6	7
For the pay off line A_2	4	–5
For the pay off line A_3	–1	–2
For the pay off line A_4	–2	5
For the pay off line A_5	7	–6

3. Representing the strategy B_1 by the scalled line B_1, the strategy B_2 by the scalled line B_2, the probability line of a unit length by P – P and drawing the pay off lines with their coordinate values as above, we get the graphic representation of the game as under:



4. **Analysis:** From the above graph it is obvious that the point P indicates the minimax where, the two pay off lines A_4 and A_5 intereset each other. These two lines represent the two optimum strategies for A. Thus, the m × 2 game is reduced to 2 × 2 game as follows:

5. Solving the above 2 × 2 matrix by the arithmetic method we get,

		B 1	2
A	4	–2	5
	5	7	–7

		B 1	2	Oddment	Propn.
A	4	–2	5	13	13/20(q_1)
	5	7	6	7	7/20 (q_2)
	Oddment	11	9	20	
	Propn.	11/20 (p_1)	9/20 (p_2)		

6. **Interpretation:** Thus, the various optimal results of the game are:

(i) Optimal strategies of A: $\left(0,0,0,\frac{13}{20},\frac{7}{20}\right)$

of B: $\left(\frac{11}{20}, \frac{9}{20}\right)$

(ii) Value of the game : $\frac{-2\times13+7\times7}{13+7} = \frac{23}{20}$.

(iii) Maximum gain for A $= \frac{23}{20}$

Minimum loss for B $= -\frac{23}{20}$.

UNIT V
NETWORK ANALYSIS
(CPM and PERT)

Meaning

Network is a series of related activities which result in some products or services which in turn contribute to the predetermined goals of an organization. Then, What is Network Analysis? 'Network analysis may be defined as a technique applied in sequencing problems of a particular project. It is suited to projects of non-routine and non-repetitive nature'. The techniques used here, are of two types, viz., CPM (Critical Path Method) and PERT (Programme Evolution and Review Technique).

Sequencing Problem

Sequencing problems are those which are concerned with minimization of some key factors of production, such as time and cost.

Project

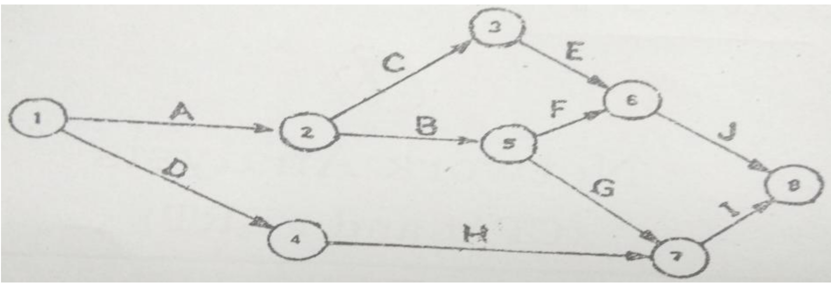
A project is a work which is proposed to be accomplished within a stipulated time and a budgeted cost. It may be a Building, a Dam or any other work involving heavy amount of resources.

Network Diagram

Network diagram is a pictorial presentation of the various events and activities relating to a particular project. It is drawn purely on the basis of two important things. viz., event and event and activity which are explained as under:

(A) Event

An event refers to the starting or ending point of an activity and it requires no time. It is represented by a figure within a circle, viz: (1), (2), (3), (4) and so on. An event cannot be achieved until all activities preceding it have been completed. In a network diagram there will be one starting event and one ending event.



(Example of a Network Diagram)

Types of Event

An event may be classified into various types, viz.: 1. Simple event, (2) Node, (3) Burst, (4) Head event and (5) Tail event.

These are explained briefly as under:

(1) **Simple event**. A simple event is one which takes place after or before one activity takes place.

(2) **Node**. A node is an event which takes place after the completion of two or more activities at a time. In other words, it is the ending event of two or more activities.

(3) **Burst**. A Burst is an event immediately after which two or more activities start. In other words, it is the beginning event of two or more activities.

(4) **Head Event**. A Head event is an event where an activity ends. Thus, if an activity, say, 'B' begins after the event (2), the event (2) will be the tail event.

(5) Tail Event. A Tail event is an event where an activity begins. Thus, if an activity, say, 'B' begins after the event (2), the event (2) will be the tail event.

It is to be noted that an activity will have two events, viz., : Head Event and Tail Event, explained as above.

(B) Activity

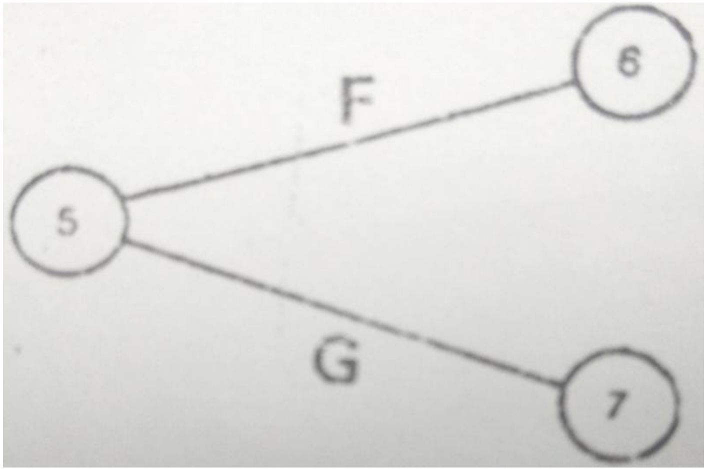
An activity refers to some action, the performance of which requires some time. It is represented by an arrow mark, viz., →. Every Activity is named by a capital letter, viz. : A, B, C, D etc., put above the arrow mark. Further, the time taken by each activity is indicated just below its arrow mark, viz.: $\frac{A}{3\ days}$. An activity may be of following types.

Types of Activities

(i) **Preceding Activity.** It is that activity which is completed before an event takes place, viz.: $\xrightarrow{C}$ (3). Here, Activity "C" is a preceding activity of the event (3).

(ii) Succeeding Activity. It is that activity which begins after an event has taken place viz.: (3) $\xrightarrow{E}$ Here, activity which is performed along with another activity of the event (3).

(iii) Concurrent Activity. It is that activity which is performed along with another activity at the same time, viz.:



Here activity "F" is a concurrent activity of the activity 'G'.

(iv) Dummy Activity. It is that activity the performance of which requires no time, nor any resources. It is a false activity indicated by the broken lines. Such activities are shown in a network diagram only to designate a precedence relationship. i.e., to show that the activities and the events in a network diagram are in the proper order. They simply help in getting a network diagram complete in an orderly manner, i.e., in sequencing a network diagram.

(v) Critical Activity. It is that activity which falls along the critical path. Any delay caused in the performance of such activity will cause delay in the completion of the project in toto and hence such activities are to be completed strictly according to the time schedule. Such activities are also called bottleneck activities.

(vi) Non-critical Activity. It is that activity which falls on a path other than the critical path. Delay in the performance of such activities does not cause delay in the completion of the project. Such activities are otherwise called as non-bottleneck activities.

Requirements for a Network Diagram

The drawal of a network diagram and its proper interpretation need the following information and a clear understanding of the following terms as well:

(A) Information Needed

(i) Sequencing of the activities, i.e., order of the activities in which they are to be performed.

(ii) Estimate of the times, to be consumed by each of the activities.

(B) Concept of the Terms Related to the Network

(i) **Path.** A path is a chain of activities that begins with the starting event and ends with ending event of a particular project. It is indicated through a Network.

(ii) **Critical Path.** It is that path that runs through a network with the maximum length of time. In other words, this path indicates the minimum possible time required for the completion of a project. It does not remain confined to any particular path. With the readjustment and improvement of the mode of operation of the different activities, any path may prove to be the critical path.

(iii) **Earliest Starting Time (EFT):** It is the earliest possible time at which an activity starts after an event. It is shortly indicated as Est and is put at the beginning point of an activity above its arrow line. The earliest starting time of the initial activities is zero. But the earliest starting time of the other activities is the greatest of the Earliest Finishing times of their respective immediately preceding activities.

(iv) **Earliest Finishing Time (EFT)** It is the earliest possible time at which at which an activity finishes. Such time is determined by adding the estimated time of an activity to its earliest starting time. It is shortly indicated as Eft and is put at the ending point of an activity above its arrow line. The Eft of a preceding activity ordinarily succeeding activity but in case of a Node event (where there are two or more activities preceding the event) the greatest of the Efts of its preceding activities becomes the Est of its succeeding activity.

Note: It is to be noted that the process of indicating the Ests and Efts above each of the arrow lines runs from the starting point to the ending point of every path of a network and such process is popularly known as forward-process.

(v) **Latest Finishing Time (LFT):** It is the latest possible time at which an activity can be finished without delaying the project beyond its stipulated time. It is shortly indicated as Lft and is shown at the ending point of an activity below its arrow line. The marking of such Lfts begins with the ending event of of a network and the process of such marking runs from the right to the left of the network. Such process is popularly known as Backward process. The greatest of the Efts of the ending activities becomes the Lfts for all those ending activities concerned. But the Lft of all other activities will be equal to the smallest of the Lst values of their immediately succeeding activities.

(vi) **Latest Starting Time (LST)**. It is the latest possible time at which an activity can start without disturbing the overall time fixed for the project. It is shortly indicated as the Lst and is put at the beginning of any activity just below its arrow line. It is determined by deducting the estimated time of an activity from its Latest finishing time thus already fixed. The Lst of an activity ordinarily becomes the Lft for all its immediate preceding activities. But at the point of a 'Burst' event (where there are two are more activities succeeding the event) the smallest of the Lsts of the succeeding activities becomes Lft of a preceding activity.

(vii) **Latest Event or Occurrence Time (ET). It is the latest possible time at which an event can occur.** It is shortly indicated as LT and is put just above or below an event. It is equal to the Lft of the activities proceeding to the event. However, the LT of the starting event of a project is zero.

(viii) **Earliest Event or Occurrence Time (ET).** It is the earliest possible time at which an event can occur. It is shortly indicated as ET and is put just above or below an event along with the LT. The ET of the starting event of a project is zero but the ET of all other events is equal to the greatest of the Efts of its preceding activities.

(ix) **Slack.** The term 'Slack' can be associated with both an event and an activity as well. In relation to an event, a slack is the difference between its latest and earliest event time. Thus, slack of an event = LT – ET. In relation to an activity, a slack will be synonymous to a float, i.e.: (Lst – Est). But, slack is ordinarily associated with an event and in such case an activity will have two slacks, viz.: Head Slack (i.e., slack of its head event) and Tail Slack (i.e., slack of its tail event) where,

Head Slack or Head Event Slack

$$= LT_H - ET_H$$

(i.e., Latest occurrence time and Earliest occurrence time of the Head Event)

and, tail Slack or Tail Event Slack

$$= LT_T - ET_T$$

(i.e., Latest occurrence time and Earliest occurrence time of the tail Event)

A Slack can be positive or negative depending upon the latest and earliest occurrence time of an event.

(*x*) **Float.** The term float refers to the slack of an activity. It is classified into three types, viz., (1) Total Float, (2) Free Float, and (3) Independent Float. These are briefly identified as under:

1. Total Float it is the time by which an activity other than the critical activity can be manipulated without delaying the overall time of the project. This can be easily calculated by taking the difference between the Lst and Est of an activity.

Thus,

$$\text{Total Float} = (Lst - Est) \text{ or } (Lft - Eft)$$

It can give either positive or negative value. But it should be noted that there will be no float for a critical activity for there is no question of delaying such an activity.

2. Free Float. It is that portion of the total float within which an activity can be manipulated without affecting the float of its subsequent activities. It is calculated by deducting the head event slack from the total float. Thus,

Free Float = Total Float – Head Event Slack or Head Slack, Such float may be either positive or negative in value.

3. Independent Float, It is that part of the total float within which an activity can be delayed for its start without affecting the floats of its preceding activities. It is found out by subtracting the Tail event slack from the free float.

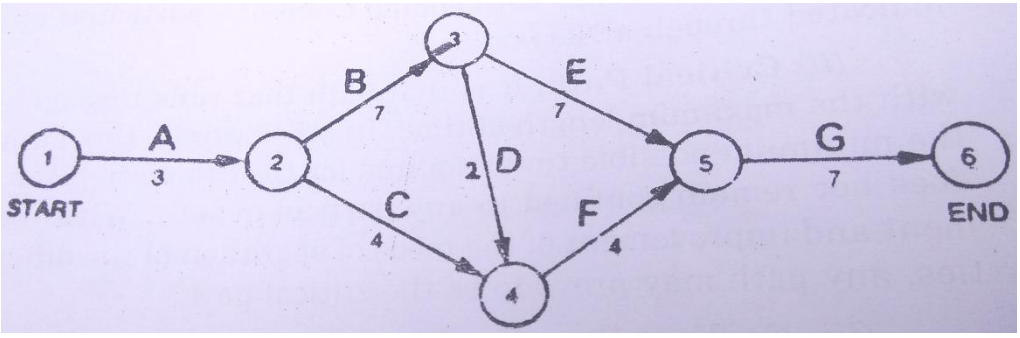
Thus,

Independent Float = Free Float – Tail event Slack or Tail Slack Independent float can be positive only and where its working obtains negative value the same should be made up to zero. However, Total float and Free float can have either positive or negative value. Positive float implies that there are idle time and resources for the activities which could be utilized otherwise. Negative float indicates that the activities concerned are short of time and resources. Both positive and negative floats are undesirable.

Illustration 1: Draw a network diagram from the following information.

B and C follow A, E follows B, F follows D and C and G follows E and F. Where A, B, C, D, E, F and G are the activities requiring 3, 7, 4, 2, 7, 4 and 7 days respectively.

Solution: Network Diagram



Here, the arrow marks represented by A, B, C, D, E, F and G are the different activities and the circles represented by 1, 2, 3 etc. are the events.

Illustration 2: Draw a Network diagram from the following.

Activity		Name of the Activity	Pre-requisite Activity	Estimated Time
Event	Event			
1	2	A	None	4
1	3	B	None	2
1	4	C	None	2
1	5	D	None	2
2	8	E	A	–
3	7	F	B	3
4	6	G	C	2
5	8	H	D	–
7	8	I	F & J	4
8	7	J	G	3

Solution:

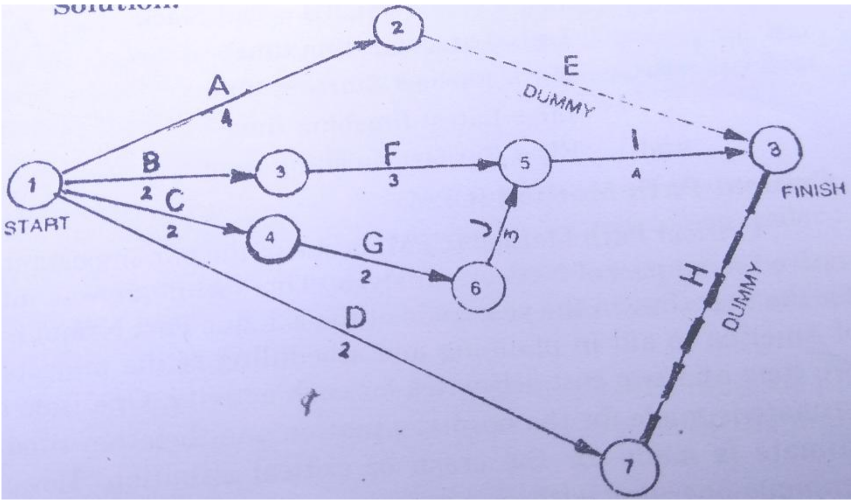


Illustration 3: From the following information draw up a Network diagram and indicate the Est, Lst, Eft and Lft of the various activities.

Name of the Activity	Pre-requisite Activity	Estimated Time (days)
A	None	2
B	None	3
C	None	4
D	A	6
E	B	7
F	C	5
G	D & E	8
H	B	9
I	H & F	5

Solution: Network Diagram

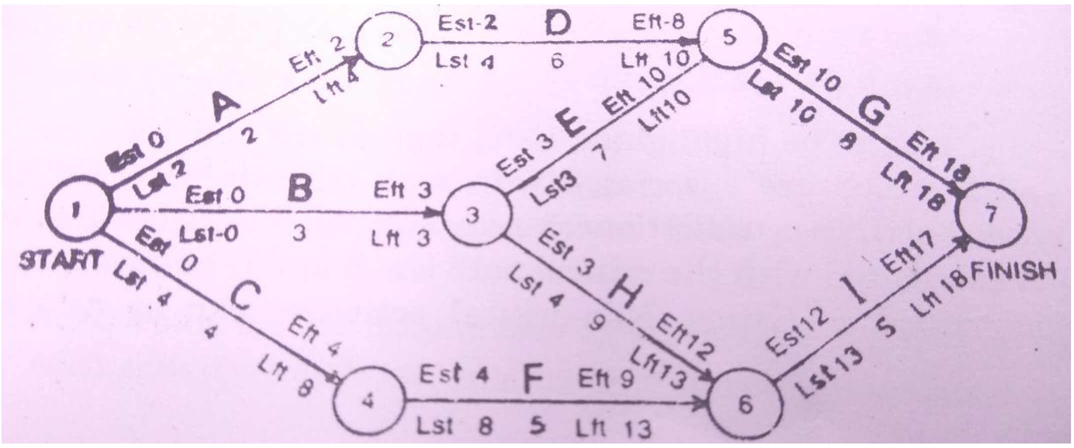


Illustration 4: From the following table of information find out the following:

a) Critical path and non–critical paths through a Network diagram.

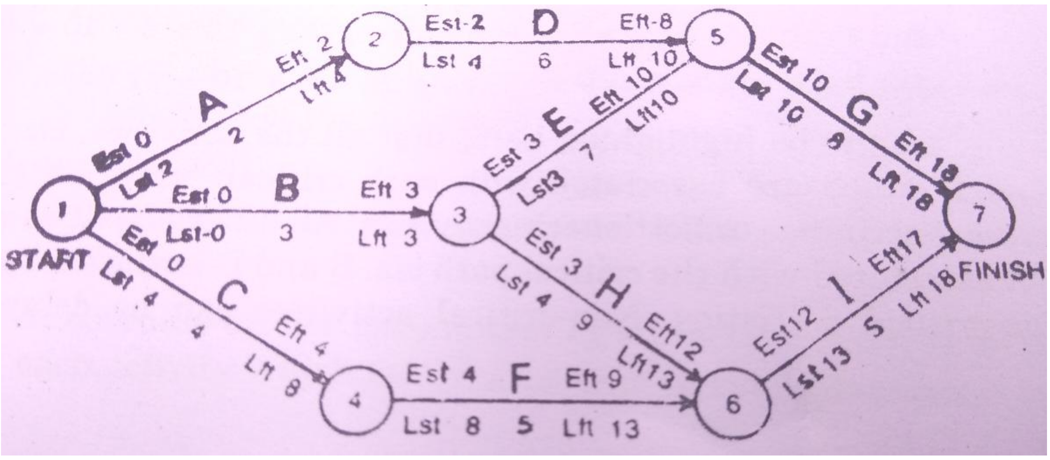
b) Latest event times, Earliest event time and slacks of the events.

c) Different slacks of the activities.

d) Different floats of the activities.

Activity:	1 – 2,	1 – 3,	2 – 3,	2 – 4,	3–4,	4–5.
Estimated Time (days):	20,	25,	10,	12,	6,	10

Solution: Network diagram



a) **Different Paths:**

(i) A → D → F requiring 20 + 12 + 10 = 42 days

(ii) A → C → E → F requiring 20 + 10 + 6 + 10 = 46 days (Maximum)

(iii) B → E → F requiring 25 + 6 + 10 = 41 days

Thus, the critical path is A → C → E → F requiring maximum number of days, i.e., 46 days and the non–critical paths are:

1) A → D → F requiring only 42 days and
2) B → E → F requiring only 41 days

(b) Latest event time, Earliest event Time and Slacks of the Events

Events	LT	ET	Slacks LT – ET
1	0	0	0
2	20	20	0
3	30	30	0
4	36	36	0
5	46	46	0

LT → Latest Event Time, ET → Earliest Event Time

(c) Slacks of the Activities

Activities	Head Slack LT_H - ET_H	Slacks LT – ET
A	**20 – 20 = 0**	**0 – 0 = 0**
B	**30 – 30 = 0**	**0 – 0 = 0**
C	**30 – 30 = 0**	**20 – 20 = 0**
D	**36 – 36 = 0**	**20 – 20 = 0**

E	**36 – 36 = 0**	**30 – 30 = 0**
F	**46 – 46 = 0**	**36 – 36 = 0**

$LT_H \rightarrow$ Latest Occurrence Time of the Head Event

$ET_H \rightarrow$ Earliest Occurrence Time of the Head Event

$LT_T \rightarrow$ Latest Occurrence Time of the Tail Event

$ET_T \rightarrow$ Earliest Occurrence Time of the Tail Event

(d) Calculation of Different floats of the Activities

Activities	**Earliest**		**Latest**		**Total Float** **Lst**–Est or Lft–Eft	**Free Float (TF**–HS)	**Independent Float (FF**–TS)
	Start	**Finish**	**Start**	**Finish**			
A	0	20	0	20	0	0 – 0 = 0	0 – 0 = 0
B	0	25	5	30	5	5 – 0 = 5	5 – 0 = 5
C	20	30	20	30	0	0 – 0 = 0	0 – 0 = 0
D	20	32	24	36	4	**4** – 0 = 4	4 – 0 = 4
E	30	36	20	36	0	0 – 0 = 0	0 – 0 = 0
F	36	46	36	46	0	0 – 0 = 0	0 – 0 = 0

Where, TF = Total float, HS = Head Slack
FF = Free Float, TS = Tail Slack
Lst = Latest starting time
Est = Earliest Starting time
Lft = Latest finishing time
and Eft = Earliest finishing time

Critical Path Method (CPM)

Critical Path Method (CPM) is one of the two important quantitative techniques of Network analysis. This technique was introduced for the first time in the year 1956 at the E.I. due Port Nemours and Co. of America to aid in planning and scheduling of the projects. It uses two time and two cost estimates for each activity. One time and Cost estimate is made for the normal situation and the other time and cost estimate is made for the crash or critical situation. However, this technique operates with an assumption of precise known time and it does not make use of the statistical analysis in determining the time estimates. It is a graphical technique which involves the drawal of a Network diagram and its analysis to indicate the critical path, non-critical paths, slacks and different floats of the various activities explained earlier. It has the potential scheduling of a project in minimum possible time and cost in the light of the given constraints.

As explained earlier, critical path is the longest path which can be easily found out from among the different possible paths indicated through the network diagram with their respective times. It is the path which takes the maximum time to complete a project. In other words this path indicates the minimum time that will be required to complete a project. In the Illustration 4 analysed above the path A $\rightarrow$ C $\rightarrow$ E $\rightarrow$ F requiring (20 + 10 + 6 + 10) 46 days in toto is the critical path for completing the project. This is because this path is the longest of all the time different path indicated in the network diagram in the said illustration 4 viz.,

1st Path: A $\rightarrow$ D $\rightarrow$ F requiring 20 + 12 + 10 = 42 days

2nd Path: A $\rightarrow$ C $\rightarrow$ E $\rightarrow$ F requiring 20 + 10 + 6 +10 = 46 days
3rd Path: B $\rightarrow$ E $\rightarrow$ F requiring 25 + 6 + 10 = 41 days

It is to be highlighted here, that all the activities, viz., A, C, E and F which are associated with such critical path are called the critical activities or bottleneck activities. All other activities which are not associated with the critical path viz. B and D in the said project are non-critical activities. Non-critical activities can be delayed to the extent of the float available without affecting the overall time stipulated for the project as they have a certain amount of spare time. But the critical activities can never be delayed as they do not have any float or spare time available.

Importance of Critical Path

Importance of the critical path and the analysis of the network diagram in the light of the critical path can be highlighted as under:

(1) The critical path indicates the activities which must be performed without any delay in order that the project may be completed within the stipulated time.

(2) The critical path indicates the activities which must be performed more rapidly in order that the completion time of the entire project may be reduced.

(3) The critical both also indicates the activities, the performance of which can be delayed to a certain extent.

(4) The critical path determined in the first instance may also aid in the determination of another path as critical through advance planning and improvement.

(5) The critical path draws attention of the management to the important facts, spots out the potential bottlenecks and avoids unnecessary pressure on other paths that do not help in the earlier completion of the project.

(6) On the basis of the different floats of the different activities found out the critical path makes it possible to make the following adjustments in the project.

a) Reduction of time estimates of the critical activities.
b) Elimination of some non-critical activities.
c) Introduction of some more resources.
d) Transfer of some resources from non-critical activities to critical activities.
e) Drawal of the revised network diagram to reduce to the completion time of the project.

Limitation of CPM

In spite of the above importance, the CPM suffers from the following limitation:

a) It spite of the above assumption of a precise known time for each activity which may not be true in real situation.
b) It does not make use of the statistical analysis in the determination of the time estimates for each activity.
c) It requires repetition of the evaluation of the entire project each time a change is introduced to the network. This is a very difficult and cumbersome process.

d) It cannot serve as a dynamic controlling device as it was introduced as a static planning model.

Illustration 5: The following table gives the information relating to the various activities concerning a project.

Name of the Activity	Pre–requisite Activity	Time Estimated (in days)
A	**None**	**2**
B	**A**	**3**
C	**A**	**4**
D	**B and C**	**6**
E	**None**	**2**
F	**E**	**8**

From the above information relating to the project:

(i) Draw up a Network diagram

(ii) Determine the critical path

(iii) Ascertain the minimum completion time of the project.

(iv) List the critical and non-critical activities.

(v) Find out the total float, Free float and independent float for the various activities

Solution: (i) Network diagram

(ii) Determination of the critical path:

Different paths are:

(a) A → B → D requiring 2 + 3 + 6 = 11 days

(b) A → C → D requiring 2 + 4 +6 = 12 days

(c) E → F requiring 2 + 8 = 10 days

From the different paths cited above, the longest path A → C → E requiring the maximum time 12 days is the critical path for the project.

(iii) The time of 12 days required for the critical path is the minimum time required for the completion of the project.

(iv) List of the critical activities:

i. A
ii. C
iii. D

List of the None-critical activities

i. B
ii. E
iii. F

(v) (a) Table showing the Slacks of the Events

Events	LT	ET	Slacks LT – ET
1	0	0	0

2	2	2	0
3	6	6	0
4	4	2	2
5	12	12	0

(b) Table showing the slacks of the activities

Activities	Head Slack LT_H - ET_H	Slacks LT – ET
A	2 – 2 = 0	0 – 0 = 0
B	6 – 6 = 0	2 – 2 = 0
C	6 – 6 = 0	2 – 2 = 0
D	12 – 12 = 0	6 – 6 = 0
E	4 – 2 = 2	0 – 0 = 0
F	12 – 12 = 0	4 – 2 =2

(iv) Computation of the Different Floats

Activities	Earliest		Latest		Total Float **Lst**–Est or Lft–Eft	Free Float (**TF**–HS)	Independent Float (**FF**–TS)
	Start	**Finish**	**Start**	**Finish**			
A	0	2	0	0	0	0 – 0 = 0	0 – 0 = 0
B	0	5	3	6	1	1 – 0 = 1	1 – 0 = 1
C	2	6	2	6	0	0 – 0 = 0	0 – 0 = 0
D	6	12	6	12	0	**0** – 0 = 0	0 – 0 = 0
E	0	2	2	4	2	2 – 0 = 2	0 – 0 = 0
F	2	10	4	12	2	2 – 0 = 2	2 – 2 = 0

Allocation and Levelling of Resources (Through CPM)

Resources allocation or scheduling of resources means the allocation of the available resources to various activities. Levelling of resources, on the other hand, means allocation of resources to various activities in such a manner that the variation in resources requirements are minimized and the resource demand is evened out.

The task of allocation of resources and their levelling is of vital importance and it involves various types of problem viz. (i) Resources levelling and (ii) Limited Resource allocation. These problems are well tackled with the help of Critical Path Method. For this the following steps to be taken up:

(A) Steps for Levelling of Resources

1) Draw up a list of the resources required
2) Draw up a profile for each of the resources
3) Determine the periods of the highest and lowest demand.
4) Alter the times of 'start' and 'finish' of the non-critical activities in order of their higher floats and repeat this process till the peaks are lowered down to the level of the troughs as far as possible without affecting the overall time limit of the project.

(B) Steps for Tackling the Problem of Limited Resources

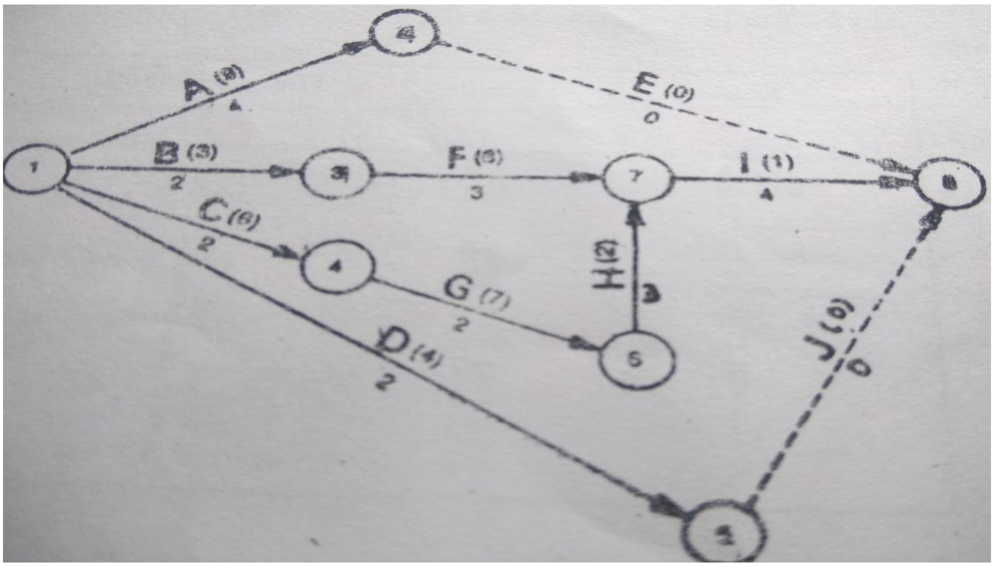
1) Allocate the resources to the various activities serially through a Network diagram.
2) When there is competition among the several activities for the same resources give preference to the activity having maximum float for the said float period.
3) If the floats of the competing activities appear to be equal, postpone the activity requiring less time for its float period.
4) Reschedule the non-critical activities if possible to free the resources for scheduling the critical activities.

The following illustrations will show how the problems of Resource levelling and limited resources allocation are tackled through Critical Path Method.

Illustration 6: (On Resource Levelling) From the data given below, make the levelling and limited of the resources as far as possible and work out the aggregate resources requirements day by day

Activity	Events	Time Estimates (in days)	Labour requirements (in units)
A	1 – 1	4	9
B	1 – 3	2	3
C	1 – 4	2	6
D	1 – 5	2	4
E	2 – 8	0	0
F	3 – 7	3	8
G	4 – 6	2	7
H	6 – 7	3	2
I	7 – 8	4	1
J	5 – 8	0	0

Solution: Net problem



From the above network diagram it appear that the longest path or critical path of the project is

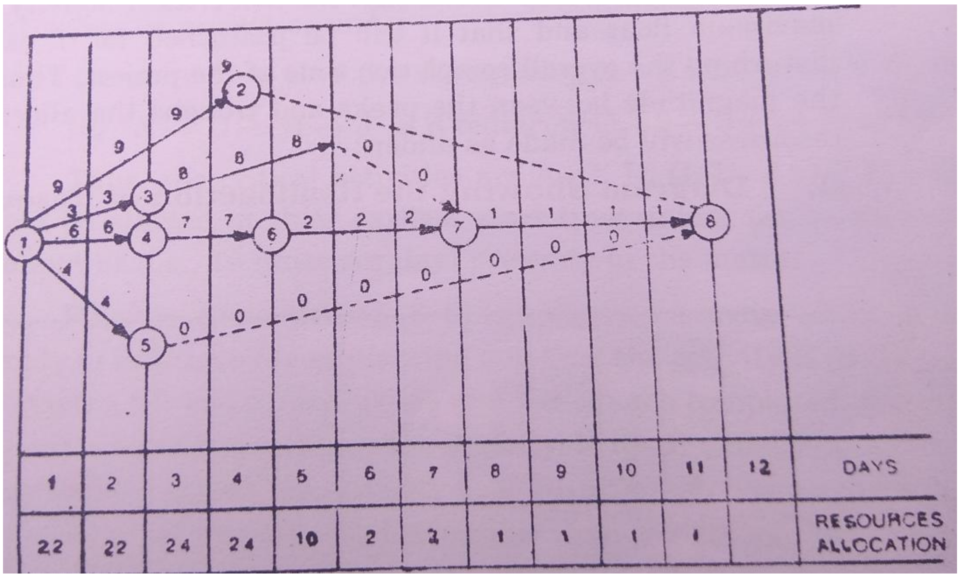
$C \rightarrow G \rightarrow H \rightarrow I$ requiring

$2 + 2+ 3 + 4 = 11$ days

and the non-critical activities are A, B, D and F.

Thus the minimum number of days required for the project is 11 days during which the resources will be used up.

1. Diagram Showing the Serial Allocation of Resources (1st Instance)



The figures put above the arrow marks of the activities indicate the units of labour resources required during each day for the part of each activity.

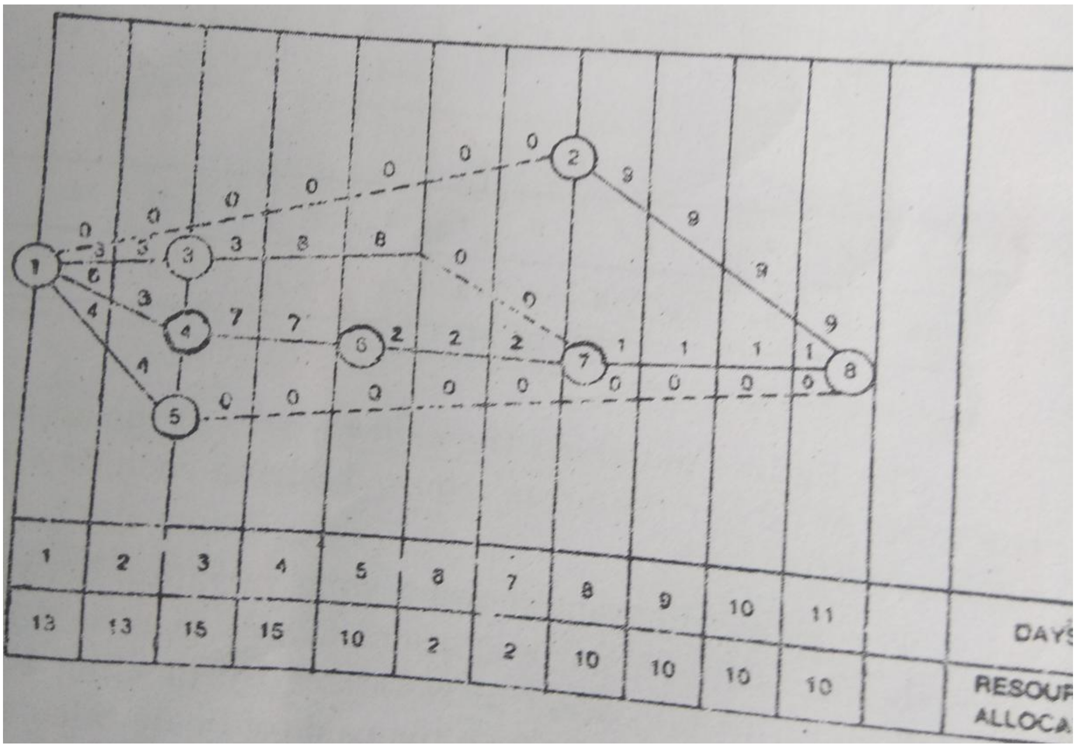
The above diagram shows a wide variation shows a wide variation between the peaks and troughs of the labour allocations i.e., 24 to 1. So it is necessary to smooth out the allocation of resources by tackling the non-critical activities viz., A, B, D and F on the basis floats calculated as under:

Calculation of the Floats

Activity		Est	Lst	Eft	Lft	Total Float
1		2	3	4	5	6
A	1–2	0	7	4	11	7
B	1–3	0	2	2	4	2
C	1–4	0	0	2	2	0
D	1–5	0	2	2	4	2
F	3–7	2	4	5	7	2
G	4–6	2	2	4	4	0
H	6–7	4	4	7	7	0
I	7–8	7	7	11	11	0

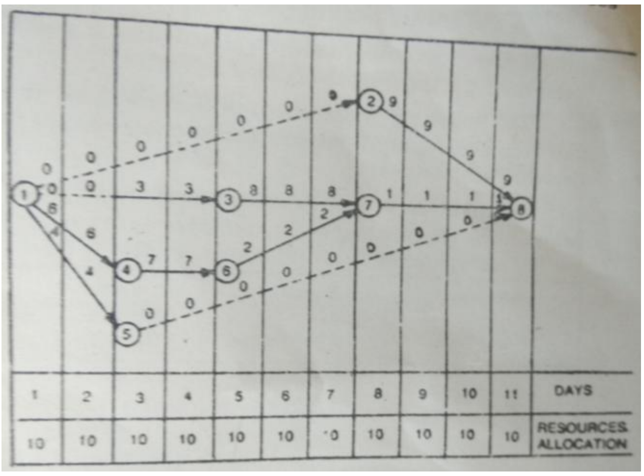
The above table shows that the non-critical activity 'A' has the maximum float and that it can be postponed for 7 days without disturbing the overall completion time of the project. Thus, to reduce the magnitude between the peaks and troughs the allocation of the resources will be made as under:

II. Diagram Showing the Reallocation of Resource



From the above modified diagram it appears that the magnitude of resource allocation between the peaks and troughs has been reduced from 1 to 24 to 2 to 15. This indicates that we can make further modifications in the resource allocation to reduce such wider magnitude. According to the float table we can delay the start of the activity ‘B’ (1-3) by 2 days without imparting the overall completion time. Thus the third modifying diagram of resource levelling will be as follows:

III. Diagram Showing the Levelling of Resource



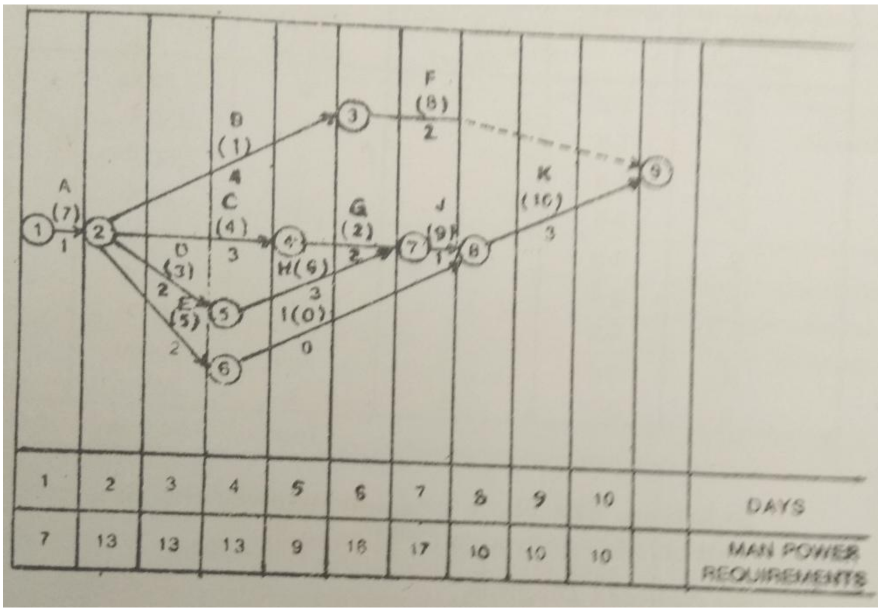
The diagram drawn above shows that the resource allocation has now been perfectly levelled up which of course may not be possible in all the cases. However, the magnitude of allocation between the peaks and trough can be reduced in the manner.

Illustration 7: (On limited resources). From the following data relating to a project show how the project is to be scheduled to be completed as soon as possible assuming that only 10 units of labour can be made available a day:

Data

Activity	Events	Time Estimates	Labour requirements per day
A	1 – 2	1	7
B	2 – 3	4	1
C	2 – 4	3	4
D	2 – 5	2	3
E	2 – 6	2	5
F	3 – 9	2	8
G	4 – 7	2	2
H	5 – 7	3	6
I	6 – 8	0	0
J	7 – 8	1	9
K	8 – 9	3	10

Solution: Net Diagram



Figures indicated within the brackets above the arrow marks are the units of manpower required per day. The above diagram shows that the manpower requirements on various days do not remain within the constraint of 10 units of labour. Hence, it needs scheduling in the following manner:

Statement Showing the Floats

Activity		Time estimate	Est	Lst	Eft	Lft	Total Float
1		**2**	**3**	**4**	**5**	**6**	**7**
A	1–2	1	0	0	1	1	0
B	2–3	4	1	4	5	8	3
C	2–4	3	1	1	4	4	0
D	2–5	2	1	1	3	3	0
E	2–6	2	1	5	3	7	4
F	3–9	2	5	8	7	10	3
G	4–7	2	4	4	6	6	0
H	5–7	3	3	3	6	6	0
I	6–8	0	3	7	3	7	4
J	7–8	1	6	6	7	7	0
K	8–9	3	7	7	10	10	0

The above net work diagram shows that there are two critical paths for the project as under:

(i) A $\rightarrow$ C $\rightarrow$ G $\rightarrow$ J $\rightarrow$ K requiring (1 + 3 + 2 + 1 + 3) = 10 days

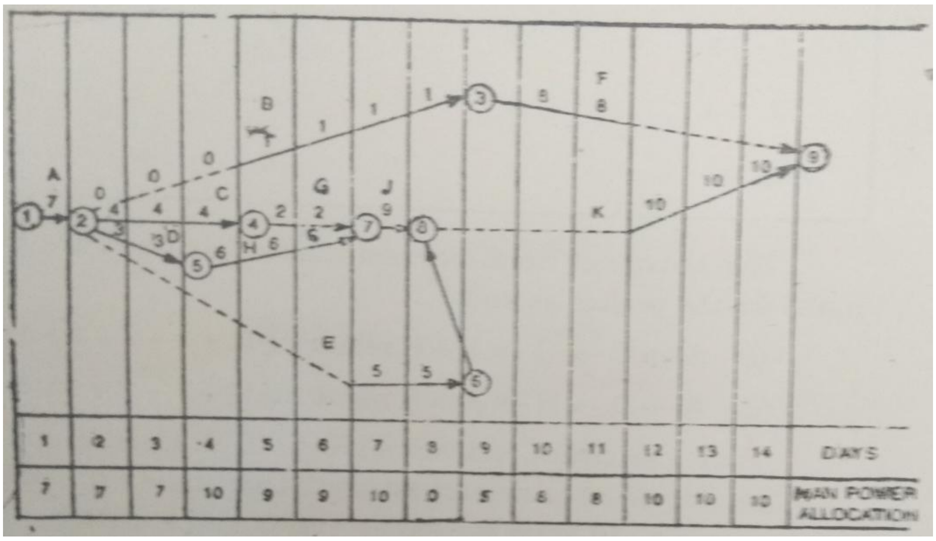
(ii) A $\rightarrow$ D $\rightarrow$ H $\rightarrow$ J $\rightarrow$ K requiring (1 + 2 – 3 + 1 + 3) = 10 days

Thus, the critical activities are A, C, D, G, H, H and K, which must be started at their earliest start time unless the constraint of manpower (i.e., 10 units per day) prevents in the matter.

Further, the activity I (6–8) is a dummy activity which is there simply to facilitate the sequencing and that the activity K (8–9) cannot be started till activities E (2–6) and I (6–8) are completed. Hence, all critical activities excepting the activity K (8–9) should be started at their earliest start time. Activity E (2–6), although has a float of 4 days cannot start on the 6th day nor earlier than the 8th day in view of the constraint of 10 units of labour available on a day. Activity I (6–8) being a dummy activity can be ignored totally irrespective of its float of 4 days. Both the activities B (2–3) and F (3–9) have floats of 3 days for which the starting time of each of them can be delayed for 3 days. Thus, the activity B (2–3) can start from the 5th day instead of the 2nd day. But the activity F (3–9) can start only from the 10th day instead of the 6th and the 9th day in view of the shortage of the manpower beyond 10 units on a day.

In view of the above arrangements in the light of the given constraints the last critical activity K (8–9) can be started only from the 12th day instead of the earlier 8th day. As such the overall completion time of the project will be increased to 14 days from 10 days. Thus, the project will be scheduled in the revised network diagram as follows:

Revised Network Diagram
(Showing the Scheduling of the project)



The above diagram keeps the manpower requirements within the constraint of 10 units of labour on a day.

Time Cost Analysis (Through CPM)

The execution of a project needs some time and fund both of which are very precious for the management. But the relation between the two is obviously an inverse one i.e., more the time less is the fund and less the time more is the fund required for a project. Thus, it will be a critical task for the manager to decide over the time-cost relationship concerning his project. If he considers the time to be more precious then he should plan out to complete the project within the shortest possible time by reducing the estimated time and investing more funds in the project. This type of execution of work is called as the execution of a work on Crash basis or hurry up basis, the manager must see that the basis thus adopted proves most economical for the management. For this he has to make comparative analysis of the various paths in the light of the time-cost trade off concept. Critical path method helps a lot in taking a wise decision in such matter. The following illustration will make the point in issue more clear.

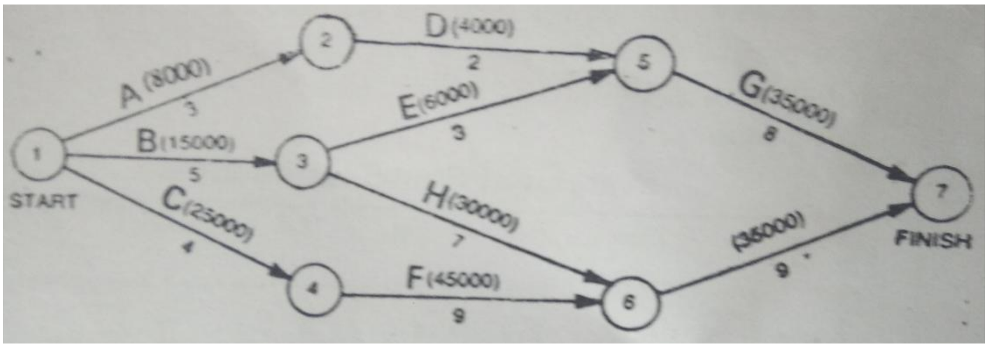
Illustration 8: From the following data determine the minimum possible time and maximum possible find that proves most economical to complete a project in issue.

Name of the Activity		Timed Estimate		Cost Estimate	
		Normal (Days)	Crash (Days)	Normal Rs.	Crash Rs.
A	1–2	3	2	8,000	10,000
B	1–3	5	3	15,000	20,000
C	1–4	4	3	25,000	32,000
D	2–5	2	1	4,000	6,000
E	3–5	3	2	6,000	8,000
F	4–6	9	8	45,000	60,000
G	5–7	8	6	35,000	50,000
H	3–6	7	6½	30,000	40,000

I	6–7	9	7	35,000	42,000
			Total	2,03,000	2,68,000

Solution: The above problem involves the time cost analysis which can be solved through CPM as under:

(a) Network Diagram



Figures above the arrows represent cost and below the arrows represent time required on normal basis.

(b) Different paths and their required time and cost.

(i) A → D → G require 13 days and Rs. 49,000

(ii) B → E → G require 16 days and Rs. 56,000

(iii) B → H → I require 21 days and Rs. 80,000

(iv) C → F → I require 22 days and Rs. 1,05,000

Thus, Path C → F ← 1 in the critical path.

(c) Table Showing the Possible Reduction in Time and Increment in Cost under Crash Programme

Name of the Critical Activities	**Reducible Time**	**Total of Incremental Cost (Rs.)**	**Incremental Cost per day (Rs.)**
C	1	7,000	7,000
F	1	15,000	15,000
I	2	7,000	3,500

Reduction in time required only in case of critical activities that fall on the critical path, viz., C → F → I. Since the non-critical activities have floats and reduction of time in respect there of would not reduce the overall time of the project.

(D) Table showing the Time-Cost Analysis (Table given below:)

Activities		**Critical Path**	**Time required (in days)**	**Cost required (Rs.)**
On Normal basis	**On Crash basis**			
All	None	C → F → 1	22	203000
A, B, C, D, E, F, G, H	I	C → F → 1	20 (−2 for I)	210000 (+ 7000 for I)

A, B, D, E, F, G, H	C, I	$C \rightarrow F \rightarrow 1$ $B \rightarrow H \rightarrow 1$	$19\begin{pmatrix}-2 \text{ for I} \\ -1 \text{ for C}\end{pmatrix}$	217000 (+ 7000 for C)
A, D, E, G, H	C, F, I fully and B for 1 day	$C \rightarrow F \rightarrow 1$	$18\begin{pmatrix}-1 \text{ for F} \\ -1 \text{ for B}\end{pmatrix}$	243500 $\begin{pmatrix}+15000 \text{ for F} \\ +2500 \text{ for B}\end{pmatrix}$

From the above analysis table it transpires that the minimum possible times is 18 days and maximum possible cost in Rs. 2,34,500 to complete the project most economically.

Note: Activity 'B' could have been reduced by 2 days to complete the path BHI within 17 days but it is not possible because even if all the activates are done on crash basis the total project time comes to 18 days and the critical path happens to be $C \rightarrow F \rightarrow I$. Hence the activates 'B' can be worked out on crash basis only for a day with an additional cost of Rs. 2,500 and the activity 'F' can be worked out for 1 day with additional cost of Rs. 15,000.

Programme Evaluation and Review Technique (PERT)

Programme Evaluation and Review Technique (PERT) is another quantitative technique of Network analysis. This technique was introduced for the first time in the year 1958 at the E.I due Pnrt Nemous & Co. of U.S.A. for use in defence projects. Now it is being used as a formidable instrument by the project planners and controllers in the determination of the expected total time a project is likely to take for its completion and for analysing and administering the large complex projects to be performed in some sequence. It is defined as an approach of multiple time estimates in which the following four time estimates are employed through statistical analysis.

(i) The most optimistic time (t_o)
(ii) The most pessimistic time (t_p)
(iii) The most likely time (t_m)
(iv) The most expected time (t_e)

Optimistic time(t_o): It refers to that shortest possible time within which an activity can be performed when everything goes perfectly without any disturbance. It is the most ideal time which rarely occurs in practice and thus its chance of taking place may be 1%.

Pessimistic time (t_p): It refers to that longest possible time which is likely to take place to complete an activity when all sorts of complicacies and unusual delays take place. It is another extreme time which rarely takes place in actual practice and thus its chance of occurrence is also 1%.

Most likely Time(t_m): It refers to that time which is most likely to take place to complete a particular activity or a project. This is the best possible estimate of time which normally occurs in the completion of an activity.

Expected Time (t_e): It refers to that time within which an activity is expected to be accomplished. It is determined for each activity on the basis of the three estimated times aforestated, viz., optimistic time, most likely time and pessimistic time. It is calculated as a weighted average of these three types of activates. The weights assigned for the purpose are:

For optimistic time – 1

For pessimistic time 1
For most likely time 4
Thus, the expected time of activity is calculated with the following formula.

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

It is to be noted here that under CPM the precise known time for each activity was assumed but under PERT it is assumed that the time to perform an activity is uncertain and as such here we calculate the expected time for each activity on the basis of the three time estimates cited above, viz., optimistic time, most likely time and pessimistic time. After the expected times for each activity are estimated in the above manner, determination of all other relevant facts, viz., Different paths of performance including the critical path, minimum time required to complete a project, earliest and latest start and finish times of the activates, earliest and latest times of occurrence of events, different, floats and slacks etc. are found out through Network diagram just on similar lines of that of the critical path method.

In addition to the above, PERT also enables us to determine the probabilities of completing an activity and the entire project within a given time. For this PERT allows us to make use of the following statistical devices to ascertain the values of the various relevant factors as follows:

1. Standard Deviation of an Activity

$$\sigma_a = \frac{t_p - t_o}{6}$$

2. Variance of an Activity

$$V_a = \left(\frac{t_p - t_o}{6}\right)^2$$

3. Standard Deviation of a Path

$$\sigma_p = \sqrt{\left(\frac{t_p - t_o}{6}\right)^2 + \left(\frac{t_{p2} - t_{o2}}{6}\right)^2 + \cdots\cdots\cdots + \left(\frac{t_{pn} - t_{on}}{6}\right)^2}$$

4. Variance of path:

$$V_p = \left(\sigma_p\right)^2 + (\sigma_{a1})^2 + (\sigma_{a2})^2 + \cdots\cdots\cdots + (\sigma_{an})^2$$

where,

a_1, a_2, a_3, ... and an represent different activates along a particular path and (p) represents a particular path.

The above formula for variance of a path is based upon the assumptions that the variance of sum of random variables equals the sum of the variance of each random variable of independent nature.

It is to be noted that for finding the standard deviation and variance of the entire project the standard deviation and variance of the critical path are to be found out as per the above formulae (3) and (4) respectively.

5. Probability of a Path being completed within a given time.

This is given by

$P_{(p)}$ = Table value of the standard normal variate (z) as per the cumulative normal distribution table (vide the Appendix at the end of the book.)

$$z = \frac{x - \bar{x}}{\sigma_{(p)}} = \frac{x_p - a_p}{\sigma_p}$$

where

Z = Standard normal variate of a path

X_p = Give time of the path

a_p = Mean or length of the given path

$\sigma_{(p)}$ = Standard deviation of the path

6. Probability of the Project being completed within a given time:

P_w = Products of the probabilities of the different independent paths

i.e., = Product of the probability of critical and that of the other independent paths.

= z(cp) × z(oip)

The following illustrations will make the procedure of PERT analysis more clear.

Illustration 9: From the following information relating to the various activities of a project find the expected time (t_e) for the different activities.

Activities	**Optimistic time (days) t_o**	**Most likely time (days) t_m**	**Pessimistic Time (days) t_p**
A	2	4	6
B	6	8	10
C	1	5	15
D	1	5	9
E	6	8	10
F	5	7	8

Solution: Expected time or

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

The expected time of the above six activities will be calculated as under:

Activity	Formulation $t_e = \frac{t_o + 4t_m + t_p}{6}$	Expected Time (in days)
A	(2 + 4 × 4 + 6)/6	4
B	(6 + 4 × 8 + 10)/6	8
C	(1 + 4 × 5 + 15)/6	6
D	(1 + 4 × 5 + 9)/6	5
E	(6 + 4 × 8 + 10)/6	8
F	(5 + 4 × 7 + 8)/6	

Illustration 10: From the following data relating to a project.

a) Find the expected duration, standard deviation and variance of each activity.

b) Draw up the project network diagram and trace out all the possible paths from it.

c) Find the expected project length.

d) Find the probability that the project will be completed within 4 weeks.

e) Calculate the Est, Eft, Lst and Lft values for each activity. Also find the total float for the activities.

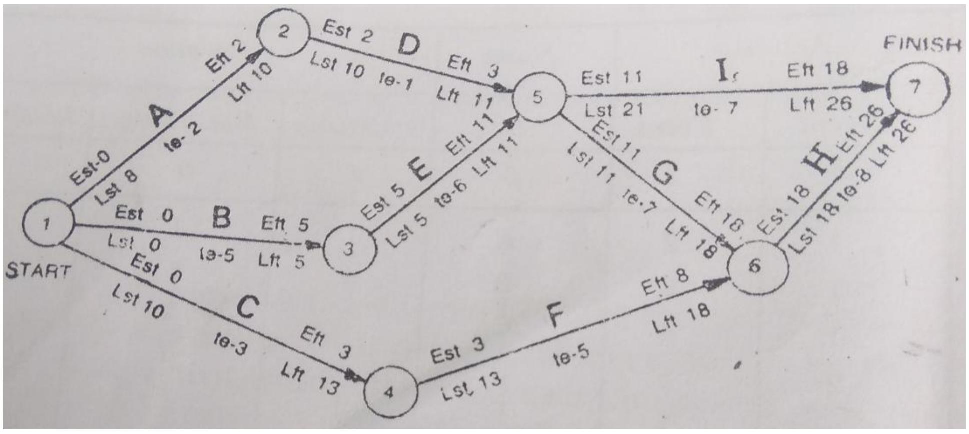
Activity		Name of Activity	Duration		
Event	Event		Optimistic	Most likely	Pessimistic
1	2	3	4	5	6
1	2	A	1	1	7
1	3	B	3	5	7
1	4	C	2	2	8
2	5	D	1	1	1
3	5	E	3	6	9
4	6	F	2	5	8
5	6	G	4	6	14
6	7	H	6	8	10
5	7	I	3	7	11

Solution: (a) Computation of Expected duration, Standard Deviation and Variance of each activity.

Name of the Activity	Expected duration (week) $t_e = \frac{t_o + 4t_m + t_p}{6}$	Standard deviation $\sigma_a = \frac{t_p - t_o}{6}$	Variance $(\sigma\alpha)^2$
A	(1 + 4 × 1 + 7)/6 = 2	(7 – 1)/6 = 1	1

B	(3 + 4 × 5 + 7)/6 = 5	(7 – 3)/6 = 67	45
C	(2 + 4 × 2 + 8)/6 = 3	(8 – 2)/6 = 1	1.00
D	(1 + 4 × 1 + 1)/6 = 1	(1 – 1)/6 = 0	0
E	(3 + 4 × 6 + 9)/6 = 6	(9 – 3)/6 = 1	1.00
F	(2 + 4 × 5 + 8)/6 = 5	(8 – 2)/6 = 1	1.00
G	(4 + 4 × 6 + 14)/6 = 7	(14 – 4)/6 = 1.67	2.79
H	(6 + 4 × 8 + 10)/6 = 8	(10 – 6)/6 = .67	.45
I	(3 + 4 × 7 + 11)/6 = 7	(11 – 3)/6 = 1.33	1.77

(b) Project Network Diagram (on the basis of the Expected duration i.e. t_e)



Possible Paths

(i) A → D → I requiring 2 + 1 + 7 = 10 weeks

(ii) A → D → G → H requiring 2 + 1 + 7 + 8 = 18 weeks

(iii) B → E → I requiring 5 + 6 + 7 = 18 weeks

(iv) B → E → G → H requiring 5 + 6 + 7 + 8 = 26 weeks

(v) C → F → H requiring 3 + 5 + 8 = 16 weeks

(c) Expected project length is 26 weeks, i.e., the length of the longest path or the critical path i.e. B → E → G → H.

(d) Probability that the project will be completed within 30 weeks.

For this we must find in the following manner, the product of the probabilities of the independent paths, viz., (1) The critical path, i.e., B → E → G → H and (2) A → D → I

Thus, probability of completing the C.P. within 30 weeks in given by

$$z_{(ep)} = \frac{X - a_{cp}}{\sigma_{cp}}$$

where,

Z_{cp} = Standard normal variate of the critical path.

X = Given time = 30 weeks

a_{cp} = Mean or length of the critical path = 26 weeks and

$\sigma_{(cp)}$ = Standard deviation of critical path.

$$\sigma_{(cp)} = \sqrt{\left(\frac{t_{pE} - t_{oB}}{6}\right)^2 + \left(\frac{t_{pE} - t_{oE}}{6}\right)^2 + \left(\frac{t_{pG} - t_{pH}}{6}\right)^2 + \left(\frac{t_{pH} - t_{oH}}{6}\right)^2}$$

$$= \sqrt{.45 + 1.00 + 2.79 + .45}$$

$$= \sqrt{4.69} = 2.16$$

Therefore,

$$Z_{(cp)} = \frac{30 - 26}{2.16} = 1.85$$

Table value of Z at 1.85 = .297684

Further, probability of the other independent path A → D → I being completed with 30 weeks.

$$Z = \frac{x - a_{(vip)}}{\sigma_{cp}}$$

where,

$Z_{(vip)}$ = Standard normal variate of the other independent path.

X = Given time = 30 weeks

$a_{(vip)}$ = Mean or length of the other independent path i.e., 10 weeks

$\sigma_{(cip)}$ = Standard deviation of the other independent path.

$= \sqrt{1 + 0 + 1.77} = \sqrt{2.77} = 1.66$

Thus, $Z_{(vip)} = \frac{30 - 10}{1.66} = 12.04$

Table value of Z at 12.04 approximately = 1. Thus the probability of the project being completed within 30 weeks.

or $P_{(w)} = Z_{(cp)} \times Z_{(vip)}$

$P_{(w)} = .96784 \times 1 = .96784$

Therefore, the probability of the project not being completed within 30 weeks.

or $q_{(w)} = 1 - P_w$

$= 1 - .96784 = .03216$

(e) Calculation of the Est, Eft, Lst, Lft and total floats of the various activities

Activity	Expected duration tE.	Est	Eft	Lst	Lft	Floats
(1–2) A	2	0	2	8	10	8

(1–3) B	5	0	5	0	5	0
(1–4) C	3	0	3	10	13	10
(2–5) D	1	2	3	10	11	8
(3–5) E	6	5	11	5	11	8
(4–6) F	5	3	8	13	18	10
(5–6) G	7	11	18	11	18	0
(6–7) H	8	18	26	18	26	0
(5–7) I	7	11	18	19	26	8

Merits and Demerits of PERT

Having thus analysed, the merits and demerits of PERT as Quantitative Technique can be summed up as follows:

Merits:

1. It enables a manager to understand easily the relationship that exists between the activities in a project.
2. It enables a manager to know in advance where the trouble may occur, where more supervision may be needed and where resources may be transferred to keep the project on schedule.
3. It compels the manager to plan carefully and study how the various activities fit in the project.
4. It draws attention of the management to the critical activities of the project.
5. It suggests areas of increasing efficiency, decreasing cost and maximizing profits.
6. It enables the use of statistical analysis.
7. It makes possible a forward looking type of control.
8. It compels the management for taking necessary action at the right time without any let up.
9. It provides upto date information through frequent reporting, data processing and programme analysis.
10. It helps in formulating a new schedule when the existing ones cannot meet the situation.
11. It helps in minimizing delays and disruptions by scheduling the time and budgeting the resources.
12. It helps in coordinating the various parts of the project and expediting the mode of operation for completing the project in time.
13. It permits more effective planning and control.

Demerits:

1. It does not lay any emphasis on the cost of a project but time only.
2. It does not help in routine planning of the recurring events.
3. Errors in time estimates under the PERT make the network diagram and the critical path etc. meaningless.
4. In the calculation of probability under the PERT it is assumed that a large number of independent activities operate on critical path and that the distribution of total time is normal. This may not hold good in a peculiar situation.

5. For effective control, PERT requires, frequent up-to-date information and revision in calculation which may be quite costly for the management.
6. It does not consider the matter of resources required for various types of activities of a project.

Comparison between CPM and PERT

The critical path method and programme Evaluation and Review Technique have some characteristics in common. These are depicted as under:

1. Both CPM and PERT are quantitative techniques of Network analysis.
2. Both CPM and PERT are used by the management, as tools for taking decisions relating to a large complex project.
3. Both CPM and PERT technique involve drawal of Network diagram and its analysis on various scores.

Contrasts between the CPM and PERT

CPM and PERT contrast each other on the following points:

CPM	PERT
1. It is a deterministic model under which result is ascertained in a manner of certainty	1. It is a probabilistic model under which result is estimated in a manner of probability
2. It deals with the activities of precise well known time	2. It deals with the activities of uncertain time.
3. It is used for repetitive jobs like residential construction.	3. It is known for non-repetitive jobs like planning and scheduling of research programmes.
4. It is activity oriented in as much as its results are calculated on the basis of the activities.	4. It is event oriented in as much as its results are calculated on the basis of events.
5. It does not make use of Dummy activities.	5. It makes use of Dummy activities to represent the proper sequencing of the activities.
6. It deals with costs of a project schedules and their minimisation.	6. It has nothing to do with cost of a project.
7. It deals with the concept of crashing.	7. It does not deal with concept of crashing.
8. Its calculation is based on one type of time estimation that is precisely known.	8. It finds out expected time of each activity on the basis of three types of estimates, viz., optimistic time, pessimistic time and most likely time.
9. It can be used as a control devices as it requires repetition of the entire evaluation of the project each time the changes are introduced to the network.	9. It is used as an important control device as it assists the management in controlling a project by constant review of the delays in activities.

10. It does not make use of the statistical devices in the determination of the time estimates.	10. It makes use of the statistical devices, viz., standard deviation variance, probability (Z), and normal distribution table in the determination of probabilities of completing or not completing a project or a path within a given time.