# B.SC. STATISTICS – I YEAR

## *DJS1D* : MATHEMATICS

### SYLLABUS

**Unit - I**

Differentiation: Tangent and Normal-Direction of the tangent-Angle of intersection of curves-subtangent and subnormal - Polar coordinates - Angle between the radius vector and the tangent-Polar subtangent and polar subnormal - Length of arc in polar coordinates. Envelope - Circle, radius and centre of curvature.

**Unit - II**

Successive differentiation –Leibnitz's Formula. Partial differentiation – Successive partial differentiation – Implicit functions – homogeneous functions – Euler's theorem. Maxima and Minima for one variable - Concavity, Convexity and points of inflexion - Maxima and Minima for two variables.

**Unit - III**

Integration- Methods of integration - Integrals of functions containing linear functions of x - Integrals of functions involving $a^2 \pm x^2$ - Integration of rational algebraic functions - $1/(ax^2+bx+c)$, $(px+q)/(ax^2+bx+c)$. Integration of irrational functions - $1/(ax^2+bx+c)^{1/2}$, $(px+q)/(ax^2+bx+c)^{1/2}$, $(px+q)\sqrt{(ax^2+bx+c)}$ - Integration by parts. Multiple integrals - Evaluation of double integrals - polar coordinates - Beta and Gamma functions and their properties.

**Unit - IV**

Differential equations: Types of first order and first degree equations. Variables separable, Homogeneous, Non-homogeneous equations and Linear equation. Equations of first order but of higher degree. Linear differential equations of second order with constant coefficients. Methods of solving homogenious linear differential equations of second order. Laplace transform and its inverse – solving ordinary differential equation with constant coeficients using Laplce transform.

**Unit - V**

Theory of Equations: Nature of roots, Formulation of equation whose roots are given. Relation between coefficients and roots - Transformation of equations - Reciprocal equations - Horner's method of solving equatios.

**UNIT - I : DIFFERENTIATION**

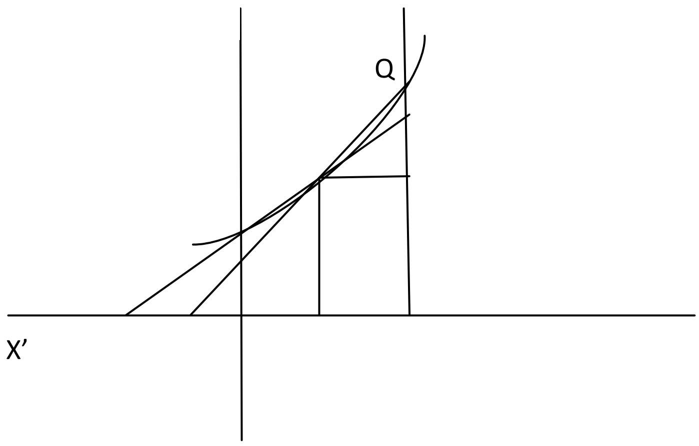
*Differentiation: Tangent and Normal-Direction of the tangent-Angle of intersection of curves-subtangent and subnormal - Polar coordinates - Angle between the radius vector and the tangent-Polar subtangent and polar subnormal - Length of arc in polar coordinates. Envelope - Circle, radius and centre of curvature.*

## Tangent and Normal

Meaning of the Derivative – Geometrical Interpretation**:**

Let us assume that P and Q are two neighbouring points on the continuous curve $y = f(x)$. From P and Q, draw PM, QN perpendicular to the x-axis and from P draw PR perpendicular to QN.

If P remains fixed and Q moves towards P along the curve and finally coincides with P, then the chord PQ becomes the tangent at P.




The inclination of the tangent at P to OX is $\angle PTX$.

Here,$\angle PTX = \lim_{Q\to P} \angle PSX$

$= \lim_{Q\to P} \angle QPR$ .

Let the coordinates of P, Q be $(x, y)$ and$(x + \Delta x, y + \Delta y)$ respectively.

Then,$\tan \angle QPR = \frac{RQ}{PR} = \frac{\Delta y}{\Delta x}$ .

and $\tan \angle PTX$ =$\log_{Q\to P} \tan \angle QPR = \lim_{Q\to P} \frac{\Delta Y}{\Delta X}$

$$= \lim_{\Delta x\to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} .$$

Hence, $\frac{dy}{dx}$ is the gradient of the tangent to the curve at the point $(x, y)$.

## The Direction of the Tangent:

We know that, the gradient of the tangent at $(x, y)$ to the curve $y = f(x)$ is $\frac{dy}{dx}$. Also, the gradient at any point of a curve is defined as the trigonometrical tangent of the angle which the geometrical tangent at the point makes with positive direction of the $x$-axis. If the angle is negative or obtuse, the gradient is negative; if the tangent is parallel to the axis of $x$, the gradient is zero.

**EXAMPLES:**

1.Find the angle which the tangent at (2, 4) to the curve $y = 6 + x - x^2$ makes with the $x$-axis.

**Solution:**

Given that, $y = 6 + x - x^2$

$$\Rightarrow \frac{dy}{dx} = 1 - 2x.$$

At (2, 4), the value of $\frac{dy}{dx} = 1 - 2(2) = -3.$

$$\therefore \tan\varphi = -3$$

Hence, $\varphi = 160° \, 34'$.

2. What is the direction of the tangent (2,1) to the curve $x^3 + y^3 = 9$ ?

**Solution:**

On differentiating the equation of the curve $x^3 + y^3 = 9$, we have

$$3x^2 + 3y^2 \frac{dy}{dx} = 0$$

$$\Rightarrow 3y^2 \frac{dy}{dx} = -3x^2$$

$$\therefore \frac{dy}{dx} = -\frac{x^2}{y^2}.$$

At the point (2, 1), the value of $\frac{dy}{dx} = -\frac{(2)^2}{(1)^2} = -4$.

Therefore, the tangent at (2,1) makes an angle,

$$\tan\varphi = -4 \qquad \Rightarrow \varphi = tan^{-1}(-4) \text{with the } x\text{-axis.}$$

**Result:**

If a curve passes through the origin, we can find the shape of that curve at the origin.

Find the shape of the curves in the following examples, where the curve passes through the origin:

(i)The parabola $y^2 = 4ax$ passes through the origin.

Here, $2y\frac{dy}{dx} = 4a(1)$

$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$.

$\therefore$ The value of $\frac{dy}{dx}$ at (0,0)is $\frac{dy}{dx} = \frac{2a}{0} = \infty$.

$\therefore \tan\varphi = \infty \qquad \Rightarrow \varphi = \frac{\pi}{2}$.

Therefore, at the point (0,0), the tangent is at right angles to the $x$-axis.

That is, the $y$-axis is the tangent to the curve at the origin.

(ii) The curve $y = \frac{x^2}{(1+x^2)}$ passes through the origin.

Consider $\frac{dy}{dx} = \frac{2x}{(1+x^2)^2}$ .

At the origin, the value of $\frac{dy}{dx} = \frac{2(0)}{(1+0)^2} = 0$.

$\therefore \ \varphi = 0$.

Thus, the curve touches the $x$-axis at the origin.

(iii) In a curve if the value $\frac{dy}{dx}$ at (0,0) is 1, then

$\therefore \tan\varphi = 1$

$\Rightarrow \varphi = 45°$.

Thus, the tangent to the curve at the origin bisects the angle between the co-ordinate axes.

3. At which point on the curve, $y = x^3 - 12x + 18$ is the tangent parallel to the $x$-axis?

**Solution:**

Given that, $y = x^3 - 12x + 18 \qquad \ldots (1)$

$$\Rightarrow \frac{dy}{dx} = 3x^2 - 12.$$

Also given that, the tangent and $x$-axis are parallel.

$\therefore \varphi = 0 \Rightarrow \tan\varphi = 0 = \frac{dy}{dx}.$

$$\Rightarrow 3x^2 - 12 = 0.$$

$\Rightarrow x^2 = \frac{12}{3} = 4$

$\Rightarrow x = \pm 2$

$\therefore x = 2$ in (1) $\Rightarrow y = 2^3 - 12(2) + 18 = 2.$

$\therefore x = -2$ in (1) $\Rightarrow y = (-2)^3 - 12(-2) + 18 = 34.$

Therefore, the corresponding values of y are 2 and 34.

$\therefore$ The Tangents at the points (2,2) and (-2,34) are parallel to the $x$-axis.

4. At which point is the tangent to the curve $x^2 + y^2 = 5$parallel to the line $2x - y + 6 = 0$?

**Solution:**

The given curve is $x^2 + y^2 = 5$ ... (1)

On differentiating the equation of the curve, we get

$$2x + 2y\frac{dy}{dx} = 0$$

$\Rightarrow \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$ ... (2)

At the point (x,y) the gradient of the tangent $= -\frac{x}{y}$.

The gradient of the line $2x - y + 6 = 0$is obtained as:

$$2x - y + 6 = 0 \Rightarrow y = 2x + 6$$

$$\Rightarrow \frac{dy}{dx} = 2 \quad \ldots (3)$$

Comparing (2) and(3), we have, $-\frac{x}{y} = 2$

$$\Rightarrow x = -2y \quad \ldots (4)$$

Substituting (4) in the given curve (1), we get

$(-2y)^2+y^2=5 \quad \Rightarrow 4y^2+y^2=5 \quad \Rightarrow 5y^2=5 \quad \Rightarrow y^2=1$

$\Rightarrow \; y = \pm 1$ … (5)

Using $y = 1$ in (4), we get $x = -2(1) = -2$.

Using $y = -1$ in (4), we get $x = -2(-1) = 2$.

Therefore, the corresponding values of $x$ are - 2 and 2.

Therefore, at the point (-2, 1) and (2, -1), the tangents to the curve $x^2 + y^2 = 5$ are parallel to $2x - y + 6 = 0$.

5. Find the equation of the tangent of the curve, $y = x^3 \; at \; (1/2, 1/8)$.

**Solution:**

The given curve is $y=x^3$ … (1)

Differentiating the equation of the curve (1), we get

$\frac{dy}{dx} = 3x^2$

$\Rightarrow \frac{dy}{dx} \; at(1/2,1/8) = 3(1/2)^2 = 3(1/4) = 3/4$ .

Therefore, the equation of the tangent to the curve at $(x_1, y_1)$ is: $y - y_1 = \frac{dy}{dx}(x - x_1)$.

Therefore, at the point (1/2, 1/8), we have

$y - 1/8 = 3/4(x - 1/2)$

$\Rightarrow \frac{8y-1}{8} = \frac{3}{4}x - \frac{3}{8}$

$\Rightarrow 8y - 1 = 6x - 3 \qquad \Rightarrow 6x - 8y - 2 = 0$

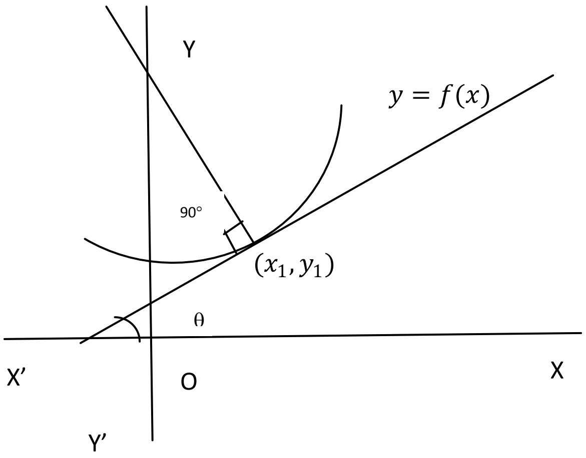
$\Rightarrow 3x - 4y - 1 = 0$ is the required equation of the tangent.

Equations of the Tangent and Normal at any Point of a Curve:

Let the equation of the curve be $y = f(x)$ and the co - ordinates of P be $(x_1, y_1)$. We know that, when the tangent at P makes an angle $\theta$ with the x-axis, the gradient of the line is $\tan\theta$. We have seen already that $\tan\theta = \frac{dy}{dx}$ at $(x_1, y_1)$.

The value of $\frac{dy}{dx}$ at $(x_1,y_1)$ is usually denoted by $\left(\frac{dy}{dx}\right)_1$ .

∴ The equation of the tangent at P is $y - y_1 = tan\,\theta\,.\,(x_1, y_1)$




$$\Rightarrow y - y_1 = \left(\frac{dy}{dx}\right)_1 (x_1, y_1).$$

The normal to a curve at any point is the straight line which passes through that point. It is also at right angles to the tangent to the curve at the point.

Any line through the point $(x_1,y_1)$ is $y-y_1=m(x-x_1)$.

This will be perpendicular to the tangent to the curve if

$m\left(\frac{dy}{dx}\right) = -1$

Hence, the normal at $(x_1,y_1)$ to the curve $y = f(x)$ is

$$(y-y_1)\left(\frac{dy}{dx}\right) + (x-x_1) = 0.$$

<u>When the curve is given in the 'Parametric From'</u>:

If the equation of the curve is given in the parametric from $x = f\,(\theta)$ and $y = \varphi(\,\theta)$,

$\left(since, \frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}\right)$,

the equation of the tangent becomes $\frac{y-\varphi(\,\theta)}{\frac{dy}{dx}} = \frac{x-f(\theta)}{\frac{dy}{dx}}$ .

and the equation of the normal becomes,

$$\{\,x - f(\theta)\,\}\frac{dx}{d\theta} + y - \varphi(\theta)\frac{dy}{d\theta} = 0.$$

<u>When the curve is given in the form $f(x, y) = 0$</u>:

Let the equation of the curve is given in the form $f(x,y) = 0$. We should calculate $\frac{dy}{dx}$ by differentiating the equation and then write down the equation of the tangent and normal at the point $(x_1,y_1)$.

**PROBLEMS:**

1.Find the equation of the tangent to the curve $y = \frac{6x}{x^2-1}$at the point (2,4).

**Solution**:

The given curveis$y = \frac{6x}{x^2-1}$.

Differentiating the equation of the given curve, we get

$$\frac{dy}{dx}=\frac{(x^2-1)6(1)-6x(2x)}{(x^2-1)^2}= \frac{6x^2-6-12x^2}{(x^2-1)^2}=\frac{-6x^2-6}{(x^2-1)^2} = \frac{-6(x^2+1)}{(x^2-1)^2}$$

$\therefore$ $\frac{dy}{dx}$at the point (2,4), we get

$$\frac{dy}{dx}= -6\frac{(4+1)}{(4-1)^2} = -\frac{10}{3}.$$

We know that, the equation of the tangent to the curve at P is $y - y_1 = \left(\frac{dy}{dx}\right)_1 (x - x_1)$

$\therefore$At $(2, 4)$, we have $y - 4 = \frac{-10}{3}(x - 2)$

$$\Rightarrow 3y - 12 + 10x - 20 = 0$$

$\Rightarrow 10x + 3y - 32 = 0$, is the required equation of the tangent.

2. Find the equation of the tangent to the ellipse$\frac{x^2}{a^2}+\frac{y^2}{b^2}$=1 at $(x_1,y_1)$.

**Solution:**

The given equation of the curveis $\frac{x^2}{a^2}+\frac{y^2}{b^2}$=1.

On differentiating, we get

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0.$$

$$\Rightarrow \frac{2y}{b^2}\frac{dy}{dx}= -\frac{2x}{a^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2x}{a^2}\frac{b^2}{2y}=\frac{-b^2x}{a^2y}.$$

$\therefore$The value of$\frac{dy}{dx}$at $(x_1,\ y_1)$ is $\frac{-b^2x_1}{a^2y_1}$.

We know that, the equation of the tangent at the point $(x_1,y_1)$ is

$$y - y_1 = \left(\frac{dy}{dx}\right)_1 (x - x_1)$$

$$\Longrightarrow (y - y_1) = \frac{-b^2 x_1}{a^2 y_1}(x - x_1)$$

$$\Longrightarrow a^2 y_1(y - y_1) = -b^2 x_1 (x - x_1)$$

$$\Longrightarrow a^2 y y_1 - a^2 y_1^2 = -b^2 x x_1 + b^2 x_1^2$$

$\Longrightarrow a^2 y y_1 + b^2 x x_1 = a^2 y_1^2 + b^2 x_1^2$.

On dividing both sides of the above equation by $a^2 b^2$, we have

$\frac{a^2 y y_1}{a^2 b^2} + \frac{b^2 x x_1}{a^2 b^2} = \frac{a^2 y^2{}_1}{a^2 b^2} + \frac{b^2 x_1^2}{a^2 b^2}$

$\Longrightarrow \frac{y y_1}{b^2} + \frac{x x_1}{a^2} = \frac{y_1^2}{b^2} + \frac{x_1^2}{a^2}$

$\Longrightarrow \frac{x x_1}{a^2} + \frac{y y_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$.

Since the point $(x_1,y_1)$ lies on the given curve, the equation of the tangent on the curveat $(x_1, y_1)$ is $\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1$.

3. Find the points in the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$, where the tangents are parallel to y=2x and prove that two of these points have the same tangent.

Solution:

Let the tangent at the point $(x_1,y_1)$ be parallel to y=2x.

On differentiating the equation of the curve, we get

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10.$$

Thegradient of the tangent at $(x_1, y_1)$ is $4{x_1}^3 - 18{x_1}^2 + 26x_1 - 10$.

Given that y=2x $\quad \Longrightarrow \frac{dy}{dx} = 2$.

Since, the tangent is parallel to y=2x, we get

$$4{x_1}^3 - 18{x_1}^2 + 26x_1 - 10 = 2$$

$\Longrightarrow 4{x_1}^3 - 18{x_1}^2 + 26x_1 - 10 - 2 = 0$

$\Longrightarrow \quad 2{x_1}^3 - 9{x_1}^2 + 13x_1 - 6 = 0$

The roots of the equation are found as follows:

| | | | | |
|---|---|---|---|---|
| 1 | 2 | -9 | 13 | -6 |
| | 0 | 2 | -7 | 6 |
| | 2 | -7 | 6 | 0 |

Therefore, $x_1$=1 is a root.

Therefore, the above equation becomes: $(x_1 - 1)\ (2x_1^2 - 7x_1 + 6\ ) = \ 0.$

Now, the roots of the equation, $(2x_1^2 - 7x_1 + 6\ )$, is obtained as follows:

$$x_1 = \frac{7 \pm \sqrt{49-48}}{4} = \frac{7 \pm 1}{4}$$

$= 2$ or $\frac{3}{2}$

Therefore, we have $x_1 = 1$ or $\frac{3}{2}$ or 2.

On substituting these values of $x_1$ in y=$x^4 - 6x^3 + 13x^2 - 10x + 5$, we have:

x=1$\Longrightarrow$ $y = 1 - 6 + 13 - 10 + 5 = 3$

x=2$\Longrightarrow$y=16-48+52-20+5=5

x=$\frac{3}{2}$ $\Longrightarrow = \frac{81}{16} - \frac{162}{8} + \frac{117}{4} - \frac{30}{2} + 5 = \frac{81-324+468-240+80}{16} = \frac{65}{16}$

$\therefore$ The points on the curve where the tangents are parallel to y=2x are

(1,3), (2,5) and $\left(\frac{3}{2}, \frac{65}{16}\right)$.

The equations of the tangent at these points are respectively:

y-2x-1=0, y-2x-1=0 and y-2x-$\frac{17}{6}$=0.

Here,we have the same tangent at the points (1,3) and (2, 5).

4. Find the equation of the tangent at the point '$\theta$' on the following curve

x=a$(\theta - sin\theta)$; y=a(1-cos $\theta$).

**Solution:**

Given that, x= $a(\ \theta - asin\theta)$and y= (a $-$ acos$\theta$).

We know that, $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$

We have, $\frac{dy}{d\theta} = \text{a} \sin\theta$ and $\frac{dx}{d\theta} = \text{a}(1 - \cos\theta)$

$\therefore \frac{dy}{dx} = \frac{a\ sin\theta}{\text{a}(1-\cos\ \theta)} = \frac{2\sin\left(\frac{\theta}{2}\right)cos\left(\frac{\theta}{2}\right)}{2\ sin^2\left(\frac{\theta}{2}\right)} = cot\ \left(\frac{\theta}{2}\right).$

The equation of the tangent at $\theta$ is

y-a(1-cos $\theta$)=cot $\left(\frac{\theta}{2}\right)$\{x-a($\theta - sin\ \theta$)\}

$\Rightarrow$y-a+acos $\theta$=cot $\left(\frac{\theta}{2}\right)$\{x-a $\theta$ +asin $\theta$ \}

$\Rightarrow$y-a+acos $\theta$=cot $\frac{\theta}{2}$ (x-a$\theta$)+cot $\left(\frac{\theta}{2}\right)$ (asin $\theta$)

$\Rightarrow$ y-a+acos $\theta$- cot $\frac{\theta}{2}$. asin $\theta$= (x-a $\theta$ ) cot $\left(\frac{\theta}{2}\right)$

$\Rightarrow$y-a+acos $\theta$ - $\frac{\cos\left(\frac{\theta}{2}\right)}{sin\left(\frac{\theta}{2}\right)}$ .asin$\frac{\theta}{2}$ cos $\left(\frac{\theta}{2}\right)$=(x-a $\theta$) cot $\left(\frac{\theta}{2}\right)$

$\Rightarrow$y-a+acos $\theta$ -2a$cos^2\left(\frac{\theta}{2}\right)$ =(x-a $\theta$) cot $\left(\frac{\theta}{2}\right)$

$\Rightarrow$y-a+acos $\theta$ - a(1+cos $\theta$) =(x-a $\theta$) cot$\left(\frac{\theta}{2}\right)$

$\Rightarrow$y-a+acos $\theta$-a-acos $\theta$ =(x-a $\theta$) cot $\left(\frac{\theta}{2}\right)$

$\Rightarrow$y-2a =(x-a $\theta$) cot $\left(\frac{\theta}{2}\right)$ , is the required equation of the tangent at the point '$\theta$'.

5. In the calenary, y=acosh$\left(\frac{x}{a}\right)$, prove that the length of the normal intercepted between the curve and the axis is $\frac{y^2}{a}$.

**Solution:**

Given that, y= a cosh$\left(\frac{x}{a}\right)$

$\Rightarrow \frac{dy}{dx}$=$a$ sinh $\left(\frac{x}{a}\frac{1}{a}\right)$ = sinh$\left(\frac{x}{a}\right)$

We know that, the length of the normal interrupted is

y $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}$= y [1 + sin$sinh^2\left(\frac{x}{a}\right)]^{\frac{1}{2}}$ =y [$cosh^{2\times\frac{1}{2}}\frac{x}{a}$]=y cosh $\frac{x}{a}$ =$\frac{y^2}{a}$.

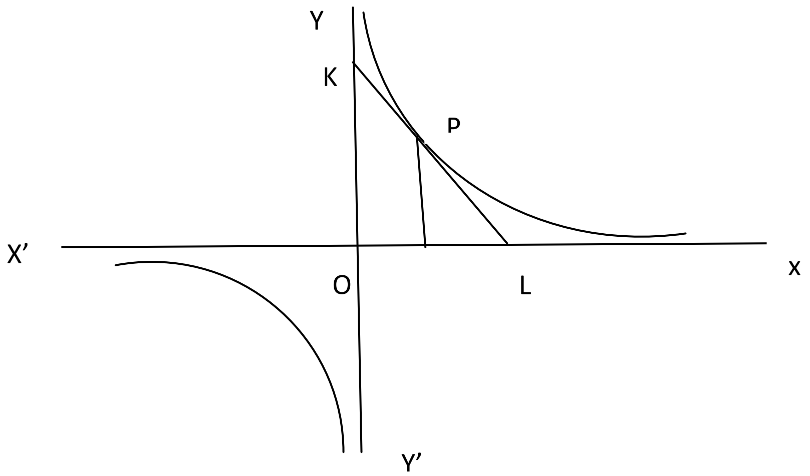
Therefore, the length of the normal intercepted between the curve and the axis is obtained as $\frac{y^2}{a}$.

**Examples of the Properties of the Tangents and Normals to the Curves from their Equations:**

1. Show that the portion of the tangent intercepted between the asymptotes of the rectangular hyperbola $xy=c^2$ is bisected by the point of contact. Show also that the area of the triangle formed by the tangent and the asymptotes is constant.

**Solution**

The above equation and the tangent can be drawn as follows:




The asymptotes of the rectangular hyperbola $xy=c^2$ are the co-ordinate axes. Let the co-ordinates of the point P be $(x_1, y_1)$.

On differentiating the equation of the curve, $xy=c^2$, we get

$$x\frac{dy}{dx}+y=0 \Longrightarrow \frac{dy}{dx}=\frac{-y}{x}.$$

Therefore, the equation of the tangent to the curve at P is

$$y-y_1=\frac{-y_1}{x_1}(x-x_1).$$

$\Longrightarrow x_1y-x_1y_1=-y_1x+y_1x_1$

$\Longrightarrow \frac{x_1y}{y_1x_1}+\frac{xy_1}{x_1y_1}=\frac{2y_1x_1}{x_1y_1}$ (on dividing $y_1x_1$)

$\Longrightarrow \quad \frac{y}{y_1}+\frac{x}{x_1}=2 \Longrightarrow \frac{x}{x_1}+\frac{y}{y_1}=2$ ... (1)

Let this tangent meet the coordinate axes in L, K. Letthe ordinate of P be MP.

To find the point where the tangent at P meets the x-axis, put y=0 in (1), we have

$\frac{x}{x_1} = 2 \Longrightarrow$ OL=$2x_1$=2 OM.

∴KL=2KP $\Longrightarrow$ P bisects KL.

To find the point where the tangent at P meet the y-axis, put x=0 in (1), we have

$\frac{y}{y_1} = 2 \Longrightarrow$ OK=$2y_1$.

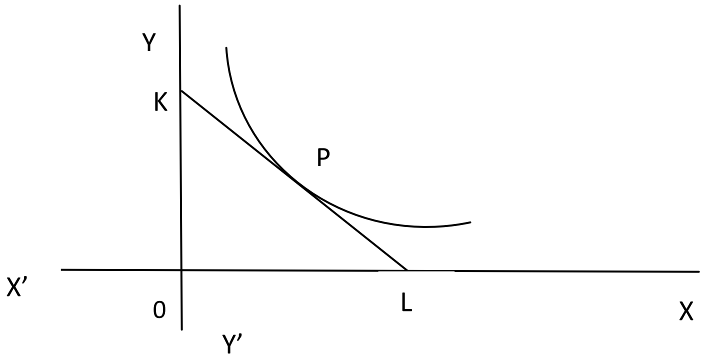
Therefore, the area of the triangle KOL =$\frac{1}{2}$ OK .OL =$\frac{1}{2}$ . $2y_1$. $2x_1$ =$2x_1y_1$ =$2c^2$.

This implies, the area of the triangle KOL does not depend on the position of P on the curve.

2. Show that for the hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ that portion of the tangent included between the coordinate axes is constant and is equal to a.

**Solution**

The above equation and the tangent can be drawn as follows:




We know that, any point on the curve can be represented as

$(a\cos^3\theta, a\sin^3\theta) \Longrightarrow x = a\cos^3\theta,\ y = a\sin^3\theta.$

Let the tangent at P meet the coordinate axes in L,K.

We know that, $\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta}$.

∴ $\frac{dy}{d\theta}$=$3a\sin^2\theta\, cos\theta$ and $\frac{dx}{d\theta}$=$3a\cos^2\theta(-\sin\theta)$ .

$\Longrightarrow \frac{dy}{dx} = -\frac{3a\sin^2\theta\, cos\theta}{3a\cos^2\theta \sin\theta} \Longrightarrow \frac{dy}{dx} = -\tan\theta.$

Therefore, the tangent at P to the curve is

$y - a\sin^3\theta = -\tan\theta(x - a\cos^3\theta)$

$\Longrightarrow y - a\sin^3\theta = -x tan\theta + a\, tan\theta. cos^3\theta$

$\Longrightarrow y - a\sin^3\theta - a\tan\theta \cdot cos^3\theta = -x\, tan\theta$

$$\Rightarrow -a(\sin^3\theta + tan\theta . cos^3\theta) = -xtan\theta - y$$

$\Rightarrow$a($\sin^3\theta + tan\theta . cos^3\theta$)= x tan$\theta + y$

$\Rightarrow$a$\left(\sin 3\theta + \frac{sin\theta}{cos\theta} . \cos 3\theta\right)$= x tan$\theta + y$

$\Rightarrow$a $\left(\frac{sin^3\theta \; cos\theta + sin\theta \; .cos^3\theta}{cos\theta}\right)$= x tan$\theta + y$

$\Rightarrow$ a $\left(\frac{sin\theta \; cos\theta \; (sin^2\theta + cos^2\theta)}{cos\theta}\right)$= x tan$\theta + y$

$\Rightarrow$a sin $\theta$.(1)= x tan$\theta + y$

$\Rightarrow$x tan$\theta + y$= a sin $\theta$.

The intercept OL on the x-axis is obtained by putting y=0 in the above equation.

i.e.,$OL = \frac{a \; sin \; \theta}{tan \;\; \theta} = a \; cos \; \theta.$

Similarly, putting x=0, we get

$OK = a \; sin \; \theta$

$\therefore KL^2 = OL^2 + OK^2 = a^2cos^2\theta + a^2sin^2\theta = a^2(cos^2\theta + sin^2\theta) = a^2$

$\Rightarrow KL = a.$

$\therefore$The hypocycloid$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ that portion of the tangent included between the coordinate axes is equal to a, which is a constant.

## ANGLE OF INTERSCTION OF CURVES:

The angle of intersection of two curves is defined as the angle between their respective tangents at the common point of intersection.

We know that, the angle between the two straight lines $y=m_1x+c_1$ and $y=m_2x+c_2$ is

$$\tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1m_2}\right).$$

Replace $m_1$and $m_2$in the formula by the values of$\frac{dy}{dx}$ for the two curves at their point of intersection. The value is the angle $\theta$ at the point where the curves cut.

**PROBLEMS:**

1. For the curves $x^2$ =4y and $y^2$=4x, find the angle of intersection.

**Solution:**

To find the point of intersection of the curves, solve the equation $x^2$=4y and$y^2$=4x.

Consider the curve $x^2$=4y.

On squaring both sides of the equation, we have

$x^4 = 16y^2 \Longrightarrow x^4 = 16(4x) \Longrightarrow x^4 - 64x = 0 \quad \Longrightarrow x(x^3 - 64) = 0$

⟹x=0 or

$x^3 - 64 = 0 \Longrightarrow x^3$=64 ⟹ x=4.

The corresponding values of y are:

x=0⟹ $y^2 = 4(0)$ ⟹ y=0

x=4⟹ $y^2 = 4(4)$ ⟹ $y^2 = 16$ ⟹y=4.

The points of intersection are (0,0) and (4,4) .

(i) Consider the point (4,4).

To find $m_1$, consider the curve $x^2 = 4y$.

On differentiating, we get

2x dx= 4dy ⟹dx=$\frac{2\,dy}{x} \Longrightarrow \frac{dy}{dx} = \frac{x}{2}$ ... (1)

$m_1$is the value of$\frac{dy}{dx}$ at the point (4,4), that is, $\frac{4}{2} = 2$.

To find $m_2$, consider the curve$y^2 = 4x$.

Ondifferentiating, we get

$\frac{dy}{dx} = \frac{2}{y}$ ... (2)

$m_2$is the value of$\frac{dy}{dx}$ at the point (4,4), that is,$\frac{2}{4} = \frac{1}{2}$

We know that, the angle between the two straight lines y=$m_1$x+$c_1$ and y=$m_2$x+$c_2$ is

$$\tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right).$$

At the point (4,4),$tan\theta = \frac{m_1-m_2}{1+m_1m_2} \quad = \frac{2-\frac{1}{2}}{1+2\times\frac{1}{2}} = \frac{2-\frac{1}{2}}{2} \quad = \frac{3}{2} \times \frac{1}{2}$

$\Rightarrow \quad \tan\theta = \frac{3}{4} \qquad \Rightarrow \theta = \tan^{-1}\left(\frac{3}{4}\right).$

(ii) Consider the point (0, 0).

$m_1$is obtained by substituting (0, 0) in (1).

i.e., $m_1 = \frac{0}{2} = 0 \Rightarrow \tan\theta = 0 \qquad \Rightarrow \theta = \tan^{-1}(0) = 0$

$m_2$is obtained by substituting (0, 0) in (2)

i.e., $m_2 = \frac{2}{0} = \infty \Rightarrow \tan\theta = \infty \qquad \Rightarrow \theta = \tan^{-1}(\infty) = \frac{\pi}{2}.$

Therefore, the values of $\psi$ for the two curves are 0 and$\frac{\pi}{2}$.

Hence the angle of intersection is$\frac{\pi}{2}$.

That is, the curves cut orthogonally.

2. Find the angle at which the curves (1)$x^2$=ay and (2)$x^3 + y^3 = 3axy$ cut eah other.

**Solution:**

Substituting the value of y from $x^2 = ay \; in \; x^3 + y^3 = 3axy$

$\Rightarrow x^2 = ay \Rightarrow \qquad y = \frac{x^2}{a}$.

On substituting the value of y in $x^3 + y^3 = 3axy$, we have

$$\Rightarrow x^3 + \left(\frac{x^2}{a}\right)^3 = 3ax\left(\frac{x^2}{a}\right) \quad \Rightarrow x^3 + \frac{x^5}{a^3} = 3x^3 \quad \Rightarrow \frac{x^5}{a^3} = 2x^3 \quad \Rightarrow \frac{x^5}{a^3} - 2x^3 = 0$$

$$\Rightarrow x^3\left(\frac{x^2}{a^3} - 2\right) = 0$$

$\therefore \quad x^3 = 0 \; or$

$\frac{x^3}{a^3} = 2 \qquad \Rightarrow x^3 = 2a^3 \qquad \Rightarrow \qquad x = (2)^{\frac{1}{3}} \; a.$

On substituting these values of $x$ in $x^2 = ay$, we have

x= 0 $\Rightarrow\frac{x^2}{a}$=y$\Rightarrow$0/a=y$\Rightarrow$ y=0

$x = 2^{1/3}a \;\Rightarrow$ y=$\frac{x^2}{a} \Rightarrow = \frac{(2^{1/3}.a)^2}{a} \Rightarrow = \frac{2^{2/3}.a^2}{a} \Rightarrow \; = 2^{2/3}\, a \Rightarrow$=a.$(4)^{1/3}$.

$\therefore$ y=0 or y=a $(4)^{\frac{1}{3}}$ .

Hence, the curve cut at the points (0,0) and $\left\{a(2)^{\frac{1}{3}}, a\ (4)^{\frac{1}{3}}\right\}$.

On differentiating $x^2 = ay$ , we get,

$$2x\,dx = a\,dy. \implies 2x = a\frac{dy}{dx} \implies \frac{dy}{dx} = \frac{2x}{a}.$$

On differentiating $x^3 + y^3 - 3axy = 0$, we get,

$$3x^2dx + 3y^2\,dy - 3ax\,dy - 3aydy = 0$$

$$\implies 3y^2\,dy - 3ax\,dy = 3aydy - 3x^2dx$$

$\implies$ dy($3y^2 - 3ax$)=(3ay-3$x^2$) dx

$\implies$ dy($y^2 - ax$)=(ay-$x^2$)dx

$\implies \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$.

(i) The value of $\frac{dy}{dx}$ at $\left\{a(2)^{\frac{1}{3}}, a(4)^{\frac{1}{3}}\right\}$for the curve is $2^{\frac{4}{3}}$

and for the second curve at the same point is 0.

$$\therefore tan\theta = 2^{\frac{4}{3}} \implies \theta = \tan^{-1}\left\{2\left(2^{\frac{1}{3}}\right)\right\}.$$

(ii) The value of $\frac{dy}{dx}$ at (0,0) for both of the two curves is 0.

That is, the two curves touch at the origin, y=0.

That is, the tangent is common to both the curves.

3. Find the condition that the curves at $ax^2 + by^2 = 1$, $a_1x^2 + b_1y^2 = 1$ shall cut orthogonally.

**Solution:**

Let the curves intersect at the point whose co-ordinates are $(x_1, y_1)$

$\therefore ax_1^2 + by_1^2 - 1 = 0$ and $a_1x_1^2 + b_1y_1^2 - 1 = 0$

$\therefore \frac{x_1^2}{b_1 - b} = \frac{y_1^2}{a - a_1} = \frac{1}{ab_1 - a_1b}$…(1)

On differentiating the equations of the curve, we get

$$2ax + 2by\frac{dy}{dx} = 0 \quad \Rightarrow \frac{dy}{dx} = -\frac{ax}{by}$$

and$2a_1x + 2b_1y\ \frac{dy}{dx} = 0 \quad \Rightarrow \frac{dy}{dx} = -\frac{a_1x}{b_1y}$

The gradients of the tangents of the two curves at the points of intersection are:

$$-\frac{ax_1}{by_1}, -\frac{a_1x_1}{b_1y_1}.$$

These curves cut each other orthogonally.

Then, we know that,

$$-\frac{ax_1}{by_1}.\frac{-a_1x_1}{b_1y_1} = -1 \quad \Rightarrow \frac{aa_1x_1^2}{bb_1y_1^2} = -1 \quad \Rightarrow \frac{x_1^2}{y_1^2} = -\frac{bb_1}{aa_1}.$$

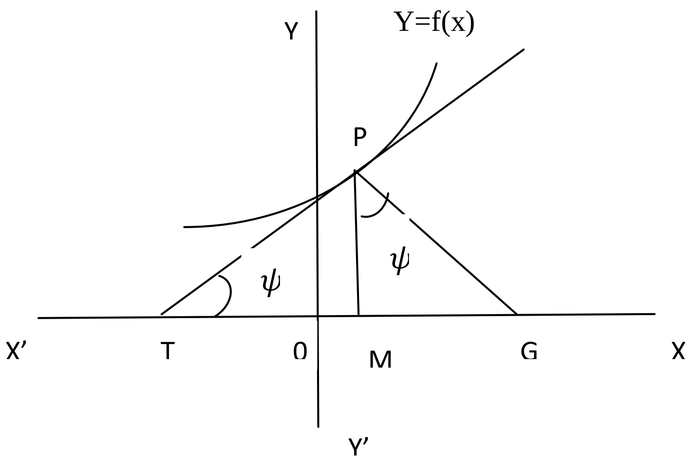
But the value of $\frac{x_1^2}{y_1^2}$ from (1) is $\frac{b_1-b}{a-a_1}$

$$\Rightarrow \frac{b_1-b}{a-a_1} = -\frac{bb_1}{aa_1} \quad \Rightarrow \frac{b_1-b}{bb_1} = \frac{a_1-a}{aa_1} \Rightarrow \frac{1}{b} - \frac{1}{b_1} = \frac{1}{a} - \frac{1}{a_1}$$

$$\Rightarrow \frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}.$$

**SUBTANGENT AND SUBNORMAL:**

Consider the following digram.




Let the tangent and normal at the point P meet the axis of x in T and G respectively. Let M be the foot of the ordinate.Then, TM is called the subtangent and MG is called the subnormal.

Subtangent=TM = MP cot $\psi = y/\frac{dy}{dx}$.

Here, MP = y $\Rightarrow$ cot $\psi = 1/\frac{dy}{dx}$.

Subnormal = MG=MP tan $\psi = y\frac{dy}{dx}$

Tangent = TP =MP cosec $\psi$=y$\sqrt{1+cot^2\psi}$

$$=y\left\{1+\frac{1}{\left(\frac{dy}{dx}\right)^2}\right\}^{\frac{1}{2}} \Rightarrow y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}}/\frac{dy}{dx}$$

Normal = PG=MP sec $\psi$ =y$\sqrt{1+tan^2\psi}$

$$= y\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{1/2}.$$

**PROBLEM:**

1.Show that, in the parabola $y^2 = 4ax$, the subtangent at any point is double the abscissa and the subnormal is constant.

**Proof:**

Differentiating the equation of the parabola,$y^2 = 4ax$, we have

$$2ydy = 4adx \qquad \Rightarrow \frac{dy}{dx} = \frac{4a}{2y} = \frac{2a}{y}.$$

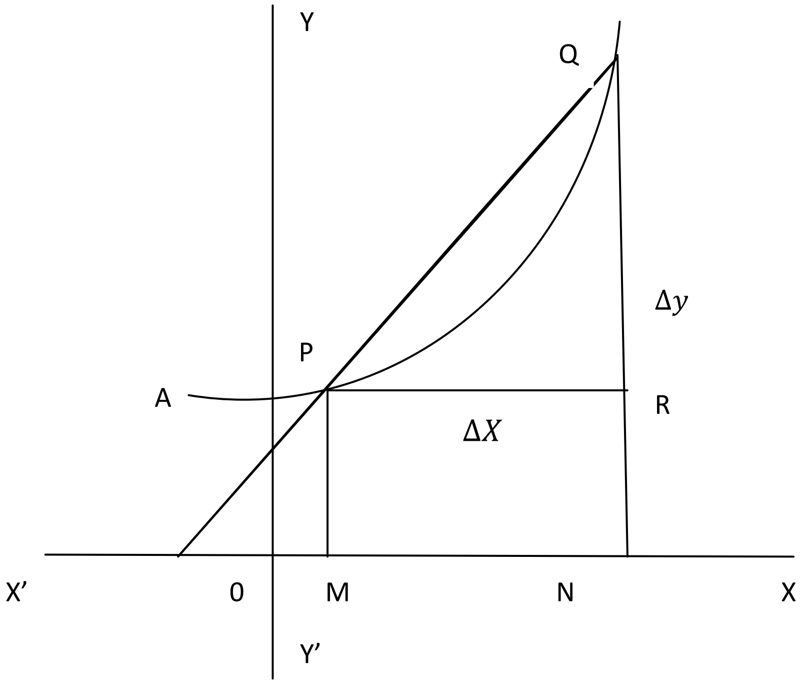
The subtangent= $\frac{y}{\frac{dy}{dx}} = \frac{y}{\frac{2a}{y}} = \frac{y^2}{2a} = 2x$

= double the abscissa

The subnormal= $y\frac{dy}{dx} = y\frac{2a}{y}$= 2a (constant).

## DIFFERENTIAL COEFFICIENT OF THE LENGTH OF AN ARC OF Y=f(x)

Consider the following diagram:




Let P be any point (x,y) on the curve y=f(x). Let Q be a point very near P, so that the coordinates of Qare $(x+\Delta x, y+\Delta y)$.

Now, let S be the length of the arc AP,where A is a fixed point on the curve. Then, s+$\Delta s$ is the length of the arc AQ, so that arc PQ=$\Delta s$.

As $\Delta x$ tends to zero, Q approaches P.Then, the chord PQ and arc PQ become almost equal. Thus, the ultimate ratio of the arc PQ to the chord PQis unity,as $\Delta s \to 0$.

Now, from the right-angled triangle PQR,

$$(chord\ PQ)^2 = (PR)^2 + (QR)^2$$

$$\Rightarrow (chord\ PQ)^2 = (\Delta x)^2 + (\Delta y)^2$$

$$\Rightarrow \left(\frac{chord\ PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$ [ On dividing both sides by $(\Delta x)^2$ ]

$$\Rightarrow \left(\frac{chord\ PQ}{arc\ PQ}\right)^2 \left(\frac{arc\ PQ}{\Delta x}\right)^2 = 1 + \left(\frac{\Delta y}{\Delta x}\right)^2$$ [multiplyand divide by $(arc\ PQ)^2$ in L.H.S]

Taking the limits as $\Delta x \to 0$, we get

$$\text{Lim}_{\Delta x \to 0} \frac{chord\ PQ}{arc\ PQ} = 1, \quad \lim_{\Delta x \to 0} \frac{arc\ PQ}{\Delta x} = \frac{ds}{dx}, \text{and} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

$$\therefore \left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2.$$

Simlarly, it may be shown that $\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2$.

NOTE:

From the figure, it easily follows that $\psi = \frac{dy}{ds}$ .

Here, $\psi$ is the angle between OX and the tangent at the point (x,y).

PROBLEM:

For the cycloid $x = a(1 - cos\theta), y = a(\theta + sin\theta)$, find $\frac{ds}{dx}$.

**Solution:**

We know that, $\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$.

Given that, $x = a(1 - cos\theta)$ $\quad and \quad$ $y = a(\theta + sin\theta)$

$$\Longrightarrow dx = a\ sin\theta \qquad and \qquad dy = a(1 + cos\theta)$$

$$\therefore \frac{dy}{dx} = \frac{a(1+\cos\theta)}{a\sin\theta} = \frac{2cos^2\left(\theta/2\right)}{2sin\left(\theta/2\right)cos\left(\theta/2\right)} = \frac{cos\left(\theta/2\right)}{sin\left(\theta/2\right)} = \cot\left(\theta/2\right)$$

$$\therefore \left(\frac{ds}{dx}\right)^2 = 1 + \cot^2 \theta/2 = \operatorname{cosec}^2 \theta/2$$

$$\Longrightarrow \frac{ds}{dx} = \operatorname{cosec}\left(\theta/2\right).$$

2. Find $\frac{ds}{dx}$ in the curve $y = a\ cosh\ \left(\frac{x}{a}\right)$.

Solution :

Given that, $y = a\ cosh\ \left(\frac{x}{a}\right)$ $\qquad$ ...(1)

We know that, $\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2$.

On differentiating (1), we get

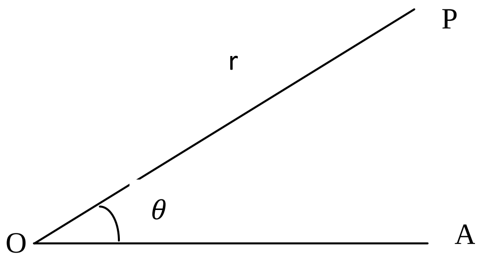
$$\frac{dy}{dx} = a\left(sin\ h\ \left(\frac{x}{a}\right) \times \left(\frac{x}{a}\right)\right) \quad = sin\ h\ \left(\frac{x}{a}\right)$$

$$\therefore \left(\frac{ds}{dx}\right)^2 = 1 + \left(\sin h\left(\frac{x}{a}\right)\right)^2 \quad = 1 + \sin^2 h\left(\frac{x}{a}\right) = \cosh^2\left(\frac{x}{a}\right)$$

$$\Rightarrow \frac{ds}{dx} = \cosh\left(\frac{x}{a}\right).$$

**POLAR COORDINATES:**

Consider the following diagram:




The position of a point P on a plane can be indicated by stating:

(1) Its distance r from a fixed point 'O'.
(2) The inclination $\theta$ of OP to a fixed straight line through 'O'.

Here, r is called the radius vector and $\theta$ the vertorial angle, O the pole and OA the initial line, where r and $\theta$ are called the polar coordinates of P.

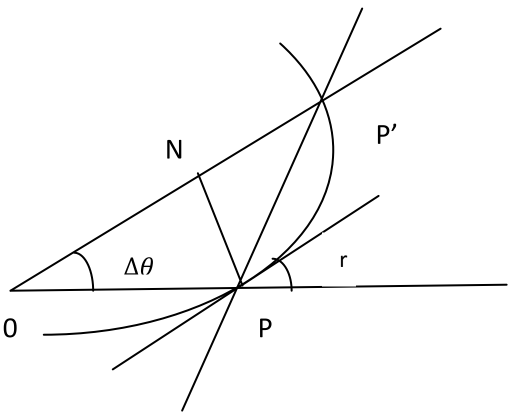
r is considered to be positive when measured away from O along the line bounding the vertorial angle and $\theta$ is considered to be positive when measured in the anticlock wise direction. It is usual to regard $\theta$ as the independent variable.

When converting polar co-ordinates to Cartesian or vice versa, it is customary to take the pole as the origin and initial line as the x-axis.

Then, the formulae for conversion are $x = r\cos\theta$ and $y = r\sin\theta$.

## ANGLE BETWEEN THE RADIUS VECTOR AND THE TANGENT

Consider the following diagram. Let P, P′ be two neighbouring points on a curve. Let (r, $\theta$) be the polar coordinates of P and $(r+\Delta r,\ \theta+\Delta\theta)$be the polar coordinates of P′.




If we join P, P′ and draw PN perpendicular to OP′, we have

$PN = OP\sin\angle PON = r\sin\Delta\theta.$

Again, $PN = OP' - ON = r + \Delta r - r\cos\Delta\theta$

$\Rightarrow \qquad = \Delta r + r\,(1-\cos\Delta\theta)$

$\Rightarrow \qquad = \Delta r + 2r\sin^2\left(\frac{\Delta\theta}{2}\right).$

Denote by $\varphi$ the angle between the radius vector OP and the tangent at P. If we now let Δθ approach the limit zero, then

(1) The point P′ will approach P

(2) The secant PP′ will become the tangent PT in the limiting positions.

(3) The angle PP′N will approach $\varphi$ as a limit.

From the above diagram, we have

$$\tan PP'O = \frac{r\sin\Delta\theta}{\Delta r + 2r\sin^2\left(\frac{\Delta\theta}{2}\right)} \qquad = r\,\frac{\frac{\sin\Delta\theta}{\Delta\theta}}{\frac{\Delta r}{\Delta\theta} + \frac{r\sin\left(\frac{\Delta\theta}{2}\right)}{\frac{\Delta\theta}{2}}\sin\frac{\Delta\theta}{2}}$$

$$\lim_{\Delta\theta\to 0}\frac{\sin\Delta\theta}{\Delta\theta} = 1\ ,\ \lim_{\Delta\theta\to 0}\sin\frac{\Delta\theta}{2}\ ,\ \lim_{\frac{\Delta\theta}{2}\to 0}\frac{\sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}} = 1$$

and $\lim_{\frac{\Delta\theta}{2}\to 0}\frac{\Delta r}{\Delta\theta} = \frac{dr}{d\theta}$

$\therefore \tan\varphi = \lim_{\Delta\theta\to 0} \tan PP'O = r \; \frac{1}{\frac{dr}{d\theta}+r.1.0} = r\frac{d\theta}{dr}.$

**PROBLEMS:**

1. Find the angle at which the radius vector cuts the curve $\frac{l}{r} = 1 + e\cos\theta$.

**Solution:**

Let $\emptyset$ be the angle between the radius vector and the tangent at the point at which the radius vector meets the curve.

On differentiating $\frac{l}{r} = 1 + e\cos\theta$ with respect to$\theta$, we get

$\frac{-l}{r^2}\frac{dr}{d\theta} = -e\sin\theta \Longrightarrow \frac{dr}{d\theta} = \frac{er^2}{l}\sin\theta.$

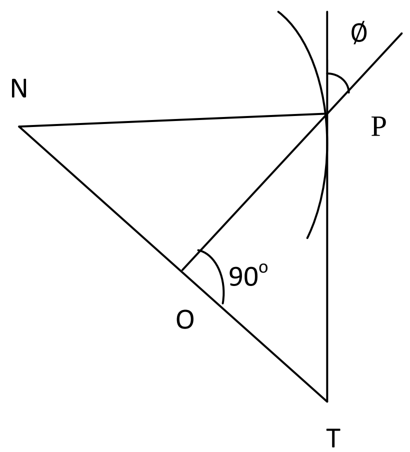
We know that, $\tan\emptyset = r\,\frac{d\theta}{dr}$ .

$\therefore \quad \tan\emptyset = \frac{r\,l}{e\sin\theta . r^2} = \frac{l}{r.e\ \sin\theta} = \frac{1+e\cos\theta}{e\ \sin\theta}$ .

$\therefore$ The required angle,$\emptyset = \tan^{-1}\left(\frac{1+e\ \cos\ \theta}{e\ \sin\ \theta}\right)$.

**POLAR SUBTANGENT AND POLAR SUBNORMAL**

Consider the following diagram:




Draw a line NT through the pole perpendicular to the radius vector of the point P on the curve. If PT is the tangent and PN the normal to the curve at P, then

OT= Length of the polar sub tangent.

ON=Length of the subnormal of the curve at P.

Polar subtangent= $OT = OP\ tan\emptyset = r.r\ \frac{d\theta}{dr} = r^2\frac{d\theta}{dr}$.

Polar subnormal = ON $=$ OP tan $\angle$ OPN $=$ OP tan $(\angle TPN - \angle TPO)$

$$= OP\ \tan\left(\pi/_2 - \emptyset\right) = r\ \cot\emptyset = \frac{r}{\tan\emptyset} = \frac{r}{r.\frac{d\theta}{dr}} = \frac{dr}{d\theta}.$$

Hence, Polar subtangent is$r^2\frac{d\theta}{dr}$ and Polar subnormal is $\frac{dr}{d\theta}$.

**Problems:**

1. Show that in the curve r=a$e^{\theta \cot\alpha}$,

(i) The polar subtangent = r tan α.

(ii)The polar subnormal = r cot α.

**Solution:**

Here, r = a$e^{\theta\cot\alpha}$

$\therefore \frac{dr}{d\theta} = ae^{\theta\cot\alpha} = r\cot\alpha.$

Hence, the polar subnormal is(rcot α).

Also,$\frac{d\theta}{dr} = \frac{1}{r\ \cot\alpha}$.

$$\therefore\ r^2\frac{d\theta}{dr} = \frac{r^2}{r\ \cot\alpha} = r\ \tan\alpha$$

Hence, the polar subtangentis $(r\ \tan\alpha)$.

2. Show that in the curve r = aθ, the polar subtangent varies as the square of the radius vector and the polar subnormal is constant.

**Solution;**

Given that, r = aθ.

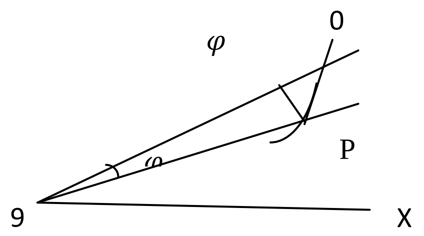
$\therefore \frac{dr}{d\theta} = $ a,which is constant.

Again,$\frac{d\theta}{dr} = \frac{1}{a} \Rightarrow r^2\frac{d\theta}{dr} = \frac{r^2}{a}$.

Thus, the polar subtangent varies asthe$r^2$.

## THE LENGTH OF ARC IN POLAR CO ORDINATES

Consider the following diagram:




Let the coordinates of a point P on the curve be (r,θ).

Then, OP = r; and ∠AOP=θ.

Let the coordinates of a point Q on the curve very close to P be $( r + \Delta r, \theta + \Delta\theta )$.

Then, OQ= r+Δr, ∠QOA = θ+Δθ and $\angle POR = \Delta\theta$.

Let s be the length of the arc BP, where B is a fixed point on the curve. Then, the length of the arc BQ is $s + \Delta s$ and the length of the arc PQ is Δs.

Now, $PR = OP\ sin\Delta\theta = r\ sin\Delta\theta$ .

$$\mathrm{OR} = \mathrm{OP}\ \cos\Delta\theta = \mathrm{r}\ \cos\Delta\theta.$$

Also, $\mathrm{QR} = \mathrm{r} + \Delta\mathrm{r} - \mathrm{r}\ \cos\Delta\theta = \mathrm{r}\,(1 - \cos\Delta\theta) + \Delta\mathrm{r} = 2\mathrm{r}\sin^2\frac{\Delta\theta}{2} + \Delta\mathrm{r}.$

$$\mathrm{PQ}^2 = \mathrm{PR}^2 + \mathrm{RQ}^2$$

$$\Rightarrow (r\,sin\,\delta\theta)^2 + \{2r\,sin^2\frac{\Delta\theta}{2} + \Delta r\}^2.$$

$$\therefore \left(\frac{\text{chord PQ}}{\Delta\theta}\right)^2 = (\mathrm{r}\ \frac{\sin\Delta\theta}{\Delta\theta})^2 + \left\{\frac{2\mathrm{r}\sin^2\frac{\Delta\theta}{2}}{\Delta\theta} + \frac{\Delta\mathrm{r}}{\Delta\theta}\right\}^2 = \mathrm{r}^2\left(\frac{\sin\Delta\theta}{\Delta\theta}\right)^2 + \left\{\frac{\mathrm{r}\ \sin\frac{\Delta\theta}{2}}{\frac{\Delta\theta}{2}}\ \sin\frac{\Delta\theta}{2} + \frac{\Delta\mathrm{r}}{\Delta\theta}\right\}^2.$$

Passing to the limit as Δθ tends to zero, we get

$\frac{\sin\Delta\theta}{\Delta\theta} \to 1, \frac{\Delta\mathrm{r}}{\Delta\theta} \to \frac{\mathrm{dr}}{\mathrm{d}\theta}, \sin\frac{\Delta\theta}{2} \to 0.$

$$\therefore \frac{\text{chord PQ}}{\Delta\theta} = \frac{\text{chord PQ}}{\text{arc PQ}} \cdot \frac{\text{arc PQ}}{\Delta\theta} \to 1 \cdot \frac{\mathrm{ds}}{\mathrm{d}\theta}$$

$$\therefore \left(\frac{\mathrm{ds}}{\mathrm{d}\theta}\right)^2 = \mathrm{r}^2 + \left(\frac{\mathrm{dr}}{\mathrm{d}\theta}\right)^2$$

In the same way, it may be shown that $\left(\frac{\mathrm{ds}}{\mathrm{dr}}\right)^2 = \left(\mathrm{r}\frac{\mathrm{d}\theta}{\mathrm{dr}}\right)^2 + 1.$

It is easily seen that $\cos\emptyset = \frac{dr}{ds}$ and $\sin\emptyset = r\,\frac{d\theta}{ds}$ .

**Problem**:

1. Find $\frac{ds}{d\theta}$ and $\frac{ds}{dr}$ for the cardioid r = a (1+cos θ).

**Solution;**

Given that r = a (1+cos θ)

On differentiating the above equation, we have

$$\frac{dr}{d\theta} = a\,(0 - \sin\theta) = -a\sin\theta.$$

Also, $r.\frac{d\theta}{dr} = \frac{r}{-a\ \sin\theta} = \frac{a\ (1+\cos\theta)}{-a\ \sin\theta} = -\frac{1+\cos\theta}{\sin\theta} = -\frac{2\cos^2\ \theta/2}{2\sin\theta/2\cos\theta/2} = -\cot\theta/2$ .

We know that, $\left(\frac{ds}{dr}\right)^2 = 1 + \left(r\frac{d\theta}{dr}\right)^2$

$= 1 + \ \cot^2(\theta/2) \ = \text{cosec}^2\ \theta/2$ .

$\therefore \frac{ds}{dr} \quad = \text{cosec}\ \theta/2.$

We know that, $\left(\frac{ds}{d\theta}\right)^2 = \ r^2 + \ \left(\frac{dr}{d\theta}\right)^2$

$$= a^2(1 + \ \cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2(1 + \cos^2\theta + 2\cos\theta) + a^2\sin^2\theta$$

$$= a^2 + a^2(\cos^2\theta + \sin^2\theta) + 2a^2\cos\theta$$

$$= a^2 + a^2 + 2a^2\cos\theta = 2a^2(1 + \cos\theta)$$

$$\Rightarrow \left(\frac{ds}{d\theta}\right)^2 \qquad = 2a^2[2\cos^2(\theta/2)] = 4\ a^2\cos^2(\theta/2).$$

$$\therefore \frac{ds}{d\theta} = 2a\cos(\theta/2).$$

2. Find $\frac{ds}{d\theta}$ and $\frac{ds}{dr}$ for the curve r = a (1-cos θ).

**Solution;**

Given that, r = a (1 - cos θ) ⇒ $\frac{dr}{d\theta} = a\sin\theta.$

Consider, $r.\frac{d\theta}{dr} = \frac{a\ (1-\cos\theta)}{a\sin\theta}$

$$= \frac{2a\, sin^2\, \theta/2}{2a\, sin\, \theta/2\ \ cos\, \theta/2} \ = tan\, \theta/2\,.$$

We know that, $\left(\frac{ds}{dr}\right)^2 = 1 + \left(r\ \frac{d\theta}{dr}\right)^2$

$= 1 + \ \tan^2 \theta/2 \ = \ \sec^2 \theta/2.$

$$\therefore \left(\frac{ds}{dr}\right) \ = \ \ \sec\theta/2$$

We know that, $\left(\frac{ds}{d\theta}\right)^2 \ = \ r^2 + \ \left(\frac{dr}{d\theta}\right)^2$

$$= \ a^2(\,1 - \ \cos\theta)^2 + a^2\sin^2\theta$$

$$= \ a^2(1 + \cos^2\theta - 2\cos\theta) + a^2\sin^2\theta$$

$$= a^2 + a^2\cos^2\theta + a^2\sin^2\theta - 2a^2\cos\theta$$

$$= \ a^2 + a^2(\cos^2\theta + \sin^2\theta) - 2a^2\cos\theta$$

$$= 2a^2 - 2a^2\cos\theta = 2a^2(1 - cos\,\theta) = 4a^2 sin^2\,\theta/2$$

$$\therefore \left(\frac{ds}{d\theta}\right) \ = 2a\sin\theta/2.$$

**Envelopes, Curvature of Plane Curves:**

**Envelopes:**

The equation $f(x, y, t) \ = \ 0$ determines a curve corresponding to each particular value of t. The totality of all such curves by gaining different values of t, is said to be a family of curves. The variable t is different for different curves; and is said to be the parameter of the family of curves.

Consider the equation $x\ \ cos\,\theta + y\, sin\,\theta = a$ , where a is constant. For different values of θ, the equation represents a family of straight lines touching the circle$x^2 + y^2 = a^2$. Here,θ is the parameter of the family of straight lines, $x\ \ cos\,\theta + y\, sin\,\theta = a$ .

Similarly,$y = mx + \frac{a}{m}$ represents a family of straight lines. Here, the parameter is m and it touches the parabola $y^2 = 4ax$.

Similarly,$(x - a)^2 + y^2 = r^2$, where r is a constant, is a family of circles. Here, the parameter isa and it touch the lines $y = \pm\, r$.

We have seen that in the above three illustrations, the family of curves touches a curve. The first case is a circle, the second case is parabola and thethird case is a pair of lines. The curve E which is touched by family of curves C is called the envelope of the family of curves C.

**Problems:**

1. Find the envelope of the family of straight lines, $y + tx = 2at + at^3$, the parameter being t.

**Solution;**

Given that, $y + tx = 2at + at^3$ … (1)

On differentiating partially with respect to t, we have

$$x = 2a + 3at^2 \quad ----(2)$$

To get the envelope, we have to eliminate t from (2) and (1). We have

$$y = -tx + 2at + at^3$$

$$\Rightarrow y = t(-x + 2a + at^2) \quad ----(3)$$

From (2), we have $3at^2 = x - 2a \Rightarrow t^2 = \frac{x-2a}{3a} \quad ----(4)$

On substituting (4) in (3), we have

$$y = t\left(-x + 2a + a\,\frac{(x-2a)}{3a}\right) = t\left(\frac{-3x + 6a + x - 2a}{3}\right) = t\left(\frac{-2x + 4a}{3}\right)$$

$$\therefore \; y = -\frac{2t}{3}(x - 2a)$$

Squaring on both sides we get,

$$y^2 = \frac{4t^2}{9}\,(x-2a)^2 \;\Rightarrow\; y^2 = \frac{4}{9}\left(\frac{x-2a}{3a}\right)(x-2a)^2$$

$\Rightarrow \; 27ay^2 = 4\,(x-2a)^3$, which is the equation of the required envelope.

This curve is called a semi – cubical parabola.

2. Find the envelop of the family of circle $(x-a)^2 + y^2 = 2a$, wherea is the parameter.

**Solution:**

Given that, $(x-a)^2 + y^2 = 2a \quad ----(1)$

On differentiating partially with respect to 'a', we have

$$2(x-a)(-1) + 0 = 2 \Rightarrow -2(x-a) = 2 \Rightarrow \; (x-a) = -1$$

$$\Rightarrow a = (x+1) \qquad ----(2)$$

On substituting (2) in (1), we have

$(-1)^2 + y^2 = 2(x+1) \Rightarrow y^2 = 2(x+1) - 1 \quad = 2x + 2 - 1 \quad = 2x + 1.$

Hence, the envelope of the family of circles $(x-a)^2 + y^2 = 2a$ is $y^2 = 2x + 1.$

3(a) Find the envelope of the family of straight lines, $y = mx + \frac{a}{m}$ , where m is the parameter.

**Solution;**

Given that, $y = mx + \frac{a}{m} ----(1)$

On differentiating partially with respect to 'm', we have

$0 = x + \left(\frac{0-a}{m^2}\right) = m^2x + (-a) = m^2x - a \Rightarrow a = m^2x \Longrightarrow x = \left(\frac{a}{m^2}\right).$

From (1), we have $y = mx + \frac{a}{m} = m\ \frac{a}{m^2} + \frac{a}{m} = \frac{a}{m} + \frac{a}{m} \Rightarrow y = \frac{2a}{m}$

$\therefore\ y^2 = \frac{4a^2}{m^2} = \frac{4a.a}{m^2} = 4a.\frac{a}{m^2} \Rightarrow y^2 = 4ax.$

Hence, the envelope of the family of straight line $y = mx + \frac{a}{m}$ is $y^2 = 4ax.$

3(b) Find the envelope of the family of straight lines, $Y = mx + am^2$, where m is the parameter.

**Solution;**

Given that, $y = mx + am^2 ----(1)$

On differentiating partially with respect to 'm', we get

$$0 = x + 2am \Rightarrow\ -2am = x \Rightarrow m = -\frac{x}{2a} ----(2)$$

From (1), we have $y = mx + am^2\ \Rightarrow\ \ y = m(x + am) ----(3)$

On substituting (2) in (3), we have

$y\ = \frac{-x}{2a}\left(x + a\ \frac{-x}{2a}\right) = \frac{-x}{2a}\left(x - \frac{x}{2}\right) = \frac{-x}{2a}\left(\frac{x}{2}\right) \therefore y = \frac{-x^2}{4a}.$

On squaring on both sides, we get

$y^2 = \frac{x^4}{16a^2} \Rightarrow\ \ 16a^2y^2 = x^4.$

Hence, the envelope of the family of straight lines $y = mx + am^2$ is $16a^2y^2 = x^4.$

3(c) Find the envelope of the family of straight lines, $x\cos^3\theta + y\sin^3\theta = a$ , where $\theta$ is the parameter.

**Solution;**

Given that, $x\cos^3\theta + y\sin^3\theta = a ----(1)$

On differentiating partially with respect to $'\theta'$, we get

$$-3x\cos^2\theta.\sin\theta + 3y\sin^2\theta.\cos\theta = 0$$

$\Rightarrow\ 3y\sin^2\theta\ \ \cos\theta = 3x\cos^2\theta.\sin\theta \Rightarrow y\sin\theta = x\cos\theta \Rightarrow \frac{\cos\theta}{y} = \frac{\sin\theta}{x}$.

$$\therefore \cos\theta = \frac{y}{\sqrt{x^2+y^2}}, \qquad \sin\theta = \frac{x}{\sqrt{x^2+y^2}} \qquad ---- (2)$$

On substituting(2) in (1), we have

$$x\left(\frac{y}{\sqrt{x^2+y^2}}\right)^3 + y\left(\frac{x}{\sqrt{x^2+y^2}}\right)^3 = a$$

$$\Rightarrow \frac{xy^3}{(\sqrt{x^2+y^2})^3} + \frac{yx^3}{(\sqrt{x^2+y^2})^3} = a$$

$$\Rightarrow xy^3 + x^3y = a\left((x^2+y^2)^{1/2}\right)^3$$

$$\Rightarrow \quad xy\,(y^2+x^2) = a\,(x^2+y^2)^{3/2}$$

$$\Rightarrow \quad xy = a\frac{(x^2+y^2)^{3/2}}{(x^2+y^2)} = a\,(x^2+y^2)^{\frac{3}{2}-1} = a(x^2+y^2)^{1/2}$$

$\therefore \quad xy = a\sqrt{x^2+y^2}$ .

Hence, the envelope of the family of straight lines $x\cos^3\theta + y\sin^3\theta = a$is $xy = \sqrt{x^2+y^2}$.

4.Find the envelope of circles drawn on the radius vectors of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ as diameter.

**Solution:**

The coordinates of any point P on the ellipse are $(a\cos\theta, b\ \sin\theta)$.

The equation of the circle on CP as diameter is$x(x - a\cos\theta) + y(y - b\sin\theta) = 0$

$$\Rightarrow x^2 + y^2 - ax\cos\theta - by\sin\theta = 0 \qquad ----(1)$$

We have to find the envelope of the family of circles (1) for different values of θ.

On differentiating partially with respect to 'θ', we have

$$-ax(-\sin\theta) - yb\cos\theta = 0 \Rightarrow ax\sin\theta - by\cos\theta = 0 ----(2)$$

Now, we have to eliminate θ from (1) and (2).

(2) $\Rightarrow ax\sin\theta = by\cos\theta$

$$\Rightarrow \frac{a\sin\theta}{y} = \frac{b\cos\theta}{x}$$

$$\Rightarrow \frac{\sin\theta}{by} = \frac{\cos\theta}{ax} = \frac{1}{\sqrt{a^2x^2+b^2y^2}}$$

$$\therefore \frac{\sin\theta}{by} = \frac{1}{\sqrt{a^2x^2+b^2y^2}} \Rightarrow \sin\theta = \frac{by}{\sqrt{a^2x^2+b^2y^2}} ----(3)$$

and $\frac{\cos\theta}{ax} = \frac{1}{\sqrt{a^2x^2+b^2y^2}} \Rightarrow \cos\theta = \frac{ax}{\sqrt{a^2x^2+b^2y^2}} ----(4)$

On substituting (3) and (4) in (1), we get

$$x^2 + y^2 - ax\left(\frac{ax}{\sqrt{a^2x^2+b^2y^2}}\right) - by\left(\frac{by}{\sqrt{a^2x^2+b^2y^2}}\right) = 0$$

$$\Rightarrow x^2 + y^2 - \frac{a^2x^2}{\sqrt{a^2x^2+b^2y^2}} - \frac{b^2y^2}{\sqrt{a^2x^2+b^2y^2}} = 0$$

$$\Rightarrow x^2 + y^2 - \left(\frac{(a^2x^2+b^2y^2)}{\sqrt{a^2x^2+b^2y^2}}\right) = 0$$

$$\Rightarrow \quad x^2 + y^2 - (a^2x^2+b^2y)^{1/2} = 0 \Rightarrow x^2 + y^2 = \sqrt{a^2x^2+b^2y^2}$$

On squaring on both sides, we have

$$(x^2+y^2)^2 = a^2x^2 + b^2y^2 ----(5).$$

Hence, (5) is the equation of the required envelope.

7. Find the envelope of the straight lines $\frac{x}{a} + \frac{y}{b} = 1$, where the parameters are related by the equation $a^2 + b^2 = c^2$, where c is a constant.

**Solution**

Let us regard a and b be the functions of t.

$$a^2 + b^2 = c^2 \qquad ----(1)$$

$\frac{x}{a} + \frac{y}{b} = 1 \qquad ----\ (2)$

On differentiating the above equations partially with respect to 't', we have:

$(1) \Longrightarrow 2a\,\frac{da}{dt} + 2b\,\frac{db}{dt} = 0 \qquad ----(3)$

$(2) \Longrightarrow \quad xa^{-1} + yb^{-1} = 1 \ \Rightarrow \ -a^{-2}\,x\,\frac{da}{dt} + \ y(-b^{-2})\frac{db}{dt} = 0$

$$\Rightarrow \quad -\frac{x}{a^2}\frac{da}{dt} - \frac{y}{b^2}\frac{db}{dt} = 0 \qquad ----(4)$$

On equating (3) and (4), we have

$$-\frac{x}{a^2} = 2a \ , -\frac{y}{b^2} = 2b \Rightarrow \ \frac{-x}{2a^3} = 1, \quad \frac{-y}{2b^3} = 1$$

$\Longrightarrow \frac{-x}{2a^3} = \frac{-y}{2b^3} \Rightarrow \ \frac{x}{a^3} = \frac{y}{b^3} \ \rightarrow \ (5)$

We have to eliminate a and b from (1), (2) and (5).

$(5) \Longrightarrow \frac{(x/a)}{a^2} = \frac{(y/b)}{b^2} = \frac{(x/a)+(y/b)}{a^2+b^2} = \frac{1}{c^2}$ .

$$\therefore \frac{x}{a^3} = \frac{1}{c^2} \ , \qquad \frac{y}{b^3} = \frac{1}{c^2} \ \Rightarrow a^3 = c^2x\,, \ \ b^3 = c^2y \ \ \Rightarrow a = (c^2x)^{1/3}, \ \ b = (c^2y)^{1/3}$$

On substituting the values of a , b in (1), we get ,

$$((c^2x)^{1/3})^2 + \ ((c^2y)^{1/3})^2 = c^2 \quad \Rightarrow (c^2x)^{2/3} + \ (c^2y)^{2/3} = \ c^2$$

$\Rightarrow \quad c^{4/3}x^{2/3} + \ c^{4/3}y^{2/3} = c^2 \Rightarrow x^{2/3} + y^{2/3} = \frac{c^2}{c^{4/3}} = c^{2-4/3}$

$$\therefore \ \ x^{2/3} + y^{2/3} = c^{2/3}$$

This is the equation of the required envelope.

This curve is known as Four cusped hypocycloid.

## **Circle, Radius and Centre of Curvature**

Let P and Q be two points on a plane curve;$\varphi$ and $\varphi + \Delta\varphi$ the angles which the tangents at P and Q make with the x-axis; s the arc measured from some fixed point A on the curve up to P and $\Delta$s the arc PQ.Let the normals at P,Q intersect at C.

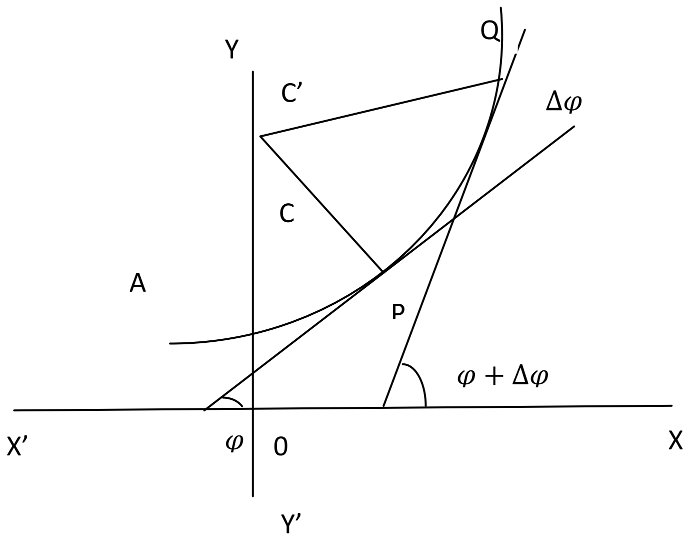
From the figure, it easily follows that $\angle PC'Q = \ \Delta\varphi$.

$\therefore \frac{PC'}{\sin PQC'} = \frac{\text{chord } PQ}{\sin PC'Q} = \frac{\text{chord } PQ}{\text{arc } PQ}\frac{\text{arc } PQ}{\sin PC'Q} = \frac{\text{chord } PQ}{\text{arc } PQ}\frac{\Delta s}{\sin \Delta\varphi} = \frac{\text{chord } PQ}{\text{arc } PQ}\frac{\Delta s}{\Delta\varphi}\frac{\Delta\varphi}{\sin \Delta\varphi}.$

Now, the limit of∠ PQC' as Q tends to P is 90° and also

$\lim_{Q\to P}\frac{\text{chord } PQ}{\text{arc } PQ} = 1$ , $\lim_{\Delta\varphi\to 0}\frac{\Delta s}{\Delta\varphi} = \frac{ds}{d\varphi}$ and $\lim_{\Delta\varphi\to 0}\frac{\sin\Delta\varphi}{\Delta\varphi} = 1$ .

As Q tends to P, limit of PC' is $\frac{ds}{d\varphi}$.




Let the limiting position of C' be C.

Then, $PC = \frac{ds}{d\varphi}$, i.e., $\frac{1}{PC} = \frac{d\varphi}{ds}$ $\Rightarrow PC = \frac{ds}{d\varphi}$ is called the radius of curvature at p.

Therefore, the circle with centre C and radius PC has the same tangent. The curvature is also same as the curve has at P.

This circle is called the circle of curvature at P. So, it can be defined as that circle which touches the given curve at the point. It has a radius equal to the radius of curvature at the point. It lies on the same side of the tangent as the curve. Its radius is PC, the radius of curvature and its centre is C, the centre of curvature at the point P. The radius of curvature is often denoted by ρ . Hence, the curvature is $\frac{1}{\rho}$ .

**Cartesian Formula for The Radius of Curvature**

The Cartesian Formula for The Radius of Curvature is, $\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$ .

**Problems:**

1. Show that the radius of curvature at any point of the catenary $y = c\left(\cosh\frac{x}{c}\right)$ is equal to the length of the portion of the normal intercepted between the curve and the axis of x.

**Solution**

Given that, $y = c\left(\cosh\frac{x}{c}\right)$ $\quad ----(1)$

$\Rightarrow \frac{dy}{dx} = c\left(\sinh\frac{x}{c} \times \frac{1}{c}\right) = \sinh\frac{x}{c}$

$\Rightarrow \frac{d^2y}{dx^2} = \cosh\frac{x}{c}\cdot\frac{1}{c}$

We know that, $\rho(x,y) = \frac{[1+\left(\frac{dy}{dx}\right)^2]^{3/2}}{\frac{d^2y}{dx^2}}$

$\Rightarrow \rho(x,y) = \frac{[1+(\sinh\frac{x}{c})^2]^{3/2}}{{}^1/_c \cosh\frac{x}{c}} = \frac{c\,[1+\sinh^2 x/c]^{3/2}}{\cosh\frac{x}{c}} = c\,\frac{(\cosh^2 x/c)^{3/2}}{\cosh\frac{x}{c}} = c\,\cosh^2\frac{x}{c}.$

From (1), we have $\rho = c\,\frac{y^2}{c^2} = \frac{y^2}{c}$.

We know that, length of the normal is $y\,[1+(\frac{dy}{dx})^2]^{1/2}$

$= y\,[1+\sinh^2\frac{x}{c}]^{1/2} = y\,(\cosh^2\frac{x}{c})^{1/2} = y\,\cosh\frac{x}{c} = y\,.\frac{y}{c} = \frac{y^2}{c}$.

Therefore, the length of the normal = $\frac{y^2}{c}$ = the radius of curvature.

2. Find the radius of curvature for the curve $xy = 30$ at the point $(3,10)$.

**Solution**

Given that, $xy = 30$

$$\therefore\ x\frac{dy}{dx} + y(1) = (0) \ \Rightarrow\ x\frac{dy}{dx} = -y \ \Rightarrow\ \frac{dy}{dx} = \frac{-y}{x}$$

$$\therefore\ \frac{dy}{dx}\ at\ (3,10) = \frac{-10}{3}$$

$$\therefore\quad \frac{d^2y}{dx^2} = \frac{-x.\frac{dy}{dx} - (-y).1}{x^2} = \frac{-x\frac{dy}{dx} + y}{x^2} = \frac{-x - \frac{y}{x} + y}{x^2} = \frac{2y}{x^2}$$

$\Rightarrow \frac{d^2y}{dx^2}$ at $(3,10) = \frac{20}{9}$.

We know that, $\rho = \frac{[1+(\frac{dy}{dx})^2]^{3/2}}{d^2y/_{dx^2}}$

$$= \frac{[1+(\frac{-10}{3})^2]^{3/2}}{d^2y/_{dx^2}} = \frac{[1+\frac{100}{9}]^{3/2}}{20/_9} = \frac{[109]^{3/2}}{9\sqrt{9}} \times \frac{9}{20} \Rightarrow \rho = \frac{(109)^{3/2}}{60}.$$

The radius of curvature for the curve is $\frac{(109)^{3/2}}{60}$.

3. If a curve is defined by the parametric equation $x = f(\theta)$ and $y = \emptyset(\theta)$, prove that the curvature is $\frac{1}{\rho} = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$, where dashes denote differentiation with respect to $\theta$.

**Solution**

Given that, $x = f(\theta)$ and $y = \emptyset(\theta)$.

On differentiating with respect to $\theta$, we have

$$x' = \frac{dx}{d\theta} \text{ and } y' = \frac{dy}{d\theta}$$

$\Rightarrow \frac{d\theta}{dx} = \frac{1}{x}$ and $\frac{d\theta}{dy} = \frac{1}{y'}$ .

$$\therefore \frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{y'}{x'}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\left(\frac{d\theta}{dx}\right) = \frac{d}{d\theta}\left(\frac{y'}{x'}\right)\left(\frac{d\theta}{dx}\right)$$

$$\Rightarrow = \frac{x'y'' - y'x''}{x'^2} \cdot \frac{1}{x'} = \frac{x'y'' - y'x''}{x'^3}$$

We know that, $\rho = \frac{[1+(\frac{dy}{dx})^2]^{3/2}}{d^2y/_{dx^2}}$

$$\Rightarrow \quad \frac{1}{\rho} = \left(\frac{d^2y/_{dx^2}}{[1 + (\frac{dy}{dx})^2]^{3/2}}\right)$$

$$\Rightarrow \left(\frac{\frac{x'y'' - y'x''}{x'^3}}{[1 + (\frac{y'}{x'})^2]^{3/2}}\right) = \left(\frac{x'y'' - y'x''}{x'^3} \Big/ [\frac{x'^2 + y'^2}{x'^2}]^{3/2}\right)$$

$$\Rightarrow = \frac{x' y'' - y' x''}{x'^{3}} \times \frac{x'^{2\times 3/2}}{(x'^{2} + y'^{2})^{3/2}}$$

∴ The required curvature is $\frac{1}{\rho} = \frac{x' y'' - y' x''}{(x'^{2}+y'^{2})^{3/2}}$.

4. Prove that the radius of curvature at any point of the cycloid, $x = a\,(\theta + \sin\theta)$ and $y = a\,(1 - \cos\theta)$ is $4a\ \cos\theta/2$.

**Solution**

Given that, $x = a\,(\theta + \sin\theta)$ and $y = a\,(1 - \cos\theta)$.

On differentiating with respect to θ, we have

$$\frac{dx}{d\theta} = a(1 + \cos\theta) = x' \quad , \quad \frac{dy}{d\theta} = a\sin\theta = y'$$

$$\Rightarrow \frac{dy}{dx} = \frac{a\ \sin\theta}{a\ (1+\cos\theta)} = \frac{y'}{x'}$$

Again differentiating, we get,

$$\frac{d^2y}{d\theta^2} = y'' = a\ \cos\theta\,, \frac{d^2x}{d\theta^2} = x'' = -\ a\sin\theta.$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-a\ \sin\theta}{a\ \cos\theta}.$$

We know that, $\frac{1}{\rho} = \frac{x' y'' - y' x''}{(x'^{2}+y'^{2})^{3/2}}$

$$= \frac{a\,(1 + \cos\theta).a\ \cos\theta - a\ \sin\theta\,(-\ a\sin\theta)}{((a(1 + \cos\theta))^2 + (a\ \sin\theta)^2)^{3/2}}$$

$$= \frac{a^2\,[\cos\theta + \cos^2\theta] + a^2\sin^2\theta}{[a^2(1 + \cos^2\theta + 2\cos\theta + \sin^2\theta)]^{3/2}}$$

$$= \frac{a^2\cos\theta + a^2(\cos^2\theta + \sin^2\theta)}{[a^2(1 + 2\cos\theta) + a^2(\cos^2\theta + \sin^2\theta)]^{3/2}}$$

$$= \frac{a^2\cos\theta + a^2}{(a^2(1 + 2\cos\theta) + a^2]^{3/2}}$$

$$= \frac{a^2(1 + \cos\theta)}{a^{2\times\frac{3}{2}}(2\cos\theta + 1 + 1)^{3/2}} = \frac{(1 + \cos\theta)}{2^{3/2}a(1 + \cos\theta)^{3/2}} = \frac{1}{2^{3/2}a}\ (1 + \cos\theta)^{1-\frac{3}{2}}$$

$$= \frac{1}{2^{3/2}\,a}\,(1+\cos\theta)^{-\frac{1}{2}} = \frac{1}{2\sqrt{2}.\,a(1+\cos\theta)^{\frac{1}{2}}} = \frac{1}{2\;\sqrt{2}\;a\,.\,(2\;\cos^2\theta/2)^{1/2}}$$

$$= \frac{1}{2\,\sqrt{2}\;a.\,2^{\frac{1}{2}}\cos^{2\times\frac{1}{2}}\theta/2} = \frac{1}{2\times 2\,.\,a\;\cos\;\;\theta/2} = \frac{1}{4a\;\cos\theta/2}$$

Therefore, the required radius of curvature is $\rho = 4a\;\cos\theta/2$.

5.Find the radius of curvature for the curve,$\sqrt{x}+\sqrt{y}=1$ at the point $({}^{1}/_{4}\,,{}^{1}/_{4})$.

**Solution;**

Given that, $\sqrt{x}+\sqrt{y}=1$

i.e.,$x^{\frac{1}{2}}+y^{\frac{1}{2}}=1$

On differentiating, we get

$$^{1}/_{2}\,x^{-\frac{1}{2}} + {}^{1}/_{2}\,y^{-\frac{1}{2}}\frac{dy}{dx}=0 \quad \Rightarrow \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{2\sqrt{x}}\times 2\sqrt{y} = \frac{-\sqrt{y}}{\sqrt{x}}$$

$$\therefore \frac{dy}{dx}\;at\;\left(\tfrac{1}{4},\tfrac{1}{4}\right)\text{is}\frac{-\sqrt{{}^{1}/_{4}}}{\sqrt{{}^{1}/_{4}}} = -1$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-x^{1/2}({}^{y}/_{2})^{-1/2}\frac{dy}{dx} - -(-y^{1/2}).{}^{1}/_{2}x^{-1/2}}{(x^{1/2})^2} = \frac{-\sqrt{x}\frac{1}{2\sqrt{y}}.\frac{-\sqrt{y}}{\sqrt{x}}+\sqrt{y}\,.\frac{1}{2\sqrt{x}}}{x} = \frac{\frac{1}{2}+\frac{\sqrt{y}}{2\sqrt{x}}}{x}$$

$$.\frac{d^2y}{dx^2}\left(\tfrac{1}{4},\tfrac{1}{4}\right) = \frac{{}^{1}/_{2}+\frac{{}^{1}/_{2}}{2\times 1/2}}{1/4} = \frac{1/2+1/2}{1/4} = \frac{1}{1/4} = 4\,.$$

We know that,$\rho = \frac{[1+(\frac{dy}{dx})^2]^{\frac{3}{2}}}{\left(\frac{d^2y}{dx^2}\right)} = \frac{(1+1)^{\frac{3}{2}}}{4} = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$.

Hence, the required radius of curvature is$\frac{1}{\sqrt{2}}$ .

**The Coordinates of the Centre of Curvature**

Let X and Y be the centre of curvature of the curve, $y=f(x)$ corresponding to the point (x, y). Let $y_1$ and $y_2$ denote $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ respectively.

Then, the coordinates ( X, Y ) of the centre of curvature is given by,

$$X = x - \frac{y_1(1+{y_1}^2)}{y_2} \text{and} \quad Y = y + \frac{(1+{y_1}^2)}{y_2}$$

**Evolute**:

The locus of the centre of curvature for a curve is called the 'Evolute' of the curve.

**Problems**:

1. Find the coordinates of the centre of curvature of the curve $xy = 2$ at the point (2,1).

**Solution**

Given that, $xy = 2$

$\Rightarrow y = \frac{2}{x} = 2x^{-1}$

$\Rightarrow \frac{dy}{dx} = -2x^{-2} = y_1$

$$\therefore \frac{dy}{dx} \ at\ (2,1)\ is\ -2(2)^{-2} = \frac{-2}{2^2} = \frac{-2}{4} \Rightarrow y_1 = \frac{-1}{2}$$

and $\frac{d^2y}{dx^2} = -2\,(-2)x^{-3} = 4x^{-3}$.

$$\therefore \frac{d^2y}{dx^2} \ at\ (2,1)\ is\ 4(2)^{-3} = \frac{4}{2^3} = \frac{4}{8} = \frac{1}{2} \Rightarrow y_2 = \frac{1}{2}$$

We know that, the coordinates of the centre of curvature is,

$X = x - \frac{y_1(1+{y_1}^2)}{y_2} \ldots$ (1) and $Y = y + \frac{(1+{y_1}^2)}{y_2} \ldots$ (2)

(1) $\Rightarrow X = 2 - \frac{\left(\frac{-1}{2}\right)(1+\left(\frac{-1}{2}\right)^2)}{1/2} = 2 + \frac{\frac{1}{2}(1+1/4)}{1/2} = 2 + \frac{\frac{1}{2}(5/4)}{1/2} = 2 + \frac{5}{8} \times 2 = 2 + \frac{5}{4}$

$\Rightarrow X = \frac{13}{4}$

(2) $\Rightarrow Y = 1 + \frac{1+(-1/2)^2}{1/2} = 1 + \frac{1+1/4}{1/2} = 1 + \frac{5/4}{1/2} = 1 + 5/2$

$\Rightarrow Y = \frac{7}{2}$.

The required coordinates of the centre of curvature of the curve xy $= 2$is, $X = \frac{13}{4}$ and $Y = \frac{7}{2}$

2. Find the coordinates of the centre of curvature of the curve $xy = c^2$ at the point ( c, c ).

**Solution**

We know that, the coordinates of the centre of curvature is,

$X = x - \frac{y_1(1+y_1{}^2)}{y_2}$ ... (1) and $Y = y + \frac{(1+y_1{}^2)}{y_2}$ ... (2)

Given that, $xy = c^2$.

On differentiating with respect to x, we have

$$x\frac{dy}{dx} + y(1) = 0 \Rightarrow x\frac{dy}{dx} = -y \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$$

$$\therefore y' = \frac{dy}{dx} \; at\ (c, c)\ is\ \frac{-c}{c} = -1 = y_1$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{x\left(\frac{-dy}{dx}\right)+y(1)}{x^2} = \frac{-x\left({}^{-y}/_{x}\right)+y}{x^2} = \frac{x\,.{}^{y}/_{x}+y}{x^2} = \frac{2y}{x^2}$$

$$\therefore \frac{d^2y}{dx^2} \; at\ (c, c)\ is\ \frac{2c}{c^2}$$

That is, $y_2 = \frac{2}{c}$

$(1) \Rightarrow X = c - \frac{(-1)(1+1)}{{}^2/_c} = c + \frac{2\times c}{2}$

$\therefore \quad X = 2c$

$(2) \Rightarrow Y = c + \frac{(1+1)}{{}^2/_c} = c + 2 \times \frac{c}{2}$

$\therefore Y = 2c.$

The required coordinates of the centre of curvature of the curve $xy = c^2$ at the point ( c, c )is, $X = 2c$ and $Y = 2c$.

3. Show that for the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ at the point $(a\cos^3 t, a\sin^3 t)$, the coordinates of the centre of curvature is $X = a\cos^3 + 3a\cos + \sin^2 t$ and $Y = a\sin^3 t + 3a\sin t\cos^2 t$.

**Solution**

We know that, the coordinates of the centre of curvature is,

$X = x - \frac{y_1(1+y_1{}^2)}{y_2}$ ... (1) and $Y = y + \frac{(1+y_1{}^2)}{y_2}$ ... (2)

Given that, x $= acos^3 t \;\; and \;\;\; y = a\, sin^3\, t$

$\Longrightarrow \frac{dx}{dt} = -3acos^2 t\; sint \quad and \quad \frac{dy}{dt} = 3a\; sin^2 t\; cos\;\; t$

$\therefore \frac{dy}{dx} = \frac{3a\; sin^2 t\; cos\;\; t}{-3acos^2\, t\; sin\; t} = -tan\, t = y_1$

$\Longrightarrow \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx} = \frac{d}{d\theta}\;(-\,tan\; t)\,\frac{1}{-3a\; cos^2 + sin\; t} = \frac{sec^2 t}{3acos^2 t\; sin\; t} = \frac{1}{cos^2\; t}\frac{1}{3a\; cos^2\; t\; sin\; t} = y_2$

$\Longrightarrow y_2 = \frac{1}{3a\; cos^4\; t\; sin\; t}.$

(1) $\Longrightarrow X = acos^3\; t - \frac{(-\,tan\;\, t)\;(1 + tan^2 t)}{{}^1\!/_3\; acos^4 t\; sin\; t}$

$$= a\; cos^3 t + \frac{sin\; t}{cos\; t} \times sec^2 t \times 3a\; cos^4 t\; sint$$

$$= a\; cos^3 t + \frac{sin\; t}{cos\; t} \times \frac{1}{cos^2 t} \times 3a\; cos^4 t + sin\, t$$

$$\therefore X = acos^3 t + 3sin^2 t\; cos\; t$$

(2) $\Longrightarrow \;\; Y = asin^3 t + \frac{(1 + tan^2 t)}{\frac{1}{3}acos^4\;\; t\sin\; t}$

$$= asin^3 t + sec^2 t \times 3a\; cos^4 t \sin t$$

$$= a\; sin^3 t + \frac{1}{cos^2 t} \times 3a\; cos^4\; t \sin t$$

$$\therefore Y = a\; sin^3 t + 3a\; cos^2 t \sin t$$

The required coordinates of the centre of curvature are:

$X = acos^3 t + 3sin^2 t\; cos\; t$and $Y = a\; sin^3 t + 3a\; cos^2 t \sin t.$

4. Show that the evolute of the cycloid $x = a(\theta - \sin\theta)$; $y = a\;(1 - \cos\theta)$ is another cycloid.

**Solution**

We know that, the coordinates of the centre of curvature is,

$X = \mathrm{x} - \frac{\mathrm{y}_1(1+\mathrm{y}_1{}^2)}{\mathrm{y}_2}$… (1) and $Y = \mathrm{y} + \frac{(1+\mathrm{y}_1{}^2)}{\mathrm{y}_2}$… (2)

Given that, $x = a(\theta - \sin\theta)\; and \;\; y = a\;(1 - \cos\theta)$

$$\therefore \frac{dx}{d\theta} = a(1-\cos\theta) \ and \ \frac{dy}{d\theta} = a \ \sin\theta$$

$$\Longrightarrow \frac{dy}{dx} = \frac{a \ \sin\theta}{a\ (1-\cos\theta)} = s \ \cot\frac{\theta}{2}$$

$$y_2 = \frac{d^2y}{dx^2} = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\left(\frac{d\theta}{dx}\right) = \frac{d}{d\theta}\ (\cot\frac{\theta}{2})\ \frac{1}{a(1-cos\theta)} = -cosec^2\frac{\theta}{2}\cdot\frac{1}{2}\cdot\frac{1}{a\ 2sin^2\theta/2}$$

$$\therefore y_2 = \frac{-cosec^4\frac{\theta}{2}}{4a} = \frac{-1}{4asin^4\frac{\theta}{2}}$$

$$(1) \Longrightarrow X \ = a(\theta-\sin\theta) - \frac{\cot\theta/2(1+\cos^2\theta/2)}{\frac{cosec^4\theta/2}{4a}}$$

$$= a(\theta-\sin\theta) + \frac{cos\frac{\theta}{2}}{sin\frac{\theta}{2}}\ .cosec^2\frac{\theta}{2}\times\frac{4a}{cosec^4\frac{\theta}{2}}$$

$$= a(\theta-\sin\theta) + \frac{cos\frac{\theta}{2}}{sin\frac{\theta}{2}}\ .cosec^2\frac{\theta}{2}\ .4a\ sin^2\frac{\theta}{2}$$

$$= a(\theta-\sin\theta) + 4\ \cos\frac{\theta}{2}\ .sin^3\frac{\theta}{2}.\ a\ \frac{1}{sin^2\frac{\theta}{2}}$$

$$= a(\theta-\sin\theta) + 2.2a\ \sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= a(\theta-\sin\theta) + 2a\ (2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$$

$$= a\theta - a\sin\theta + 2a\ sin2\frac{\theta}{2}$$

$$= a\theta + 2a\ \sin\theta - a\sin\theta = a\theta + a\ \sin\theta$$

$$\therefore X = a(\theta+\sin\theta)$$

$$(2) \Longrightarrow Y = a(1-\cos\theta) + \frac{(1+cot^2\theta/2)}{\frac{-cosec^4\frac{\theta}{2}}{4a}}$$

$$= a(1-\cos\theta) + cosec^2\frac{\theta}{2}\times\frac{-4a}{cosec^4\frac{\theta}{2}}$$

$$= a(1-\cos\theta) - 4asin^2\frac{\theta}{2}$$

$$= a - a\cos\theta - 4a\left(\frac{1-\cos\theta}{2}\right)$$

$$= a - a\cos\theta - 2a(1-\cos\theta)$$

$$= -a + a\cos\theta$$

$\therefore\ Y = -a\,(1-\cos\theta)$

The required coordinates of the centre of curvature are:

$X = a(\theta + \sin\theta)$and $Y = -a\,(1-\cos\theta)$

Therefore, the locus of (X,Y) is, x= $a(\theta + \sin\theta)$ and $y = -a\,(1-\cos\theta)$.

The curve represented by these equations is also a cycloid.

**UNIT – II : SUCCESSIVE DIFFERENTIATION**

*Successive differentiation –Leibnitz's Formula. Partial differentiation – Successive partial differentiation – Implicit functions – homogeneous functions – Euler's theorem. Maxima and Minima for one variable - Concavity, Convexity and points of inflexion - Maxima and Minima for two variables.*

## SUCCESSIVE DIFFERENTIATION

## Introduction:

We have seen that, in general, the derivative of a function of $x$ is also a function of $x$. If the new function is differentiable, then the derivative of the first derivative is called the second derivative of the original function. Similarly, the derivative of the second derivative is called the third derivative; and so on the $n^{th}$ derivative.

Consider, $y = 4x^5$

Then, $\frac{dy}{dx} = 20x^4$

$\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d^2y}{dx^2} = 80x^3$

$\frac{d}{dx}\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\} = 240x^2$, etc.

The symbols of the successive derivatives are usually abbreviated as follows:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D^2y$$

$$\frac{d}{dx}\left\{\frac{d}{dx}\left(\frac{dy}{dx}\right)\right\} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = D^3y$$

.
.
.

$$\frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = D^ny.$$

If $y = f(x)$, the successive derivatives are then denoted by

$f'(x), f''(x), \dots, f^n(x)$ or

$y', y'', \dots, y^{(n)}$ or

$y_1, y_2, \dots, y_n$.

**The $n^{th}$ derivative**

For certain functions, we can find a general expression for the $n^{\text{th}}$ derivative involving $n$.For this purpose, find the number of successive derivatives necessary to discover their law of formation. Then, write down the $n^{\text{th}}$ derivative by induction.

**Example:**

Let $y = e^{ax}$

$\Rightarrow \quad \frac{dy}{dx} = ae^{ax}$

$\Rightarrow \quad \frac{d^2y}{dx^2} = a^2e^{ax}$

Then, we can write $\frac{d^ny}{dx^n} = a^ne^{ax}$.

**Standard Results:**

1. Let $y = (ax+b)^m$,

then, $y_1 = m\,a(ax+b)^{m-1}$
$y_2 = m(m-1)a^2(ax+b)^{m-2}$
$y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3}$

.
.
.

$y_n = m(m-1)\ldots(m-n+1)a^n(ax+b)^{m-n}$.

When $m = -1$, we have

$D^n(ax - b)^{-1} = (-1)^n n!\, a^n (ax+b)^{-n-1}$. … (1)

2. Let $y = log\,(ax+b)$,

then, $y_1 = a(ax+b)^{-1}$.

We have $y_n = a\,\frac{d^{n-1}}{dx^{n-1}}(ax+b)^{-1} = a\,D^{n-1}(ax - b)^{-1}$.

Replace $n$ by $(n-1)$ in (1), we have

$$y_n = a(-1)^{n-1}(n-1)!\; a^{n-1}\,(ax+b)^{-n}$$

$= (-1)^{n-1}(n-1)!\; a^n\,(ax+b)^{-n}$.

3. Let $y = sin(ax+b)$,

then, $y_1 = a\,cos(ax+b)$
$= a\,sin\left(\frac{\pi}{2} + ax + b\right)$.

Thus, in this case, the effect of a differentiation is to multiply by $a$ and increase the angle by $\frac{\pi}{2}$.

$$\therefore y_2 = a^2 \cos\left(\frac{\pi}{2} + ax + b\right) = a^2 \sin\left(\frac{2\pi}{2} + ax + b\right)$$

Similarly, $y_3 = a^3 \sin\left(\frac{3\pi}{2} + ax + b\right)$

In general, $D^n \sin(ax + b) = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$ … (2)

4. Similarly, $D^n \cos(ax + b) = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$ … (3)

**Corollaries:**

Let $a = 1,\ b = 0.$

From (2) and (3), we have

$$D^n(\sin x) = \sin\left(\frac{n\pi}{2} + x\right)$$

and $D^n(\cos x) = \cos\left(\frac{n\pi}{2} + x\right).$

5. Let $y = e^{ax} \sin(bx + c)$,

then, $y_1 = e^{ax}\, b \cos(bx + c) + ae^{ax} \sin(bx + c).$

Put $a = r\cos\phi \quad and \quad b = r\sin\phi.$

We have, $y_1 = re^{ax} \sin(bx + c + \phi).$

Thus, in this case, the effect of a differentiation is to multiply by $r$ and increase the angle by $\phi$.

Similarly, $y_2 = r^2 e^{ax} \sin(bx + c + 2\phi), \ldots$

In general,

$$D^n\{e^{ax} \sin(bx + c)\} = r^n e^{ax} \sin(bx + c + n\phi),$$

where $r = (a^2 + b^2)^{\frac{1}{2}}$ and $\phi = \tan^{-1}\left(\frac{b}{a}\right).$

6. Similarly,

$$D^n\{e^{ax} \cos(bx + c)\} = r^n e^{ax} \cos(bx + c + n\phi),$$

where $r$ and $\phi$ have the same meanings as before.

**Fractional expressions of the form $\frac{f(x)}{\phi(x)}$ , where both functions are being algebraic and rational:**

The above form can be differentiated $n$ times by splitting them into partial fractions.

**Problems:**

1. Find $y_n$, where $y = \frac{3}{(x+1)(2x-1)}$ .

**Solution:**

Given that,$= \frac{3}{(x+1)(2x-1)}$ .

Resolving into partial fractions, we obtain$y = \frac{2}{2x-1} - \frac{1}{x+1}$

We know that,$D^n(ax - b)^{-1} = (-1)^n n!\, a^n (ax + b)^{-n-1}$

$$\Rightarrow \quad y_n = \frac{2(-1)^n.2^n.n!}{(2x-1)^{n+1}} - \frac{(-1)^n n!}{(x+1)^{n+1}}$$

$$= (-1)^n n! \left\{\frac{2^{n+1}}{(2x-1)^{n+1}} - \frac{1}{(x+1)^{n+1}}\right\}.$$

2. Find $y_n$, when $y = \frac{x^2}{(x-1)^2(x+2)}$ .

**Solution:**

Consider,$y = \frac{x^2}{(x-1)^2(x+2)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)}$

We have $x^2 = A(x-1)(x+2) + B(x+2) + C(x-1)^2$.

$x = 1 \quad \Rightarrow 1 = A(0) + B(3) + C(0) \quad \Rightarrow B = \frac{1}{3}$

$x = -2 \quad \Rightarrow 4 = A(0) + B(0) + C(-2-1)^2 \quad \Rightarrow C = \frac{4}{9}$

$x = 0 \quad \Rightarrow 0 = A(-1)(2) + B(2) + C$

$\Rightarrow 0 = -2A + \left(\frac{1}{3}\right)2 + \frac{4}{9} \Rightarrow \frac{-10}{9} = -2A \quad \Rightarrow A = \frac{5}{9}$

$\therefore y = \frac{5}{9}.\frac{1}{(x-1)} + \frac{1}{3}.\frac{1}{(x-1)^2} + \frac{4}{9}.\frac{1}{(x+2)}$ .

Hence, $y_n = \frac{5}{9}.\frac{n!\,(-1)^n}{(x-1)^{n+1}} + \frac{(n+1)(-1)^n}{3(x-1)^{n+2}} + \frac{4}{9}.\frac{(-1)^n\, n!}{(x+2)^{n+1}}$

$$\Longrightarrow\ y_n = (-1)^n\, n! \left\{\frac{5}{9(x-1)^{n+1}} + \frac{(n+1)}{3(x-1)^{n+2}} + \frac{4}{9(x+2)^{n+1}}\right\}.$$

3. Find $y_n$, when $y = \frac{1}{x^2+a^2}$ .

**Solution:**

Consider, $y = \frac{1}{x^2+a^2} = \frac{1}{2ai}\left[\frac{1}{x-ai} - \frac{1}{x+ai}\right]$ $\left(\because \frac{1}{x^2+a^2} = \frac{1}{x-ai} - \frac{1}{x+ai}\right)$

$$\therefore y_n = \frac{(-1)^n n!}{2ai}\left[\frac{1}{(x-ai)^{n+1}} - \frac{1}{(x+ai)^{n+1}}\right]$$

## Trigonometrical Transformation:

It is possible to break up products of powers of sines and cosines into a sum by trigonometrical methods.

**Problems:**

1. Find the $n^{\text{th}}$ differential coefficient of $cosx.cos2x.cos3x$.

**Solution:**

Consider, $cosx.cos2x.cos3x = \frac{1}{2}\, cos2x\,(cos4x + cos2x)$

$$= \frac{1}{2} cos2x\, cos4x + \frac{1}{2} cos^2 2x$$

$$= \frac{1}{4}(cos2x + cos\,6x) + \frac{1}{4}(1 + cos4x)$$

$$= \frac{1}{4} + \frac{1}{4}(cos2x + cos4x + cos6x)$$

$$\therefore D^n(cosx\, cos2x\, cos3x) = \frac{1}{4}\left\{2^n cos\left(\frac{n\pi}{2} + 2x\right) + 4^n cos\left(\frac{n\pi}{2} + 4x\right) + 6^n cos\left(\frac{n\pi}{2} + 6x\right)\right\}.$$

2. Find the $n^{\text{th}}$ differential coefficient of $\cos^5\theta\,.\sin^7\theta$ .

**Solution:**

Let $x = cos\theta + i\, sin\theta$ … (1)

$\Longrightarrow \frac{1}{x} = cos\theta - i\, sin\theta$ … (2)

(1) + (2) $\Longrightarrow x + \frac{1}{x} = 2\, cos\theta$

(1) - (2) $\Longrightarrow \quad x - \frac{1}{x} = 2isin\theta$

Applying De Moivre's theorem, we have

$$x^n \; = \; cos\, n\theta + isin\, n\theta \quad and \quad \frac{1}{x^n} = \; cos\, n\theta - isin\, n\theta$$

$$\Longrightarrow x^n + \frac{1}{x^n} = 2\, cos\, n\theta \qquad and \qquad x^n - \frac{1}{x^n} \; = 2i\, sin\, n\theta \; \ldots(3)$$

Using (3), we have

$$2^5 \cos^5\theta = \left(x \; + \frac{1}{x}\right)^5 \text{and} \quad 2^7\, i^7 \sin^7\theta = \left(x - \frac{1}{x}\right)^7.$$

On multiplying the respective sides of the equations, we have

$$2^{12}\, i^7 \cos^5\theta \sin^7\theta = \left(x \; + \frac{1}{x}\right)^5 . \left(x - \frac{1}{x}\right)^7$$

$$= \left(x^2 - \frac{1}{x^2}\right)^5 . \left(x - \frac{1}{x}\right)^2$$

$$= \left(x^{10} - \; 5x^6 + 10x^2 - \frac{10}{x^2} + \frac{5}{x^6} - \frac{1}{x^{10}}\right)\left(x^2 - 2 + \frac{1}{x^2}\right)$$

$$= x^{12} - \; 5x^8 + 10x^4 - 10 + \frac{5}{x^4} - \frac{1}{x^8}$$

$$- 2x^{10} + \; 10x^6 - 20x^2 + \frac{20}{x^2} - \frac{10}{x^6} + \frac{2}{x^{10}}$$

$$+ x^8 - \; 5x^4 + 10 - \frac{10}{x^4} + \frac{5}{x^8} - \frac{1}{x^{12}}$$

$$= \left(x^{12} - \frac{1}{x^{12}}\right) - 2\left(x^{10} - \frac{1}{x^{10}}\right) - 4\left(x^8 - \frac{1}{x^8}\right) + 10\left(x^6 - \frac{1}{x^6}\right)$$

$$+ 5\left(x^4 - \frac{1}{x^4}\right) - 20\left(x^2 - \frac{1}{x^2}\right)$$

Using (3), we have

$$2^{12} i^7 \cos^5\theta \sin^7\theta = \; 2i\, [sin\, 12\theta \;\; - \; 2sin\, 10\theta - \; 4sin\, 8\theta \; + \; 10sin\, 6\theta \; + \; 5sin\, 4\theta \; - \; 20sin\, 2\theta]$$

$$\Longrightarrow -2^{11} \cos^5\theta \sin^7\theta = \; sin\, 12\theta \;\; - \; 2sin\, 10\theta - \; 4sin\, 8\theta \; + \; 10sin\, 6\theta \; + \; 5sin\, 4\theta \; - \; 20sin\, 2\theta$$

$\therefore D^n(\cos^5\theta \sin^7\theta) = \frac{-1}{2^{11}}\left\{12^n \sin\left(\frac{n\pi}{2} + 12\theta\right) - 10^n\, 2 \sin\left(\frac{n\pi}{2} + 10\theta\right) - 8^n 4 \sin\left(\frac{n\pi}{2} + \right.\right.$
$8\theta+$ $\quad 6n\ 10sinn\pi 2+6\theta \quad +4n\ 5\sin n\pi 2+4\theta\ -2n\ 20$
$\sin n\pi 2+2\theta$.

## **Formation of Equations Involving Derivatives:**

Let there exists a relation between $x$and $y$. Then, in many cases, we can deduce a relationship between the variables $x$ , $y$ and the derivatives of $y$ with respect to x, using the given relationship.

**Problems:**

1. If $xy = ae^{x} + be^{-x}$, prove that $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0$.

**Solution:**

Given that, $xy = ae^{x} + be^{-x}$

On differentiating both sides with respect to $x$, We have

$y + x\frac{dy}{dx} = ae^{x} - be^{-x}$.

On differentiating both sides of the equation once again, we get

$$\frac{dy}{dx} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} = ae^{x} + be^{-x}$$

$\Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy$

$\Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0$, which is the required result.

2. Prove that if $y = sin\,(m\ \sin^{-1} x)$, then$(1 - x^2)y_2 - xy_1 + m^2y = 0$.

**Solution:**

Given,$y = sin\,(m\ \sin^{-1} x) \Rightarrow \sin^{-1} y = m\ \sin^{-1} x$

On differentiating both sides with respect to $x$, we get,

$$\frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = \frac{m}{\sqrt{1-x^2}}$$

On squaring and transposing, we have

$(1 - x^2)\left(\frac{dy}{dx}\right)^2 = m^2(1 - y^2)$.

On differentiating the above equation with respect to x, we get

$(1 - x^2)\ 2\ \frac{dy}{dx}.\frac{d^2y}{dx^2} - 2x\left(\frac{dy}{dx}\right)^2 = -2m^2y\frac{dy}{dx}$.

Cancelling the common factor 2 $\frac{dy}{dx}$throughout, we get

$$(1 - x^2)\,.\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0.$$

3. If $x = sin\theta$, $y = cos\, p\theta$, prove that $(1-x^2)y_2 - xy_1 + p^2y = 0$.

**Solution:**

Given that, $x = sin\theta$ and $y = cosp\theta$

$$\Rightarrow \frac{dx}{d\theta} = cos\theta \quad and \quad \frac{dy}{d\theta} = -p\, sinp\theta$$

$$\therefore \frac{dy}{dx} = -p.\frac{sin\ p\theta}{cos\ \theta} = -p.\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}.$$

Since, $y = cosp\theta \Rightarrow y^2 = cos^2 p\theta \Rightarrow 1-y^2 = 1-cos^2 p\theta = sin^2 p\theta$

$\Rightarrow \sin p\theta = \sqrt{1-y^2}$.

Similarly, $\cos\theta = \sqrt{1-x^2}$.

Squaring both sides $\Rightarrow \left(\frac{dy}{dx}\right)^2 = p^2\left(\frac{1-y^2}{1-x^2}\right)$

$$\Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 = p^2(1-y^2)$$

On differentiating the above equation and cancelling the common factor $2\frac{dy}{dx}$, we get

$(1-x^2)\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + p^2y = 0$, which is the required equation.

**Leibnitz Formula for the $n^{\text{th}}$ Derivative of a Product:**

This formula expresses the $n^{\text{th}}$ derivative of the product of two variables in terms of the variables themselves and their successive derivatives.

If $u$ and $v$ are functions of $x$, we have,

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$i.e., D(uv) = v\,Du + u\,Dv.$$

On differentiating again with respect to $x$, we get

$$D^2(uv) = D(v.Du) + D(u.Dv)$$
$$= v\,D^2u + 2\,Du.Dv + u.D^2v$$

Similarly, $D^3(uv) = v\,D^3u + 3D^2u\,Dv + 3Du.D^2v + u\,D^3v.$

From the above, it can be observed that the numerical coefficients follow the same law as that of the Binomial theorem; and the indices of the derivatives correspond to the exponents of the Binomial theorem. Hence,

$$\frac{d^n}{dx^n}(uv) = \frac{d^nu}{dx^n}.v + nC_1\frac{d^{n-1}u}{dx^{n-1}}.\frac{dv}{dx} + nC_2\frac{d^{n-2}u}{dx^{n-2}}.\frac{d^2v}{dx^2} + \cdots + nC_{n-1}\frac{du}{dx}.\frac{d^{n-1}v}{dx^{n-1}} + u.\frac{d^nv}{dx^n}.$$

This theorem is particularly useful whenone of the factors is a small integral multiple of $x$. Further, ifthis factor is taken as vin the proceeding formula, its differential coefficienst and the series will consist of only a few terms.

**Problems:**

1. Find the $n^{th}$ differential coefficient of $x^2 logx$.

**Solution:**

Taking $v = x^2 \ and \ u = log x$, we have

$$\frac{d^n}{dx^n}(x^2 logx) = \frac{d^n}{dx^n}(logx)(x^2) + nC_1\frac{d^{n-1}}{dx^{n-1}}(logx)\frac{d}{dx}(x^2) + \ nC_2\frac{d^{n-2}}{dx^{n-2}}(logx)\frac{d^2}{dx^2}(x^2).$$

All the other terms will be zero,since the successive derivatives of $x^2$after the second derivative vanish.

$$\therefore D^n(x^2 logx) = \frac{(-1)^{n-1}(n-1)!}{x^n}x^2 + \frac{n\,(-1)^{n-2}(n-2)!}{x^{n-1}}.2x + \frac{n(n-1)(-1)^{n-3}(n-3)!\,2}{2x^{n-2}}$$

$$= \frac{(-1)^{n-3}2.(n-3)!}{x^{n-2}}.$$

2. If $y = \ sin(m\ \sin^{-1}x)$, then prove that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0.$$

**Solution:**

We know that, $(1-x^2)y_2 = xy_1 - m^2y$.

Taking the $n^{th}$ derivative of each term by Leibnit'z theorem, we have

$y_{n+2}\,(1-x^2) + nC_1y_{n+1}(-2x) + nC_2y_n(-2) = y_{n+1}\,x\ + nC_1y_n - m^2y_n$

$\Rightarrow y_{n+2}(1-x^2) - 2nxy_{n+1} - n(n-1)y_n = xy_{n+1} + ny_n - m^2y_n$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0.$

# Partial Differentiation

**Successive Partial Derivatives:**

Consider the function, $u = f(x,y)$. Then, in general $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are functions of both $x$ and $y$ and may be differentiated again with respect to either of the independent variables giving rise to 'Successive Partial Derivatives'.

Let $x$ alone as varying, then the successive partial derivatives are denoted by

$$\frac{\partial^2 \mathrm{u}}{\partial x^2}, \frac{\partial^3 \mathrm{u}}{\partial x^3}, \ldots, \frac{\partial^n \mathrm{u}}{\partial x^n}.$$

Let $y$ alone as varying, then the successive partial derivatives are denoted by

$$\frac{\partial^2 \mathrm{u}}{\partial y^2}, \frac{\partial^3 \mathrm{u}}{\partial y^3}, \ldots, \frac{\partial^n \mathrm{u}}{\partial y^n}.$$

If we differentiate $u$ with respect to $x$ regarding $y$ as constant and then this result is differentiated with respect to $y$ regarding $x$ as constant, we obtain $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$, which we denote by $\frac{\partial^2 \mathrm{u}}{\partial y \partial x}$ .Similarly, if we differentiate $u$ twice with respect to $x$ and then once with respect to $y$, the result is denoted by the symbol $\frac{\partial^3 \mathrm{u}}{\partial y\, \partial^2 x}$ .

The partial differential coefficient of $\frac{\partial u}{\partial y}$ with respect to $x$ considering y as a constant is denoted by $\frac{\partial^2 \mathrm{u}}{\partial x \partial y}$.

Generally, in the ordinary functions which we come across, we have $\frac{\partial^2 \mathrm{u}}{\partial y \partial x} = \frac{\partial^2 \mathrm{u}}{\partial x \partial y}$ .

**Function of Function Rule:**

This rule is very useful in Partial Differentiation.

Let $z$ be a function of $u$, where $u$ is a function of two independent variables $x$ and $y$.

$$\text{Then, } \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y}.$$

Let $x$ and $y$ receive arbitrary increments $\Delta x$ and $\Delta y$ and let the corresponding increments in $u$ and $z$ be $\Delta u$ and $\Delta z$ respectively.

$$\text{Then, } \frac{\Delta z}{\Delta x} = \frac{\Delta z}{\Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

Proceeding to the limit when $\Delta x \to 0$, we have $\frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x}$ .

**Note:**

The straight limit '$d$' is used in $\frac{dz}{dx}$, where$z$ is a function of only one variable $u$. The curved '$\partial$' is used in $\frac{\partial u}{\partial x}$, where$u$ is a function of two independent variables.

**Problems:**

1. Find the partial differential coefficients of $u = sin\,(ax + by + cz)$.

**Solution:**

Given that, $u = sin\,(ax + by + cz)$

$\Longrightarrow \quad \frac{\partial u}{\partial x} = a\,cos\,(ax + by + cz);$

$\frac{\partial u}{\partial y} = b\,cos\,(ax + by + cz)$

and $\quad \frac{\partial u}{\partial z} = c\,cos\,(ax + by + cz).$

2. If $u = \frac{xy}{x+y}$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$

**Solution:**

Given that, $u = \frac{xy}{x+y}$

$$\Longrightarrow \quad \frac{\partial u}{\partial x} = \frac{(x+y)y - xy}{(x+y)^2}$$

$$= \frac{xy + y^2 - xy}{(x+y)^2}$$

$$= \frac{y^2}{(x+y)^2}$$

Similarly, $\quad \frac{\partial u}{\partial y} = \frac{(x+y)x - xy}{(x+y)^2}$

$$= \frac{x^2 + xy - xy}{(x+y)^2}$$

$$= \frac{x^2}{(x+y)^2}$$

$$\Longrightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x.\frac{y^2}{(x+y)^2} + y.\frac{x^2}{(x+y)^2}$$

$$= \frac{x\,y^2 + y\,x^2}{(x+y)^2}$$

$$= \frac{xy(x+y)}{(x+y)^2} \quad = \frac{xy}{x+y}$$

$$\Longrightarrow \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u.$$

3. If $= \tan^{-1}\frac{x^3+y^3}{x-y}$, then prove that $x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$.

**Solution:**

Given that, $u = \tan^{-1}\frac{x^3+y^3}{x-y}$

$$\Rightarrow \tan u = \frac{x^3+y^3}{x-y}$$

On differentiating w. r. to $x$ alone, we have

$$\sec^2 u\frac{\partial u}{\partial x} = \frac{(x-y)3x^2 - (x^3+y^3)}{(x-y)^2}$$

$$= \frac{3x^3 - 3x^2y - x^3 - y^3}{(x-y)^2}$$

$$\Rightarrow \sec^2 u\frac{\partial u}{\partial x} = \frac{2x^3-3x^2y-y^3}{(x-y)^2} \qquad \ldots (1)$$

Similarly,

$$\sec^2 u\frac{\partial u}{\partial y} = \frac{x^3+3xy^2-2y^3}{(x-y)^2} \qquad \ldots (2)$$

From (1) and (2), we have

$$\sec^2 u\left(x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y}\right) = \frac{x(2x^3-3x^2y-y^3) + y(x^3+3xy^2-2y^3)}{(x-y)^2}$$

$$= 2\frac{x^3+y^3}{x-y} = 2\,tan\,u$$

$$\therefore x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 2\frac{\tan u}{\sec^2 u}$$

$$= 2\tan u\cos^2 u$$

$$= 2\frac{\sin u}{\cos u}.\cos^2 u$$

$$= 2\sin u.\cos u = sin2u$$

$$\therefore\ x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = sin2u.$$

4. If $V = (x^2+y^2+z^2)^{\frac{-1}{2}}$, show that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$.

**Solution:**

Given that, $V = (x^2+y^2+z^2)^{\frac{-1}{2}}$

On differentiating $v$ with respect to $x$ alone, we get

$$\frac{\partial V}{\partial x} = \frac{-1}{2}\,(x^2+y^2+z^2)^{\frac{-3}{2}}.2x$$

$$= -x.\;(x^2+y^2+z^2)^{\frac{-3}{2}}$$

On differentiating once again with respect to x alone,

$$\frac{\partial^2 V}{\partial x^2} = \frac{3}{2}x(x^2+y^2+z^2)^{\frac{-5}{2}}.2x - (x^2+y^2+z^2)^{\frac{-3}{2}}$$

$$= 2.\frac{3}{2}x(x^2+y^2+z^2)^{\frac{-5}{2}}.x - (x^2+y^2+z^2)^{\frac{-3}{2}}$$

$$= \frac{3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

$$= \frac{3x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \frac{(x^2+y^2+z^2)}{(x^2+y^2+z^2)^{\frac{3}{2}+1}}$$

$$= \frac{3x^2-x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$$= \frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

Similarly, $\frac{\partial^2 V}{\partial y^2} = \frac{2y^2-z^2-x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$

and $\frac{\partial^2 V}{\partial z^2} = \frac{2z^2-x^2-y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$

$\therefore \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{2x^2-y^2-z^2+2y^2-z^2-x^2+2z^2-x^2-y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$

$$= \frac{2x^2-2x^2+2y^2-2y^2+2z^2-2z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

$\Longrightarrow \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$, which is the required result.

5. Prove that $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y \partial x}$ when u is equal to $log\,\frac{x^2+y^2}{xy}$.

**Solution:**

Given that, $u = log\,\frac{x^2+y^2}{xy}$

$$= \log(x^2+y^2) - \log x - \log y$$

On differentiating with respect to x alone, we have

$$\frac{\partial u}{\partial x} = \frac{2x}{x^2+y^2} - \frac{1}{x}$$

On differentiating once again with respect to y alone, we have

$\frac{\partial u^2}{\partial x\,\partial y} = \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2} - \frac{1}{x}\right) = \frac{-4xy}{(x^2+y^2)^2}$.

On differentiating with respect to y alone, we have

$$\frac{\partial u}{\partial y} = \frac{2y}{x^2+y^2} - \frac{1}{y}$$

On differentiating once again with respect to x alone, we have

$$\frac{\partial u^2}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2} - \frac{1}{y}\right)$$

$$= \frac{-4xy}{(x^2+y^2)^2}$$

$$\Rightarrow \frac{\partial u^2}{\partial x\,\partial y} = \frac{\partial u^2}{\partial y \partial x}.$$

**Total Differential Coefficient:**

Let $u$ be a continuous function of $x$ and $y$. If $x$ and $y$ receive small increments $\Delta x$ and $\Delta y$, then $u$ will receive, in turn, a small increment $\Delta u$. Then,

$$\Delta u \;=\; f(x+\Delta x, y+\Delta y) - f(x,y).$$

The quantity $\Delta u$ is called the total increment of $u$.

$\therefore\ \Delta u \;=\; f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) + f(x, y+\Delta y) - f(x,y).$

Applying the 'Mean Value Theorem' to each of the two differences on the R.H.S., we have

$$f(x+\Delta x, y+\Delta y) - f(x, y+\Delta y) \;=\; f_x{}'(x+\theta_1\,\Delta x, y+\Delta y)\,\Delta x$$

$$\Rightarrow f'(x, y+\Delta y) - f(x,y) = f_y'(x, y+\theta_2\Delta y)\Delta y,$$

where $f_x{}'$ and $f_y{}'$ denote the partial differential coefficients with respect to $x$ and $y$ respectively. Here, $\theta_1$ and $\theta_2$ are positive fractions.

We have $\Delta u \;=\; f_x{}'(x+\theta_1\Delta x, y+\Delta y)\,\Delta x + f_y{}'(x, y+\theta_2\Delta y)\,\Delta y.$

Let $x$ and $y$, and therefore, also $u$ are continuous functions of some other variable $t$. Also, let $\Delta x, \Delta y$ and $\Delta u$ are the increments of $x, y$ and $u$ due to an increment $\Delta t$ of $t$.

On dividing $\Delta u$ by $\Delta t$, we have

$$\frac{\Delta u}{\Delta t} = f_x{}'(x+\theta_1\Delta x_1, y+\Delta y)\,\frac{\Delta x}{\Delta t} + f_y{}'(x, y+\theta_2\Delta y)\,\frac{\Delta y}{\Delta t}.$$

Now let $\Delta t \to 0$, we have

$$\frac{du}{dx} = f_x{}'(x,y)\frac{dx}{dt} + f_y{}'(x,y)\frac{dy}{dt}$$

$$\Rightarrow \quad \frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}.$$

It can be expressed in the differential form as:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}.dy,$$

where $du$ is called the total differential of $u$.

In the same way, if $u \;=\; f(x,y,z)$ and $x, y, z$ are all functions of $t$, we get,

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}.$$

Similarly, if $u = f(x_1, x_2, \ldots, x_n)$, where $x_1, x_2, \ldots, x_n$ are known functions of a variable t, we have the relation:

$$\frac{du}{dt} = \frac{\partial u}{\partial x_1}.\frac{dx_1}{dt} + \frac{\partial u}{\partial x_2}.\frac{dx_2}{dt} + \cdots + \frac{\partial u}{\partial x_n}.\frac{dx_n}{dt}$$

(or) $$du = \frac{\partial u}{\partial x_1}.dx_1 + \frac{\partial u}{\partial x_2}.dx_2 + \cdots + \frac{\partial u}{\partial x_n}dx_n.$$

**A Special Case:**

If $u = f(x, y)$, where $x$ and $y$ are functions of $t$, we get

$$\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt}$$

If we take $t$ to be $x$, we get $u$ as a function of $x$ and $y$, where $y$ is a function of $x$.

Since, $\frac{dx}{dx}$ is now unity, this relation becomes

$$\frac{du}{dt} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dx}$$

The quantities $\frac{du}{dx}$ and $\frac{\partial u}{\partial x}$ are quite distinct.

**Example:**

Let $u = x^2 + 2xy + y^2$; and let $y$ be a function of $x$.

$\Rightarrow \frac{\partial u}{\partial x} = 2x + 2y \quad and \qquad \frac{\partial u}{\partial y} = 2x + 2y.$

We know that, $\frac{du}{dt} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}.\frac{dy}{dX}$

$\Rightarrow \frac{du}{dx} = 2x + 2y + (2x + 2y)\frac{dy}{dx}$

and the value of $\frac{dy}{dx}$ will depend on the relation between $x$ and $y$.

**Implicit Functions:**

Let the relation between $x$ and $y$ be given in the form, $f(x, y) = c$, where $c$ is a constant. Then, the total differential coefficient with respect to $x$ is zero, since the differential coefficient of a constant is zero.

Hence, $0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = -\left[\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}\right].$

This gives an alternate method to find the differential coefficient of $y$ with respect to $x$ when $y$ is given as an implicit function of $x$.

**Problems:**

1. Find $\frac{du}{dt}$, where $u = x^2 + y^2 + z^2, x = e^t,\ y = e^t sint, z = e^t\ cost.$

**Solution:**

We know that, $\frac{du}{dt} = \frac{\partial u}{\partial x}.\frac{dx}{dt} + \frac{\partial u}{\partial y}.\frac{dy}{dt} + \frac{\partial u}{\partial z}.\frac{dz}{dt}$ ... (1)

Given that, $u = x^2 + y^2 + z^2$

$\Rightarrow \frac{\partial u}{\partial x} = 2x; \quad \frac{\partial u}{\partial y} = 2y; \quad \frac{\partial u}{\partial z} = 2z;$

$$\frac{dx}{dt} = e^t;\ \frac{dy}{dt} = e^t \cos t + \sin t\ e^t = e^t(\cos t + \sin t);$$

and $\frac{dz}{dt} = e^t(-\sin t) + \cos t e^t = e^t(\cos t - \sin t).$

$$(1) \Rightarrow \frac{du}{dt} = 2x.e^t + 2y.e^t(\cos t + \sin t) + 2ze^t(cost - sint)$$

$$= 2e^t(x + y\cos t + y\sin t + z\cos t - z\sin t)$$

$$= 2e^t(e^t + \cos t(y+z) + sint(y-z))$$

$$= 2e^t\left(e^t + \cos t\,(e^t \sin t + e^t \cos t) + \sin t\,(e^t \sin t - e^t \cos t)\right)$$

$$= 2e^t(e^t + e^t \cos t \sin t + e^t cos^2 t + e^t sin^2 t - e^t sint \cos t)$$

$$= 2e^t(e^t + e^t)$$

$\therefore \frac{du}{dt} = 4\,e^{2t}.$

2. If $x^3 + y^3 - 3axy$, find $\frac{dy}{dx}$.

**Solution:**

We know that, $\frac{dy}{dx} = -\left[\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)}\right].$

Given that, $f(x,y) = x^3 + y^3 - 3axy$

$$\Rightarrow \frac{\partial}{\partial x}(x^3 + y^3 - 3axy) = 3x^2 - 3ay$$

$$\Rightarrow \frac{\partial}{\partial y}(x^3 + y^3 - 3axy) = 3y^2 - 3ax$$

Consider, $\frac{dy}{dx} = \frac{-3x^2+3ay}{3y^2-3ax}$

$\Longrightarrow \frac{dy}{dx} = \frac{ay-x^2}{y^2-ax}$

3. Find $\frac{du}{dx}$, when $u = x^2 + y^2$, $y = \frac{1-x}{x}$.

**Solution:**

We know that, $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} . \frac{dy}{dx}$

Given that, $u = x^2 + y^2$

$$\Longrightarrow \frac{\partial u}{\partial x} = 2x; \qquad \frac{\partial u}{\partial y} = 2y$$

$\therefore \frac{dy}{dx} = \frac{x(-1)-(1-x)(1)}{x^2}$

$\Longrightarrow \frac{dy}{dx} = \frac{-x-1+x}{x^2} = \frac{-1}{x^2}$

Consider, $\frac{du}{dx} = 2x + 2y\left(-\frac{1}{x^2}\right) = 2x - \frac{2y}{x^2} = \frac{2x^3-2y}{x^2}$

$= 2\left(\frac{x^3 - y}{x^2}\right) = 2\left(\frac{x^3 - \left(\frac{1-x}{x}\right)}{x^2}\right)$

$$\therefore \frac{du}{dx} = 2\left(\frac{x^4-1-x}{x^3}\right).$$

4. Find $\frac{du}{dt}$, where $u = x^3y + z^2$, $x = t^2$ $y = t^3$ $z = t^4$.

**Solution:**

We know that, $\frac{du}{dt} = \frac{\partial u}{\partial x} . \frac{dx}{dt} + \frac{\partial u}{\partial y} . \frac{dy}{dt} + \frac{\partial u}{\partial z} . \frac{dz}{dt}$

Given that, $u = x^3y + z^2$ and $x = t^2$ $y = t^3$ $z = t^4$

$\Longrightarrow \frac{\partial u}{\partial x} = 3x^2y^4z^2 ; \frac{\partial u}{\partial y} = 4y^3x^3z^2$

$\therefore \frac{dx}{dt} = 2t; \qquad \frac{dy}{dt} = 3t^2$

and $\frac{\partial u}{\partial z} = 2zx^3y^4; \qquad \frac{dz}{dt} = 4t^3.$

Consider, $\frac{du}{dt} = 3x^2y^4z^2 2t + 4y^3x^3z^2 3t^2 + 2zx^3y^4 4t^3$

$= 6x^2y^4z^2t + 12y^3x^3z^2t^2 + 8zx^3y^4t^3$

$\therefore \frac{du}{dt} = 26(t^{25}).$

5. Find$\frac{du}{dt}$, where $u = sin\, xy^2, x = logt, y = e^t$.

**Solution:**

We know that, $\frac{du}{dt} = \frac{\partial u}{\partial x} . \frac{dx}{dt} + \frac{\partial u}{\partial y} . \frac{dy}{dt}$

Given, $u = sin\, xy^2$ and $x = logt, y = e^t$

$\Rightarrow \frac{\partial u}{\partial x} = cosxy^2 . y^2;\quad \frac{\partial u}{\partial y} = cosxy^2 2xy$

and$\frac{dx}{dt} = \frac{1}{t}\;$ ; $\quad \frac{dy}{dt} = e^t$

Consider, $\frac{du}{dt} = cos\, xy^2 . y^2 . \frac{1}{t} + cosxy^2 2xye^t$

$= cos\, xy^2 . y^2 . \frac{1}{t} + cosxy^2 2xy$y

$\therefore \frac{du}{dt} = cos\, xy^2 . y^2 \left(\frac{1}{t} + 2x\right)$.

6. Find $\frac{du}{dt}$,where $u = x^2 + y^2 + a^2 \& x^3 + y^3 = a^3$.

**Solution:**

We know that, $\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} . \frac{dy}{dx}$

Given that, $u = x^2 + y^2 + a^2 \Rightarrow \frac{\partial u}{\partial x} = 2x$

Also, given that, $x^3 + y^3 = a^3$.

On differentiating with respect to $x$, we have

$$3x^2 + 3y^2 \frac{dy}{dx} = 0$$

$\Rightarrow \frac{dy}{dx} = \frac{-x2}{y^2}$an

Consider, $\frac{du}{dx} = 2x + 2y \left(\frac{-x^2}{y^2}\right) = 2x - \frac{2x^2}{y} = \frac{2xy - 2x^2}{y}$

$$\therefore \frac{du}{dx} = 2x \left(\frac{y-x}{y}\right).$$

**Homogeneous Function:**

Let us consider,$f(x, y) = a_0 x^n + a_1 x^{n-1} y + \cdots + a_n y^n$. In this expression, the sum of indices of the variables x and y in each term is 'n'. Such an expression is called a homogeneous function of degree 'n'.

This expression can be written as follows:

$$f(x,y) = x^n\left(a_0 + a_1\left(\frac{y}{x}\right) + \ ... + \ a_n\left(\frac{y^n}{x^n}\right)\right)$$
$$= x^n\left(a\ function\ of\ \frac{y}{x}\right) = x^n F\left(\frac{y}{x}\right).$$
Similarly, a homogeneous function of degree $n$ consisting of $m$ variables$x_1 + x_2 + \cdots + x_m$ can be written as:
$$x_r{}^n.F\left(\frac{x_1}{x_r}, \frac{x_2}{x_r}, \dots, \frac{x_m}{x_r}\right).$$

**Euler's Theorem:**

If $f(x,y)$ is a homogeneous function of degree $n$, then
$$x\ \frac{\partial f}{\partial x} + y\ \frac{\partial f}{\partial y} = nf.$$
This is known as Euler's theorem on homogeneous functions.

**Proof:**
Let us assume that
$f(x,y) = a_0x^n + a_1x^{n-1}y + \cdots + a_ny^n$
$$= x^n F\left(\frac{y}{x}\right)$$
Consier, $\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[x^n\ F\left(\frac{y}{x}\right)\right]$
$$= nx^{n-1}\ F\left(\frac{y}{x}\right) - x^n\ F'\left(\frac{y}{x}\right)\frac{y}{x^2}$$
$\frac{\partial f}{\partial x} = nx^{n-1}\ F\left(\frac{y}{x}\right) - x^{n-2}\ y\ F'\left(\frac{y}{x}\right).$
Again, consier$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x^n\ F\left(\frac{y}{x}\right)\right]$
$$= x^n\ F'\left(\frac{y}{x}\right)\frac{1}{x}$$
$$= x^{n-1}\ F'\left(\frac{y}{x}\right)$$
$\therefore x\ \frac{\partial f}{\partial x} + y\ \frac{\partial f}{\partial y} = \ nx^{n-1}\ F\left(\frac{y}{x}\right) - x^{n-2}\ y\ F'\left(\frac{y}{x}\right) + x^{n-1}\ F'\left(\frac{y}{x}\right)$
$$= nx^n\ F\left(\frac{y}{x}\right)$$
$$\Longrightarrow \quad x\ \frac{\partial f}{\partial x} + y\ \frac{\partial f}{\partial y} = nf.$$

**Note:**

In general, if $f(x_1, x_2, \dots, x_m)$ is a homogeneous function of degree $n$, then
$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \cdots + x_m\frac{\partial f}{\partial x_m} = nf.$$

**Problems:**

1.Verify Euler's theorem when $u = x^3 + y^3 + z^3 + 3xyz.$

**Solution:**

We know that Euler's theorem is given by

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} \qquad \ldots (1)$$

Given that,$u = x^3 + y^3 + z^3 + 3xyz$

$\Longrightarrow \frac{\partial u}{\partial x} = 3x^2 + 3yz; \qquad \frac{\partial u}{\partial y} = 3y^2 + 3xz; \qquad \frac{\partial u}{\partial z} = 3z^2 + 3xy.$

On substituting these in (1), we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = x(3x^2 + 3yz) + y(3y^2 + 3xz) + z(3z^2 + 3xy)$$

$$= 3x^3 + 3xyz + 3y^3 + 3xyz + 3z^3 + 3xyz$$

$$= 3(x^3 + y^3 + z^3 + 3xyz)$$

$$\Longrightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 3u.$$

2. If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, prove that $x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \sin 2u.$

**Solution:**

Given that, $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$

$\Longrightarrow \tan u = \left(\frac{x^3+y^3}{x-y}\right) = \frac{x^3\left(1+\left(y/x\right)^3\right)}{x(1-y/x)} \quad = x^2 f\left(y/x\right),$

which is a homogeneous function of degree 2.

We know that Euler's theorem is given by$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$.

Let $v = \tan u$. Then, $v$ is a homogeneous function of $x$ & $y$ of degree 2.

$\therefore x.\frac{\partial v}{\partial x} + y.\frac{\partial v}{\partial y} = 2v$

$\Longrightarrow x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$

$\Longrightarrow x.sec^2 u\frac{\partial u}{\partial x} + y.sec^2 u\frac{\partial u}{\partial y} = 2\tan u$

$\Longrightarrow x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = 2\frac{\sin u}{\cos u}cos^2 u = 2\,sin\,u\,cos\,u.$

$\therefore x.\frac{\partial u}{\partial x} + y.\frac{\partial u}{\partial y} = \sin 2u$,which is a required equation.

**Partial Derivative of a Function of Two Functions:**

**Problems:**

1. If $z = f(x,y), and\ x = rcos\theta, y = rsin\theta$, prove that$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2$.

**Solution:**

Given that, $x = r\,cos\theta$ ; $y = r\,sin\,\theta$

$\Longrightarrow \frac{\partial x}{\partial r} = cos\theta$ ; $\frac{\partial y}{\partial r} = \sin\theta$;

$$\Longrightarrow \frac{\partial x}{\partial \theta} = -r\sin\theta \;;\; \frac{\partial y}{\partial \theta} = \cos\theta$$

We know that, $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial z}{\partial x}\,cos\theta + \frac{\partial z}{\partial y}\sin\theta ----(1)$

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial z}{\partial x}(-r\sin\theta) + \frac{\partial z}{\partial y}(r\cos\theta) ----(2)$$

Consider R.H.S.:

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left(\frac{\partial z}{\partial x}\right)^2 cos^2\theta + \left(\frac{\partial z}{\partial y}\right)^2 sin^2\theta + 2\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) sin\theta\,cos\theta + \frac{1}{r^2}\left(\frac{\partial z}{\partial x}\right)^2 r^2\,sin^2\theta + \frac{1}{r^2}\left(\frac{\partial z}{\partial y}\right)^2 r^2\,cos^2\theta$$

$$-2\frac{1}{r^2}\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) sin\theta\cos\theta . r^2$$

$= \left(\frac{\partial z}{\partial x}\right)^2 (cos^2\theta + sin^2\theta) + \left(\frac{\partial z}{\partial y}\right)^2 (cos^2\theta + sin^2\theta)$

$=\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 =$ L.H.S.

$$\Longrightarrow \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = \left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2.$$

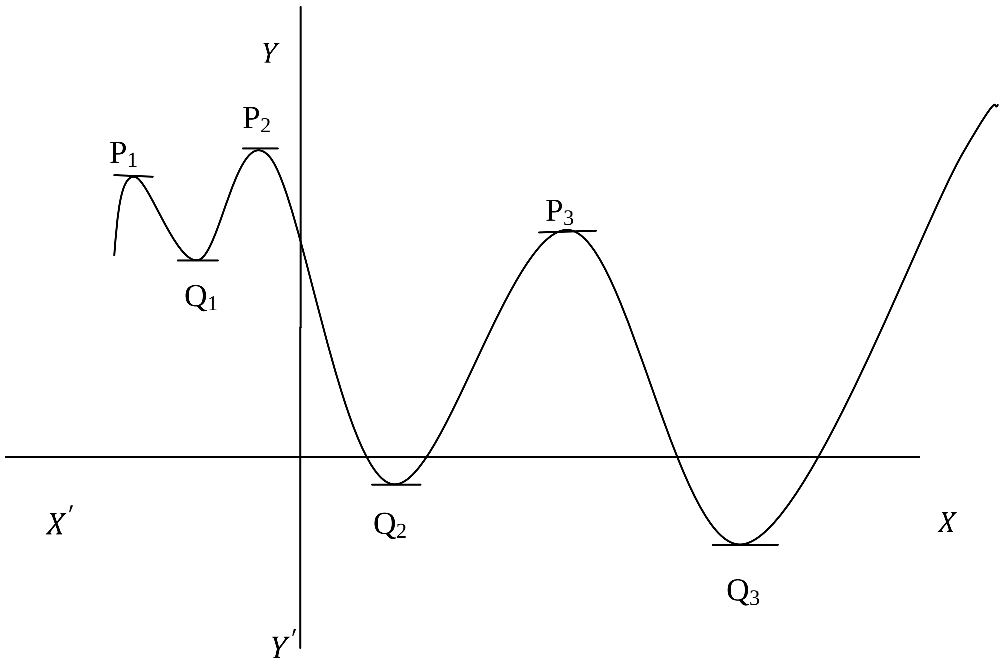
**Maxima and Minima for One Variable:**

If a continuous function increases up to a certain value and then decreases, then this value is called a 'Maximum Value' of the function. Similarly, if a continuous function decreases up to a certain value and then increases, then this value is called a 'Minimum Value' of the function.

We say that the value $f(a)$assumed by $f(x)$ at $x = a$ is a maximum, if $f(x)$is in the immediate neighbourhood of $x = a$.That is, if we can find an interval $(a - h, a + h)$ of values of $x$ such that $f(a) > f(x)$when $a - h < x < a + h$ and $a < x < a + h,$ where $h$ is an arbitrary small positive number.

Similarly, we define a minimum as: if in the interval$(a - h, a + h)$, $f(a) < f(x)$, $f(a)$ is said to be a minimum value of $f(x)$.

Thus, in the figure, the points P correspond to maxima, and the points Q to minima of the function,$f(x)$. The graph of $f(x)$is shown below:




It is to be noticed that:

(1) a maximum value is not necessarily the greatest value of the function can have; nor a minimum the least.

(2) the maxima and minima values occur alternately.

**Theorem 1:**

A necessary condition for a maximum or a minimum value of $f(x)$ at $x = a$ is that $f'(a) = 0$.

**Theorem 2:**
If $f'(a) = 0$ and $f''(a) \neq 0$ then $f(x)$ has a maximum if $f''(a) < 0$ and the minimum if $f''(a) > 0$.

**Rule for determining the maxima and minima values of $f(x)$, when $f(x)$ and $f'(x)$ are continuous:**

The roots of the equation $f'(x) = 0$ are, in general, the values of $x$ which make $f(x)$, a maximum or a minimum. Let 'a' be a root of $f'(x) = 0$, then $f(a)$ will be maximum value of $f(x)$ if $f''(a)$ is negative and a minimum value if $f''(a)$ is positive.

**Problems:**

1. Find the maxima and minima of the function $2x^3 - 3x^2 - 36x + 10$.

**Solution:**

Let $f(x) = 2x^3 - 3x^2 - 36x + 10$.

We know that, at a maximum or minimum, $f'(x) = 0$

$\Rightarrow f'(x) = 6x^2 - 6x - 36 = 0$

$\Rightarrow f'(x) = 6(x-3)(x+2) = 0$

$\therefore x = 3$ and $x = -2$ give maximum or minimum.

<u>To distinguish between the maximum and the minimum:</u>

Consider, $f''(x) = 12x - 6$

$= 6(2x - 1)$

When $x = 3$, $f''(3) = 6(6-1) = 6(5) = 30$ $\Rightarrow f''(x)$ is positive.

When $x = -2$, $f''(2) = 6(-5) = -30$ $\Rightarrow f''(x)$ is negative.

$\therefore x = -2$gives the maximum

and$x = 3$ gives the minimum.

Now, the 'Maximum and Minimum' values of the function, $f(x)$, are obtained by substituting these values of $x$ in $f(x)$.

Given that, $f(x) = 2x^3 - 3x^2 - 36x + 10$.

Maximum value: $x = -2 \Rightarrow f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 10$

$\Rightarrow f(-2) = 2(-8) - 3(4) + 72 + 10 = 54.$

The function $f(x)$ has maximum value 54 at $x = -2$.

Minimum value: $x = 3 \Rightarrow f(3) = 2(3)^3 - 3(3)^2 - 36(3) + 10 - 71$

$\Rightarrow f(3) = 54 - 27 - 108 - 61 = -142.$

The function $f(x)$ has minimum value -142 at $x = 3$.

2. Find the maximum value of $\frac{\log x}{x}$ for positive values of $x$.

**Solution:**

Let $f(x) = \frac{\log x}{x}$

$$\Rightarrow f'(x) = \frac{1 - \log x}{x^2}$$

$$\Rightarrow f''(x) = \frac{-3 + 2\log x}{x^3}$$

We know that, at a maximum or minimum,$f'(x) = 0$.

$\therefore 1 - logx = 0$ $\Rightarrow x = e$

On substituting $x = e$, we have $f''(x) = \frac{-3+2log\, x}{x^3} = \frac{-1}{(e)^3}$.

$\Rightarrow f''(x)$ is negative.

$\Rightarrow f(x)$ attains maximum at $x = e$.

$\therefore$ The Maximum value of the function, $f(x) = \frac{\log e}{e} = \frac{1}{e}$.

3. Show that the least value of $a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x \quad is \quad (a+b)^2$.

**Proof:**

Let $f(x) = a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x$

$\Longrightarrow f'(x) = 2a^2 \sec^2 x \, tanx - 2b^2 \operatorname{cosec}^2 x \, cotx$

$$= \frac{2(a^2 sin^4 x - b^2 cos^4 x)}{cos^3 x \, sin^3 x}$$

$$\Longrightarrow f''(x) = 2\frac{4.\sin^4 x.\cos^4 x.(a^2 \sin^2 x + b^2 \cos^2 x)}{\cos^6 x \sin^6 x} - 2\frac{(a^2 \sin^4 x - b^2 \cos^4 x)\frac{d}{dx}(\sin^3 x.\cos^3 x)}{\cos^6 x \sin^6 x}.$$

We know that, at the maximum or minimum, $f'(x) = 0$.

Consider, $f''(x) = \frac{8.\sin^4 x.\cos^4 x.(a^2 \sin^2 x + b^2 \cos^2 x)}{\cos^6 x \sin^6 x}$

$= 8(a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x)$, is positive for any value of x.

$\therefore f'(x) = a^2 \sin^4 x - b^2 \cos^4 x = 0$ gives a minimum. It is obtained as follows:

We have, $f'(x) = a^2 \sin^4 x - b^2 \cos^4 x = 0$.

$$\Longrightarrow a^2 \sin^4 x = b^2 \cos^4 x \qquad \Longrightarrow \frac{\sin^4 x}{\cos^4 x} = \frac{b^2}{a^2} \Longrightarrow \tan^2 x = \frac{b}{a}.$$

The least value of $f(x)$ is obtained when $\tan^2 x = \frac{b}{a}$; and it is obtained as follows:

Consider, $f(x) = (a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x)$

$$= a^2(1 + \tan^2 x) + b^2(1 + \cot^2 x)$$

On substituting $\tan^2 x = \frac{b}{a}$, we have

$$f(x) = a^2\left(1 + \frac{b}{a}\right) + b^2\left(1 + \frac{a}{b}\right)$$

$$= a^2\left(\frac{a+b}{a}\right) + b^2\left(\frac{b+a}{b}\right)$$

$$= a(a+b) + b(a+b)$$

$$= a^2 + ab + b^2 + ab.$$

$\therefore$ The least value of $a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x \;\; is \; (a+b)^2$, obtained at $\tan^2 x = \frac{b}{a}$.

4. The greatest value of $ax + by$ where $x \; and \; y$ are positive and $x^2 + xy + y^2 = 3k^2$ is

$2k\sqrt{a^2 - ab + b^2}$.

**Solution:**

Let $u = ax + by$.

We know that, $u$ attains a maximum or minimum, when $\frac{du}{dx} = 0$ $and$ $\frac{d^2u}{dx^2}$is –ve or + ve.

Consider, $u = ax + by$ $\Rightarrow \frac{du}{dx} = a + b\frac{dy}{dx} = 0$ … (1)

Given that, $x^2 + xy + y^2 = 3k^2$.

On differentiating the above equation with respect to x, we get

$$2x + y + x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$\Rightarrow (2x + y) + (x + 2y)\frac{dy}{dx} = 0$ … (2)

$\Rightarrow (x + 2y)\frac{dy}{dx} = -(2x + y)$

$\Rightarrow \frac{dy}{dx} = -\frac{(2x+y)}{(x+2y)}$ …. (3)

From (1), we have $\frac{dy}{dx} = -\frac{a}{b}$.

On substituting this value in (3) , we have

$$-\frac{a}{b} = -\frac{(2x + y)}{(x + 2y)}$$

$$\Rightarrow y = \frac{a-2b}{b-2a}x \qquad ...(4)$$

On differentiating equation (2) once again, we get,

$$2 + 2\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 + (x + 2y)\frac{d^2y}{dx^2} = 0$$

On substituting the value of $\frac{dy}{dx}$ from (1), and $y$ from (3), we get

$$\frac{d^2y}{dx^2} = \frac{2}{3} \cdot \frac{a^2-ab+b^2}{b^2} \cdot \frac{b-2a}{x}$$

$\Rightarrow \frac{d^2y}{dx^2}$is negative $\Rightarrow$ y attains a maximum.

Here, $\frac{a^2-ab+b^2}{b^2}$ is positive $\Rightarrow \frac{b-2a}{x}$ is negative.

Consider, $x^2 + y^2 + xy = 3k^2$.

On substituting the value for $y$ from (4), we get

$$x^2 + \left(\frac{a-2b}{b-2a}\right)^2 x^2 + x^2\left(\frac{a-2b}{b-2a}\right) = 3\,k^2$$

$\Rightarrow\ x^2\left\{1 + \left(\frac{a-2b}{b-2a}\right)^2 + \left(\frac{a-2b}{b-2a}\right)\right\} = 3\,k^2$

$\Rightarrow x^2\left\{\frac{b^2+4a^2-4ab+a^2+4b^2-4ab+ab+4ab-2b^2-2a^2}{(b-2a)^2}\right\} = 3\,k^2$

$\Rightarrow x^2\left\{\frac{3b^2+3a^2-3ab}{(b-2a)^2}\right\} = 3\,k^2$

$\Rightarrow x^2(b^2+a^2-ab) = k^2(b-2a)^2$

$\Rightarrow x\sqrt{a^2-ab+b^2} = -k\,(b-2a)$ ... (5)

We take the negative sign, since, $\frac{b-2a}{x}$ is –ve.

On substituting (4) in u, we have

$ax + by = ax + \frac{b(a-2b)}{b-2a}x$

$$= -2(a^2-ab+b^2).\frac{x}{b-2a}$$

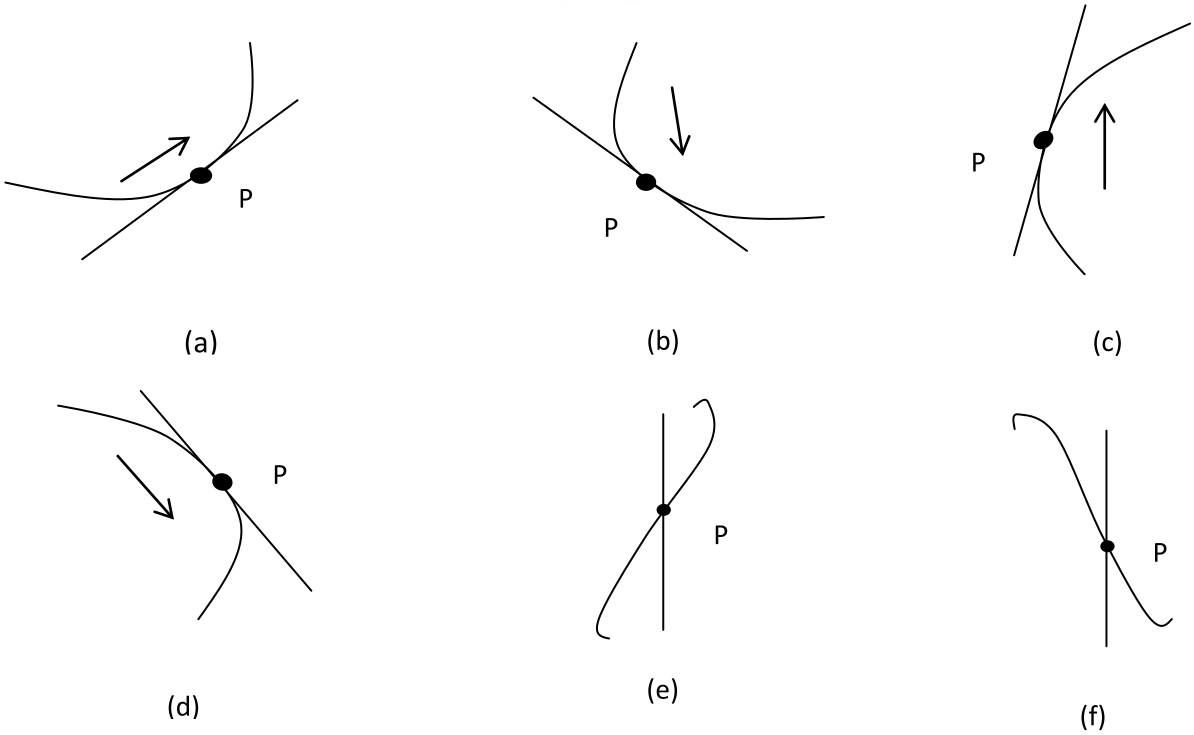
$= 2k\sqrt{a^2-ab+b^2}$ [from (5)]

∴ The greatest value is, $2k\sqrt{a^2-ab+b^2}$ .

**Concavity and Convexity, Points of Inflexion:**

If in the neighbourhood of a point P on a curve is above the tangent at P [as in figures (a) and (b)], it is said to be concave upwards; if the curve is below the tangent at P [as in figures (c) and (d)], it is said to be concave downwards or convex upwards.

If at a point P, a curve changes its concavity from upwards to downwards or vice versa [as in figures (e) and (f)], P is called a point of inflexion.




From this definition, it is seen that (i) the curve crosses its tangent at the point of inflexion, and (ii) a point of inflexion lies between a maximum and a minimum.

As a point on the curve in the figures (a) and (b) moves to the right (the direction of arrows), the tangent turns about its point of contact anti – clockwise and therefore the angle which it makes with the $x$– axis increases. Hence, we get all points in the neighbourhood of P on the curve when it is concave upwards; the slope of the curve, that is, $\frac{dy}{dx}$ increases as $x$ increases. Therefore, its differential coefficient is positive, that is, $\frac{dy}{dx}$ is positive.

Similarly, if at all points in the neighbourhood of P, the curve is concave downwards, then the slope $\frac{dy}{dx}$ decreases as $x$ increases. Therefore, its differential coefficient $\frac{d^2y}{dx^2}$ is negative.

The concavity or convexity of a curve is determined from the sign of the second differential coefficient. That is, if it is negative, the curve is concave downwards or convex upwards. At the pint of inflexion, the curve changes from concave upwards to convex upwards or vice versa. So, $\frac{d^2y}{dx^2}$ changes sign and if it is continuous it is zero at that point.

Hence, the conditions for the point of inflexion are:

1) $\frac{d^2y}{dx^2} = 0$ at the point

2) $\frac{d^2y}{dx^2}$ changes its sign as $x$ increases through the values at which $\frac{d^2y}{dx^2} = 0$,

$i.e., \frac{d^3y}{dx^3} \neq 0.$

**Problems:**

1. For what values of $x$ is the curve $y = 3\,x^2 - 2x^3$ concave upwards and when is it convex upwards?

**Solution:**

Given that, $y = 3\,x^2 - 2\,x^3$

$\Rightarrow \frac{dy}{dx} = 6x - 6x^2$

$\Rightarrow \frac{d^2y}{dx^2} = 6 - 12\,x = 6(1-2x) = -6(2x-1).$

If $x > \frac{1}{2}$, then $\frac{d^2y}{dx^2}$ is –ve $\Rightarrow$ the curve is convex upwards.

If $x < \frac{1}{2}$, then $\frac{d^2y}{dx^2}$ is +ve $\Rightarrow$ the curve is concave upwards.

If $x = \frac{1}{2}$, then $\frac{d^2y}{dx^2} = 0$, $\frac{d^3y}{dx^3} = -12 \Rightarrow$ there is a point of inflexion at $x = \frac{1}{2}$.

When $x = \frac{1}{2}$ $\Rightarrow y = 3\,x^2 - 2\,x^3 = 3\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right)^3 = \frac{1}{2}$.

$\therefore$ The point of inflexion is at the point: $\left(\frac{1}{2}, \frac{1}{2}\right)$.

2. Find the points of inflexion on the cubic $y = \frac{a^2x}{x^2+a^2}$ and show that they lie on a straight line.

**Solution:**

Given that, $y = \frac{a^2x}{x^2+a^2}$

Then, $\frac{dy}{dx} = \frac{a^2(a^2-x^2)}{(x^2+a^2)^2}$

$\Rightarrow \frac{d^2y}{dx^2} = -\frac{2a^2x(3a^2-x^2)}{(x^2+a^2)^3}$

At the points of inflexion, $\frac{d^2y}{dx^2} = 0$

$\therefore x\,(3a^2 - x^2) = 0$

$\Rightarrow x = 0$ (or) $x = \pm\sqrt{3}\,.\,a$

Consider, $\frac{d^3y}{dx^3} = -\frac{6a^2(x^4+a^4-6a^2x^2)}{(x^2+a^2)^4}$

At the points $x = 0$ (or) $x = \pm\sqrt{3}\,.\,a,$ $\frac{d^3y}{dx^3} \neq 0$

Let $x = 0 \quad \Rightarrow \quad y = 0.$

Let $x = \sqrt{3}\,.\,a \quad \Rightarrow \quad y = \frac{\sqrt{3}\,.a}{4}.$

Let $x = -\sqrt{3}\,.\,a \quad \Rightarrow \quad y = -\frac{\sqrt{3}\,.a}{4}.$

The points of inflexion are: (0, 0),$\left(\sqrt{3}\,.\,a\,,\frac{\sqrt{3}\,.a}{4}\right)$,$\left(-\sqrt{3}\,.\,a\,,-\frac{\sqrt{3}\,.a}{4}\right)$.

We know that, equation of the straight line is, $y = \left(\frac{y_2-y_1}{x_2-x_1}\right)x + \left[y_1 - \left(\frac{y_2-y_1}{x_2-x_1}\right)x_1\right]$

Consider, the two points, $(x_1, y_1)$= (0,0) and $(x_2, y_2)$ = $\left(\sqrt{3}\,.\,a\,,\frac{\sqrt{3}\,.a}{4}\right)$, then we have

$$y = \left(\frac{\frac{\sqrt{3}\,.a}{4}}{\sqrt{3}\,.a}\right)x + \left[0 - \left(\frac{\frac{\sqrt{3}\,.a}{4}}{\sqrt{3}\,.a}\right)0\right] = \left(\frac{1}{4}\right)x \qquad \Rightarrow x = 4y.$$

On substituting the point $\left(-\sqrt{3}\,.\,a\,,-\frac{\sqrt{3}\,.a}{4}\right)$ in the straight line, we have

$$-\sqrt{3}\,.\,a = 4\left(-\frac{\sqrt{3}\,.\,a}{4}\right) = -\sqrt{3}\,.\,a$$

$\Rightarrow$The point $\left(-\sqrt{3}\,.\,a\,,-\frac{\sqrt{3}\,.a}{4}\right)$ also lie on the line $x = 4y$.

That is, these three points of inflexion lie on the straight line $x = 4y$.

**Maxima and Minima of Functions of Two Variables:**

**Working Rule:**

Consider a function $u = f(x, y)$.To find the maxima and minima, we proceed as follows:

(i) Find $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$, equate them to zero which give the points x= $a_1, a_2, \ldots$ and $y = b_1, b_2, \ldots$, where the maxima and minima exists.

(ii) Find the value of $\frac{\partial^2 u}{\partial x^2}\,.\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = rt - s^2$ at the points, x= $a_1, a_2, \ldots$ and $y = b_1, b_2, \ldots$.See on what points the expression is +ve, which will be the possible points of maxima and minima.

(iii) If $rt - s^2 < 0$, then there is no maxima or minima at these points. Such points are called "Saddle Points".

(iv) If $rt - s^2 = 0$, nothing can be said about the maxima or minima. It requires further investigation.

(v) Find the value of $r$ and $t$ separately at the point where maxima or minima are possible. If $r$ is negative for one or more points, then those points are the points of maxima of the function; and if positive, then of minima.

(vi) If $r = 0$, nothing can be said about the maxima or minima. It requires further investigation.

**Problems:**

1. Find the maximum or minimum values of $2(x^2 - y^2) - x^4 + y^4$.

**Solution:**

Let $u = 2(x^2 - y^2) - x^4 + y^4$

$\Rightarrow \quad \frac{\partial u}{\partial x} = 4x - 4x^3 \qquad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 4 - 12x^2$

$\Rightarrow \quad \frac{\partial u}{\partial y} = -4y + 4y^3 \qquad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = -4 + 12y^2$

$\Rightarrow \quad \frac{\partial^2 u}{\partial x\, \partial y} = 0$

For a maxima or a minima, we have

$\frac{\partial u}{\partial x} = 0 \quad \Rightarrow 4x - 4x^3 = 0 \quad \Rightarrow 4x(1 - x^2) = 0 \quad \Rightarrow \quad x = 0, \pm 1.$

$\frac{\partial u}{\partial y} = 0 \quad \Rightarrow -4y + 4y^3 = 0 \quad \Rightarrow 4y(y^2 - 1) = 0 \quad \Rightarrow \quad y = 0, \pm 1.$

Consider, $\frac{\partial^2 u}{\partial x^2} . \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x\, \partial y}\right)^2 = 16(1 - 3x^2).(3y^2 - 1)$

$$= \begin{cases} 16.(3y^2 - 1), & \text{when } x = 0 \\ -32.(3y^2 - 1) & \text{when } x = \pm 1 \end{cases}.$$

Consider, $rt - s^2$:

$x = 0,\ y = 0,\ rt - s^2 = -ve$

$x = 0,\ y = \pm 1,\ rt - s^2 = +ve$

$x = \pm 1,\ y = 0,\ rt - s^2 = +ve$

$x = \pm 1,\ y = \pm 1,\ rt - s^2 = -ve$

Hence, the function attains a maxima or minima at the points (0, $\pm1$) and ($\pm1$,0).

Since,$rt - s^2$ is negative, the points (0,0) and($\pm1, \pm1$) are saddle points.

At (0, $\pm1$), $\frac{\partial^2 u}{\partial x^2} = +ve$

At ($\pm1$,0), $\frac{\partial^2 u}{\partial x^2} = -ve$

Hence, the function attains a minimum at (0, $\pm1$) and a maximum at ($\pm1$,0).

The minimum value = -1 and the maximum value = +1.

2. Discuss the maxima and minima of the function $x^3y^2(6-x-y)$.

**Solution:**

Let $u = x^3y^2(6-x-y)$.

$\Rightarrow \quad \frac{\partial u}{\partial x} = x^2y^2(18-4x-3y)$

$$\Rightarrow \frac{\partial u}{\partial y} = 2x^3y(6-x-3)$$

$$= x^3y(12-2x-3y)$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x\,\partial y} = x^2y(36-8x-9y)$$

$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = 6xy^2(6-2x-y)$

$\Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = x^3(12-2x-6y)$

For a maxima or a minima,$\frac{\partial u}{\partial x} = 0 \quad \Rightarrow x^2y^2(18-4x-3y) = 0.$

$$\Rightarrow \quad x = 0 \quad or \quad y = 0 \quad or \quad 4x+3y = 18.$$

Consider, $\frac{\partial u}{\partial y} = 0 \quad \Rightarrow \quad x^3y(12-2x-3y) = 0$

$$\Rightarrow x = 0 \quad or \quad y = 0 \quad or \quad 2x+3y = 12\,.$$

$\therefore \frac{\partial^2 u}{\partial x^2} . \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2 = 6xy^2(6-2x-y).2x^2(6-x-3y) - x^4y^2(36-8x-9y)^2$

$$= x^4y^2\{12\,(6-2x-y).(6-x-3y) - (36-8x-9y)^2\}.$$

For$x = 0, y = 0$ , the expression is zero.

On solving $4x + 3y = 18$ and $2x + 3y = 12$, we obtain: $x = 3 \; and \; y = 2$.

At (3, 2), $rt - s^2 = (3)^4(2)^4\{12(6-6-2)(6-3-6) - (36-24- 18)^2\}$

$= (3)^4(2)^4\{12(-2)(-3) - (-6)^2\}$

$= (3)^4(2)^4(72 - 36)$.

$\Longrightarrow rt - s^2$ispositive.

Hence, the function attain s a maximum or minimum at $x$ = 3, $y$ = 2.

At (3, 2), $\frac{\partial^2 u}{\partial x^2} = 6 \,.\, (3).\,(2)^2 \,.\, (6-6-2) = -74$

At (3, 2), $\frac{\partial^2 u}{\partial x^2}$is – $ve$.

Hence, the function attains a maximum at $(3, 2)$.

The maximum value of the function is obtained by substituting $(3,2)$ in $u = x^3y^2(6 - x - y$

$\therefore u = (3)^3.\,(2)^2.\,(6-3-2) = (27).\,(4)\,.\,(1)$= 27 (4) = 108.

That is, the maximum value of the function is 108 at $(3, 2)$.

3. Show that, if the perimeter of a triangle is constant, the triangle has a maximum area when it is equilateral.

**Solution:**

Let a + b + c = constant (2k)

Then the area Δ of the triangle is given by the formula:

$$\Delta = \sqrt{k(k-a)(k-b)(k-c)}$$

Δis at maximum when $\Delta^2$ is at a maximum.

$i.e.$, When $(k-a)(k-b)(k-c)$ is maximum.

Consider, $(k-a)(k-b)(k-c) = (k-a)(k-b)\{k-(2k-a-b)\}$

$= (k-a)(k-b)(a+b-k)$

Let $f(a,b)$= $(k-a)(k-b)(a+b-k)$

$\Longrightarrow \quad \frac{\partial f}{\partial a} = (k-b)(2k-2a-b);$

$\frac{\partial f}{\partial b} = (k-a)(2k-a-2b);$

$$\frac{\partial^2 f}{\partial a^2} = -2(k-b);$$

$$\frac{\partial^2 f}{\partial a\,\partial b} = -3k + 2a + 2b;$$

and $\frac{\partial^2 f}{\partial b^2} = -2(k-a)$.

For a maximum or a minimum, $\frac{\partial f}{\partial a} = 0, \quad \frac{\partial f}{\partial b} = 0$

$$and \quad \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2 > 0.$$

$$\Rightarrow \; -2(k-b)(-2k+2a) - (-3k+2a+2b)^2 > 0$$

$$\Rightarrow \; 4(k-b)(k-a) - (9k^2 + 4a^2 + 4b^2 + 8ab - 6ak - 6kb) > 0$$

$$\Rightarrow 4(k^2 - kb - ak + ab) - 9k^2 - 4a^2 - 4b^2 - 8ab + 6ak + 6kb > 0$$

$$\Rightarrow \; 4k^2 - 9k^2 - 4kb + 6kb - 4ak + 6ak + 4ab - 8ab - 4a^2 - 4b^2 > 0$$

$$\Rightarrow \; 5k^2 + 2kb + 2ka - 4ab - 4a^2 - 4b^2 > 0$$

$$\Rightarrow \; k = b \; (or) \frac{2a+b}{2} (or) \quad k = a \; or \; \frac{a+2b}{2}$$

i.e.,

1) $b = k, a = k$ $\quad (or)$

2) $2a + b = 2k, a = k$ $\quad (or)$

3) $b = k, a + 2b = 2k$ $\quad (or)$

4) $2a + b = 2k, a + 2b = 2k.$

i.e.,

(1) $\quad c = 0 \quad (or)$

(2) $\quad b = 0 \; (or)$

(3) $\quad a = 0 \; (or)$

(4) $\quad a = \frac{2k}{3}, b = \frac{2k}{3}.$

Hence, $a = \frac{2k}{3}, b = \frac{2k}{3}$ will give either a maximum or a minimum, since the other values will make the triangle degenerate into a straight line.

When, $a = \frac{2k}{3}, b = \frac{2k}{3}$, we have

$$\frac{\partial^2 f}{\partial a^2} = -\frac{2k}{3}; \quad \frac{\partial^2 f}{\partial b^2} = -\frac{2k}{3}; \quad \frac{\partial^2 f}{\partial a\,\partial b} = -\frac{k}{3}.$$

$$\therefore \frac{\partial^2 u}{\partial x^2} \cdot \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x\,\partial y}\right)^2 = \left(-\frac{2k}{3}\right)\left(-\frac{2k}{3}\right) - \left(\frac{k}{3}\right)^2 = \frac{4k^2}{9} - \frac{k^2}{9} = \frac{k^2}{3}$$

That is, $rt - s^2$ is positive.

$\therefore f(a,b)$ attains a maximum when $a = \frac{2k}{3}, b = \frac{2k}{3}$.

$\therefore a + b + c = 2k \Longrightarrow c = \frac{2k}{3}$.

Hence, the area of the triangle is at a maximum when a = b = c.

That is, the area of the triangle is at a maximum when the triangle is equilateral.

**UNIT - III : INTEGRATION**

*Integration- Methods of integration - Integrals of functions containing linear functions of x - Integrals of functions involving $a^2 \pm x^2$ - Integration of rational algebraic functions - $1/(ax^2+bx+c)$, $(px+q)/(ax^2+bx+c)$. Integration of irrational functions - $1/(ax^2+bx+c)^{1/2}$, $(px+q)/(ax^2+bx+c)^{1/2}$, $(px+q)\sqrt{(ax^2+bx+c)}$ - Integration by parts. Multiple integrals - Evaluation of double integrals - polar coordinates - Beta and Gamma functions and their properties.*

## INTEGRATION

<u>Introduction:</u>

We have so far considered the problem of differentiation, that is, finding $\frac{dy}{dx}$, when $y = f(x)$ is given. Now, let us consider the problem of integration, which may be regarded as the inverse of differentiation.

That is, given $\frac{dy}{dx} = f(x)$, find y in terms of x.

The process of finding y, in such a way, is called 'Integration'. It is symbolically written as $y=\int f(x)dx$ .

$'\int'$ isthe sign of integration and the above statement is read as: " integral of $f(x)$ with respect to $x$" or shortly "integral f(x) dx". f(x) is called the integrand; x is called the variable of integration. Hence, $\int f(x)dx$ is called the indefinite integral of $f(x)$ with respect to x.

Hence, by definition, the problem of evaluating $\int f(x)dx$ is to find F(x), a function of x whose derivative with respect to x shall be the integrand f(x).

That is, F'(x)=f(x).

<u>Example</u>:

Consider $\int 2x\, dx$.

We know that, $\frac{d}{dx}(x^2) = 2x$.

We know that, from the definition of the integral, $\int 2x\, dx = x^2$.

We may also add an arbitrary constant c to $x^2$ as follows:

$$\frac{d}{dx}(x^2 + c) = 2x.$$

$$\Rightarrow \int 2x\, dx = x^2 + c.$$

Here, an arbitrary constant of integration is present. Therefore, the integral is called an 'indefinite integral'.

## Definite and Indefinite Integrals:

### Definite Integral:

Let $\int f(x)dx = F(x) + c$, where c is an arbitrary constant on integration.The value of the integral when x=b is F (b)+c and when x=a isF (a) +c.

Therefore, we have

(the value of the integral when x=b)– (the value of the integral when x=a)

=F (b) + c -F (a)- c = F (b)-F (a).

The symbol $\int_a^b f(x)dx$ denotes the value of the integral when x=b, minus the value of the integral when x=a.

$\int_a^b f(x)dx$is called the definite integral, where a&b are called the ‘limits of integration’. Here, ‘a’ is the lower limit and ‘b’ is the upper limit.

### Indefinite Integral:

$\int_a^b f(x)dx$is a definite constant. But,$\int_a^x f(x)dx$, is a function of the variable, x. Then, the integral$\int_a^x f(x)dx$ is called an indefinite integral. The upper limit is ‘x’, which is a variable and not constant. For this reason, this is called an indefinite integral.

The following formulae for integrals are based directly on the results of differentiation, which have been studied earlier:

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c$
2. If $n = -1, \int \frac{dx}{x} = \log x + c$
3. $\int e^x \, dx = e^x + c$
4. $\int \sin x \, dx = -\cos x + c$
5. $\int \cos x \, dx = \sin x + c$
6. $\int sec^2 dx = \tan x + c$
7. $\int cosec^2 dx = -\cot x + c$
8. $\int \sec x \tan x \; dx = \sec x + c$
9. $\int \operatorname{cosec} x \cot x \; dx = -\operatorname{cosec} x$
10. $\int \cosh x \, dx = sinh \, x + c$
11. $\int \sinh x \, dx = \cosh x + c$
12. $\int \frac{dx}{1+x^2} = tan^{-1}x \; (or) - cot^{-1}x$
13. $\int \frac{dx}{\sqrt{1-x^2}} = sin^{-1}x \; (or) - cos^{-1}x$

14. $\int \frac{dx}{\sqrt{x^2-1}} = cosh^{-1}x\ (or)\ \log(x + \sqrt{x^2 - 1})$

15. $\int \frac{dx}{\sqrt{x^2+1}} = sinh^{-1}x\ (or)\ \log(x + \sqrt{x^2 + 1})$

16. $\int \frac{dx}{x\sqrt{x^2-1}} = sec^{-1}x\ (or) - cosec^{-1}x$

17. $\int c\, f(x)dx = c\ \int f(x)dx,\ where\ c\ is\ constant.$

18. $\int (u \pm v)dx = \int u\, dx\ \pm \int v\, dx,\ where\ u\ \&\ v\ are\ functions\ of\ x.$

Example:

Evaluate $\int x^{-4}dx$

Solution:

We know that, $\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$

n = -4 $\Longrightarrow$ $\int x^{-4}\, dx = \frac{x^{-4+1}}{-4+1} + c = \frac{-1}{3x^3} + c.$

Methods of Integration:

The rules in the integral calculus are more or less similar to the various rules in the differential calculus, namely, for differentiating sums, products and functions of functions. These give rise to the following methods of integration:

- The Method of Substitution
- The Method of Decomposition into a sum
- The Method of Integration by parts
- The Method of Successive reduction

**I. The Method of Substitution:**

(i) To evaluate, $\int f(x)dx$, we put $x = \varphi(t)$

$$\Longrightarrow \frac{dx}{dt} = \varphi'(t)\text{(or)}dx=\varphi'(t)dt.$$

Then, we have $\int f(x)dx = \int f\{\varphi(t)\}\, \varphi'(t)dt.$

Proof:

$\frac{d}{dx}(L.H.S.) = f(x),$ ... (1)( by definition )

$$\frac{d}{dx}(R.H.S.) = \frac{d}{dt}(R.H.S) \times \frac{dt}{dx}$$

$$= f\{\varphi(t)\}\, \varphi'(t) \times \frac{1}{\varphi'(t)}$$

$= f(x) \quad \ldots (2)$

On comparing (1) and (2), we have $\int f(x)dx = \int f\{\varphi(t)\}\, \varphi'(t)dt$.

(ii) Integrals of Functions Containing Linear Function of X, that is, $f(ax+b)$:

Evaluate, $\int f(ax+b)dx$

Solution:

Put $ax + b = t$.

$\Longrightarrow \ a\, dx = dt$

$$\therefore \int f(ax+b)dx = \int f(t).\frac{1}{a}dt$$

$= \frac{1}{a}\int f(t)dt.$

Here, $\int f(t)dt,$ can be evaluated in the usual manner. Finally, substitute the value of t.

(iii) Derived Formulae:

The following formulae are derived from the existing formulae for integration.

1. Evaluate $\int (ax+b)^n dx$, for $n \neq -1$

Solution:

Put $t = ax + b \Longrightarrow dt = a\, dx$

$\therefore \int (ax+b)^n dx = \frac{1}{a}\int t^n dt = \frac{t^{n+1}}{a(n+1)}$

Substitute $t = ax + b$ in the above, we have

$\boldsymbol{\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)}}$.

2. Evaluate $\int \frac{dx}{ax+b}$

Solution:

Put $t = ax + b \qquad \Longrightarrow dt = a\, dx$

$\therefore \int \frac{dx}{ax+b} = \frac{1}{a}\int \frac{dt}{t} = \frac{1}{a}\log t$

$$\Rightarrow \int \frac{dx}{ax+b} = \frac{1}{a}\log(ax+b)$$

3. $\int e^{ax+b}\, dx = \frac{1}{a} e^{ax+b}$

4. $\int \sin(ax+b)\, dx = -\frac{1}{a}\cos(ax+b)$

5. $\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b)$

6. $\int sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)$

7. $\int cosec^2(ax+b)dx = -\frac{1}{a}\cot(ax+b)$

8. $\int \sec(ax+b)\tan(ax+b)dx = \frac{1}{a}\sec(ax+b)$

9. $\int \mathrm{cosec}(ax+b)\cot(ax+b)\, dx = -\frac{1}{a}\mathrm{cosec}(ax+b)$

Problems:

1. Evaluate $\int sin^2 x\, dx$.

Solution:

Consider, $\int sin^2 x\, dx = \int \left(\frac{1-cos\,2x}{2}\right) dx$

$$= \int \frac{dx}{2} - \frac{1}{2}\int cos\, 2x\, dx$$

We know that, $\int \cos(ax+b)\, dx = \frac{1}{a}\sin(ax+b)$

$$\therefore \int sin^2 x\, dx = \frac{x}{2} - \frac{1}{2}\frac{sin2x}{2} + c$$

$$\Rightarrow \int sin^2 x\, dx = \frac{1}{2}\left[x - \frac{sin\,2x}{2} + c\right].$$

2. Evaluate $\int \frac{1}{sin^2\ x\ cos^2 x} dx$

<u>Solution:</u>

Consider, $\int \frac{1}{sin^2\ x\ cos^2 x} dx\ = \int \frac{sin^2 x + cos^2 x}{sin^2 x\ cos^2 x} dx$

$$= \int \frac{sin^2 x}{sin^2 x.cos^2 x} dx + \int \frac{cos^2 x}{sin^2 x.cos^2 x} dx$$

$$= \int \frac{dx}{cos^2 x} + \int \frac{dx}{sin^2 x}$$

$$= \int sec^2 x\ dx + \int cosec^2 x\ dx$$

We know that, $\int sec^2(ax+b)dx = \frac{1}{a}\tan(ax+b)$

and $\int cosec^2(ax+b)dx = -\frac{1}{a}\cot(ax+b)$

$\therefore \int \frac{1}{sin^2\ x\ cos^2 x} dx = \tan x - \cot x +\ c$

$\Rightarrow \int \frac{1}{sin^2 x\ cos^2 x} dx = \tan x - \cot x + c$

3.Evaluate $\int \frac{x^2}{(a+bx)^3}\ dx$

<u>Solution:</u>

Put $a + bx = t$ $\Rightarrow dt = b\ dx$

$\Rightarrow\ x = \frac{t-a}{b} \Rightarrow dx = \frac{1}{b} dt$

$$\therefore \int \frac{x^2}{(a+bx)^3}\ d\,x = \int \frac{((t-a)/b)^2}{t^3} \frac{1}{b}\ dt$$

$$= \frac{1}{b^3} \int \frac{(t-a)^2}{t^3}\ dt$$

$$= \frac{1}{b^3} \int \frac{t^2 + a^2 - 2at}{t^3}\ dt$$

$$= \frac{1}{b^3}\left\{\int \frac{t^2}{t^3} dt + \int \frac{a^2}{t^3}\ dt - 2\int \frac{at}{t^3}\ dt\right\}$$

$$= \frac{1}{b^3}\left\{\int \frac{1}{t} dt + \int \frac{a^2}{t^3}\ dt - 2\int \frac{a}{t^2}\ dt\right\}.$$

We know that, $\int \frac{dx}{x} = \log x + c$

$$\therefore \int \frac{x^2}{(a+bx)^3}\,d\,x = \frac{1}{b^3}\left\{\log t - \frac{a^2}{2t^2} + \frac{2a}{t}\right\}$$

$$\Longrightarrow \int \frac{x^2}{(a+bx)^3}\,d\,x = \frac{1}{b^3}\log(a+bx) - \frac{a^2}{2b^3}\frac{1}{(a+bx)^2} + \frac{2a}{b^3}\frac{1}{a+bx}\,.$$

4. Evaluate $\int (3-2x)^3\,dx$.

Solution:

Let, $t = 3-2x$ $\Longrightarrow dt = -2\,dx$

$\Longrightarrow x = \frac{3-t}{2}$ $\Longrightarrow dx = \frac{-dt}{2}$

Consider, $\int (3-2x)^3\,dx = \int t^3\left(\frac{-dt}{2}\right) = -\frac{1}{2}\int t^3\,dt$

$$= -\frac{1}{2}\left(\frac{t^4}{4} + c\right) = -\frac{t^4}{8} + c = -\frac{(3-2x)^4}{8} + c$$

$$\therefore \int (3-2x)^3\,dx = -\frac{(3-2x)^4}{8} + c.$$

5.Evaluate $\int \frac{dx}{(2x+1)^{3/2}}$ .

Solution:

Put $t = 2x+1$, then $dt = 2\,dx \Longrightarrow dx = \frac{1}{2}\,dt$

$$\therefore \int \frac{dx}{(2x+1)^{3/2}} = \frac{1}{2}\int \frac{dt}{t^{3/2}} = \frac{1}{2}\left(\frac{t^{-1/2}}{-1/2}\right) + c = -\frac{1}{\sqrt{t}} + c$$

On substituting $t = 2x+1$, we have $\int \frac{dx}{(2x+1)^{3/2}} = \frac{-1}{(2x+1)^{1/2}} + c.$

(iv) Integrals of Functions Involving $\mathbf{a^2 \pm b^2}$

Evaluate $\int \frac{dx}{\sqrt{a^2-x^2}}$

Solution:

Put $x = a\sin\theta$, $then\ dx = a\cos\theta\,d\theta$

$$\therefore \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2-(a\sin\theta)^2}} = \int \frac{a\cos\theta\, d\theta}{\sqrt{a^2(1-\sin^2\theta)}} = \frac{a\cos\theta\, d\theta}{a\cos\theta} = \int d\theta = \theta$$

$$\Rightarrow \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right).$$

Similarly, we have the following formulae:

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right) \text{or} \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)$$

Examples

**1)**Evaluate $\int \frac{dx}{\sqrt{4-9x^2}}$

Solution:

Consider, $\int \frac{dx}{\sqrt{4-9x^2}} = \int \frac{dx}{3\sqrt{\frac{4}{9}-x^2}} = \frac{1}{3}\int \frac{dx}{\sqrt{\left(\frac{2}{3}\right)^2-x^2}}$

We know that, $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

$\therefore \int \frac{dx}{\sqrt{4-9x^2}} = \frac{1}{3}\sin^{-1}\frac{3x}{2}.$  $\left[\because \frac{x}{2/3} = \frac{3x}{2}\right]$

**2)** Evaluate $\int \frac{dx}{4-25x^2}$ .

Solution:

Consider, $\int \frac{dx}{4-25x^2} = \frac{1}{25}\int \frac{dx}{\frac{4}{25}-x^2}$

$$= \frac{1}{25}\int \frac{dx}{\left(\frac{2}{5}\right)^2-x^2}$$

We know that, $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left(\frac{a+x}{a-x}\right)$

$$\Rightarrow \int \frac{dx}{4-25x^2} = \frac{1}{25}\frac{1}{2\left(\frac{2}{5}\right)} log \frac{\left(2/5\right)+x}{\left(2/5\right)-x}$$

$$= \frac{5}{4}\frac{1}{25} log \frac{2+5x}{2-5x}$$

$$\therefore \int \frac{dx}{4-25x^2} = \frac{1}{20} log \frac{2+5x}{2-5x} .$$

**II. The Method of Decomposition into a Sum (Partial Fractions):**

(i) Integration of Rational Algebraic Functions:

Fractions whose numerator and denominator contain positive integer power of $x$ with constant coefficients.

**Type -1:** $\int \frac{dx}{ax^2+bx+c}$

Rule: Denominator is of the second degree and does not resolve into rational factors.

Following are the examples of this type:

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\frac{x-a}{x+a}$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\frac{a+x}{a-x}$$

Problems:

1.Evaluate $\int \frac{dx}{x^2+2x+5}$ .

Solution:

Consider, $\int \frac{dx}{x^2+2x+5} = \int \frac{dx}{x^2+2x+1+4} = \int \frac{dx}{(x+1)^2+(2)^2}$

Now, we use

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

That is, $\int \frac{dx}{x^2+2x+5} = \frac{1}{2}\tan^{-1}\frac{x+1}{2}$ .

2.Solve $\int \frac{dx}{x^2+8x-7}$ .

Solution:

Consider, $\int \frac{dx}{x^2+8x-7} = \int \frac{dx}{x^2+8x+16-16-7} = \int \frac{dx}{x^2+8x+16-23} = \int \frac{dx}{(x+4)^2-23}$

Now, we use $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \frac{x-a}{x+a}$

$$\Longrightarrow \int \frac{dx}{x^2+8x-7} = \int \frac{dx}{(x+4)^2-(\sqrt{23})^2} = \frac{1}{2\sqrt{23}} \log \frac{x+4-\sqrt{23}}{x+4+\sqrt{23}} .$$

3.Solve $\int \frac{dx}{3x^2+13x-10}$

Solution:

Consider, $\int \frac{dx}{3x^2+13x-10} = \frac{1}{3} \int \frac{dx}{x^2+\left(\frac{13}{3}x\right)-\frac{10}{3}} = \frac{1}{3} \int \frac{dx}{x^2+\frac{13}{3}x+\frac{169}{36}-\frac{169}{36}-\frac{10}{3}}$

$$= \frac{1}{3} \int \frac{dx}{x^2+2\left(\frac{13}{6}\right)x+(\frac{13}{6})^2-\frac{289}{36}} = \frac{1}{3} \int \frac{dx}{(x+\frac{13}{6})^2-(\frac{17}{6})^2}$$

We know that, $\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \log \frac{x-a}{x+a}$

$$\therefore \int \frac{dx}{3x^2+13x-10} = \frac{1}{3} \frac{6}{2(17)} \log \frac{\frac{6x+13}{6}-\frac{17}{6}}{\frac{6x+13}{6}+\frac{17}{6}} = \frac{1}{17} \log \frac{6x-4}{6x+30} = \frac{1}{17} \log \frac{2(3x-2)}{6(x+5)}$$

$$\Longrightarrow \int \frac{dx}{3x^2+13x-10} = \frac{1}{17} \log \frac{3x-2}{3(x+5)} .$$

4. Solve $\int \frac{dx}{1+x-x^2}$

Solution:

Consider, $\int \frac{dx}{1+x-x^2} = \int \frac{dx}{-x^2+x+1} = \int \frac{dx}{-(x^2-x)+1} = \int \frac{dx}{1-(x^2-x)}$

$$= \int \frac{dx}{1-\left[x^2-2\left(\frac{1}{2}\right)x+(\frac{1}{2})^2\right]+(\frac{1}{2})^2} = \int \frac{dx}{1+\frac{1}{4}-\left(x-\frac{1}{2}\right)^2} = \int \frac{dx}{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2} = \int \frac{dx}{(\frac{\sqrt{5}}{2})^2-(x-\frac{1}{2})^2}$$

Now, we use $\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \log \frac{a+x}{a-x}$ .

$$\therefore \int \frac{dx}{1+x-x^2} = \frac{1}{2(\frac{\sqrt{5}}{2})} \log \frac{\frac{\sqrt{5}}{2}+\left(x-\frac{1}{2}\right)}{\frac{\sqrt{5}}{2}-\left(x-\frac{1}{2}\right)} = \frac{1}{\sqrt{5}} \log \frac{2x-1+\sqrt{5}}{\sqrt{5}-2x+1}.$$

**Type-II:** $\int \frac{px+q}{ax^2+bx+c}$

Rule: If $ax^2 + bx + c$ has no rational factors, express the numerator as A(derivative of denominator) + B. Then, integrate each part separately

Problems:

**1.** Solve $\int \frac{2x+3}{x^2+x+1}dx$

**Solution:**

We take, $2x + 3 = A\left[\frac{d}{dx}(x^2 + x + 1)\right] + B$

$\Rightarrow 2x + 3 = A(2x + 1) + B$ ... (1)

Take, $2x + 1 = 0 \Rightarrow 2x = -1 \Rightarrow x = -1/2.$

Substitute $x = -1/2$ in (1), we get

$-1 + 3 = A(0) + B \Rightarrow B = 2$

Substitute $x = 0$ in (1), we get

$2x + 3 = A(2x + 1) + B$

$\Rightarrow 2(0) + 3 = A(2(0) + 1) + 2 \Rightarrow 3 = A + 2 \Rightarrow A = 1$

(1) $\Rightarrow 2x + 3 = (2x + 1) + 2$

$$\therefore \int \frac{2x+3}{x^2+x+1}dx = \int \frac{2x+1}{x^2+x+1}dx + 2\int \frac{dx}{x^2+x+1}$$

$$= \log(x^2 + x + 1) + 2\int \frac{dx}{x^2+x+\frac{1}{4}+\frac{3}{4}}$$

$$= \log(x^2 + x + 1) + 2\int \frac{dx}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$$

$$= \log(x^2 + x + 1) + 2\left(\frac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c$$

$$\therefore \int \frac{(2x+3)}{x^2+x+1}dx = \log(x^2 + x + 1)) + \frac{4}{\sqrt{3}}\tan^{-1}(\frac{2x+1}{\sqrt{3}}) + c.$$

**2.**Solve $\int \frac{2x+3}{-1+x-2x^2} dx$

**Solution:**

We take $2x+3 = A\left[\frac{d}{dx}(-1+x-2x^2)\right] + B$

$\Rightarrow 2x+3 = A\,(1-4x) + B \ldots (1)$

Let $1-4x = 0 \Rightarrow x = \frac{1}{4}$

Substitute $x = \frac{1}{4}$ in (1), we get

$$2\left(\frac{1}{4}\right) + 3 = A\left(1-4\left(\frac{1}{4}\right)\right) + B \Rightarrow \frac{1}{2} + 3 = B \Rightarrow B = \frac{7}{2}$$

Substitute$x = 0$ in (1), we get

$$2(0) + 3 = A\left(1-4(0)\right) + \frac{7}{2} \Rightarrow 3 - \frac{7}{2} = \text{A} \Rightarrow \text{A} = \frac{-1}{2}$$

Now, we have $2x+3 = \frac{-1}{2}(1-4x) + \frac{7}{2}$

$$= \frac{1}{2}(4x-1) + \frac{7}{2}$$

$\therefore \int \frac{(2x+3)}{-1+x-2x^2} dx = \frac{1}{2}\int \frac{(4x-1)dx}{-1+x-2x^2} + \frac{7}{2}\int \frac{dx}{-1+x-2x^2}$

Now, let $t = -1+x-2x^2 \Rightarrow \frac{dt}{dx} = 1-4x \Rightarrow dt = (1-4x)dx$

$\therefore -dt = (4x-1)dx$

$\therefore \int \frac{(2x+3)dx}{-1+x-2x^2} = \frac{1}{2}\int \frac{-dt}{t} + \frac{7}{2}\int \frac{dx}{-2x^2+x-1} = \frac{-1}{2}\int \log t + \frac{-7}{4}\int \frac{dx}{x^2-\frac{x}{2}+\frac{1}{2}}$

$$= \frac{-1}{2}\int \log t + \frac{-7}{4}\int \frac{dx}{x^2 - 2\left(\frac{1}{4}\right)x + \left(\left(\frac{1}{4}\right)^2\right) - \left(\left(\frac{1}{4}\right)^2\right) + \frac{1}{2}}$$

$$= \frac{-1}{2}\log(-1+x-2x^2) - \frac{7}{4}\int \frac{dx}{(x-\frac{1}{4})^2 + (\frac{\sqrt{7}}{4})^2}$$

$$= -\frac{1}{2}\log(-2x^2+x-1) - \frac{7}{4}\frac{4}{\sqrt{7}}\tan^{-1}(\frac{4x-1}{\sqrt{7}})$$

$$= -\frac{1}{2}\log(-2x^2+x+1) - (\frac{7}{\sqrt{7}})\tan^{-1}(\frac{4x-1}{\sqrt{7}})$$

$$\Rightarrow \int \frac{(2x+3)dx}{-1+x-2x^2} = -\frac{1}{2}\log(-2x^2+x-1) - \sqrt{7}\tan^{-1}(\frac{4x-1}{\sqrt{7}})$$

(ii) Integration of Irrational Function:

Rule: Irrational expressions should berationalized by a suitable change of variable.

It has been already shown that,

$$\mathbf{1.} \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$$

$$\mathbf{2.} \int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}\frac{x}{a}\,(or)\,\log(x+\sqrt{x^2+a^2})$$

$$\mathbf{3.} \int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a}\,(or)\,\log(x+\sqrt{x^2-a^2})$$

Allied Integrals:

1.Evaluate $\int \sqrt{a^2-x^2}\,dx$ .

Solution:

Put $x = a\sin\theta$, then $dx = a\cos\theta\, d\theta$

$$\therefore \int \sqrt{a^2-x^2}\,dx = a^2\int cos^2\theta\, d\theta = \frac{a^2}{2}\int(1+cos2\theta)\,d\theta = \frac{a^2}{2}\left(\theta + \frac{sin2\theta}{2}\right)$$

$$= \frac{a^2}{2}(\theta + sin\theta\, cos\theta) = \frac{a^2}{2}\left\{\sin^{-1}\left(\frac{x}{a}\right) + \frac{x}{a}(1-\frac{x^2}{a^2})^{\frac{1}{2}}\right\}$$

$$\therefore \int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{2}.$$

2. Evaluate $\int \sqrt{a^2+x^2}\,dx$

Solution:

Put $x = \text{a}\sinh\theta \quad \Rightarrow \; dx = \text{a}\cos\theta\, d\theta.$

$$\therefore \int \sqrt{a^2+x^2}\,dx = \int \sqrt{a^2+a^2sinh^2\theta}\; acosh\theta\, d\theta$$

$$= \int a\sqrt{1+sinh^2\theta}\; acosh\theta\, d\theta$$

$$= a^2 \int cosh\theta \ \ cosh\theta \ d\theta = a^2 \int cosh^2\theta \ \ d\theta$$

$$= a^2 \int \left(\frac{1 + cosh2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2} \int (1 + cosh2\theta) d\theta$$

$$= \frac{a^2}{2} \int \left(\theta + \frac{sinh2\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{4} 2sinh\theta \ cosh\theta$$

$$= \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2}\frac{x}{a}\left(1 + \frac{x^2}{a^2}\right)^{1/2}$$

$$= \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + \ \frac{ax}{2}\frac{(a^2 + x^2)^{1/2}}{a}$$

$$\therefore \int \sqrt{a^2 + x^2} \ dx \ \ = \frac{a^2}{2} \sinh^{-1}\left(\frac{x}{a}\right) + \ \ \frac{x(a^2+x^2)^{1/2}}{2}.$$

3. Evaluate $\int \sqrt{x^2 - a^2} \ dx$

Solution:

Put $x = acosh\theta \Longrightarrow \ dx = asinh\theta \ d\theta$

$$\therefore \int \sqrt{x^2 - a^2} \ dx = \ \int \sqrt{a^2cosh^2\theta - a^2} \ \text{asinh}\,\theta \ d\theta$$

$$= a^2 \int \sinh\theta \sinh\theta \ d\theta = a^2 \int sinh^2\theta \ \ d\theta$$

$$= \frac{a^2}{2} \int (cosh2\theta - 1) d\theta \ \ = \frac{a^2}{2} \int \left(\frac{sinh2\theta}{2} - \theta\right) d\theta$$

$$= \frac{a^2}{2} sinh\theta \ cosh\theta - \frac{a^2}{2}\theta$$

$$= \frac{a^2}{2}\frac{x}{a}\left(\frac{x^2}{a^2} - 1\right)^{1/2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$$

$$= \ \frac{ax}{2}\frac{(x^2 - a^2)^{1/2}}{a} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \int \sqrt{x^2 - a^2}\, dx = \frac{x(x^2 - a^2)^{1/2}}{2} - \frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)$$

**Problems of the Type:** $\int \frac{\boldsymbol{dx}}{\sqrt{(\boldsymbol{ax^2+bx+c})}}$ **.**

<u>Rule:</u> Divide the expression under the root by the numerical value of the coefficient of $x^2$. Complete the square of the terms which contain x. The integral reduces to one of the forms above.

1.Solve $\int \frac{dx}{\sqrt{2-3x+x^2}}$ .

Solution:

Consider, $\int \frac{dx}{\sqrt{2-3x+x^2}} = \int \frac{dx}{\sqrt{x^2-3x+2}}$

$$= \int \frac{dx}{\sqrt{x^2 - 2\left(\frac{3}{2}\right)x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 + 2}} = \int \frac{dx}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{9}{4}\right) + 2}}$$

$$= \int \frac{dx}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \frac{9+8}{4}}} = \int \frac{dx}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}$$

$$\therefore \int \frac{dx}{\sqrt{2-3x+x^2}} = \cosh^{-1}\left(\frac{2x-3}{2(1/2)}\right).$$

2.Solve $\int \frac{dx}{\sqrt{3x-x^2-2}}$

Solution:

Consider, $\int \frac{dx}{\sqrt{3x-x^2-2}} = \int \frac{dx}{\sqrt{-(x^2-3x+2)}}$

$$= \int \frac{dx}{\sqrt{-\left\{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right\}}}$$

$$= \int \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}}$$

$$= \sinh^{-1}\left(\frac{\frac{2x-3}{2}}{(1/2)}\right) = \sinh^{-1}(2x-3)$$

$\therefore \int \frac{dx}{\sqrt{3x-x^2-2}} = \sinh^{-1}(2x-3).$

**Problems of the Type:** $\int \frac{px+q}{\sqrt{(ax^2+bx+c)}}dx$ **.**

<u>Rule</u>: Let

Numerator = [A*(differential coefficient of the expression under the radical sign) + B], where A&B are constants.

$$\text{i.e., } px+q = A(2ax+b)+B.$$

The values of A and B can be easily determined.

1. Solve $\int \frac{xdx}{\sqrt{x^2+x+1}}$

Solution:

Let $x = A\left(\frac{d}{dx}(x^2+x+1)\right)+B$

$\Longrightarrow \quad x = A(2x+1)+ B \qquad \qquad$ ... (1)

Put, $2x+1=0 \implies 2x=-1 \implies x=-\frac{1}{2}$

Substitute$x = -\frac{1}{2}$ in (1), we get

$$-\frac{1}{2} = A(0) + B \implies B = -\frac{1}{2}$$

Substitute $x=0$ in (1), we get

$0 = A(0+1) - \frac{1}{2} \implies A = \frac{1}{2}$

$$\therefore \int \frac{xdx}{\sqrt{x^2+x+1}} = \frac{1}{2}\int \frac{(2x+1)dx}{\sqrt{x^2+x+1}} - \frac{1}{2}\int \frac{xdx}{\sqrt{x^2+x+1}}$$

$$= \sqrt{x^2+x+1} - \frac{1}{2}\int \frac{xdx}{\sqrt{x^2+2\left(\frac{1}{2}\right)x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}}$$

$$= \sqrt{x^2 + x + 1} - \frac{1}{2}\int \frac{xdx}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= \sqrt{x^2 + x + 1} - \frac{1}{2}\sinh^{-1}\left(\frac{\frac{2x+1}{2}}{\left(\frac{\sqrt{3}}{2}\right)}\right)$$

$\therefore \int \frac{xdx}{\sqrt{x^2+x+1}} = \sqrt{x^2 + x + 1} - \frac{1}{2}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$.

2.Solve $\int \frac{(6x+5)dx}{\sqrt{6+x-2x^2}}$ .

Solution:

Let $6x + 5 = A\left(\frac{d}{dx}(6 + x - 2x^2)\right) + B$

$\Longrightarrow 6x + 5 = A(1 - 4x) + B$

Let $1 - 4x = 0 \Longrightarrow x = \frac{1}{4}$

Put $x = \frac{1}{4}$ in equation (1), we get

$$6\left(\frac{1}{4}\right) + 5 = A(0) + B \Longrightarrow B = \frac{13}{2}$$

Substitute $x = 0$ in equation (1), we get

$$6(0) + 5 = A\big(1 - 4(0)\big) + \frac{13}{2}$$

$$\Longrightarrow 5 - \frac{13}{2} = A \Longrightarrow A = -\frac{3}{2}$$

$$\therefore \int \frac{(6x+5)dx}{\sqrt{6+x-2x^2}} = \frac{13}{2}\int \frac{dx}{\sqrt{6+x-2x^2}} - \frac{3}{2}\int \frac{(1-4x)dx}{\sqrt{6+x-2x^2}}$$

$$= \frac{13}{2}\int \frac{dx}{\sqrt{-2(x^2 - \frac{1}{2}x - 3)}} - 3\sqrt{6 + x - 2x^2}$$

$$= \frac{13}{2\sqrt{2}}\int \frac{dx}{\sqrt{-2(x^2 - 2\left(\frac{1}{4}\right)x + \left(\frac{1}{4}\right)^2 - \left(\frac{1}{4}\right)^2 - 3)}} - 3\sqrt{6 + x - 2x^2}$$

$$= \frac{13}{2\sqrt{2}} \int \frac{dx}{\sqrt{-\left[\left(x - \frac{1}{4}\right)^2 - \left(\frac{1+48}{16}\right)\right]}} - 3\sqrt{6 + x - 2x^2}$$

$$= \frac{13}{2\sqrt{2}} \int \frac{dx}{\sqrt{\left(\frac{7}{4}\right)^2 - \left(x - \frac{1}{4}\right)^2}} - 3\sqrt{6 + x - 2x^2}$$

$\therefore \int \frac{(6x+5)dx}{\sqrt{6+x-2x^2}} = \frac{13}{2\sqrt{2}} \sinh^{-1}\left(\frac{4x-1}{7}\right) - 3\sqrt{6 + x - 2x^2}$ .

**Formulae of the Type:** $\boldsymbol{(px+q)\sqrt{ax^2+bx+c}}$ **and** $\boldsymbol{\sqrt{ax^2+bx+c}}$

1. $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2} x\sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sinh^{-1}\left(\frac{x}{a}\right)$

2. $\int \sqrt{x^2 + a^2}\, dx = \frac{1}{2} x\sqrt{x^2 + a^2} + \frac{1}{2} a^2 \sinh^{-1}\left(\frac{x}{a}\right)$

3. $\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2} x\sqrt{x^2 - a^2} + \frac{1}{2} a^2 \cosh^{-1}\left(\frac{x}{a}\right)$

Problems:

1. Evaluate $\int \sqrt{x^2 + 2x + 10}\, dx$

Solution:

Consider, $\int \sqrt{x^2 + 2x + 10}\, dx = \int \sqrt{x^2 + 2x + 1 + 9}\, dx$

$$= \int \sqrt{(x+1)^2 - 3^2}\, dx$$

We know that, $\int \sqrt{x^2 + a^2}\, dx = \frac{1}{2} x\sqrt{x^2 + a^2} + \frac{1}{2} a^2 \sinh^{-1}\left(\frac{x}{a}\right)$

$\therefore \int \sqrt{x^2 + 2x + 10}\, dx = \frac{1}{2}(x+1)\sqrt{x^2 + 2x + 10} + \frac{9}{2} \sinh^{-1}\left(\frac{x+1}{9}\right)$.

2. Evaluate $\int (3x-2)\sqrt{x^2+x+1}\,dx$

Solution:

Let $3x-2 = A\left(\frac{d}{dx}(x^2+x+1)\right)+B$

$\Rightarrow \quad 3x-2 = A(2x+1)+ B$ ... (1)

Put $2x+1=0 \Rightarrow 2x=-1 \Rightarrow x=-\frac{1}{2}$

Substitute $x=-\frac{1}{2}$ in (1), we get

$$3\left(-\frac{1}{2}\right)-2 = A(0)+ B \Rightarrow B = -\frac{3}{2}-2 = -\frac{3}{2}-\frac{4}{2} \Rightarrow B = -\frac{7}{2}$$

Substitute $x=0$ in (1), we get

$$-2 = A(0+1)-\frac{7}{2} \Rightarrow A = -2+\frac{1}{2} \Rightarrow A=\frac{3}{2}$$

Now, we have $3x-2 = \frac{3}{2}(2x+1)-\frac{7}{2}$

$\therefore \int (3x-2)\sqrt{x^2+x+1}\,dx = \frac{3}{2}\int (2x+1)\sqrt{x^2+x+1}\,dx - \frac{7}{2}\int \sqrt{x^2+x+1}\,dx$ ... (2)

Put $x^2+x+1 = t \Rightarrow (2x+1)dx = dt$

Consider the first integral.

$$\frac{3}{2}\int (2x+1)\sqrt{x^2+x+1}\,dx = \frac{3}{2}\int (t)^{1/2} = \left(\frac{3}{2}\right)\frac{(t)^{3/2}}{\frac{3}{2}} + c$$

$$\frac{3}{2}\int (2x+1)\sqrt{x^2+x+1}\,dx = (t)^{3/2} + c$$

$\therefore \frac{3}{2}\int (2x+1)\sqrt{x^2+x+1}\,dx = (x^2+x+1)^{3/2} + c$ ... (3)

Consider the second integral.

$$-\frac{7}{2}\int \sqrt{x^2+x+1}\,dx = -\frac{7}{2}\int \sqrt{x^2+2\left(\frac{1}{2}\right)x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}\;dx$$

$$= -\frac{7}{2}\int \sqrt{\left(x+\frac{1}{2}\right)^2-\left(\frac{\sqrt{3}}{2}\right)^2}\;dx$$

$$= -\frac{7}{2}\left[\frac{1}{2}\frac{2x+1}{2}\sqrt{x^2+x+1} + \frac{1}{2}\frac{3}{4}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right]$$

$$-\frac{7}{2}\int\sqrt{x^2+x+1}\,dx = -\frac{7(2x+1)\sqrt{x^2+x+1}}{8} - \frac{21}{16}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \quad \dots (4)$$

Substitute (3) and(4) in (2), we get

$$\int(3x-2)\sqrt{x^2+x+1}\,dx = (x^2+x+1)^{3/2} - \frac{7(2x+1)\sqrt{x^2+x+1}}{8}\frac{21}{16}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right).$$

## <u>III. The Method of Integration by Parts</u>:

Let u and v are functions of x.

Then, $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$ [by Product Rule]

Integrating both sides with respect to x, we have

$$\int\frac{d}{dx}(uv)dx = \int u\frac{dv}{dx}dx + \int v\frac{du}{dx}dx$$

$$\therefore uv = \int udv + \int vdu$$

Thus, $\int \boldsymbol{udv} = \boldsymbol{uv} - \int \boldsymbol{vdu}$ **.**

<u>Problems</u>:

1.Solve $\int xe^x dx$

Solution:

Let $u = x \text{ and } dv = e^x dx$

$\Longrightarrow du = dx \text{ and } v = e^x$

We know that, $\int udv = uv - \int vdu$

$$\Longrightarrow \int xe^x dx = xe^x - \int e^x dx = xe^x - e^x = e^x(x-1)$$

$\therefore \int xe^x dx = e^x(x-1)$ .

2.Solve $\int x^n logx\, dx$

Solution:

Let $u = logx$ and $dv = x^n dx.$

Then, $du = \frac{dx}{x}$ $and$ $v = \frac{x^{n+1}}{n+1}$.

We know that, $\int udv = uv - \int vdu$

$$\Rightarrow \int x^n logx\, dx = \frac{x^{n+1}}{n+1} logx - \int \frac{x^{n+1}}{n+1} \frac{dx}{x}$$

$$= \frac{x^{n+1}}{n+1} logx - \frac{1}{n+1} \int x^n\, dx$$

$$= \frac{x^{n+1}}{n+1} logx - \frac{1}{n+1} \frac{x^{n+1}}{n+1}$$

$\therefore \int x^n logx\, dx = \frac{x^{n+1}}{n+1}\left(\log x - \frac{1}{n+1}\right)$.

3.Solve $\int e^x \frac{x+1}{(x+2)^2}\, dx$.

Solution:

Consider, $\int e^x \frac{x+1}{(x+2)^2}\, dx = \int e^x \frac{(x+2-1)}{(x+2)^2}\, dx$

$$= \int \frac{e^x}{x+2}\, dx - \int \frac{e^x}{(x+2)^2}\, dx = I_1 - I_2$$

Now, $I_1 = \int \frac{e^x}{x+2}\, dx$

Let $u = \frac{1}{x+2}$ $and$ $dv = e^x dx$

Then, $du = \frac{-1}{(x+2)^2}$ $and$ $v = e^x$

We know that, $\int udv = uv - \int vdu$

$$\Rightarrow I_1 = \frac{e^x}{x+2} - \int e^x \frac{-1}{(x+2)^2}\, dx = \frac{e^x}{x+2} + \int \frac{e^x}{(x+2)^2}\, dx$$

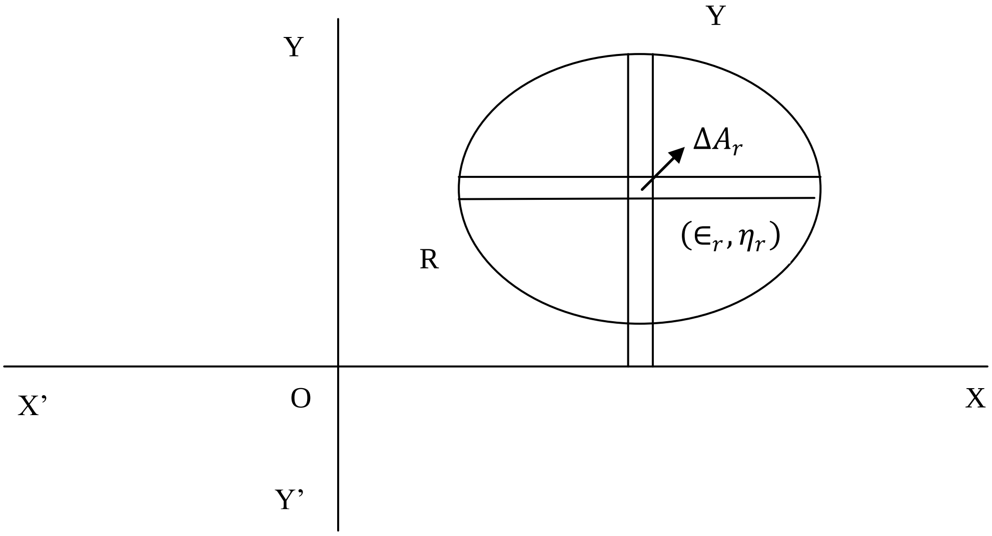
$\therefore I_1 - I_2 = \frac{e^x}{x+2} + \int \frac{e^x}{(x+2)^2}\, dx - \int \frac{e^x}{(x+2)^2}\, dx = \frac{e^x}{x+2}$

$\therefore \int e^x \frac{x+1}{(x+2)^2}\, dx = \frac{e^x}{x+2}$.

# MULTIPLE INTEGRALS

## Double Integral:

Let $f(x,y)$ be a continuous and single valued function of $x$ $and$ $y$ within a region R bounded by a closed curve C and upon the boundary C. Let the Region R be subdivided in any manner into n sub-regions of areas$\Delta A_1, \Delta A_2, \dots , \Delta A_n$.




Let $(\in_r, \eta_r)$ be any point in the sub-region of Area $\Delta A_r$ and consider the sum $\sum_{r=1}^{n} f\left(\in_\rho, \eta_r\right) \Delta A_r$ . The limit of this sum as $n \to \infty$ $and$ $\Delta A_r \to 0$, is defined as the double integral of $f(x,y)$; $(r = 1,2, \dots)$ over the region R.

Thus $\iint_R f(x,y)dA = \lim_{n\to\infty} \sum_{r=1}^{n} f\left(\in_r, \eta_\rho\right) \Delta A_r$ .

The region R is called the region of integration corresponding to interval of integration $(a,b)$ in the case of the simple integral.

This integral is sometimes written as$\iint_{\boldsymbol{R}} \boldsymbol{f(x,y)dxdy}$.

Note:

(1) Let the region of integration is a rectangle between the lines

$$x = a \quad to \quad x = b \quad and \quad y = c \quad to \quad y = d.$$

Then, $\int_a^b \int_c^d f(x,y)dy\,dx = \int_c^d \int_a^b f(x,y)dx\,dy.$

Thus, for constant limits, the order of integration can be changed.

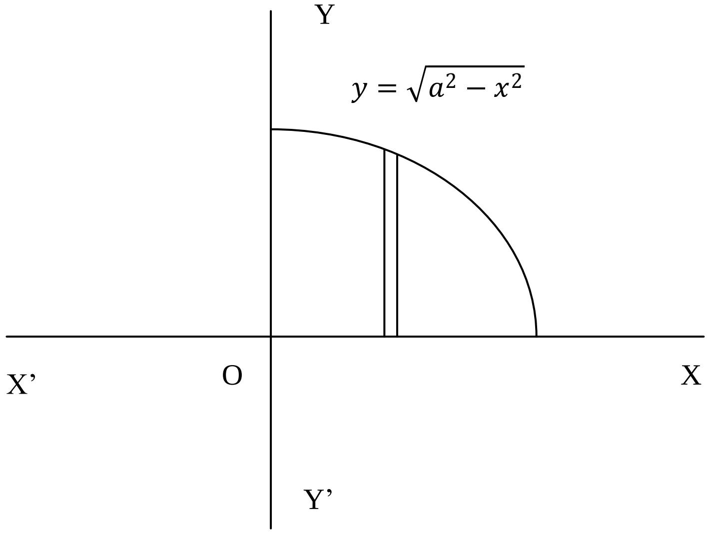
(2) Integration may be considered as the process of summation.

<u>Problems</u>:

1. Evaluate $\iint xy\, dxdy$, taken over the positive quadrant of the circle, $x^2 + y^2 = a^2$.

Solution:

Let the given curve, $x^2 + y^2 = a^2$, be represented as in the following diagram:




Let us keep x as constant then y will vary from 0 to $\sqrt{a^2 - x^2}$. To cover the whole area, x varies from 0 to a.

$$\therefore \iint xy\, dxdy = \int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dx\, dy \; = \int_0^a x \left[\frac{y^2}{2}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a x \left[\frac{a^2 - x^2}{2}\right] dx = \int_0^a \frac{xa^2}{2} dx - \int_0^a \frac{x^3}{2} dx$$

$$= \frac{a^2}{2}\int_0^a xdx - \frac{1}{2}\int_0^a x^3 dx = \frac{a^2}{2}\left[\frac{a^2}{2}\right] - \frac{1}{2}\left[\frac{a^4}{4}\right]$$

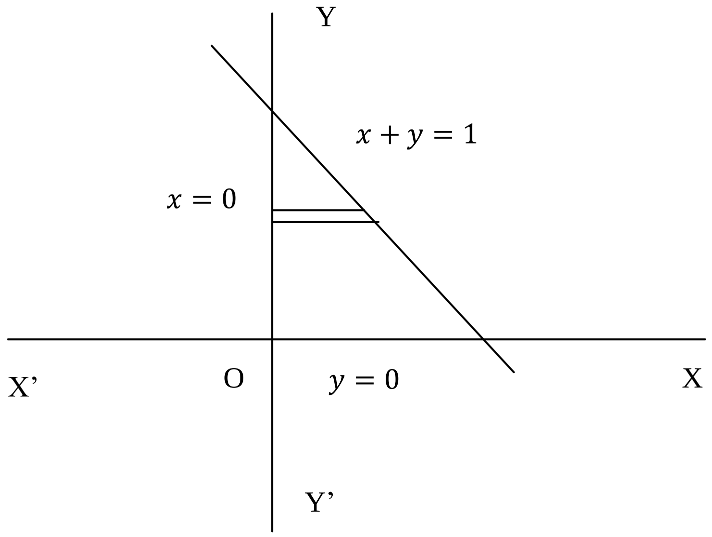
$$= \frac{a^4}{4} - \frac{1}{8}a^4 = \frac{2a^2 - a^4}{8}$$

$$\Rightarrow \iint xy\, dxdy = \frac{a^4}{8}.$$

2. Evaluate $\iint (x^2+y^2)\,dxdy$ over the region for which x,y are each $\geq 0$ and$x+y\leq 1$.

Solution:

The region is triangle formed by the lines,$x=0, y=0, x+y=1$.




$$\therefore \iint (x^2+y^2)dxdy = \int_0^1\int_0^{1-x}(x^2+y^2)dy\,dx$$

$$=\int_0^1 [x^2y+\frac{y^3}{3}]_0^{1-x}\,dx$$

$$=\int_0^1\left[x^2(1-x)+\frac{(1-x)^3}{3}\right]dx$$

$$=\int_0^1 x^2dx-\int_0^1 x^3dx-\int_0^1\frac{(1-x)^3}{3}dx$$

$$=\left[\frac{x^3}{3}\right]_0^1-\left[\frac{x^4}{4}\right]_0^1+\frac{1}{3}\left[\frac{(1-x)^4}{4}\right]_0^1$$

$$=\left[\frac{x^3}{3}-\frac{x^4}{4}+\frac{1}{12}(1-x)^4\right]_0^1$$

$$=\frac{1}{12}[4x^3-3x^4+(1-x)^4]_0^1$$

$$=\frac{1}{12}[4+0-3+0+0+1]=\frac{2}{12}$$

$$\Rightarrow \iint (x^2+y^2)dxdy=\frac{1}{6}.$$

3.Evaluate $\int_0^a \int_0^b (x^2+y^2)dx\,dy$ .

Solution:

Consider,

$$\int_0^a \int_0^b (x^2+y^2)dx\,dy = \int_0^a [x^2y+\frac{y^3}{3}]_0^b\,dx$$

$$= \int_0^a \left[x^2b+\frac{b^3}{3}\right]dx$$

$$= \int_0^a x^2b dx \; + \int_0^a \frac{b^3}{3}dx$$

$$= b\int_0^a x^2 dx \; + \frac{b^3}{3}\int_0^a dx$$

$$= b\left[\frac{x^3}{3}\right]_0^a + \frac{b^3}{3}[x]_0^a$$

$$= \frac{ba^3}{3}+\frac{ab^3}{3}$$

$\Longrightarrow \int_0^a \int_0^b (x^2+y^2)dx\,dy = \frac{ab\left(a^2+b^2\right)}{3}$.

4. Solve the double integral, $\int_0^a \int_0^x (x^2+y^2)dy\,dx$.

Solution:

$$\int_0^a \int_0^x (x^2+y^2)dy\,dx = \int_0^a \left[x^2y+\frac{y^3}{3}\right]_0^x dx$$

$$= \int_0^a [x^3+\frac{x^3}{3}]dx$$

$$= \int_0^a [\frac{(3x^3+x^3)}{3}]dx$$

$$= \frac{4}{3}\int_0^a x^3 dx \; = \frac{4}{3}\left[\frac{x^4}{4}\right]_0^a \qquad = \frac{4}{3}\frac{a^4}{4}$$

$\Longrightarrow \int_0^a \int_0^x (x^2+y^2)dx\,dy = \frac{a^4}{3}$.

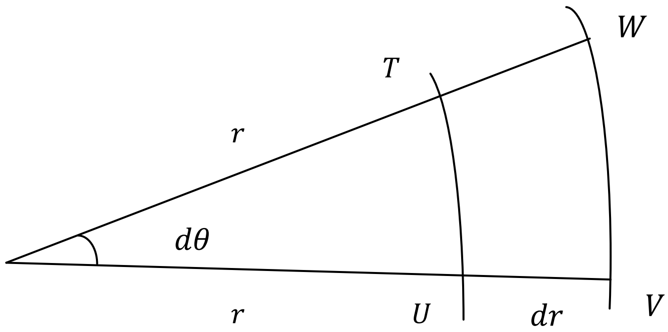
**Double Integral in Polar Co-ordinates**:

The double integral can also be evaluated using polar co-ordinates.

To evaluate the double integral in polar co-ordinates, let us first integrate $[r. f(r,\theta)]$, with respect to r, keeping $\theta$as constant. The limits of integral becomes$r = f_1(\theta)$and $r = f_2(\theta)$. Now, integrate the remaining expression with respect to $\theta$,in the limits$\theta = \alpha$ and $\theta = \beta$.

That is, $$\iint_R r\, f(r,\theta)\, dr\, d\theta = \int_\alpha^\beta \int_{r=f_1(\theta)}^{r=f_2(\theta)} r\, f(r,\theta)\, dr\, d\theta.$$

The above can be represented in the diagram given bellow:




We know that, the double integral of $f(x,y)$ over the region R can be represented as

$$\iint_R f(x,y)dA = \lim_{n\to\infty} \sum_{r=1}^{n} f\left(\epsilon_r, \eta_\rho\right)\Delta A_r \ .$$

Here, dA can be regarded as a rectangle. Its area will be the products of a pair of adjacent sides, say, TU and UV.

That is, $dA = (r\, d\theta)dr = r\, dr\, d\theta.$

Hence, the double integral given in the Cartesian form is transformed into the corresponding Polar form as follows:

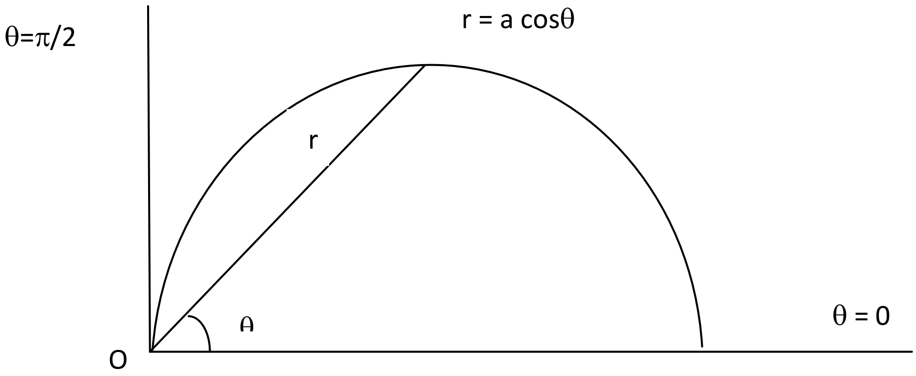
$$\iint_R f(x,y)dxdy = \iint_R f(r\cos\theta,\, r\sin\theta)\, r\, drd\theta.$$

Problems:

1. Evaluate $\iint r\sqrt{a^2 - r^2}\, drd\theta$ over the upper half of the circle $r = a\cos\theta$.

Solution:

Let us represent the given curve, $r = a\cos\theta$, in the following diagram:




The limits of the integral are, $r = 0$ $to$ $a\cos\theta$ and 0 $to$ $\frac{\pi}{2}$.

Therefore, the given integral is, $\int_0^{\frac{\pi}{2}}\int_{r=0}^{r=a\cos\theta} r\sqrt{a^2-r^2}\,dr d\theta$

$$= \int_0^{\frac{\pi}{2}}\left[\int_{r=0}^{r=a\cos\theta} r\sqrt{a^2-r^2}\,dr\right]d\theta$$

Put $= a^2 - r^2 \quad \Rightarrow dt = -2rdr \quad \Rightarrow rdr = -\frac{dt}{2}$.

Therefore, the integral becomes, $\int_{\theta=0}^{\frac{\pi}{2}}\left[\int_{r=0}^{r=a\cos\theta} -\frac{dt}{2}\sqrt{t}\right]d\theta$

$$= \int_{\theta=0}^{\frac{\pi}{2}}\left[-\frac{1}{2}\right]\left[\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{a\cos\theta} d\theta = \left[-\frac{1}{2}\right]\int_{\theta=0}^{\frac{\pi}{2}}\left[\frac{2}{3}t^{\frac{3}{2}}\right]_0^{a\cos\theta} d\theta\ .$$

On substituting $t = a^2 - r^2$, the integral becomes,

$$= \left[-\frac{1}{2}\right]\int_{\theta=0}^{\frac{\pi}{2}}\left[\frac{2}{3}(a^2-r^2)^{\frac{3}{2}}\right]_0^{a\cos\theta} d\theta = -\frac{1}{3}\int_{\theta=0}^{\frac{\pi}{2}}[(a^2-a^2\cos^2\theta)^{\frac{3}{2}} - (a^2-0)^{\frac{3}{2}}]d\theta$$

$$= -\frac{1}{3}\int_{\theta=0}^{\frac{\pi}{2}}[(a^2-a^2\cos^2\theta)^{\frac{3}{2}} - a^3]d\theta$$

$$= -\frac{1}{3}\int_{\theta=0}^{\frac{\pi}{2}} a^3(1-\cos^2\theta)^{\frac{3}{2}} - a^3 d\theta$$

$$= -\frac{1}{3}\int_{\theta=0}^{\frac{\pi}{2}} a^3(\sin^2\theta)^{\frac{3}{2}} - a^3 d\theta$$

$$= -\frac{1}{3}\int_{\theta=0}^{\frac{\pi}{2}} a^3\sin^3\theta - a^3 d\theta$$

$$= -\frac{a^3}{3}\int_{\theta=0}^{\frac{\pi}{2}} \sin^3\theta + \frac{a^3}{3}\int_{\theta=0}^{\frac{\pi}{2}} d\theta$$

$$= -\frac{a^3}{3}[\frac{2}{3}]_0^{\frac{\pi}{2}} + \frac{a^3}{3}[\theta]_0^{\frac{\pi}{2}}$$

$$= -\frac{2a^3}{9} + \frac{a^3}{3}[\frac{\pi}{2} - 0]$$

$$= -\frac{2a^3}{9} + \frac{\pi a^3}{6}$$

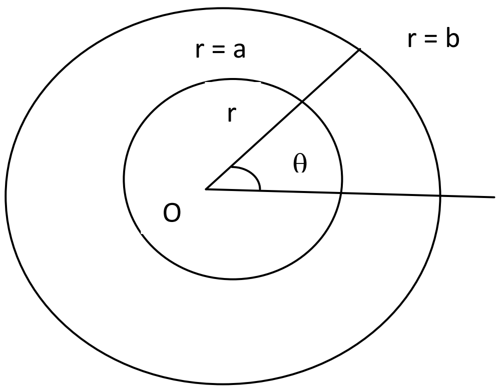
$$= \frac{-4a^3 + 3a^3\pi}{18}$$

$\therefore \iint r\sqrt{a^2 - r^2}\, drd\theta = \frac{a^3}{18}(3\pi - 4).$

2. By transforming into polar co-ordinates, evaluate $\iint \frac{x^2y^2}{x^2+y^2} dx\, dy$ over the annular region between the circles $x^2 + y^2 = c^2$ and$x^2 + y^2 = b^2,\ (b > a)$.

Solution:

Let us represent the two curves, $x^2 + y^2 = c^2$and $x^2 + y^2 = b^2,\ (b > a)$, in the following diagram:




By transforming into polar co-ordinates, the two circles becomes

$$r = a \quad \text{and} \quad r = b.$$

We know that, $\iint_R f(x,y)dxdy \quad = \quad \iint_R f(r\cos\theta,\ r\sin\theta)r\ drd\theta.$

$$\Longrightarrow \iint_R \frac{x^2y^2}{x^2+y^2}dxdy = \iint_R \frac{r^2\cos^2\theta r^2\sin^2\theta}{r^2(\cos^2\theta+\sin^2\theta)} r\, drd\theta$$

$$= \iint_R r^3\cos^2\theta\sin^2\theta \, drd\theta = \int_0^{2\pi}\int_a^b r^3\cos^2\theta\sin^2\theta \, drd\theta$$

$$= \int_0^{2\pi}[\frac{r^4}{4}]_a^b \cos^2\theta\sin^2\theta d\theta = \frac{b^4-a^4}{4}\left[\int_0^{2\pi}\cos^2\theta\sin^2\theta d\theta\right]$$

$$\Longrightarrow \iint_R \frac{x^2y^2}{x^2+y^2}dxdy = \pi\frac{b^4-a^4}{16}.$$

3. By changing into polar co-ordinates, evaluate $\int_0^a\int_0^{\sqrt{a^2-x^2}}(x^2y+y^3)\,dy\,dx$ .

Solution:

We know that, $\iint_R f(x,y)dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)r\,dr\,d\theta,$

where $x = r\cos\theta \quad and \quad y = r\sin\theta$ … (1)

$\Longrightarrow \quad dx\,dy = r\,dr\,d\theta.$

The given limits of the integral represents the semi-circle, $x^2+y^2=a^2$ (above x-axis). Now, the region of integration in terms of polar co-ordinates is obtained as follows:

Consider, $x^2+y^2=a^2$

(1) $\Longrightarrow r^2\cos^2\theta + r^2\sin^2\theta = a^2 \quad \Longrightarrow r^2 = a^2 \quad \Longrightarrow r = \pm a, \quad i.e., r = -a \; to \; a$

Since, $r > 0$, we have $r = 0 \; to \; a$ .

(1) $\Longrightarrow x = r\cos\theta.$

Given that, the limits of $x = 0 \; and \; x = a.$

$\therefore 0 = r\cos\theta \Longrightarrow \theta = \frac{\pi}{2}$ and $a = r\cos\theta \Longrightarrow a = a\cos\theta \Longrightarrow \theta = 0.$

i.e., $\theta = 0 \; to \; \frac{\pi}{2}.$

$$\Longrightarrow \int_0^a\int_0^{\sqrt{a^2-x^2}}(x^2y+y^3)\,dydx = \int_0^{\frac{\pi}{2}}\int_0^a (r^2\cos^2\theta(r\sin\theta) + (r\sin\theta)^3 rdrd\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^a (r^3\cos^2\theta\sin\theta + r^3\sin^3\theta)rdrd\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^a r^3(\cos^2\theta\sin\theta + \sin^3\theta)rdrd\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^a r^4(\cos^2\theta\sin\theta + \sin^3\theta)drd\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=0}^{a} r^4 dr(\cos^2\theta\, sin\theta + sin^3\theta) d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{r^5}{5}\right]_0^a sin\theta(\cos^2\theta + sin^2\theta) d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \sin\theta\, d\theta \;.[\frac{a^5}{5}]$$

$$= \frac{a^5}{5}[-cos\theta]_0^{\frac{\pi}{2}}$$

$$= \frac{a^5}{5}\left[cos\frac{\pi}{2} - \cos(0)\right] = \frac{a^5}{5}(0+1)\ .$$

$$\Longrightarrow \int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2y + y^3)\, dy\, dx \;= \frac{a^5}{5}.$$

## BETA AND GAMMA FUNCTIONS

Beta function:

The Beta function is defined and denoted by,

$\beta(m,n)= \int_0^1 x^{m-1}(1-x)^{n-1}dx$ , for $m > 0, n > 0.$

Gamma function

The Gamma function is defined and denoted by,

$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$ ,for $n > 0.$

Properties of Gamma Function

(i) $\Gamma(n)$ converges for $n > 0$.

**(ii) $\Gamma(1)$=1**

**Proof:**

By definition, $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$ .

Put $n = 1$, we have $\Gamma(1) = \int_0^\infty x^{1-1}e^{-x}dx = \int_0^\infty e^{-x}dx = \left[\frac{e^{-x}}{x}\right]_0^\infty = -\frac{1}{\infty} + 1 = 1$

$\therefore \Gamma(\mathbf{1}) = \mathbf{1}$

**(iii)** $\Gamma(\boldsymbol{n}+\mathbf{1}) = \boldsymbol{n}\Gamma(\boldsymbol{n})$

**Proof:**

We know that, $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$

Put $n = n + 1$, we have $\Gamma(n+1) = \int_0^\infty x^{n+1-1}e^{-x}dx$

$$= \int_0^\infty x^n e^{-x}dx = \lim_{x\to\infty}\int_0^\infty x^n e^{-x}dx$$

Let us evaluate this integral by using 'Integration by Parts' formula as follows:

Let $u = x^n$ $\Rightarrow du = nx^{n-1}dx$

Let $\int dv = \int e^{-x}$ $\Rightarrow v = -e^{-x}$

$\therefore \Gamma(n+1) = \lim_{x\to\infty}[x^n(-e^{-x})]_0^x - \int_0^\infty -e^{-x}nx^{n-1}dx$

$= \lim_{x\to\infty}[x^n(-e^{-x})]_0^x - n\int_0^\infty -e^{-x}x^{n-1}dx$

$= \lim_{x\to\infty}[x^n(-e^{-x}) - (\mathrm{n}(-e^0)] - n\int_0^\infty -e^{-x}x^{n-1}dx$

$= \infty^n(-e^{-\infty}) - 0 + n\int_0^\infty e^{-x}x^{n-1}dx$

$= 0 - 0 + n\int_0^\infty x^{n-1}e^{-x}dx = n\int_0^\infty x^{n-1}e^{-x}dx$

$\therefore \Gamma(\boldsymbol{n}+\mathbf{1}) = \boldsymbol{n}\Gamma(\boldsymbol{n})$

iii) $\Gamma(\boldsymbol{n}+\mathbf{1}) = \boldsymbol{n}!$

**Proof:**

Consider, $\Gamma(n+1) = n\Gamma(n)$

$= n(n-1)\Gamma(n-1)$

$= n(n-1)(n-2)\Gamma(n-2)$

$= n(n-1)(n-2)(n-3) \ldots 3.2.1$

$= 1.2.3 \ldots\ (n-3)(n-2)(n-1)n$

$\therefore \boldsymbol{\Gamma(n+1) = n!}$

Corollary:

$\Gamma(n+a) = (n+a-1)(n+a-2) \ldots a\,\Gamma(a)$, when n is a positive integer.

(iv)$\Gamma(n)$can also be expressed as $\boldsymbol{\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx}$, for $n > 0$.

<u>Properties of Beta Function</u>

(i)$\boldsymbol{\beta(m,n) = \beta(n,m)}$

**Proof:**

We know that, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$

Put$x = y - 1 \quad \Rightarrow y = 1 - x \Rightarrow\ dx = -dy$

Put $x = 0 \quad \Rightarrow y = 1$

Put $x = 1 \quad \Rightarrow y = 0$

$\therefore \beta(m,n) = \int_1^0 (1-y)^{m-1}(y)^{n-1}(-dy)$

$= \int_0^1 y^{n-1}(1-y)^{m-1}dy$

$= \int_0^1 (1-x)^{n-1}(x)^{m-1}dx$

$\therefore \boldsymbol{\beta(m,n) = \beta(n,m).}$

(ii) $\boldsymbol{\beta(m,n)}$can be expressed as a definite integral with limits 0 to ∞.

That is, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$ .

(iii)$\beta(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}x\ cos^{2n-1}x\ dx$

(iv) The Relation between Beta and Gamma Functions

$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ … (1)

(v) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proof:

When $= n = \frac{1}{2}$, we have $\beta(m,n) = \beta\left(\frac{1}{2},\frac{1}{2}\right) = \pi$

When $= n = \frac{1}{2}$, (1) $\Rightarrow \beta\left(\frac{1}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}$

$$\Rightarrow \frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma(1)} = \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = \pi$$

$$\Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

(vi) Express $\int_0^1 x^m(1-x^n)^p\,dx$, in terms of Gamma function.

Solution:

The given integral is, $\int_0^1 x^m(1-x^n)^p\,dx$

Put $x^n = y \Rightarrow nx^{n-1}dx = dy$

$$\therefore \int_0^1 x^m(1-x^n)^p\,dx = \int_0^1 y^{\frac{m}{n}}(1-y)^p \frac{dy}{n.y^{\frac{n-1}{n}}}$$

$$= \frac{1}{n}\int_0^1 y^{\frac{m-n+1}{n}}(1-y)^p\,dy$$

$$= \frac{1}{n}\beta\left(\frac{m-n+1}{n}+1, p+1\right) = \frac{1}{n}\beta\left(\frac{m+1}{n}, p+1\right)$$

$$\Rightarrow \int_0^1 x^m(1-x^n)^p\,dx = \frac{1}{n}\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$$

**Problems:**

1. Evaluate $\int_0^1 x^m\left(\log\frac{1}{x}\right)^n dx$

Solution:

The given integral can be evaluated using Gamma function,

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx \text{ ,for} n > 0.$$

Put $\log\left(\frac{1}{x}\right) = t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t}\, dt$

$\therefore \int_0^1 x^m \left(\log\frac{1}{x}\right)^n\, dx = \int_0^\infty (e^{-t})^m\, t^n\, (-e^{-t} dt) = \int_0^\infty e^{-(m+1)t}\, t^n\, dt$ ... (1)

Put $(m+1)t = y \quad \Rightarrow dt = \frac{1}{m+1}\, dy$

On substituting in (1),we have

$$\int_0^\infty \frac{e^{-y} y^n}{(m+n)^n} \frac{1}{m+n}\, dy = \frac{1}{(m+1)^{n+1}} \int_0^\infty e^{-y}\, y^{(n+1)-1}\, dy = \frac{1}{(m+1)^{n+1}} \Gamma(n+1)$$

$\therefore \int_0^1 x^m \left(\log\frac{1}{x}\right)^n\, dx = \frac{1}{(m+1)^{n+1}} \Gamma(n+1).$

2. Evaluate $\int_0^1 e^{-x^2}\, dx$

Solution:

The given integral can be evaluated using Gamma function,

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \text{ ,for} n > 0.$$

Put $x^2 = t \qquad \Rightarrow 2x\, dx = dt \qquad \Rightarrow dx = \frac{1}{2}\frac{1}{\sqrt{t}}\, dt$

$\therefore \int_0^1 e^{-x^2}\, dx = \int_0^1 e^{-t} \frac{1}{2\sqrt{t}}\, dt = \frac{1}{2}\int_0^1 e^{-t} t^{-\frac{1}{2}}\, dt = \frac{1}{2}\int_0^1 e^{-t} t^{-\frac{1}{2}+1-1}\, dt = \frac{1}{2}\int_0^1 e^{-t} t^{\frac{1}{2}-1}\, dt$

$= \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}$.

3. Evaluate (i) $\int_0^1 x^7 (1-x)^8\, dx$

(ii) $\int_0^{\frac{\pi}{2}} sin^7\theta\, cos^5\theta\, d\theta$

(iii) $\int_0^{\frac{\pi}{2}} sin^{10}\theta\, d\theta$

and (iv) $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta$

Solution :

(i) The given integral can be evaluated using Beta function,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx \text{ , for} m > 0, n > 0.$$

Also, we know that, $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ .

From the given integral, we have

$m - 1 = 7 \quad \Rightarrow m = 8$ and $\quad n - 1 = 8 \quad \Rightarrow n = 9$

$\therefore \int_0^1 x^7(1-x)^8 \, dx = \beta(8\,,9) = \frac{\Gamma(8)\Gamma(9)}{\Gamma(17)} = \frac{7!\,8!}{16!} = \frac{1}{102960}$ .

(ii) We know that, Beta integral has the following form also:

$$\frac{1}{2}\beta(m,n) = \int_0^{\frac{\pi}{2}} sin^{2m-1}x\; cos^{2n-1}x\; dx \;.$$

From the given integral, we have

$2m - 1 = 7 \quad \Rightarrow m = 4 \quad$ and $\quad 2n - 1 = 5 \quad \Rightarrow n = 3$

$\therefore \int_0^{\frac{\pi}{2}} sin^7\theta\; cos^5\theta\; d\theta = \frac{1}{2}\;\beta(4\,,3) = \frac{1}{2}\frac{\Gamma(4)\Gamma(3)}{\Gamma(7)} = \frac{1}{2}\frac{3!\;\;2!}{6!} = \frac{1}{120}$ .

(iii) Given that, $\int_0^{\frac{\pi}{2}} sin^{10}\theta \;\; d\theta = \int_0^{\frac{\pi}{2}} sin^{10}\theta\; cos^0\theta\; d\theta$

We know that, Beta integral has the following form also:

$$\frac{1}{2}\beta(m,n) = \int_0^{\frac{\pi}{2}} sin^{2m-1}x\; cos^{2n-1}x\; dx \;.$$

Here, $2m - 1 = 10 \quad \Rightarrow m = \frac{11}{2} \quad$ and $\quad 2n - 1 = 0 \quad \Rightarrow n = \frac{1}{2}$

$\therefore \int_0^{\frac{\pi}{2}} sin^{10}\theta \;\; d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{11}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{11}{2}+\frac{1}{2}\right)} = \frac{\frac{1}{2}\frac{9}{2}\frac{7}{2}\frac{5}{2}\frac{1}{2}\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma(6)} = \frac{9.7.5.3..(\sqrt{\pi})^2}{5!\;2^6} = \frac{63}{512}\;\pi$ .

(iv) Given that, $\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\ d\theta = \int_0^{\frac{\pi}{2}} sin^{\frac{1}{2}}\theta\ cos^{\frac{-1}{2}}\theta\ d\theta$

We know that, Beta integral has the following form also:

$$\frac{1}{2}\beta(m,n) = \int_0^{\frac{\pi}{2}} sin^{2m-1}x\ cos^{2n-1}x\ dx\ .$$

Here, $2m-1 = \frac{1}{2} \quad \Rightarrow m = \frac{3}{4}$ and $2n-1 = -\frac{1}{2} \quad \Rightarrow n = \frac{1}{4}$

$\therefore \int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\ d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{1}{2}\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{\pi}{2\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}\ .$

4. Evaluate the integral $\int_0^1 x^5(1-x^3)^{10}\ dx$.

Solution:

Given that, $\int_0^1 x^5(1-x^3)^{10}\ dx$

We know that, $\int_0^1 x^m(1-x^n)^p\ dx = \frac{1}{n}\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}$

Here, $m = 5, n = 3, p = 10$

$\therefore \qquad \int_0^1 x^5(1-x^3)^{10}\ dx = \frac{1}{3}\frac{\Gamma\left(\frac{5+1}{3}\right)\Gamma(10+1)}{\Gamma\left(\frac{5+1}{3}+10+1\right)} = \frac{1}{3}\frac{\Gamma(2)\Gamma(11)}{\Gamma(13)} = \frac{1}{3}\frac{1!10!}{12!}$

$\Rightarrow \qquad \int_0^1 x^5(1-x^3)^{10}\ dx = \frac{1}{396}\ .$

UNIT - IV : DIFFERENTIAL EQUATIONS

*Differential equations: Types of first order and first degree equations. Variables separable, Homogeneous, Non-homogeneous equations and Linear equation. Equations of first order but of higher degree. Linear differential equations of second order with constant coefficients. Methods of solving homogenious linear differential equations of second order. Laplace transform and its inverse – solving ordinary differential equation with constant coeficients using Laplce transform.*

## DIFFERENTIAL EQUATIONS OF THE FIRST ORDER

**Definitions**:

**Differential Equations:**

A differential equation is an equation in which differential coefficients occur.

**Types of Differential Equations:**

There are two types of 'Differential Equations'. They are:

(i) Ordinary differential equations
(ii) Partial differential equations.

An ordinary differential equation is one, in which a single independent variable enters, either explicity or implicity.

Consider, $\frac{dy}{dx} = 2\sin x$, $\frac{d^2x}{dr^2} + m^2x = 0$ and $x^2\frac{d^2y}{dx^2} + 2xy\frac{dy}{dx} + y = sinx$.

These are some examples of 'Ordinary Differential Equations'.

A partial differential equation is one, in which at least two(two or more) independent variables occur. Here, the partial differential coefficients occurring in them have reference to any one of these variables.

Consider, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$ and $\frac{\partial^2 z}{\partial x^2} \cdot \frac{\partial^2 z}{\partial y^2} = \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$.

These are some examples of 'Partial Differential Equations'.

**Order and Degree of an Ordinary Differential Equation:**

The **order** of an ordinary differential equation is the order of the highest derivative occurring in it.

If a differential equation can be expressed as a polynomial equation in the derivatives, the degree of the highest derivative is called the **degree** of the equation. Here, the differential coefficients should have been cleared of radicals and fractions.

The term "degree" is not applied to differential equations which cannot be expressed as polynomial equations in the derivatives.

Example – 1:

Consider the differential equation: $[1 + (\frac{dy}{dx})]^{\frac{3}{2}} = a\frac{d^2y}{dx^2}$.

In order to clear the radicals and fractions, we have

$$\left\{\left[1 + \left(\frac{dy}{dx}\right)\right]^{\frac{3}{2}}\right\}^2 = \left\{a\frac{d^2y}{dx^2}\right\}^2 \qquad \Longrightarrow [1 + (\frac{dy}{dx})]^3 = a^2\left\{\frac{d^2y}{dx^2}\right\}^2.$$

Here, the highest derivative is $\frac{d^2y}{dx^2}$. Therefore, the order is 2.

Here, the highest power of the highest derivative is obtained from the term: $\left\{\frac{d^2y}{dx^2}\right\}^2$. It is, 2.

Therefore, the degree of the equation is, 2.

Example – 2:

Consider, $2\frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 = X$.

Here, the highest derivative is $\frac{d^2y}{dx^2}$. Therefore, the order is 2.

Here, the highest power of the highest derivative is obtained from the term: $\frac{d^2y}{dx^2}$. It is, 1.

Therefore, the degree of the equation is, 1.

**Solution of the Differential Equations:**

The **solution** of a differential equations is also called 'Integral of a differential equation'. It is defined as a relation that exists between the variable, (without their differential coefficients) by means of which and the derivatives obtained there from, the equation is satisfied.

This solution is also called the **primitive** of the differential equation.

Example:

Consider the differential equation, $\frac{d^2y}{dx^2} + n^2y = 0$ ...(1)

A solution to the above differential equation (1)is,

$y = A\ cos\, nx$, where, A is an arbitrary constant.

Now, the second derivative of 'y' is obtained as: $\frac{d^2y}{dx^2} = -An^2\, cos\, nx$.

$$\therefore (1) \Rightarrow \frac{d^2y}{dx^2} + n^2y = -An^2\, cos\, nx + An^2\, cos\, nx = 0.$$

Another solution to the differential equation (1) is,

$y = B\sin nx$, where B is an arbitrary constant.

We can find a general solution for (1) as,

$$y = A\ cos\, nx + B\ sin\, nx\ .$$

A solution is called the **'General Solution'** or **'Complete Integral'**, if the number of arbitrary constants occurring is the same as the order of the equation,.

Thus, $Y = A\, cos\, nx + B\, sin\, nx$ is the general solution of $\frac{d^2y}{dx^2} + n^2y = 0$. By giving particular values to the constants occurring in the general solution, we get particular solutions or integrals.

For example, $y = 2\, cos\, nx - sin\, nx$ is a particular integral of the above equation.

**Formation of Differential Equations :**

We shall consider the derivation of differential equations when their solutions are given.

(i)<u>Ordinary Differential Equation</u>:

Suppose, $y = A\, sin\, x + B\, cos\, x$ ... (1)

where, A and B are arbitrary constants.

Our aim is to form the differential equation, whose solution is (1).

On differentiating(1), we have

$\frac{dy}{dx} = A\, cos\, x - B\, sin\, x.$

Again differentiating, we have

$\frac{d^2y}{dx^2} = -A\, sin\, x - B\, cos\, x = -y.$

On eliminating A and B from (1),we get $\frac{d^2y}{dx^2} + y = 0$. This is the differential equation associated with (1).

Thus, to eliminate the number of constants from a given solution, we differentiate it as often as is necessary and then eliminate the constants. The differential equation is thus obtained.

(ii)Partial Differential Equation:

Consider, $\varphi(x, y, a) = 0 \qquad \dots (1)$

Where 'a' is an arbitrary constant.

On differentiating (1) with respect to x, we have

$$\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{dy}{dx} = 0 \qquad \dots (2)$$

On eliminating 'a' between (1) and (2),we get a differential equation of the form:

$$f\left(x, y, \frac{dy}{dx}\right) = 0 \qquad \dots (3)$$

Equation (3) includes all the values of y arising from (1) corresponding to the various values of 'a'. Therefore, (3) represents the differential equation of which (1) is the general solution.

Problems:

1. Form the differential equation by eliminating $\alpha$ *and* $\beta$ from,

$$(x-\alpha)^2 + (y-\beta)^2 = \gamma^2.$$

Solution:

Given that, $(x-\alpha)^2 + (y-\beta)^2 = r^2 \qquad \dots (1)$

On differentiating (1) with respect to x, we have

$$(x-\alpha) + (y-\beta)\frac{dy}{dx} = 0 \qquad \dots (2)$$

Again, differentiating with respect to x, we have

$$\Rightarrow 1 + (y-\beta)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0 \qquad \dots (3)$$

On eliminating $\alpha$ between (1) and (2), we have

$$(y-\beta)^2\left[\left(\frac{dy}{dx}\right)^2 + 1\right] = r^2 \qquad \dots (4)$$

On eliminating $\beta$ between (3)&(4), we have the differential equation as:

$$r^2(\frac{d^2y}{dx^2})^2 = [1 + (\frac{dy}{dx})^2]^3.$$

2. Form the differential equation that represents all parabolas each of which has a latus rectum 4a and whose axes are parallel to the x-axis.

Solution:

If the vertex of any parabola of the system be $(\alpha, \beta)$, the equation of the parabola is,

$$(y - \beta)^2 = 4a(x - \alpha) \qquad \text{... (1)}$$

We have to eliminate $\alpha \; and \; \beta$ from this equation to get the differential equation.

On differentiating (1), with respect to x, we have

$$(y - \beta)\frac{dy}{dx} = 2a \qquad \text{----------(2)}$$

Again differentiating with respect to x, we have

$$(y - \beta)\frac{d^2y}{dx^2} + (\frac{dy}{dx})^2 = 0 \text{-----------(3)}$$

On eliminating$\beta$ between (2) and (3), we have

$$2a\,\frac{d^2y}{dy^2} + (\frac{dy}{dx})^2 = 0.$$

This is the required differential equation.

**Equations of the First Order and the First Degree:**

Type-A:Variables Separable

Consider an equation is of the from,

$$f(x)\,dx + F(y)\,dy = 0.$$

We can directly integrate this equation, and the solution is:

$$\int f(x)dx + \int F(y)dy = c, \quad \text{where c is an arbitrary constant.}$$

Problems:

1. Solve: $\frac{dy}{dx} + \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = 0.$

Solution:

Given that, $\frac{dy}{dx} + \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$ $\quad\Rightarrow dy\left(\sqrt{1-x^2}\right) = -dx(\sqrt{1-y^2})$

$\Rightarrow \frac{dy}{\sqrt{1-y^2}} = -\frac{dx}{\sqrt{1-x^2}}$

$\Rightarrow \frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0$

On integrating, we have

$\sin^{-1} y + \sin^{-1} x = c.$

2. Solve: $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$

Solution:

Given that, $ydx - xdy + 3x^2y^2e^{x^3}dx = 0$

On dividing by, $y^2$ we have

$\frac{ydx - xdy}{y^2} + 3x^2e^{x^3}dx = 0$

We know that, $d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}$.

Therefore, we have $d\left(\frac{x}{y}\right) + d(e^{x^3}) = 0$

On integrating, we have, $\frac{x}{y} + e^{x^3} = c.$

3.The stress p in thick cylinders is given by $r\frac{dp}{dr} + 2p = 2c$,where c is a constant. Find p in terms of r.

Solution:

Given that, $r\frac{dp}{dr} = 2(c-p)$

$\Rightarrow \frac{dp}{c-p} = \frac{2dr}{r}$

On integrating, we have

$- \, log(c-p) = 2\, log\, r - log\, A$, where A is an arbitrary constant.

$\Rightarrow \log(c-p) + 2\, log\, r - log\, A = 0 \qquad \Rightarrow \log(c-p)r^2 - log\, A = 0$

$\Rightarrow \log(c-p)r^2 = log\, A \qquad \Rightarrow e^{\, log\,(c-p)r^2} = e^{log\, A}$.

$\therefore (c\ -\ p)r^2$= A is the solution.

Type B:Homogeneous Equations

Consider, $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$,where $f_1$ *and* $f_2$ are homogeneous functions of the same degree in x and y.

Therefore,$f_1(x,y)$ can be written as $x^n \varphi\left(\frac{y}{x}\right)$and $f_2(x,y)$as$x^n\ \varphi\left(\frac{y}{x}\right)$, if n is the degree of homogeneity.

Put $y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$.

Therefore, the given equation becomes $v + x\frac{dv}{dx} = \frac{\varphi(v)}{\psi(v)}$.

The variables can be separated as: $\frac{dx}{x} + \frac{\psi(v)dv}{v\psi(v)-\varphi(v)} = 0$

On integrating, we have$logx + \int \frac{\psi(v)dv}{v\psi(v)-\varphi(v)} = c$.

The solution is obtained by substituting $v = \frac{y}{x}$ , after the integration is carried out.

**Problems:**

1.Solve $y^2 + x^2\frac{dy}{dx} = \ xy\frac{dy}{dx}$ .

Solution:

Let $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ .

The variables can be separated by rewriting the equation as: $\frac{dv(1-v)}{v} + \frac{dx}{x} = 0$.

On integrating, we have $\log v - v + \log x = \log c$ .

$\Rightarrow vx = ce^v \qquad ...(1)$

We have $y = vx \quad \Rightarrow \quad v = \frac{y}{x}$

$\therefore (1) \Rightarrow y = ce^{\frac{y}{x}}$, which is the solution.

2.Solve: $xdy - ydx = \sqrt{x^2 + y^2}dx$

Solution:

Put $y = vx \Rightarrow dy = vdx + xdv$

The variables can be separated by rewriting the equation as: $\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$

On integrating, we have $\log(v + \sqrt{1 + v^2}) = \log x + \log c$

$\Rightarrow \left(v + \sqrt{1 + v^2}\right) = xc$ ... (1)

We have $y = vx \Rightarrow v = \frac{y}{x}$

$\therefore (1) \Rightarrow y + \sqrt{x^2 + y^2} = cx^2$

This can be reduced to the form $c^2x^2 = 1 + 2cy$, which is the solution.

<u>Type C:Non-homogeneous equations of the first degree in x and y</u>

Consider, $(ax + by + c)\frac{dy}{dx} = Ax + By + C$,

where a ,b ,c, A,B,C are constants.

Put $x = X + h$ and $y = Y + k$ ,the given equation becomes:

$(aX + bY + ah + bk + c)\frac{dY}{dX} = AX + BY + Ah + Bk + C$

If $h, k$ be chosen to satisfy, $a\,h + b\,k + c = 0$------(1)

and $A\,h + B\,k + C = 0$ , --------(2)

the above equation reduces to $(aX + bY)\frac{dY}{dX} = AX + BY$.

This is homogeneous in X and Y and can be solved by putting $Y = v\,X$.

**Problems:**

1.Solve $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$ .

Solution:

Given that, $\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3}$…………..(1)

Put $x = X + h\ ;\ y = Y + k$

The equation becomes $\frac{dY}{dX} = \frac{X+2Y}{Y+2X}$…………..(2)

We know that, $h, k$ be chosen to satisfy,

$$a\,h + \, b\,k\, + c = 0 \text{and}\ \ A\,h + \, B\,k + \, C = 0.$$

(1) $\Rightarrow h + 2k - 3 = 0\ and\ 2h + k - 3 = 0.$

On solving these equations, we have $h = 1\ and\ k = 1.$

Hence $x = X + 1\ and\ y = Y + 1$

To solve (2),put $Y = vX.$

The variables can be separated by rewriting the equation as: $\frac{dv\,(2+v)}{1-v^2} = \frac{dX}{X}$

$\Rightarrow \left(\frac{1}{2}\frac{1}{1+v} + \frac{3}{2}\frac{1}{1-v}\right) dv = \frac{dX}{X}$

On integrating, we have$1 + v = \ C^2X^2(1 - v)^3$

Hence, $X + Y = \ C^2(X - Y)^3.$

$\therefore\ x + y - 2 = \ C^2(x - y)^3$, is the solution.

2.Solve $(2x - 4y + 3)\ + \ \frac{dy}{dx}\ + (x - 2y + 1) = 0.$

Solution:

Given that,$(2x - 4y + 3)\ + \ \frac{dy}{dx}\ + (x - 2y + 1) = 0$

Put $x - 2y = v$, the equation becomes

$(4v + 5)\ dx - (2v + 3)dv = 0$

$\Rightarrow dx - \ \frac{2v+3}{4v+5} dv = 0$

$\Rightarrow dx\ = \left(\frac{1}{2} + \frac{1}{2}\frac{1}{4v+5}\right) dv$

On integrating, we have $\frac{v}{2} + \frac{1}{8}\log(4v + 5) = x + c$

$\therefore$The solution is: $\log\{4(x-2y)+5\} = 4(x+2y)+C$.

3.Solve $(2x-y+5)dy \;=\; (x-2y+3)dx$.

Solution:

Given that, $(2x-y+5)dy \;=\; (x-2y+3)dx$

$\Longrightarrow 2xdy + 2ydx + (-y+5)dy \;=\; (x+3)dx$

$\Longrightarrow 2(xdy+ydx) + (-y+5)dy = (x+3)dx$

On integrating, we have $2xy\; - \frac{y^2}{2} + 5y = \frac{x^2}{2} + 3x + c$.

<u>Type D:Linear Equation</u>

**Definition:**

A differential equation is said to be linear when the dependent variable and its derivatives occur only in the $1^{st}$ degree and no products of these occur. The linear equation of the $1^{st}$ order is of the form:

$$\frac{dy}{dx} + Py = Q, \ldots (1)$$

where P and Q are functions of x only.

Consider, $\frac{dy}{dx} + Py = 0$, that is,$\frac{dy}{y} + Pdx = 0$.

$\therefore$The solution is,$ye^{\int Pdx} \;= C$,

On differentiating, we have$e^{\int Pdx}\left(\frac{dy}{dx} + Py\right) = 0$.

This shows that $\frac{dy}{dx} + Py = 0$renders an exact differential equation. That is, the differential equation which can be deduced from its primitive by mere differentiation and no further operation, on multiplying it by the factor, $e^{\int Pdx}$.Such a factor as this is called an ‘Integrating Factor’.

Thus, an integrating factor is one which changes a differential equation into an exact differential equation.

From the above, we can say that $e^{\int Pdx}$ is an integrating factor of(1).

On multiplying (1) by $e^{\int Pdx}$,we have

$$\left(\frac{dy}{dx} + Py\right) e^{\int Pdx} = Qe^{\int Pdx}.$$

On integrating, we have

$$ye^{\int Pdx} = \int Qe^{\int Pdx} dx + c.$$

This is the solution of (1).

Problems:

1. Solve $\frac{dy}{dx} + ycosx = \frac{1}{2}sin\ 2x$

Solution:

Let $P = cosx$; $Q = \frac{1}{2}sin2x$.

$\Longrightarrow \int Pdx = sinx \qquad \Longrightarrow e^{\int pdx} = e^{sinx}$.

We know that, the solution is, $ye^{\int Pdx} = \int Qe^{\int Pdx} dx + c$

$\Longrightarrow ye^{sinx} = \int \frac{1}{2} sin2xe^{sinx}\ dx + c$

$= e^{\int pdx} sinx\ cosx\ dx + c$

$= \int e^z zdz + c$, (on putting $z = sin\ x$)

$= e^z(z-1) + c$

$\therefore ye^{sinx} = e^{sinx}(sinx - 1) + c.$

2. Solve $x\ \frac{dy}{dx} + y\ log\ x = e^x x^{1-\frac{1}{2}log\ x}$.

Solution:

Given that, $x\ \frac{dy}{dx} + y\ log\ x = e^x x^{1-\frac{1}{2}log\ x}$

$\Longrightarrow \frac{dy}{dx} + \frac{y}{x}\log x = e^x x^{-1/2 \log x}$

Here, $P = \frac{\log x}{x}$; $Q = e^x x^{-1/2 \log x}$

$\Longrightarrow \int Pdx = \int \frac{\log x\ dx}{x} = \frac{(\log x)^2}{2}$

$\Longrightarrow e^{\int P\,dx} = e^{\frac{(x\log x)^2}{2}} = (e^{\log x})^{\frac{\log x}{2}} = x^{\frac{\log x}{2}}.$

We know that, the solution is, $ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c.$

$\Longrightarrow y\,x^{\frac{\log x}{2}} = \int e^x x^{-1/2\log x}\, x^{1/2\log x}dx + c = \int e^x\,dx + c = e^x + c$

$\therefore y\,x^{\frac{\log x}{2}} = e^x + c.$

3. Solve (1-$x^2$)$\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$, given that $y = 0$when $x = 0$.

Solution:

Given that, $(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$

Let$P = \frac{2x}{1-x^2}$ ; $Q = \frac{x}{\sqrt{1-x^2}}$

$\Longrightarrow\int Pdx = \int\frac{2x\,dx}{1-x^2}$ = - log $(1-x^2)$ = log $(1-x^2)^{-1}$

$\Longrightarrow e^{\int P\,dx} = e^{\log(1-x^2)^{-1}} = (1-x^2)^{-1}$

We know that, the solution is, $ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c$

$\Longrightarrow\frac{y}{1-x^2} = \int\frac{x\,dx}{(1-x^2)^{3/2}} + c$

$= \int\frac{\sin\theta\,d\theta}{\cos^2\theta} + c$ [put $x = \sin\theta$]

$= \int\tan\theta\sec\theta\,d\theta + c$

$= \sec\theta + c$

$\Longrightarrow\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + c.$

Giventhat, $x = 0 \Longrightarrow y = 0.$

On substituting these values in the above equation, we have $0 = \frac{1}{1} + c \Longrightarrow c = -1$

$\therefore \frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} - 1.$

**Bernoulli'sEquation :**

Consider$\frac{dy}{dx} + Py = Qy^n$, where P and Q are functions of x only. This can be reduced to the linear form, as follows:

On dividing by $y^n$, we have $y^{-n}\frac{dy}{dx} + Py^{1-n} = Q$.

Put $z = y^{1-n}$, this equation reduces to $\frac{dz}{dx} + Pz(1-n) = Q(1-n)$.

This being linear in Z, can be integrated by the method of linear equation. Hence, y can be obtained.

Problems:

**1. Solve $\frac{dy}{dx} - y\ tanx = \frac{sinx\ cos^2x}{y^2}$.**

**Solution:**

Given that, $\frac{dy}{dx} - y\ tanx = \frac{sinx\ cos^2x}{y^2}$.

Multiplying by$y^2$, *we have* $y^2\frac{dy}{dx} - y^3 tanx = sinx\ cos^2x$

Put$z = y^3 \Rightarrow \frac{dz}{dx} - 3z\ tanx = 3\ sinx\ cos^2\ x$.

Here,$P = -3\ tanx\ ; Q = 3\ sinx\ cos^2\ x$.

$\therefore \int Pdx = 3\log cosx = \log cos^3\ x$.

$\Rightarrow e^{\int Pdz} = cos^3x$.

We know that, the solution is, $ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c$

$\Rightarrow zcos^3x = 3\int \sin x\ cos^5x\ dx + c = \frac{-cos^6x}{2} + c$

$$\therefore\ y^3cos^3x = \frac{-cos^6x}{2} + c.$$

**2. Solve $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}$.**

**Solution:**

Given that, $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}$

Multiplying by $e^y$ and putting $z = e^y$, we have

$$(x+1)\frac{dz}{dx} + z = 2$$

$\Rightarrow \frac{dz}{dx} + \frac{z}{x+1} = \frac{2}{x+1}$

Here, $P = \frac{1}{x+1}$ ; $Q = \frac{2}{x+1}$

$\Rightarrow \int P dx = \log(x+1)$

$\Rightarrow e^{\int P dx} = e^{\log(x+1)} = x+1$

We know that, the solution is, $ye^{\int P dx} = \int Q e^{\int P dx} dx + c$

$\Rightarrow z(x+1) = \int \frac{2}{x+1}(x+1)dx + c = 2x + c$

$\therefore\ e^y(x+1) = 2x + c$

**Equations of the First Order, but of Higher Degree:**

Type A: Equations solvable for $\frac{dy}{dx}$.

We shall denote $\frac{dy}{dx}$ here after by 'p'.

Let the equation of the first order and of $n^{th}$ degree in 'p' be,

$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_n = 0,$ … (1)

where, $P_1, P_2, \cdots, P_n$ denote functions of x and y.

Suppose the first number of (1) can be resolved into factors of the first degree of the form

$(p - R_1)(p - R_2)(p - R_3) \ldots (p - R_n)$.

Any Relation between $x$ and $y$ which makes any of these factors vanish, is a solution of (1).

Let the primitives of $p - R_1 = 0$, $p - R_2 = 0$, etc., be

$$\phi_1(x, y, c_1) = 0, \phi_2(x, y, c_2) = 0, \ldots, \phi_n(x, y, c_n) = 0, \text{ respectively.}$$

Here, $c_1, c_2, \ldots, c_n$ are arbitrary constants.

Without any loss of generality, we can replace $c_1, c_2, \ldots, c_n$ by c, where c is an arbitrary constant.

Hence, the solution of (1) is

$$\phi_1(x,y,c).\phi_2(x,y,c)\ldots\ \phi_n(x,y,c) = 0.$$

**Problems:**

**1. Solve $x^2p^2 + 3xyp + 2y^2 = 0$.**

**Solution:**

Given that, $x^2p^2 + 3xyp + 2y^2 = 0$.

Solving for p, we have $p = \frac{-3xy \pm \sqrt{(3xy)^2 - 4(x^2)(2y^2)}}{2x^2}$

$\Longrightarrow p = \frac{-y}{x} \quad or \ = \frac{-2y}{x}$.

Case(i): $p = \frac{dy}{dx} = -\frac{y}{x}$

On integrating, we have $\phi_1(x,y,c) = xy = c$ …………….(1)

Case(ii): $p = \frac{dy}{dx} = -\frac{2y}{x}$

On integrating, we have $\phi_2(x,y,c) = yx^2 = c$ …………….(2)

We know that, the solution shall be obtained in the following form:

$$\phi_1(x,y,c).\phi_2(x,y,c)\ldots\ \phi_n(x,y,c) = 0$$

∴The solution is, $(xy - c)(yx^2 - c) = 0$.

**2. Solve $p^2 + \left(x + y - \frac{2y}{x}\right)p + xy + \frac{y^2}{x^2} - y - \frac{y^2}{x} = 0$**

**Solution:**

Given that, $p^2 + \left(x + y - \frac{2y}{x}\right)p + xy + \frac{y^2}{x^2} - y - \frac{y^2}{x} = 0$

Solving for p, we have p $= \frac{y}{x} - y \ \ (or) \frac{y}{x} - x$

$\Longrightarrow p = \frac{dy}{dx} = \frac{y}{x} - y \quad (or) \ p = \frac{dy}{dx} = \frac{y}{x} - x$

$\Longrightarrow \frac{dy}{y} = \left(\frac{1}{x} - 1\right)dx \ \ (or) \frac{dy}{dx} - \frac{y}{x} = -x$

Case (i): $\log \frac{y}{x} = -x + log\ C \Longrightarrow \phi_1(x,y,c) = y = c\ xe^{-x}$.

Case (ii): We know that, $ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c$

Here, $P = -\frac{1}{x}$ ; $Q = -x$

$\therefore y\, e^{-\int \frac{dx}{x}} = - \int x e^{-\int \frac{dx}{x}}\, dx + c$

$\Rightarrow \frac{y}{x} = -x + c$

$\Rightarrow \phi_2(x, y, c) = y = -x^2 + cx.$

We know that, the solution shall be obtained in the following form:

$$\phi_1(x, y, c).\, \phi_2(x, y, c) \ldots\ \phi_n(x, y, c) = 0$$

$\therefore$The general solution is, $(y - c\, x e^{-x})(y + x^2 - cx) = 0.$

Type B:

Let the differential equation, $\int f(x)dx + \int F(y)dy = c$, be put in the form $f(x, y, p) = 0$. When it can not be resolved into rational linear factors, namely,

$(p - R_1)\ (p - R_2)\ (p - R_3)\ \ldots\ (p - R_n)$, it may be solved for y or x.

(i) Equations solvable for y

Let $f(x, y, p) = 0$can be put in the form, $y = F(x, p)$ …(1)

On differentiating with respect to x, we have

$$p = \varphi\left(x, p, \frac{dp}{dx}\right).$$

This is an equation in two variables p and x. It can be integrated by any of the known methods. Hence, we obtain

$$\psi(x, p, c) = 0 \qquad \text{...(2)}$$

On eliminating p between (1) and (2), the solution is obtained.

(ii) Equations solvable for x

Let $f(x, y, p) = 0$can be put in the form, $x = F(y, p)$ …(1)

On differentiating with respect to y, we have

$$\frac{1}{p} = \varphi\left(y, p, \frac{dp}{dy}\right).$$

On integrating, we have

$$\psi(y, p, c) = 0 \qquad \ldots(2)$$

On eliminating pbetween (1) and (2), the solution of (1) is obtained.

**Problems:**

**1.Solve $x^2p^2 - 2yp + x = 0$.**

**Solution:**

Given that, $x^2p^2 - 2yp + x = 0 \qquad \ldots (1)$

Solving for y, we have $y = x\frac{(p^2+1)}{2p}$

On differentiating with respect to x, we have

$$p = \frac{p^2+1}{2p} + x\frac{p^2+1}{2p^2}.\frac{dp}{dx}$$

$\Longrightarrow \frac{p^2-1}{2p} = \frac{dp}{dx}x\frac{(p^2-1)}{2p^2}$

$$\therefore \frac{dx}{x} = \frac{dp}{p}$$

On integrating, we have $p = cx \qquad \ldots (2)$

On eliminating p between (1) and (2), we have

$2cy = c^2x^2 + 1$,which is the solution.

**2. Solve $x = y^2 + log\, p$**

**Solution:**

Given that, $x = y^2 + log\, p \qquad \ldots(1)$

On differentiating with respect to y, we have $\frac{1}{p} = 2y + \frac{dp}{p\, dy}$

$\Longrightarrow \frac{dp}{dy} + 2py = 1$.This equation is linear in p.

We know that, $ye^{\int Pdx} = \int Qe^{\int Pdx}dx + c$

Here, $P = y\,;\ Q = 1$.

Hence, p$e^{y^2} = \int e^{y^2}\,dy + C$ …………(2)

Here, the integral on R.H.S., cannot be evaluated in finite number of terms. Hence, an explicit solution cannot be obtained by eliminating p.

However, it is enough to express x and y in terms of the parameter p.

**Clairaut's form:**

The following form of the equation is known as Clairaut's form:

$$y = px + f(p) \qquad ...(1)$$

On differentiating with respect to x, we have

$$p = p + \{x + f'(p)\}\frac{dp}{dx}.$$

$\Rightarrow \frac{dp}{dx} = 0$ or $x + f'(p) = 0$.

i.e., $\frac{dp}{dx} = 0 \quad \Rightarrow p = c$, a constant.

∴ The solution of (1) is: $y = cx + f(c) \qquad ...(2)$

We have to replace p in clairaut's equation by c.

A solution of (1) is obtained by eliminating p, from the other factor $x + f'(p) = 0$, along with (1). But this solution is not included in the general solution (2). Such a solution as this is called a 'Singular Solution'.

**Problems:**

**1. Solve $y = (x - a)p - p^2$.**

**Solution:**

The given equation is, $y = (x - a)p - p^2$, is in Clairaut's form.

We know that, the solution for the Clairaut's form is: $y = px + f(p)$ is, $y = cx + f(c)$.

Therefore, we put $p = c$, then we have $y = (x - a)c - c^2$, which is the solution.

**2. Solve $y = 2px + y^2p^2$**

**Solution:**

**We know that, $y = 2px + y^2p^2$.**

Put $X = 2x \Rightarrow dX = 2dx$ ;

and$Y = y^2 \Longrightarrow dY = 2ydy$ ;

$$\therefore P = \frac{dY}{dX} = yp$$

The equation is transformed into $Y = XP + P^2$,which is the Clairaut's form.

We know that, the solution for the Clairaut's form,$y = px + f(p)$, is given by

$$y = cx + f(c).$$

Therefore, we put $P = c$, then we have $Y = cX + c^2$.

Hence, the solution is $y^2 = 2xc + c^2$.

**Extended Form of Clairaut's Equation:**

The extended form of Clairaut's Equation is of the Type

$$xf(p) + \varphi(p) \ldots (1)$$

On differentiating with respect to x, we have

$$p = f(p) + [xf'(p) + \varphi'(p)]\frac{dp}{dx}$$

$$\Longrightarrow \frac{dx}{dp} + \frac{xf'(p)}{f(p)-p} = \frac{\varphi'(p)}{p-f(p)} \qquad \ldots\ldots\ldots\ldots.(2)$$

This is linear in x and hence gives $F(x, p, c) = 0$

On eliminating p between (2) and (1) give the solution of (1).

**Problem:**

**Solve**$\boldsymbol{y = xp + x(1 + p^2)^{1/2}}$

**Solution:**

On differentiating the given equation, $\boldsymbol{y = xp + x(1 + p^2)^{1/2}}$, with respect to x, we have

$$p = p + (1 + p^2)^{1/2} + \frac{dp}{dx}\left[x + \frac{xp}{\sqrt{1+p^2}}\right], \qquad \ldots \quad (1)$$

whence, $dp\frac{\left(\sqrt{1+p^2}+p\right)}{(1+p^2)} + \frac{dx}{x} = 0.$

On integrating, we have$\int \frac{dp}{\sqrt{1+p^2}} + \int \frac{pdp}{1+p^2} + \int \frac{dx}{x} = \log c$

$\Rightarrow \log\left(p+\sqrt{1+p^2}\right)+\frac{1}{2}\log(1+p^2)+logx=\log c$

$\Rightarrow \log\left[\left(p\sqrt{1+p^2}\right)+(1+p^2)\right)x\right]=\log c$

$\Rightarrow \left\{p\sqrt{1+p^2}+(1+p^2)\right\}x=c$ ... (2)

The solution is obtained by eliminating ‘p’ between (1) and (2).

**Linear Differential Equations of Second Order with Constant Coefficients:**

Definition:

A Linear differential equation of the second order with constant coefficients is given by

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=X, \ldots (1)$$

where a,b,c are constants and X is a function of x.

Let $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0.$ … (2)

The solution of this equation (2) is called the “Complementary Function” of (1).

To solve (2), assume as a trial solution, $y=e^{mx}$, for some value of m.

Now, $\frac{dy}{dx}=me^{mx}$ and $\frac{d^2y}{dx^2}=m^2e^{mx}$.

On substituting these values in (2), we get

$$e^{mx}(am^2+bm+c)=0 \quad \ldots\ldots\ldots.(3)$$

Hence, m satisfies $am^2+bm+c=0$.

The equation (3) is called the “Auxiliary Equation” and it is a function of m. The solution of an auxiliary equation can arise in three cases.

**Case (i):**

Let the auxiliary equation (3) has two real and distinct roots $m_1$ and $m_2$.

$\therefore y=e^{m_1x}$ and $y=e^{m_2x}$ are the two solutions of (2) .

Hence, $Ae^{m_1x}$ and $Be^{m_2x}$ are solutions of (2), where A and B are arbitrary constants.

Thus, $y=Ae^{m_1x}+Be^{m_2x}$ is the most general solution of (2). Here, the number of constants occurring in this solution is two, which is equal to the order of the differential equation.

**Case (ii):**

Let the auxiliary equation (3) has two roots equal and real.

Let $m_1 = m_2$ . The solution is: $y = Ae^{m_1x} + Be^{m_2x}$

$\Longrightarrow (A+B)e^{m_1x} = c\, e^{m_1x}$, ...(4)

where $c$ is a single arbitrary constant equal to A + B.

Thus, the number of constants is reduced to one, which is one short of the order of the differential equation (2). Therefore, (4) ceases to represent the general solution.

Hence, we proceed as follows:

Let us put $m_2 = m_1 + \in$ and allow $\in$ to tend to zero.

The solution is, $y = Ae^{m_1x} + Be^{(m_1 + \in)x}$

$$= e^{m_1x}(A + Be^{\in x})$$

$$= e^{m_1x}\left[A + B\left(1 + \in x + \frac{\in^2 x^2}{2} + \cdots\right)\right] \text{ [by the exponential theorem]}$$

$$= e^{m_1x}(A + B + \in Bx)\text{, the other terms tending to zero as } \in \to 0.$$

We can choose $B$ sufficiently large, so as to make $(\in B)$ finite, as $\in \to 0$ and A large with opposite sign to B, so that A + B is finite.

If A + B = C and $(\in B) = D$, the solution corresponding to two equal roots, $m_1$ , is

$$e^{m_1x}(C + Dx).$$

**Case (iii):**

Let the auxiliary equation (3) has imaginary roots.

As imaginary roots occur in pairs, let $m_1 = \alpha + i\beta$, where $\alpha$ and $\beta$ are real.Then, $m_2 = \alpha - i\beta$.

The solution is, $y = Ae^{(\alpha + i\beta)x} + \text{B}\, e^{(\alpha - i\beta)x} = e^{\alpha x}\left[Ae^{(i\beta)x} + \text{B}\, e^{-(i\beta)x}\right]$

$$= e^{\alpha x}\{A(\cos\beta x + i\sin\beta x) + B(\cos\beta x - i\sin\beta x)\} \quad \text{[by Euler's formula]}$$

$$= e^{\alpha x}[C\ \cos\beta x + D\sin\beta x],\ \text{ where C and D are arbitrary constants.}$$

This can also be written as,

$$y = Ae^{\alpha x}\cos(\ \beta x + B),$$

where A and B are arbitrary constants.

**Problems:**

1. Solve $\frac{d^2y}{dx^2} + 5\ \frac{dy}{dx} + 4y = 0.$

**Solution:**

The given auxiliary equation is,

$$m^2 + 5m + 4 = 0.$$

On solving, we have$m = 1\ and\ 4.$

Here, the roots are real and distinct, that is,$m_1\ = 1\ and\ m_2 =\ 4.$

We know that, the solution is given by, $y = Ae^{m_1 x} + Be^{m_2 x}.$

$$\therefore y = Ae^{x} + Be^{4x}.$$

2. Solve $\frac{d^2y}{dx^2} + 2\ \frac{dy}{dx} + y = 0.$

**Solution:**

The given auxiliary equation is,$m^2 + 2m + 1 = 0.$

On solving, weget, $(m + 1)^2 = 0, m = -1\ twice.$

Here, the roots are real and equal, that is, $m_1\ = m_2 = -1.$

We know that, the solution is given by, $y = e^{m_1 x}(C + Dx).$

$$\therefore y = e^{-x}(C + Dx)\ .$$

3. Solve $\frac{d^2y}{dx^2} - 3\ \frac{dy}{dx} + 5y = 0.$

**Solution:**

The given auxiliary equation is, $m^2 - 3m + 5 = 0.$

On solving, we have= $\frac{3\ \pm\sqrt{9-20}}{2} = \frac{3\ \pm i\sqrt{11}}{2}$ .

Here, the roots are imaginary, that is, $m_1 = \frac{3+i\sqrt{11}}{2}\ and\ \ m_2 = \frac{3\ -i\sqrt{11}}{2}$

We know that, the solution is given by,

$$y = Ae^{\alpha x}\cos(\ \beta x + B),\ \text{where}\alpha = \frac{3}{2}\ \text{and}\ \beta = \frac{\sqrt{11}}{2}\ .$$

$$\therefore y = e^{\frac{3x}{2}}\left\{A\sin\left(\frac{\sqrt{11}}{2}x\right) + B\sin\left(\frac{\sqrt{11}}{2}x\right)\right\}.$$

**LINEAR DIFFERENTIAL EQUATIONS**

**Linear Equations with Variable Coefficients:**

We shall first consider the homogeneous linear equation. A homogeneous linear equation of the second order is of the form:

$$ax^2\frac{d^2y}{dx^2} + bx\ \frac{dx}{dy} + cy = X,\quad \ldots(1)$$

where $a, b, c$ are constants and $X$ is a function of $x$.

**Methods of Solving Linear Equations with Variable Coefficients:**

**Method 1:**

By putting $z = \log x$ or $x = \ e^z$, this equation can be transformed into one with constant coefficients.

We introduce here an operator, $\theta = x\frac{d}{dx}$.

Consider, $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{d}{dz}$ ; $\ x = \frac{dy}{dz} = Dy$, where $D$ stands for $\frac{d}{dz}$ .

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2z}{dx^2}\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right) = \frac{D}{x^2}(D-1)y$$

$$\therefore x^2\frac{d^2y}{dx^2} = D(D-1)y,$$

where $D = \frac{d}{dz} = x\frac{d}{dx} = \theta.$

Hence, putting $x = e^z$ in (1), the equation (1) becomes

$$\{aD(D-1) + bD + c\}y = Z, \qquad \ldots.(2)$$

where $Z$ is a function of $z$ into which $X$ has been transformed. This equation (2) is a linear equation with constant coefficients and hence the foregoing method can be adopted.

**Method 2:**

Without transforming (1) into a linear equation with constant coefficients, an independent method may be given.

To find the complementary function of (1), we have to solve

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0.$$

If $x^m$, for some value of $m$, be taken as a tentative solution, then on substitution, we get

$$am(m-1) + bm + c = 0.$$

This, being an equation of the second degree in $m$, has two roots $m_1, m_2$. Hence, the 'Complementary Function' (C.F.) of (1) is $C_1x^{m_1} + C_2x^{m_2}$, taking the two roots to be distinct.

If however, a root $m_1$ be repeated twice, then putting $m_2 = m_1 + \epsilon$, where $\epsilon \to 0$, the corresponding C.F., is

$$x^{m_1}(C_1 + C_2\, x\, \epsilon) = x^{m_1}(C_1 + C_2 e^{\epsilon \log x})$$

$$= x^{m_1}\left\{C_1 + C_2\left(1 + \epsilon \log x + \frac{\epsilon^2(\log x)^2}{2}\ etc.,\right)\right\}.$$

Here, $\epsilon^2$ being neglected,as $\epsilon \to 0$. Putting $(C_2\,\epsilon) = B$ and $C_1 + C_2 = A$, the part of the C.F., arising from the two equal roots $m_1$ is: $x^{m_1}(A + B\log x)$.

**Equations Reducible to the Linear Homogeneous Equation:**

Consider an equation of the form:

$$(a+bx)^2\frac{d^2y}{dx^2} + (a+bx)p_1\frac{dy}{dx} + p_2y = X,$$

where $p_1, p_2$ are constants and $X$ is any function of $x$.

Putting $z = a + b\,x$, the equation transforms into

$$z^2 \frac{d^2y}{dz^2} + \frac{p_1 z\, dy}{b\, dz} + \frac{p_2}{b^2} y = \frac{1}{b^2} X\left(\frac{z-a}{b}\right).$$

This is a linear homogeneous equation and can be solved by the method outlined in the linear equations with variable coefficients.

**Example:**

Solve $(5+2x)^2 \frac{d^2y}{dx^2} - 6(5+2x)\frac{dy}{dx} + 8y = 6x.$

**Solution:**

Putting $z = 5 + 2x$, the given equation becomes

$$4z^2 \frac{d^2y}{dz^2} - 12z\frac{dy}{dz} + 8y = 3(z-5).$$

Putting $u = \log z$ and$D = \frac{d}{du}$, the equation is now transformed into

$$(4D^2 - 16D + 8)y = 3(e^u - 5).$$

The auxiliary equation is: $m^2 - 4m + 2 = 0.$

$$\therefore m = 2 \pm \sqrt{2}$$

$$\therefore C.F. = e^{2u}\left(Ae^{\sqrt{2}u} + Be^{-\sqrt{2}u}\right)$$

$$= z^2\left(Az^{\sqrt{2}} + B\,z^{-\sqrt{2}}\right)$$

$$= (5+2x)^2\left\{A(5+2x)^{\sqrt{2}} + B(5+2x)^{-\sqrt{2}}\right\}.$$

The particular integral is: $P.I. = \frac{3(e^u - 5)}{4(D^2 - 4D + 2)}$

$$= -\frac{3}{4}e^u - \frac{15}{8}$$

$$= -\frac{3}{4}z - \frac{15}{8} = -\frac{3}{2}x - \frac{45}{8}.$$

$$\therefore y = C.F. + P.I.$$

**The Laplace Transforms:**

The Laplace transform has been widely adopted by scientists and engineers as an efficient tool for solving linear differential equations. Here, we shall discuss the fundamental properties of Laplace transforms, Inverse Laplace transforms and we shall see how they are used to solve linear differential equations.

**Definition:**

If a function $f(t)$ is defined for all positive values of the variable $t$ and if $\int_0^\infty e^{-st} f(t)\,dt$ exists and is equal to $F(s)$, then $F(s)$ is called the Laplace transform of $f(t)$ and is denoted by the symbol $L\{f(t)\}$.

That is, $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$.

The operator L that transforms $f(t)$ into $F(s)$ is called the "Laplace Transform Operator".

**Result:** $\text{Lim}_{s\to\infty} F(s) = 0$.

**Piecewise Continuity:**

A function $f(t)$ is said to be piecewise continuous in a closed interval $[a, b]$ if itis defined on that interval [a,b]and is such that the interval can be broken up into a finite number of sub-intervals in each of which$f(t)$ can have only ordinary finite discontinuities in the interval.

**Exponential Order:**

A function $f(t)$is said to be of exponential order if $\lim_{s\to\infty} e^{-st} f(t) = 0$, or if for some number. $S_0$, the product $|f(t)| < M$ for $t > T$. i.e., $e^{-s_0 t}|f(t)|$ is bounded for large value of t, say for $t > T$.

**Sufficient conditions for the existence of the Laplace Transform:**

1. $f(t)$ is continuous or piecewise continuous in the closed interval $[a, b]$, where $a > 0$.
2. It is of exponential order.
3. $t^n f(t)$is bounded near $t = 0$ for some number $n > 1$.

**Properties of Laplace Transforms:**

i) $L[f(t)+\ \varphi(t)] = L[f(t)] + L[\varphi(t)]$

ii) $L[cf(t)] = cL[f(t)]$, where ‘c’ is constant.

iii) $L[f'(t)] = s\,L[f(t)] - f(0)$

iv) $L[f''(t)] = \ s^2L[f(t)] - sf(0) - sf'(0)$

In general,

$$L[f^n(t)] = \ s^n L[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{n-1}(0)$$

v) If $L[f(t)] = F(s), then$

a) $\lim_{t\to 0} f(t) = \ \lim_{s\to\infty} sF(s)$ [Initial value theorem]

b) $\lim_{t\to\infty} f(t) = \ \lim_{s\to 0} sF(s)$ [Final value theorem]

vi) $L[e^{-at}] = \frac{1}{s+a};\ provided s + a > 0.$

Similarly,$L[e^{at}] = \frac{1}{s-a};\ provided s - a > 0.$

**Corollary**:$L(\cosh at) = L\left(\frac{e^{at}+e^{-at}}{2}\right) \ = \frac{s}{s^2-a^2}$

Similarly, $L(\sinh at) = \ \frac{a}{s^2-a^2}$

vii) $L(\cos at) = \ \frac{s}{s^2+a^2}$

viii) $(L \sin at) = \ \frac{a}{(s^2+a^2)}.$

ix) $L(t^n) = \ \frac{\Gamma(n+1)}{s^{n+1}}$

$$n = 0 \ \Longrightarrow \quad L(1) = \ \frac{1}{s}$$

$$n = 1 \ \Longrightarrow \quad L(t) = \ \frac{1}{s^2}$$

$$n = 2 \ \Longrightarrow \quad L(t^2) = \ \frac{2}{s^3}$$

$$n = \frac{1}{2} \ \Longrightarrow \quad L\left(t^{\frac{1}{2}}\right) = \ \frac{\sqrt{\pi}}{2\,s^{\frac{3}{2}}}$$

$$n = -\frac{1}{2} \ \Longrightarrow \quad L\left(t^{-\frac{1}{2}}\right) = \frac{\sqrt{\pi}}{2\,s^{\frac{1}{2}}}$$

**Problems:**

1. Find $L(t^2+2t+3)$.

Solution:

Consider, $L(t^2+2t+3) = L(t^2) + 2\,L(t) + 3L(1)$

We know that, $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$.

$\therefore L(t^2) + 2\,L(t) + 3L(1) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}$.

2. Find $L(\sin^2 2t)$

Solution:

We know that, $\sin^2 2t = \left(\frac{1-\cos\,4t}{2}\right)$.

$\Longrightarrow\;\; L(\sin^2 2t) = L\left(\frac{1-\cos\,4t}{2}\right)$

We know that, $L(\cos at) = \frac{s}{s^2+a^2}$ and $L(1) = \frac{1}{s}$

$\therefore L(\sin^2 2t) = \frac{1}{2}L(1) - \frac{1}{2}L(\cos 4t)$

$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+4^2} = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+16}\right)$

$\Longrightarrow L(\sin^2 2t) = \frac{8}{s(s^2+16)}$.

3. Find $L(\sin^3 2t)$

Solution:

We know that, $\sin 6t = 3\;\sin 2t - 4\sin^3 2t$.

We know that, $(L\sin at) = \frac{a}{(s^2+a^2)}$.

$\therefore L(\sin^3 2t) = L\left(\frac{3\sin\,2t - \sin\,6t}{4}\right)$

$= \frac{3}{4}L(\sin 2t) - \frac{1}{4}L(\sin 6t)$

$= \frac{3}{4}\frac{2}{(s^2+2^2)} - \frac{1}{4}\frac{6}{(s^2+6^2)}$

$\therefore L(\sin^3 2t) \ = \frac{48}{(s^2+4)(s^2+36)}$

4.Find $L(f(t))$, where$f(t) = \begin{cases} 0, \text{when} \ \ 0 < t \leq 2 \ , \\ 3, \qquad \text{when} \ t > 2 \ . \end{cases}$

Solution:

We know that, $L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$

$$= \int_0^2 e^{-st} f(t) dt + \int_2^{\infty} e^{-st} f(t) dt = \int_0^2 e^{-st}(0) dt + \int_2^{\infty} e^{-st}(3) dt$$

$$= 3\int_2^{\infty} e^{-st} dt = \frac{3}{s} e^{-2s}.$$

**Some General Theorems:**

**Theorem 1:**If $L\{f(t)\} \ = \ F(s)$, then $L\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$

For example, (1) $L(cos\ at) \ = \frac{1}{a} F\left(\frac{s}{a}\right) = \frac{s}{s^2+a^2}$

(2) $L(sinh\ at) \ = \frac{1}{a} F\left(\frac{s}{a}\right) = \frac{a}{s^2-a^2}$

**Theorem 2:** $L\{e^{-at}\ f(t)\} \ = \ F(s+a), where\ F(s) = L\{f(t)\}$

**Problems:**

We have,

(1) $L(1) = \frac{1}{s} \Longrightarrow \mathrm{L}(e^{-\mathrm{at}}) = \frac{1}{\mathrm{s+a}}$.

(2) L(cosbt) $= \frac{s}{s^2+b^2}$

$\Longrightarrow \ L(e^{-at} \cos bt) \ = \frac{\mathrm{s+a}}{(\mathrm{s+a})^2+b^2}$.

(3) $L(sin\ bt) \ = \frac{b}{(s-a)^2+b^2}$

$\Longrightarrow \ L(e^{at} \sin bt) = \frac{\mathrm{b}}{(\mathrm{s-a})^2+b^2}$.

(4) $L(t^n) \ = \frac{n!}{s^{n+1}}$, if n is a positive integer.

$\Longrightarrow \ L(e^{-at} t^n) \ = \frac{\mathrm{n!}}{(\mathrm{s+a})^{\mathrm{n+1}}}$

and$L(e^{at} t^n) \ = \frac{\mathrm{n!}}{(\mathrm{s-a})^{\mathrm{n+1}}}$.

**Theorem 3:** If $L\{f(t)\} = F(s)$ , then $L\{tF(t)\} = -\frac{d}{ds}F(s)$.

**Corollary:** $\therefore L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} L\{f(t)\}$

In general, $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} [L\{f(t)\}]$

This result can also be written as follows:

If $L[f(t)] = F(s)$, then,

$F'(s) = L[-tf(t)]$ ,

$F''(s) = L[(-t)^2 f(t)]$,

.
.
.

$F^n(s) = L[(-t)^n f(t)]$.

Problems:

1.Find $L(t\, e^{-at})$

Solution:

We know that, $L\{tf(t)\} = -\frac{d}{ds}F(s)$

$\therefore L(t.e^{-at}) = -\frac{d}{ds}L(e^{-at})$

$= -\frac{d}{ds}\frac{1}{s+a} = \frac{1}{(s+a)^2}$.

2. Find $L(t^2 e^{-3t})$

Solution:

We know that, $L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2} L\{f(t)\}$

$\therefore L(t^2 e^{-3t}) = (-1)^2 \frac{d^2}{ds^2} L(e^{-3t})$

$= \frac{d^2}{ds^2}\frac{1}{s+3} = \frac{2}{(s+3)^2}$ .

3. Find $L(tsinat)$

Solution:

We know that, $L\{tf(t)\} \quad = \quad -\frac{d}{ds}F(s)$

$\therefore L(tsinat) = -\frac{d}{ds}L(sinat)$

$$= -\frac{d}{ds}\frac{a}{a^2+s^2} = \frac{2as}{a^2+s^2}.$$

4. Find $L(te^{-t}sint)$

Solution:

We know that,$L\{tf(t)\} \quad = \quad -\frac{d}{ds}F(s)$

$$L(t\, e^{-t}\, sin\, t) = -\frac{d}{ds}L(e^{-t}\, sin\, t)$$

$$= -\frac{d}{ds}\, F(s+1)$$

where, $F(s) = \; L(sin\, t) = \frac{1}{1+s^2}$.

$\therefore L(t\, e^{-t}\, sin\, t) = -\frac{d}{ds}\frac{1}{1+(s+1)^2} = \frac{2(s+1)}{(s^2+2as+2)^2}$.

**Theorem 4:** If$L\{f(t)\} \; = \; F(s)$ and if$\frac{f(t)}{t}$ has a limit as $t \to \; 0$

Then, $L\;\{\frac{f(t)}{t}\} = \; \int_s^{\infty} F\,(s)\, ds.$

Problems:

1. Find $L\left(\frac{1-e^t}{t}\right)$.

Solution:

We know that,, $L\;\{\frac{f(t)}{t}\} = \; \int_s^{\infty} F\,(s)\, ds$

$\therefore L\left(\frac{1-e^t}{t}\right) = \; \int_s^{\infty} L(\,1-e^t)\, dt$

Since, $\lim_{t\to 0}\frac{1-e^t}{t} = -1$

$\Rightarrow \; L\left(\frac{1-e^t}{t}\right) = \int_s^{\infty}\left(\frac{1}{s}-\frac{1}{s-1}\right)\, dt = \left[\log\frac{s}{s-1}\right]_s^{\infty} = \log\frac{s-1}{s}.$

2. Find $L\left(\frac{\sin at}{t}\right)$

Solution:

We know that, , $L\ \{\frac{f(t)}{t}\} = \int_s^{\infty} F\ (s)\ ds$

$$\therefore L\left(\frac{\sin at}{t}\right) = \int_s^{\infty} L(\sin at)\ ds.$$

Since, $\lim_{t\to 0}\frac{\sin at}{t} = a$, we have

$$L\left(\frac{\sin at}{t}\right) = \int_s^{\infty}\frac{a}{a^2+s^2}\ ds = tan^{-1}\infty - tan^{-1}\frac{s}{a} = \frac{\pi}{2} - tan^{-1}\frac{s}{a}$$

$\therefore L\left(\frac{\sin at}{t}\right) = cot^{-1}\frac{s}{a}$ .

## The Inverse Laplace Transforms:

**Definition:**

Let the Laplace transform of the function $f(t)$ is given by, $L\{f(t)\} = F(s)$.Then, the symbol $L^{-1}\{F(s)\}$ denotes the "Inverse Laplace Transform" of the function$F(s)$.

That is, $L^{-1}\{L[f(t)]\} = f(t)$.

## List of Certain Functions, $f(t)$, and its Laplace Transform, F(s):

The inverse transform of a given function can be obtained from the second column of the following table associated with the Laplace transform in the last column.

| Sl. No. | f(t) | F(s) |
|---|---|---|
| 1 | $e^{at}$ | 1/s-a |
| 2 | cosh at | $s/s^2-a^2$ |
| 3 | sinh at | $a/s^2-a^2$ |
| 4 | cos at | $s/s^2+a^2$ |
| 5 | sin at | $s/s^2+a^2$ |
| 6 | 1 | 1/s |
| 7 | t | $1/s^2$ |
| 8 | $t^n$ | $n!/ s^{n+1}$; [n is +ve] |
| 9 | $t\ e^{at}$ | $1/ (s-a)^2$ |
| 10 | $t^2\ e^{at}$ | $2/(s-a)^3$ |
| 11 | $t^n\ e^{at}$ | $n!/ (s-a)^{n+1}$; [n is +ve] |

| | | |
|---|---|---|
| 12 | $e^{-at}$sinbt | $b/(s+a)^2+b^2$ |
| 13 | $e^{-at}$cosbt | $(s+a)/(s+a)^2+b^2$ |
| 14 | t sinat | $2as/(s^2+a^2)^2$ |
| 15 | t cos at | $s^2-a^2/(s^2+a^2)^2$ |

We can modify the results that we have obtained in finding the Laplace transforms of functions to get the inverse transforms of functions.

**Theorem 1:**

If L{f(t)}=F(s), then L{$e^{-at}$f(t)} = F(s+a).

Hence, we get $L^{-1}${F(s+a)} = $e^{-at}$ $L^{-1}$ F(s).

Problems:

1. $L^{-1}\left(\frac{1}{(s+a)^2}\right) = e^{-at}L^{-1}\left(\frac{1}{s^2}\right) = e^{-at}\ t.$

2. $L^{-1}\left(\frac{1}{(s+2)^2+16}\right) = e^{-2t}L^{-1}\left(\frac{1}{s^2+4^2}\right) = e^{-2t}\frac{\sin 4t}{4}.$

3. $L^{-1}\left(\frac{s-3}{(s-3)^2+4}\right) = e^{3t}L^{-1}\left(\frac{s}{s^2+2^2}\right) = e^{3t}\cos 2t.$

4. $L^{-1}\left(\frac{s}{s+2s+5}\right) = L^{-1}\left(\frac{s}{(s+1)^2+2^2}\right)$

$$= L^{-1}\left(\frac{(s+1)-1}{(s+1)^2+2^2}\right)$$

$$= L^{-1}\left(\frac{s+1}{(s+1)^2+2^2}\right) - L^{-1}\left(\frac{1}{(s+1)^2+2^2}\right)$$

$$= e^{-t}L^{-1}\left(\frac{s}{s^2+2^2}\right) - e^{-t}L^{-1}\left(\frac{1}{s^2+2^2}\right)$$

$$= e^{-t}\cos 2t - e^{-t}\frac{\sin 2t}{2}$$

$\therefore L^{-1}\left(\frac{s}{s+2s+5}\right) = \frac{e^{-t}}{2}(2\cos 2t - \sin 2t).$

**Theorem 2:**

If$L[f(t)] = F(s)$, then $L[f(at)] = \frac{1}{a} F(\frac{s}{a})$.

This result can be written in the form:

$L^{-1}\left[\frac{1}{a} F\left(\frac{s}{a}\right)\right] = f(at)$, where $f(t) = L^{-1}[F(s)]$.

Put$\frac{1}{a} = k$, we have,

$L^{-1}[F(ks)] = \frac{1}{k} f\left(\frac{t}{k}\right)$, where $f(t) = L^{-1}[F(s)]$.

Problems:

1. Find $L^{-1}\left(\frac{s}{s^2a^2+b^2}\right)$.

Solution:

Consider, $\frac{s}{s^2a^2+b^2} = \frac{1}{a}\frac{sa}{s^2a^2+b^2} = \frac{1}{a} F(sa)$,

where, $F(sa) = \frac{sa}{s^2a^2+b^2} \Longrightarrow F(s) = \frac{s}{s^2+b^2}$.

$\therefore L^{-1}\left(\frac{s}{s^2a^2+b^2}\right) = L^{-1}\left(\frac{s}{s^2a^2+b^2}\right)$

$$= \frac{1}{a} L^{-1}[F(as)] = \frac{1}{a}.\frac{1}{a} f\left(\frac{t}{a}\right),$$

where, $f(t) = L^{-1}[F(s)] = L^{-1}\left(\frac{s}{s^2+b^2}\right) = \cos bt$

$$\therefore f\left(\frac{t}{a}\right) = \cos\left(\frac{bt}{a}\right).$$

Hence, $L^{-1}\left(\frac{s}{s^2a^2+b^2}\right) = \frac{1}{a^2}\cos\left(\frac{bt}{a}\right)$.

**Theorem 3:**

$If\ L[f(t)] = F(s), then\ L[t\, f(t)] = -F'(s).$

Hence, we get the result, $L^{-1}[F'(s)] = -t\, f(t) = -t\, L^{-1}[F(s)]$.

Problems:

1. Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Solution:

Consider, $F'(s) = \frac{s}{(s^2+a^2)^2}$

$$\therefore F(s) = \int \frac{s\, ds}{(s^2+a^2)^2} = -\frac{1}{2(s^2+a^2)}$$

We know that, $L^{-1}\,[F'(s)] = -t\, L^{-1}[F(s)]$

$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = -t\, L^{-1}\left[-\frac{1}{2(s^2+a^2)}\right]$

$$= \frac{t}{2} L^{-1}\left(\frac{1}{(s^2+a^2)}\right)$$

$\Longrightarrow L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t}{2a}\sin at$ .

2. Find $L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right]$

Solution:

Consider, $F'(s) = \frac{s+2}{(s^2+4s+5)^2}$

$$\therefore F(s) = -\frac{1}{2\,(s^2+4s+5)}$$

$$\therefore L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = -t\, L^{-1}\left[\frac{-1}{2(s^2+4s+5)}\right]$$

$$= \frac{t}{2} L^{-1}\left[\frac{1}{(s^2+4s+5)}\right]$$

$$= \frac{t}{2} L^{-1}\left(\frac{1}{(s+2)^2+1^2}\right)$$

$$= \frac{t}{2} e^{-2t} L^{-1} \left( \frac{1}{s^2 + 1^2} \right)$$

$\therefore L^{-1} \left[ \frac{s+2}{(s^2+4s+5)^2} \right] = \frac{t}{2} e^{-2t} \frac{\sin t}{2}$.

**Theorem 4:**

If $L\{f(t)\} = F(s), then\ L[t\, f(t)] = -F'(s)$.

This theorem can be used to get inverse transforms of certain functions.

Problem:

1. Find$L^{-1} \left[ \log \frac{s+1}{s-1} \right]$

Solution:

Let $f(t) = L^{-1} \left[ log \frac{s+1}{s-1} \right]$.

Then,$L\{f(t)\} = \log \frac{s+1}{s-1}$

$$\therefore L\{t.f(t)\} = -\frac{d}{ds} \log \frac{s+1}{s-1} = -\frac{d}{ds} [\log(s+1) - \log(s-1)]$$

$$= -\frac{1}{s+1} + \frac{1}{s-1}$$

$$\therefore t.f(t) = L^{-1} \left[ \frac{1}{s-1} \right] - L^{-1} \left[ \frac{1}{s+1} \right] = e^t - e^{-t} = 2 \sinh t.$$

$\therefore f(t) = \frac{2 \sinh t}{t}$.

**Theorem 5:**

The method of partial fractions can be used to find the inverse transform of certain functions.

<u>Problems</u>:

1. Find $L^{-1} \left[ \frac{1}{s(s+1)(s+2)} \right]$.

Solution:

We can split $\frac{1}{s(s+1)(s+2)}$ into partial fractions as: $\frac{1}{2} . \frac{1}{s} - \frac{1}{s+1} + \frac{1}{2} \frac{1}{s+2}$ .

$\therefore L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}\left(L^{-1}\left(\frac{1}{s+2}\right)\right)$

$= \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$

2. Find $L^{-1}\left[\frac{1}{s(s+1)(s^2+2s+2)}\right]$.

Solution:

Splitting into partial fractions, the function can be expressed as:

$\frac{1}{s(s+1)(s^2+2s+2)} = \frac{1}{(s+1)} - \frac{s+1}{(s^2+2s+2)}$

$\therefore L^{-1}\left[\frac{1}{s(s+1)(s^2+2s+2)}\right] = L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{s+1}{s^2+2s+2}\right)$

$= e^{-t} - L^{-1}\left(\frac{s+1}{(s+1)^2+1}\right)$

$= e^{-t} - e^{-t}L^{-1}\left(\frac{s}{s^2+1}\right)$

$= e^{-t} - e^{-t}\cos t$

$= e^{-t}(1 - \cos t).$

3. Find $L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right]$

Solution:

Splitting into partial fractions, the function can be expressed as:

$$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{1}{3}.\frac{(s+2)^2-(s-1)^2}{(s+2)^2(s-1)^2}$$

$= \frac{1}{3}.\left[\frac{1}{(s-1)^2} - \frac{1}{(s+2)^2}\right]$

Hence, $L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right] = \frac{1}{3}.L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3}L^{-1}\left[\frac{1}{(s+2)^2}\right]$

$= \frac{1}{3}(e^t.t) - \frac{1}{3}(e^{-2t}.t)$

$= \frac{t}{3}(e^t - e^{-2t}).$

**Solving Ordinary Differential Equations with Constant Coefficients Using Laplace Transform:**

<u>Problems:</u>

1. Solve the equation $\frac{d^2y}{dt^2}+2\frac{dy}{dx} - 3y = \sin t$, given that y=$\frac{dy}{dt}$ =0,when $t = 0$.

Solution:

The given equation can be written in the form: $y'' + 2y' - 3y = sin\ t$

Applying Laplace transforms to both sides, we have

$$L(y'' + 2y' - 3y) \ = \ L(sin\ t)$$

$$\Rightarrow L(y'') \ + \ 2L(y') \ - \ \ 3L(y) = \frac{1}{s^2+1}$$

We know that, $L[f'(t)] = s\ L[f(t)] - f(0)$

and $L\left[f''(t)\right] = \ s^2 L[f(t)] - sf(0) - sf'(0)$

$\therefore \ s^2(L(y)) - s\ y(0) \ - \ y'(0) + 2[s\ L(y) - \ y(o)] - 3L(y) = \frac{1}{s^2+1}$

Substituting the values of y(0) and y'(0) in the above equation, we have

$s^2\bar{y} + 2s\bar{y} - 3\bar{y} = \frac{1}{s^2+1}$, where $\bar{y} = L(y)$

$\Rightarrow (s^2 + 2s - 3)\bar{y} = \frac{1}{s^2+1}$

$\Rightarrow \bar{y} = \frac{1}{(s^2+2s-3)(s^2+1)} = \frac{1}{(s+3)(s-1)(s^2+1)}$

$\Rightarrow$ y = $L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right]$.

On splitting into partial fractions, we get,

$$y = L^{-1}\left[\frac{{}^{-1}/_{40}}{s+3} + \frac{{}^{1}/_{8}}{s-1} + \frac{{}^{-1}/_{-10}{}^{3} - {}^{1}/_{5}}{s^2+1}\right]$$

$$= {}^{-1}/_{40}\, L^{-1}\left(\frac{1}{s+3}\right) + {}^{1}/_{8}\, L^{-1}\left(\frac{1}{s-1}\right) - {}^{1}/_{10}\, L^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{5} L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= {}^{-1}/_{40}\, e^{-3t} + {}^{1}/_{8}\, e^{t} - {}^{1}/_{10} \cos t - \frac{1}{5}\sin t.$$

2. Show the solution of the differential equation $\frac{d^2y}{dt^2} + 4y = A\sin kt$, which is such that $y = 0$ and $\frac{dy}{dt} = 0$, when $t = 0$, is y = A $\frac{\sin kt - \frac{k}{2}\sin 2t}{4-k^2}$, $if\ k \neq 2$. If $k = 2$, show that $y \ = \ \frac{A\,(\sin 2t - 2t\cos 2t)}{8}$.

Solution:

Given that, $y'' + 4y = A\sin kt$

$$\Rightarrow L(y'') + 4L(y) = AL(sin\,kt)$$

We know that, $L[f'(t)] = s\ L[f(t)] - f(0)$

and $L\left[f''(t)\right] = \ s^2 L[f(t)] - sf(0) - sf'(0)$

$\therefore s^2\bar{y} - sy(0) - y'(0) + 4\bar{y} = A\frac{k}{s^2+k^2}$, where $L(y) = \bar{y}$

Since, $y(0) = 0, y'(0) = 0$, we have

$(s^2 + 4)\bar{y} = A\frac{k}{s^2+k^2}$

$\therefore\ \bar{y} = A\frac{k}{(s^2+4)(s^2+k^2)}$

$\Rightarrow y = Ak\, L^{-1}\left[\frac{1}{(s^2+4)(s^2+k^2)}\right]$

**Case (i)**: $k \neq 2$,

We have $y = Ak\, L^{-1}\left\{\frac{\left[\frac{1}{s^2+4}\right]-\left[\frac{1}{(s^2+k^2)}\right]}{(k^2-4)}\right\}$

$= \frac{Ak}{(k^2-4)}\left\{L^{-1}\left[\frac{1}{s^2+4}\right] - L^{-1}\left[\frac{1}{(s^2+k^2)}\right]\right\}$

$= \frac{Ak}{(k^2-4)}\left[\frac{\sin 2t}{2} - \frac{\sin kt}{k}\right]$

$\therefore y = \frac{Ak}{(4-k^2)}\left(\sin kt - \frac{k}{2}\sin 2t\right)$

**Case(ii)**: $k = 2$.

We have, $y = 2\,AL^{-1}\left[\frac{1}{(s^2+4)(s^2+2^2)}\right]$

$= 2\,A\,L^{-1}\left(\frac{1}{(s^2+2^2)^2}\right)$

$= 2A\,\frac{1}{2(2)^3}(sin\,2t - 2t\,cos\,2t)$

$\therefore y = \frac{A}{8}(sin\,2t - 2t\,cos\,2t)$.

<u>Advantage of the method</u>:

The special advantage of solving differential equations using Laplace Transform is that, the initial conditions are satisfied automatically. It is unnecessary to find the general solution and determine the constants using the initial conditions.

**UNIT – V : THEORY OF EQUATIONS**

*Theory of Equations: Nature of roots, Formulation of equation whose roots are given. Relation between coefficients and roots - Transformation of equations - Reciprocal equations - Horner's method of solving equatios.*

## THEORY OF EQUATIONS

General Algebraic Equation:

Consider the equation,

$$f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_rx^{n-r} + \cdots + p_{n-1}x + p_n = 0,$$

where, the coefficients $p_0, p_1, p_2, \dots, p_n$ are real and $p_0 \neq 0$. This equationis called the 'General Algebraic Equation' of degree 'n'.

Roots of an Algebraic Equation:

We know that, from the fundamental theorem of algebra, every equation of the above type has atleast one root. Let the root be $\alpha_1$. Then, we have $(\alpha_1) = 0$ . Hence, $f(x)$ is exactly divisible by $x - \alpha_1$.

That is, $f(x) = (x - \alpha_1)\,\varphi_1(x)$,

where, $\varphi_1(x)$ is a rational integral function of degree $(n-1)$ of the form$p_0x^{n-1} + \cdots$ .

Since, $\varphi_1(x)$ is also an algebraic equation,$\varphi_1(x)$=0 has a root. Let the root be $\alpha_2$.

Hence, $\varphi_1(x) = (\mathrm{x} - \alpha_2)\varphi_2(x)$,

where,$\varphi_2(x)$is a rational integral function of degree $(n-2)$ of the form $p_0x^{n-2} + \cdots$ .

$$\therefore f(x) = (x - \alpha_1)(x - \alpha_2)\varphi_2(x).$$

By continuing the process, we obtain,

$$f(x) = (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)\varphi_n(x),$$

where, $\varphi_n(x)$is of degree $n - n = 0$. Thus,$\varphi_n(x)$is a constant,that is, $\varphi_n(x) = p_0$.

$$\therefore f(x) = p_0(x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n).$$

Hence, the equation $f(x) = 0$ has '$n$' roots,since, $f(x)$ will vanish, when *x* has any one of the values $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$.

Since,$f(x)$ does not vanish for any other value, the equation$f(x) = 0$ has only n roots. All the roots of $f(x) = 0$, may not be real and they need not all be distinct.

Properties of Roots of an Algebraic Equation:

**(i) Consider an equation with real coefficients. The imaginary roots of such equation will occur in pairs**.

Let the equation be $f(x) = 0$ and let $\alpha + i\beta$ be an imaginary root of the equation.

We shall show that $(\alpha - i\beta)$ is also a root.

We have, $(x - \alpha - i\beta)(x - \alpha + i\beta) = (x-\alpha)^2 + \beta^2$ … (1)

If $f(x)$ is divided by $(x-\alpha)^2 + \beta^2$, let the quotient be P(x) and the remainder be $Qx + R$.

If $f(x)$ is of degree n, then P(x) is of degree $(n-2)$.

$\therefore f(x) = \{(x-\alpha)^2 + \beta^2\}P(x) + Q(x) + R.$ ... (2)

On substituting $x = \alpha + i\beta$ in the equation (1), we get

$$f(\alpha + i\beta) = \{(\alpha + i\beta - \alpha)^2 + \beta^2\}P(\alpha + i\beta) + Q(\alpha + i\beta) + R$$

$$= Q(\alpha + i\beta) + R$$

Given that, $\alpha + i\beta$ is a root of $f(x) = 0$ $\Rightarrow f(\alpha + i\beta) = 0.$

$$\therefore Q(\alpha + i\beta) + R = 0.$$

Equating to zero, the real and imaginary parts, we have

$Q\alpha + R = 0$ and $Q\beta = 0$

Since, $\beta \neq 0$, we have Q= 0 $\Rightarrow$ R=0.

$(2) \Rightarrow f(x) = \{(x-\alpha)^2 + \beta^2\}P(x) + 0 + 0$

$\Rightarrow f(x) = \{(x-\alpha)^2 + \beta^2\}P(x).$

We know that, $f(x) = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)\varphi_n(x),$ … (3)

where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are the roots of the equation $f(x) = 0$.

From (1)& (3), we have $(\alpha - i\beta)$ is also a root of $f(x) = 0$.

Problems:

1. Solve the equation $x^4 - 4x^2 + 8x + 35 = 0$, given that $2 + i\sqrt{3}$ is a root of it.

Solution:

Let the given equation be denoted as $f(x) = x^4 - 4x^2 + 8x + 35 = 0$ …(1)

Here, the coefficients of the equation are real.

Therefore, we know that, "In an equation with real coefficients, the imaginary roots occur in pairs".

Also, given that $2 + i\sqrt{3}$ is a root of the given equation.

$\Rightarrow 2 - i\sqrt{3}$ is also a root of the given equation.

$$\therefore \left[x - \left(2 + i\sqrt{3}\right)\right]\left[x - \left(2 - i\sqrt{3}\right)\right] = (x - 2 - i\sqrt{3})\,(x - 2 + i\sqrt{3})$$

$$= (x-2)^2 + 3$$

$$= x^2 + 4 - 4x + 3$$

$= x^2 - 4x + 7.$

On dividing $f(x)$by $x^2 - 4x + 7$, we have the quotient, $(x^2 + 4x + 5)$.

$$\therefore \; x^4 - 4x^2 + 8x + 35 = (x^2 - 4x + 7)(x^2 + 4x + 5) \qquad \ldots (2)$$

Now, let us find the roots of $(x^2 + 4x + 5)$, using the formula, $x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

$$\therefore x = \frac{-4\pm\sqrt{4^2-20}}{2} = \frac{-4\pm\sqrt{-4}}{2} = \frac{-4\pm 2i}{2} = \frac{2(-2\pm i)}{2}$$

$\Rightarrow x = -2 \pm i.$

Hence,$-2 + i$ , $-2 - i$ are also roots of the given equation.

The given roots are: $2 + i\sqrt{3}, 2 - i\sqrt{3}$.

$\therefore x^4 - 4x^2 + 8x + 35 = \left[x - \left(2 + i\sqrt{3}\right)\right]\left[x - \left(2 - i\sqrt{3}\right)\right][x - (-2 + i)][x - (-2 - i)] = 0.$

2. Form a rational cubic equation, which shall have for roots 2, $3 + \sqrt{-2}$ .

Solution:

Here, let the coefficients of the cubic equation to be formed, is real. Then, the imaginary roots occur in pairs.

Given that,$3 + \sqrt{-2}$ is a root of the equation.

$\Rightarrow 3 - \sqrt{-2}$ is also the root of the equation.

Therefore, the roots of the required equation are:$2, 3 + \sqrt{-2}, 3 - \sqrt{-2}$ .

Hence, the required equation is

$(x-2)\ (x-3-\sqrt{-2})\ (x-3+\sqrt{-2})=0$

$\Rightarrow (x-2)\ ((x-3)^2+2)=0$

$\Rightarrow (x-2)\ (x^2-6x+11)=0$

$\Rightarrow x^3-8x^2+23x-22=0$ … (1)

Therefore, (1) is the required equation.

3. Show that $\frac{a^2}{x-\alpha}+\frac{b^2}{x-\beta}-x+\gamma=0$ has only real roots, if $a, b, \alpha, \beta, \gamma$ are all real.

Solution:

Let us assume that, the given equation has imaginary roots also.Then, in order to prove that the given equation has only real roots, we have to prove that our assumption is wrong.

We know that, "In an equation with real coefficients, the imaginary roots occur in pairs".

That is,$p+iq$ is a root. Then,$p-iq$ is also a root ... (1)

$\frac{a^2}{p+iq-\alpha}+\frac{b^2}{p+iq-\beta}-p-iq+\gamma=0$ … (2)

$\frac{a^2}{p-iq-\alpha}+\frac{b^2}{p-iq-\beta}-p+iq+\gamma=0$ … (3)

$$(3)-(2)\Rightarrow \frac{-2a^2iq}{(p-\alpha)^2+q^2}-\frac{2b^2iq}{(p-\beta)^2+q^2}-2iq=0$$

$\Rightarrow -2iq\{\frac{a^2}{(p-\alpha)^2+q^2}+\frac{b^2}{(p-\beta)^2+q^2}+1\}=0$ … (4)

Equation (4) is true, only if $q=0$, since, the other factors cannot be zero.

On substituting $q=0$, in (1), we have

$$p+i(0)=p \quad and\ p-i(0)=p$$

That is, the imaginary part, q, of our assumed root $p\pm iq$, is zero.

$\Rightarrow$Our assumption that the given equation has imaginary roots is wrong.

Therefore, the given equation has only real roots.

4.Solve the equation,$x^4+4x^3+5x^2+2x-2=0$, which has a root $-1+\sqrt{-1}$ .

Solution:

Let the given equation be denoted as $f(x)=x^4+4x^3+5x^2+2x-2=0$ …(1)

Here, the coefficients of the equation are real.

Therefore, we know that, "In an equation with real coefficients, the imaginary roots occur in pairs".

Also, given that $-1+\sqrt{-1}= -1+i$ is a root of the given equation.

$\Rightarrow -1-i$ is also a root of the given equation.

$\therefore (x+1-i)(x+1+i)=(x+1)^2-i^2$

$=x^2+2x+1+1$

$=x^2+2x+2$

On dividing $f(x)$by $x^2+2x+2$, we have the quotient, $(x^2+2x-1)$.

$$\therefore (x^4+4x^3+5x^2+2x-2)=(x^2+2x+2)(x^2+2x-1) \quad \ldots(2)$$

Now, let us find the roots of $(x^2+4x+5)$, using the formula: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$.

$$\therefore x=\frac{-2\pm\sqrt{4-4(-1)(1)}}{2}=\frac{-2\pm\sqrt{8}}{2}=-1\pm\sqrt{2}.$$

$\Rightarrow x=-1\pm\sqrt{2}$are also roots of the given equation.

Therefore, the roots of the given equation are: $-1\pm i$and $-1\pm\sqrt{2}$.

**(ii) Consider an equation with rational coefficients. The irrational roots of such equation will occur in pairs**.

**Problems:**

1.Solve the equation, $x^4-6x^3+11x^2-10x+2=0$, given that $2+\sqrt{3}$ is a root of the equation.

Solution:

Given that, $f(x)=x^4-6x^3+11x^2-10x+2=0$ … (1)

Here, the coefficients of the equation are rational.

We know that, "In an equation with rational coefficients, the irrational roots occur in pairs".

Also given that, $2+\sqrt{3}$ is a root. Therefore, $2-\sqrt{3}$is also a root.

Consider, $\left(x-2-\sqrt{3}\right)\left(x-2+\sqrt{3}\right) \;= (x-2)^2-3$

$$= x^2+4-4x-3$$

$$= \; x^2-4x+1$$

On dividing $f(x)$by $x^2-4x+1$, we have the quotient, $(x^2-2x+2)$.

$\therefore x^4-6x^3+11x^2-10x+2=(x^2-4x+1)(x^2-2x+2)$

Now, let us find the roots of $(x^2-2x+2)$, using the formula: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$$\therefore x=\frac{2\pm\sqrt{4-8}}{2}=\frac{2\pm\sqrt{-4}}{2}=\frac{2\pm 2i}{2}=1\pm i$$

$\Rightarrow 1\pm i$ are also roots of the given equation.

Therefore, the roots of the given equation are: $2\pm\sqrt{3}$and $1\pm i$.

2.Form the equation with rational coefficients whose roots are $1+\sqrt{2},\;3$

Solution:

Let the coefficients of the equation to be formed, is rational. Then, the irrationalroots occur in pairs.

Given that,$1+\sqrt{2}$ is a root of the equation.

$\Rightarrow 1-\sqrt{2}$ is also the root of the equation.

Therefore, the roots of the required equation are: $1+\sqrt{2}, 1-\sqrt{2},\; 3$.

Hence, the required equation is

$$\left(x-1-\sqrt{2}\right)\left(x-1+\sqrt{2}\right)(x-3)=0$$

$$\Rightarrow \quad \{(x-1)^2-2\}(x-3)=0$$

$$\Rightarrow \quad (x^2-2x-1)(x-3)=0$$

$$\Rightarrow \quad x^3-5x^2+5x+3=0 \qquad \ldots(1)$$

Therefore, (1) is the required equation.

3.Solve the equation, $x^4 - 2x^3 - 22x^2 + 62x - 15 = 0$, given that one of the root is $2 + \sqrt{3}$.

Solution:

Given that, $f(x) = x^4 - 2x^3 - 22x^2 + 62x - 15 = 0$ … (1)

Here, the coefficients of the equation are rational.

We know that, "In an equation with rational coefficients, the irrational roots occur in pairs".

Also given that, $2 + \sqrt{3}$ is a root. Therefore, $2 - \sqrt{3}$is also a root.

Consider, $(x - 2 - \sqrt{3})(x - 2 + \sqrt{3}) = (x - 2)^2 - 3$

$$= x^2 + 4 - 4x - 3$$

$$= x^2 - 4x + 1.$$

On dividing $f(x)$by $x^2 - 4x + 1$, we have the quotient, $(x^2 - 2x - 15)$.

$\therefore x^4 - 6x^3 + 11x^2 - 10x + 2 = (x^2 - 4x + 1)(x^2 - 2x - 15)$

Now, let us find the roots of $(x^2 - 2x - 15) = 0$.

Consider, $x^2 + 2x - 15 = 0 \implies (x + 5)(x - 3) = 0 \implies x = -5 \; or \; 3$.

$\implies -5 \; and \; 3$ are also roots of the given equation.

Therefore, the roots of the given equation are: $2 \pm \sqrt{3}$, $-5$ and 3.

**(iii) Relation between the Coefficients and the Roots of an Algebraic Equation**

(1) Consider the cubic equation, $x^3 + px^2 + qx + r = 0$.

Let$\alpha, \beta, \gamma$ are the roots of the equation.

$\therefore x^3 + px^2 + qx + r = (x - \alpha)(x - \beta)(x - \gamma)$

$$= x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$$

Equating the coefficients of $x^2$, we have $\alpha + \beta + \gamma = -p$ …(1a)

Equating the coefficients of $x$, we have $\alpha\beta + \beta\gamma + \gamma\alpha = q$ …(1b)

Equating the constants, we have $\alpha\beta\gamma = -r$ …(1c)

The equations (1a), (1b) and (1c) gives the relation between the coefficients, $p, q, r$ and the roots $\alpha, \beta, \gamma$.

(2) Consider the bi - quadratic equation, $x^4 + px^3 + qx^2 + rx + s = 0$

Let $\alpha, \beta, \gamma, \delta$ are the roots of the equation.

$\therefore x^4 + px^3 + qx^2 + rx + s = (x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$

$$= x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)x^2 - (\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma)x + \alpha\beta\gamma\delta$$

Equating the coefficients of $x^3$, we have $\alpha + \beta + \gamma + \delta = -p$ … (2a)

Equating the coefficients of $x^2$, we have $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$ … (2b)

Equating the coefficients of $x$, we have $\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma = -r$ … (2c)

Equating the constants, we have $\alpha\beta\gamma\delta = s$ … (2d)

The equations (2a), (2b), (2c) and (2d) gives the relation between the coefficients, $p, q, r, s$ and the roots $\alpha, \beta, \gamma, \delta$.

Problems:

1. If $\alpha, \beta, \gamma$ are the roots of the cubic equation, $x^3 + px^2 + qx + r = 0$, find the values of

(i) $\alpha^2 + \beta^2 + \gamma^2$

(ii) $\alpha^2\beta + \alpha\beta^2 + \beta\gamma^2 + \beta^2\gamma + \gamma\alpha^2 + \gamma^2\alpha$

(iii) $(\beta + \gamma - 3\alpha)(\gamma + \alpha - 3\beta)(\alpha + \beta - 3\gamma)$

Solution:

Given that, $\alpha, \beta, \gamma$ are the roots of the equation, $x^3 + px^2 + qx + r = 0$ .

We know that, the relationship between the coefficients and roots of the equation are:

$$\alpha + \beta + \gamma = -p$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = q$$

$$\alpha\beta\gamma = -r$$

(i) Consider, $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$$= (-p)^2 - 2(q) = p^2 - 2q$$

$\therefore \alpha^2 + \beta^2 + \gamma^2 = p^2 - 2q.$

(ii)Consider, $\alpha^2\beta + \alpha\beta^2 + \beta\gamma^2 + \beta^2\gamma + \gamma\alpha^2 + \gamma^2\alpha$

$$= \alpha\beta(\alpha+\beta) + \beta\gamma(\beta+\gamma) + \gamma\alpha(\alpha+\gamma)$$

$$= \alpha\beta(\alpha+\beta+\gamma-\gamma) + \beta\gamma(\alpha+\beta+\gamma-\alpha) + \gamma\alpha(\alpha+\beta+\gamma-\beta)$$

$$= \alpha\beta(-p-\gamma) + \beta\gamma(-p-\alpha) + \gamma\alpha(-p-\beta)$$

$$= -p(\alpha\beta+\beta\gamma+\gamma\alpha) - 3\alpha\beta\gamma = -pq + 3r$$

$$\therefore \alpha^2\beta + \alpha\beta^2 + \beta\gamma^2 + \beta^2\gamma + \gamma\alpha^2 + \gamma^2\alpha = -pq + 3r.$$

(iii) Consider,$(\beta+\gamma-3\alpha)(\gamma+\alpha-3\beta)(\alpha+\beta-3\gamma)$

$$= (\alpha+\beta+\gamma-4\alpha)(\alpha+\beta+\gamma-4\beta)(\alpha+\beta+\gamma-4\gamma)$$

$$= (p-4\alpha)(-p-4\beta)(-p-4\gamma)$$

$$= -\{p^3 + 4p^2(\alpha+\beta+\gamma) + 16p(\alpha\beta+\beta\gamma+\gamma\alpha) + 64\,\alpha\beta\gamma\}$$

$$= -\{p^3 + 4p^2(-p) + 16pq - 64r\}$$

$$= 3p^3 - 16pq + 64r$$

$$\therefore (\beta+\gamma-3\alpha)(\gamma+\alpha-3\beta)(\alpha+\beta-3\gamma) = 3p^3 - 16pq + 64r.$$

2. If $\alpha, \beta, \gamma$ be the roots of the equation $x^3 + px^2 + qx + r = 0$, form the equation whose roots are $\alpha\beta, \beta\gamma$ and $\gamma\alpha$

Solution:

The given equation is, $x^3 + px^2 + qx + r = 0$.

We know that, the relation between the coefficients, $p, q, r$ and the roots, $\alpha, \beta, \gamma$ are:

$\alpha + \beta + \gamma = -p$ … (1)

$\alpha\beta + \beta\gamma + \gamma\alpha = q$ … (2)

$\alpha\beta\gamma = -r$ … (3)

Let$S_1$ be the sum of the roots of an equation taken one at a time.

Let$S_2$ be the sum of the roots of an equation taken two at a time.

Let$S_3$ be the sum of the roots of an equation taken three at a time.

Then, the corresponding equation with the coefficients $S_1, S_2, S_3$is obtained by using (1), (2) and (3) as:

$$x^3 - S_1x^2 + S_2x - S_3 = 0 \qquad \text{... (4)}$$

Given that, $\alpha\beta, \beta\gamma\ and\ \gamma\alpha$ arethe roots of the equation (4).

Using (1), we have from (4), $S_1 = \alpha\beta + \beta\gamma + \gamma\alpha.$

But, from (2), we have $S_1 = \alpha\beta + \beta\gamma + \gamma\alpha = q$

Using (2), we have from (4), $S_2 = (\alpha\beta)(\beta\gamma) + (\beta\gamma)(\gamma\alpha) + (\alpha\beta)(\gamma\alpha)$

$= \alpha\beta^2\gamma + \alpha\beta\gamma^2 + \alpha^2\beta\gamma$

$$\Rightarrow S_2 = \alpha\beta\gamma(\alpha + \beta + \gamma).$$

But, from (3), we have$\alpha\beta\gamma = -r$

and from (1), we have $(\alpha + \beta + \gamma) = -p$

$$\therefore S_2 = -r(-p) = pr$$

Using (3), we have from (4), $S_3 = (\alpha\beta)(\beta\gamma)(\gamma\alpha)$

$= \alpha^2\beta^2\gamma^2 = (\alpha\beta\gamma)^2$

But, from (3), we have $\alpha\beta\gamma = -r$

$$\Rightarrow \quad S_3 = (-r)^2 = r^2.$$

On substituting the values of $S_1, S_2, S_3$ in (4), we have

$x^3 - (q)x^2 + (pr)x - (r^2) = 0.$

Hence, the required equation is, $x^3 - qx^2 + prx - r^2 = 0.$

3. Find the condition for the roots of the equation $x^3 + px^2 + qx + r = 0$ to be in geometric progression and hence solve the equation $3x^3 - 26x^2 + 52x - 24 = 0$

Solution:

Let the roots of the equation be in geometric progression, namely,$\frac{k}{r}, k\ and\ kr.$

We know that, for the equation, $x^3 + px^2 + qx + r = 0,$ the relation between the coefficients, $p, q, r$ and the roots, $\alpha, \beta, \gamma$ are:

$\alpha + \beta + \gamma = -p \qquad \dots (1)$

$\alpha\beta + \beta\gamma + \gamma\alpha = q$ ... (2)

$\alpha\beta\gamma = -r$ ... (3)

Here, $= \frac{k}{r}$, $\beta = k$ and $\gamma = kr$.

(1) $\Rightarrow \frac{k}{r} + k + kr = -p$ ... (4)

$\Rightarrow k\left(\frac{1}{r} + 1 + r\right) = -p \quad \Rightarrow \left(\frac{1}{r} + 1 + r\right) = -\frac{p}{k}$

(2) $\Rightarrow \frac{k^2}{r} + k^2 r + k^2 = q$ ... (5)

$\Rightarrow k^2\left(\frac{1}{r} + 1 + r\right) = q \quad \Rightarrow k^2\left(-\frac{p}{k}\right) = q \quad \Rightarrow k = \frac{-q}{p}$

(3) $\Rightarrow k^3 = -r$ ... (6)

$\Rightarrow \left(\frac{-q}{p}\right)^3 = -r \qquad \Rightarrow q^3 = p^3 r$

Therefore, the condition for the roots of the equation, $x^3 + px^2 + qx + r = 0$ to be in geometric progression is obtained as: $q^3 = p^3 r$.

It is required to solve the equation, $3x^3 - 26x^2 + 52x - 24 = 0$, using the condition $q^3 = p^3 r$.

Consider, $3x^3 - 26x^2 + 52x - 24 = 0 \qquad \Rightarrow x^3 - \frac{26}{3}x^2 + \frac{52}{3}x - 8 = 0.$

Here, $p = -\frac{26}{3}, \quad q = \frac{52}{3}, \quad r = -8.$

(4) $\Rightarrow \frac{k}{r} + k + kr = \frac{26}{3}$ ...(7)

(5) $\Rightarrow \frac{k^2}{r} + k^2 + k^2 r = \frac{52}{3}$ ... (8)

(6) $\Rightarrow k^3 = 8 \;\Rightarrow\; k = 2$

Using $k = 2$ in (7), we have $\frac{1}{r} + 1 + r = \frac{13}{3}$

$\Rightarrow\; 3r^2 - 10r + 3 = 0$

$\Rightarrow (r - 3)\left(r - \frac{1}{3}\right) = 0$

$\therefore r = 3\ (or)\ \frac{1}{3}.$

Therefore, the roots of the equation are obtained as:

$k = 2 \; and \; r = 3 \Longrightarrow \frac{k}{r} = \frac{2}{3}; k = 2; kr = 2(3) = 6.$

$k = 2 \; and \; r = \frac{1}{3} \Longrightarrow \frac{k}{r} = \frac{2}{\left(\frac{1}{3}\right)} = 2(3) = 6; k = 2; kr = 2\left(\frac{1}{3}\right) = \frac{2}{3}.$

Therefore, the roots are $\frac{2}{3}$, 2,6.

4. Solve the equation, $2x^3 - 11x^2 + 10x + 8 = 0$, given that one of the roots is double of any one of other roots.

Solution:

Let the roots be $\alpha, 2\alpha, \beta$.

We know that, for the equation, $x^3 + px^2 + qx + r = 0$, the relation between the coefficients, $p, q, r$ and the roots, $\alpha, \beta, \gamma$ are:

$\alpha + \beta + \gamma = -p$ … (1)

$\alpha\beta + \beta\gamma + \gamma\alpha = q$ … (2)

$\alpha\beta\gamma = -r$ … (3)

Here, $\alpha = \alpha, \beta = 2\alpha, \gamma = \beta$.

The given equation is, $2x^3 - 11x^2 + 10x + 8 = 0$

$$\Rightarrow x^3 - \frac{11}{2}x^2 + 5x + 4 = 0$$

$$\therefore p = -\frac{11}{2}; q = 5; r = 4$$

$\Rightarrow \alpha + 2\alpha + \beta = \frac{11}{2} \quad \Rightarrow \quad 3\alpha + \beta = \frac{11}{2} \quad \Rightarrow \beta = \frac{11}{2} - 3\alpha$ …(4)

$\Rightarrow \alpha(2\alpha) + (2\alpha)\beta + \beta\alpha = 5 \quad \Rightarrow \quad 2\alpha^2 + 3\alpha\beta = 5$ …(5)

$\Rightarrow \alpha(2\alpha)\beta = -4 \quad \Rightarrow \quad 2\alpha^2\beta = -4$ …(6)

On substituting the value of $\beta$ from (4) in (5), we get

$$2\alpha^2 + 3\alpha\left(\frac{11}{2} - 3\alpha\right) = 5$$

$$\Rightarrow \; 14\,\alpha^2 - 33\alpha + 10 = 0$$

$$\Rightarrow (\alpha - 2)(14\alpha - 5) = 0$$

$\Rightarrow \alpha = 2\ (or) \dfrac{5}{14}$

$when, \alpha = 2\ , we\ have\ from\ (4),\ \beta = -\frac{1}{2}.$

$when, \alpha = \frac{5}{14}\ , we\ have\ from\ (4),\ \beta = \frac{31}{7}\ .$

But, $\alpha = \frac{5}{14}, \beta = \frac{31}{7}$ does not satisfy equation (6).

Therefore, the roots of the equation are: $2; 2\alpha = 4; \beta = -\frac{1}{2},$ that is, $2, 4, -\frac{1}{2}.$

5. Solve the equation,$x^3 - 12x^2 + 32x - 28 = 0$, whose roots are in arithmetical progression.

Solution:

Let the roots of the equation are:$\alpha - k, \alpha, \alpha + k.$

We know that, for the equation, $x^3 + px^2 + qx + r = 0$, the relation between the coefficients, $p, q, r$ and the roots, $\alpha, \beta, \gamma$ are:

$\alpha + \beta + \gamma = -p$ ... (1)

$\alpha\beta + \beta\gamma + \gamma\alpha = q$ ... (2)

$\alpha\beta\gamma = -r$ ... (3)

Here, $\alpha = \alpha - k, \beta = \alpha, \gamma = \alpha + k.$

The given equation is,$x^3 - 12x^2 + 32x - 28 = 0.$

$\Rightarrow p = -12; q = 32; r = -28.$

(1) $\Rightarrow \alpha - k + \alpha + \alpha + k = 12 \Rightarrow 3\alpha = 12 \Rightarrow \alpha = 4$

(2) $\Rightarrow (\alpha - k) + \alpha(\alpha + k) + (\alpha - k)(\alpha + k) = 39$

$\Rightarrow 3\alpha^2 - k^2 = 39 \Rightarrow k^2 = 9 \Rightarrow k = \pm 3$

(3) $\Rightarrow (\alpha - k)\alpha(\alpha + k) = 28.$

Therefore, the roots of the equation are obtained as:

$k = 3\ and\ \alpha = 4 \Rightarrow \alpha - k = 4 - 3 = 1;\ \alpha = 4;\ \alpha + k = 4 + 3 = 7$

$k = -3\ and\ \alpha = 4 \Rightarrow \alpha - k = 4 - (-3) = 7;\ \alpha = 4;\ \alpha + k = 4 - 3 = 1$

Therefore, the roots are:$1, 4, 7.$

6. Solve $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$, given that the sum of two roots is equal to the sum of the other two.

Solution:

Let $\alpha,\ \beta,\ \gamma,\ \delta$ be the roots of the equation, $x^4 - 8x^3 + 14x^2 + 8x - 15 = 0$.

We know that, for the equation, $x^4 + px^3 + qx^2 + rx + s = 0$, the relationship between the coefficients, $p, q, r, s$ and the roots, $\alpha, \beta, \gamma, \delta$ are:

$\alpha + \beta + \gamma + \delta = -p$ … (1)

$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = q$ … (2)

$\alpha\beta\gamma + \alpha\gamma\delta + \beta\gamma\delta + \alpha\beta\delta = -r$ … (3)

$\alpha\beta\gamma\delta = s$ … (4)

From the given equation, we have $p = -8; q = 14; r = 8$ and $s = -15$.

Also, given that, $\alpha + \beta = \gamma + \delta$ … (5)

From (1) and (5), we have $2(\alpha + \beta) = 8$

$\Rightarrow \alpha + \beta = 4$ … (6)

From (2), we have $\alpha\beta + \gamma\delta + (\alpha + \beta)(\gamma + \delta) = 14$

$\Rightarrow \alpha\beta + \gamma\delta + (\alpha + \beta)^2 = 14$

$\Rightarrow \alpha\beta + \gamma\delta = 14 - 16$

$\Rightarrow \alpha\beta + \gamma\delta = -2$ …(7)

From (4) and (7), on factorization, we have two cases:

Case(i): $\alpha\beta = -5,\ \gamma\delta = 3$ (o)r Case(ii): $\alpha\beta = 3,\ \gamma\delta = -5$.

Case(i):

$\alpha + \beta = 4,\ \alpha\beta = -5$ and $\gamma + \delta = 4,\ \gamma\delta = 3$.

$\Rightarrow \alpha = 5,\ \beta = -1,\ \gamma = 3,\ \delta = 1$

Case(ii):

$$\alpha + \beta = 4,\ \alpha\beta = 3 \text{ and } \gamma + \delta = 4,\ \gamma\delta = -5$$

$\Rightarrow \alpha = 3,\ \beta = 1,\ \gamma = 5,\ \delta = -1$.

Therefore, the roots are: $5, -1, 3, 1$.

Transformation of Equations:

An equation can be transformed into another equation. The roots of the transformed equation will bear a certain specified relation with those of the given equation. Such transformation helps us to solve certain type of equations easily.

Below are some of the important elementary transformations of equations:

Transformation – I:

**To transform a given equation into another, whose roots are the roots of the given equation with their signs changed.**

Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be the roots of the equation,

$$f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0.$$

$$\therefore p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = p_0(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$$

Put $x = -y$, then we have

$$P_0(-y)^n + P_1(-y)^{n-1} + \cdots + P_n = P_0(-y-\alpha_1)(-y-\alpha_2)\dots(-y-\alpha_n)$$

$$= (-1)^n P_0(y+\alpha_1)(y+\alpha_2)\dots(y+\alpha_n)$$

Hence, the roots of the equation, $P_0(-y)^n + P_1(-y)^{n-1} + \cdots + P_n = 0$ are:

$$-\alpha_1, -\alpha_2, \dots, -\alpha_n.$$

That is, the roots of the given equation are: $\alpha_1, \alpha_2, \dots, \alpha_n$.

When the transformation, $x = -y$, is used,

the roots of the transformed equation are: $-\alpha_1, -\alpha_2, \dots, -\alpha_n$.

Transformation – II:

**To transform a given equation into another whose roots are the roots of the given equation multiplied by a constant 'm'.**

Let $\alpha_1, \alpha_2, \dots \alpha_n$ be the roots of the given equation,

$$f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0.$$

$$\therefore p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = p_0(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)$$

Put $x = \frac{y}{m}$, then we have

$$P_0\left(\frac{y}{m}\right)^n + P_1\left(\frac{y}{m}\right)^{n-1} + P_2\left(\frac{y}{m}\right)^{n-2} + \cdots + P_{n-1}\left(\frac{y}{m}\right) + P_n$$

$$= P_0\left(\frac{y}{m} - \alpha_1\right)\left(\frac{y}{m} - \alpha_2\right) \dots (\frac{y}{m} - \alpha_n)$$

$$\Longrightarrow P_0\left(\frac{y}{m}\right)^n + P_1\left(\frac{y}{m}\right)^{n-1} + P_2\left(\frac{y}{m}\right)^{n-2} + \cdots + P_{n-1}\left(\frac{y}{m}\right) + P_n$$

$$= \frac{1}{m^n} P_0(y - m\alpha_1)(y - m\alpha_2) \dots (y - m\alpha_n)$$

$$\Longrightarrow p_0 y^n + P_1 m y^{n-1} + P_2 m^2 y^{n-2} + \cdots + P_{n-1} m^{n-1} y + P_n m^n$$

$$= P_0(y - m\alpha_1)(y - m\alpha_2) \dots (y - m\alpha_n)$$

Hence, the roots of the transformed equation are:$m\alpha_1, m\alpha_2, \dots\ m\alpha_n$.

That is, the roots of the given equation are: $\alpha_1, \alpha_2, \dots, \alpha_n$.

When the transformation, $= \frac{y}{m}$ , is used,

the roots of the transformed equation are: $m\alpha_1, m\alpha_2, \dots, m\alpha_n$.

Transformation – III:

**To transform a given equation into another whose roots are the roots of the given equation diminished (or increased) by a constant 'h'.**

Let the roots of the equation,$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$, be $\alpha_1, \alpha_2, \dots, \alpha_n$.

Consider, the transformation,$x = y + h$.

The transformed equation is, $p_0(y + h - \alpha_1)(y + h - \alpha_2) \dots (y + h - \alpha_n) = 0$.

The roots of the transformed equation are: $\alpha_1 - h, \alpha_2 - h, \dots, \alpha_n - h$.

The simplified form of the transformed equation is,

$$a_0 y^n + a_1 y^{n-1} + a_2 y^{n-2} + \cdots + a_n = 0,$$

where$a_0, a_1, a_2, \dots, a_n$ are successive remainders when the left side of the equation is divided by $x - h$and$a_0 = p_0$.

For this purpose, we can write the series of operations as follows:

| | | | | | |
|---|---|---|---|---|---|
| $p_0$ | $p_1$ | $p_2$ | $\cdots$ | $p_{n-1}$ | $p_n$ |
| | $p_0h$ | $b_1h$ | | $b_{n-2}h$ | $b_{n-1}h$ |
| | $b_1$ | $b_2$ | | $b_{n-1}$ | $R = a_n$ |

Problem:

1. Diminish the roots of the equation, $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$ by 1 and hence solve the equation.

Solution:

Given that, $x^4 - 4x^3 - 7x^2 + 22x + 24 = 0$.

Here, $p_0 = 1,\ p_1 = -4, p_2 = -7, p_3 = 22, p_4 = 24$ and $h = 1$

Consider, the transformation, $x = y + h$.

We know that, the transformed equation is,

$$a_0y^4 + a_1y^3 + a_2y^2 + a_3y + a_4 = 0 \qquad \ldots (1)$$

Divide the expression on the left side of the equation by $x - 1$.

We have $a_4, a_3, a_2, a_1$ as the successive remainders, and $a_0 = 1$.

We know that, the series of operations can be written as follows:

| | | | | | |
|---|---|---|---|---|---|
| $p_0$ | $p_1$ | $p_2$ | $\cdots$ | $p_{n-1}$ | $p_n$ |
| | $p_0h$ | $b_1h$ | | $b_{n-2}h$ | $b_{n-1}h$ |
| | $b_1$ | $b_2$ | | $b_{n-1}$ | $R = a_n$ |

That is,

| | | | | |
|---|---|---|---|---|
| 1 | -4 | -7 | 22 | 24 |
| | 1 | -3 | -10 | 12 |
| | -3 | -10 | 12 | 36 |

The quotient is: $x^3 - 3x^2 - 10x + 12$;        The remainder is: 36 = $a_4$.

Again, on dividing the above quotient by $x - 1$, the reminder is $a_3$.

The series of operations can be written as follows:

| | | | |
|---|---|---|---|
| 1 | -3 | -10 | 12 |
| | 1 | -2 | -12 |
| | -2 | -12 | 0 |

The quotient is: $x^2 - 2x - 12$;        The remainder is: 0 = $a_3$.

Again, on dividing the above quotient by $x - 1$, the reminder is $a_2$.

The series of operations can be written as follows:

| | | |
|---|---|---|
| 1 | -2 | -12 |
| | 1 | -1 |
| | -1 | -13 |

The quotient is: $x - 1$.;     The remainder is: $-13 = a_2$.

Again, on dividing the above quotient by $x - 1$, the reminder is $a_1$.

The series of operations can be written as follows:

| | |
|---|---|
| 1 | -1 |
| | 1 |
| | 0 |

The remainder is: $0 = a_1$.

$\therefore a_0 = 1, a_1 = 0, a_2 = -13, a_3 = 0, a_4 = 36.$

From (1), the transformed equation is, $y^4 - 13y^2 + 36 = 0$

The operations can be consolidated as follows:

| | | | | |
|---|---|---|---|---|
| 1 | -4 | -7 | 22 | 24 |
| | 1 | -3 | -10 | 12 |
| | -3 | -10 | 12 | 36 |
| | 1 | -2 | -12 | |
| | -2 | -12 | 0 | |
| | 1 | -1 | | |
| | -1 | -13 | | |
| | 1 | | | |
| | 0 | | | |

The transformed equation is $(1)y^4 + (0)y^3 + (-13)y^2 + (0)y + 36 = 0$

$$\Rightarrow y^4 - 13y^2 + 36 = 0.$$

The roots of the transformed equation is obtained as follows:

Consider, $y^4 - 9y^2 - 4y^2 + 36 = 0$

$\Longrightarrow \quad y^2(y^2 - 9) - 4(y^2 - 9) = 0$

$\Longrightarrow \quad (y^2 - 9)(y^2 - 4) = 0$

$\Longrightarrow \quad (y^2 - 3^2)(y^2 - 2^2) = 0$

$\Longrightarrow (y+3)(y-3)(y+2)(y-2) = 0.$

Hence, the roots of the transformed equation are: $-3, 3, -2, 2$.

The roots of the original equation are obtained as follows:

$\alpha_1 - h = \alpha_1 - 1 = -3 \Longrightarrow \alpha_1 = -3 + 1 = -2;$

$\alpha_2 - h = \alpha_2 - 1 = 3 \Longrightarrow \alpha_2 = 3 + 1 = 4;$

$\alpha_3 - h = \alpha_3 - 1 = -2 \Longrightarrow \alpha_3 = -2 + 1 = -1$

$\alpha_4 - h = \alpha_4 - 1 = 2 \Longrightarrow \alpha_4 = 2 + 1 = 3.$

Therefore, the roots of the original equation are: $\alpha_1 = -2, \alpha_2 = 4, \alpha_3 = -1, \alpha_4 = 3.$

Transformation –IV: (**Reciprocal Equations)**

**1. To transform an equation into another whose roots are the reciprocal of the roots of the given equation.**

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the roots of the equation,

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1} x + p_n = 0.$$

We have,

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1} x + p_n = (x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n).$$

Put $= \frac{1}{y}$, we have

$$\left(\frac{1}{y}\right)^n + P_1 \left(\frac{1}{y}\right)^{n-1} + P_2 \left(\frac{1}{y}\right)^{n-2} + \cdots + P_n = \left(\frac{1}{y} - \alpha_1\right)\left(\frac{1}{y} - \alpha_2\right) \ldots \left(\frac{1}{y} - \alpha_n\right).$$

Multiplying throughout by $y^n$, we have

$$p_n y^n + p_{n-1} y^{n-1} + \cdots + p_1 y + 1 = (1 - \alpha_1 y)(1 - \alpha_2 y) \ldots (1 - \alpha_n y)$$

$$= (\alpha_1 \alpha_2 \ldots \alpha_n)\left(\frac{1}{\alpha_1} - y\right)\left(\frac{1}{\alpha_2} - y\right) \ldots \left(\frac{1}{\alpha_n} - y\right)$$

Hence, the transformed equation is,

$$p_n y^n + p_{n-1} y^{n-1} + \cdots + p_1 y + 1 = 0.$$

The roots of the transformed equation are: $\frac{1}{\alpha_1}, \frac{1}{\alpha_2}, \dots, \frac{1}{\alpha_n}$.

**2. If an equation remains unaltered when it is changed into its reciprocal, it is called a reciprocal equation.**

Let the reciprocal equation be,

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \cdots + p_{n-1} x + p_n = 0. \qquad \dots (1)$$

When $x$ is changed into $\frac{1}{x}$, we get the transformed equation as

$$\left(\frac{1}{x}\right)^n + p_1 \left(\frac{1}{x}\right)^{n-1} + p_2 \left(\frac{1}{x}\right)^{n-2} + \cdots + p_{n-1} \left(\frac{1}{x}\right) + p_n = 0.$$

On multiplying both sides by $x^n$, we have

$$p_n x^n + p_{n-1} x^{n-1} + \cdots + p_1 x + 1 = 0$$

On dividing both sides by $p^n$, we have

$$x^n + \frac{p_{n-1}}{p_n} x^{n-1} + \cdots + \frac{p_1}{p_n} x + \frac{1}{p_n} = 0 \qquad \dots (2)$$

Since (1) is reciprocal equation, it must be the same as (2).

$$\therefore \frac{p_{n-1}}{p_n} = p_1, \frac{p_{n-2}}{p_n} = p_2, \dots, \frac{p_1}{p_n} = p_{n-1}, \frac{1}{p_n} = p_n \qquad \dots (3)$$

Consider, $\frac{1}{p_n} = p_n \Rightarrow {p_n}^2 = 1 \Rightarrow p_n = \pm 1$.

Case(i): $p_n = 1$

(3) $\Rightarrow p_{n-1} = p_1, p_{n-2} = p_2, p_{n-3} = p_3, \cdots$.

In this case, the coefficients of the terms at equidistant from the beginning and the end are equal in magnitude and have the same sign.

Case(ii): $p_n = -1$

(3) $\Rightarrow p_{n-1} = -p_1, p_{n-2} = -p_2, p_{n-3} = -p_3, \cdots$.

In this case, the coefficients of the terms at equidistant from the beginning and the end are equal in magnitude and have the opposite sign.

## 3. Standard Form of a Reciprocal Equation

Let $\alpha_1$ be a root of a reciprocal equation. Then, $\frac{1}{\alpha_1}$ must also be a root, because it is a root of the transformed equation. Also, the transformed equation is identical with the first equation.

Hence, the roots of a reciprocal equation occur in pairs $\left(\alpha_1, \frac{1}{\alpha_1}\right), \left(\alpha_2, \frac{1}{\alpha_2}\right), \dots$ .

When the degree is odd, one of its roots must be its own reciprocal.

$\therefore \frac{1}{\alpha_p} = \alpha_p \quad \Rightarrow {\alpha_p}^2 = 1 \quad \Rightarrow \alpha_p = \pm 1.$

Case(i): If all the coefficients have like signs, then -1 is a root.

Case(ii): If the coefficients of the terms at equidistant from the first and last have opposite signs, then +1 is a root.

In either case, the degree of an equation can be depressed by unity if we divide the equation by $x+1$ or by $x-1$. The depressed equation is always a reciprocal equation of even degree with like signs for its coefficients.

If the degree of a given reciprocal equation is even, say, $n = 2m$ and first and last have opposite signs, then $P_m = -P_m$. That is, $P_m = 0$, so that in this type of reciprocal equations, the middle term is absent. Such an equation may be written as:

$$x^{2m} - 1 + P_1 x(x^{2m-2} - 1) + \cdots = 0.$$

Dividing by $x^2 - 1$, this reduces to a reciprocal equation of like signs of even degree.

Hence, all reciprocal equations may be reduced to an even degree reciprocal equation with like signs.

**Hence, an even degree reciprocal equation, with like signs is considered as the standard form of reciprocal equations.**

That is, the standard form of reciprocal equations is given by

$$x^{2m} - 1 + P_1 x(x^{2m-2} - 1) + \cdots = 0 .$$

Problems:

1. Solve the equation $6x^5 + 11x^4 - 33x^3 - 33x^2 + 11x + 6 = 0$.

Solution:

The given equation is a reciprocal equation of odd degree with like signs.

Hence, $(x+1)$ is a factor of the expression on the left – hand side of the equation.

The equation can be written as follows:

$$6x^5 + 6x^4 + 5x^4 + 5x^3 - 38x^3 - 38x^2 + 5x^2 + 5x + 6x + 6 = 0$$

$\Rightarrow 6x^4(x+1) + 5x^3(x+1) - 38x^2(x+1) + 5x(x+1) + 6(x+1) = 0$

$\Rightarrow (x+1)\,(6x^4 + 5x^3 - 38x^2 + 5x + 6) = 0$

$\therefore (x+1) = 0 \Rightarrow x = -1$ is a root.

Consider the equation, $(6x^4 + 5x^3 - 38x^2 + 5x + 6) = 0$.

To solve the equation, divide the equation by$x^2$. We have,

$\Rightarrow 6\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 38 = 0$ ... (1)

Put $x + \frac{1}{x} = t$ $\Rightarrow x^2 + \frac{1}{x^2} = t^2 - 2$

(1) $\Rightarrow 6(t^2 - 2) + 5t - 38 = 0$

$\Rightarrow 6t^2 + 5t - 50 = 0$

$\Rightarrow (3t + 10)(2t - 5) = 0$

$$\Rightarrow t = \frac{-10}{3} \ (or)\ \frac{5}{2}$$

i.e., $x + \frac{1}{x} = \frac{-10}{3}\ (or)\ x + \frac{1}{x} = \frac{5}{2}$.

$\Rightarrow 3x^2 + 10x + 3 = 0\ (or)\ 2x^2 - 5x + 2 = 0$

$\Rightarrow (3x+1)(x+3) = 0\ (or)\ (2x-1)(x-2) = 0$

$\Rightarrow x = \frac{-1}{3}\ (or)\ -3\ (or)\ \frac{1}{2}\ (or)\ 2$

Hence, the roots of the equation are:$-1, \frac{-1}{3}, -3, \frac{1}{2}, 2$.

2. Solve the equation,$2x^5 - 15x^4 + 37x^3 - 37x^2 + 15x - 2 = 0$.

Solution:

This is a reciprocal equation of odd degree with unlike signs.

Hence,$(x-1)$ is a factor of the expression on the L.H.S., of the equation.

Therefore, the equation can be written as follows:

$$2x^5 - 2x^4 - 13x^4 + 13x^3 + 24x^3 - 24x^2 - 13x^2 + 13x + 2x - 2 = 0$$

$\Rightarrow 2x^4(x-1) + 13x^3(x-1) + 24x^2(x-1) - 13x(x-1) + 2(x-1) = 0$

$\Rightarrow (x-1)\ (2x^4 - 13x^3 + 24x^2 - 13x + 2) = 0.$

Now, let us solve the equation, $(2x^4 - 13x^3 + 24x^2 - 13x + 2) = 0.$

On dividing the equation by $x^2$, we have

$2(x^2 + \frac{1}{x^2}) - 13\ (x + \frac{1}{x}) + 24 = 0$ ... (1)

Put $x + \frac{1}{x} = t \Rightarrow x^2 + \frac{1}{x^2} = t^2 - 2.$

(1) $\Rightarrow 2(t^2 - 2) - 13t + 24 = 0$

$\Rightarrow 2t^2 - 13t + 20 = 0$

$\Rightarrow (t-4)(2t-5) = 0$

$\Rightarrow t = 4\ (or)\ \frac{5}{2}$

$$\Rightarrow x + \frac{1}{x} = 4\ (or)\ x + \frac{1}{x} = \frac{5}{2}$$

$\Rightarrow x^2 - 4x + 1 = 0\ (or)\ 2x^2 - 5x + 2 = 0 \Rightarrow (2x-1)(x-2) = 0.$

$\therefore x = 2 \pm \sqrt{3} (or) \frac{1}{2}\ (or)\ 2.$

Hence, the roots of the given equation are: $1, 2 \pm \sqrt{3}, \frac{1}{2}, 2.$

3. Solve the equation, $3x^6 + x^5 - 27x^4 + 27x^2 - x - 3 = 0.$

Solution:

This is a reciprocal equation of even degree with unlike signs.

Hence, $(x^2 - 1)$ is a factor of the expression on the L.H.S., of the equation.

The equation can be written as follows:

$3(x^6 - 1) + x(x^4 - 1) - 27x^2(x^2 - 1) = 0$

$\Rightarrow 3\ (x^2 - 1)(x^4 + x^2 + 1) + x(x^2 - 1)(x^2 + 1) - 27x^2(x^2 - 1) = 0$

$\Rightarrow (x^2 - 1)\ (3x^4 + x^3 - 24x^2 + x + 3) = 0$

$\Rightarrow (x^2 - 1) = 0$ or $3x^4 + x^3 - 24x^2 + x + 3 = 0$

$\Rightarrow (x^2 - 1) = 0 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$

Let us solve the equation, $3x^4 + x^3 - 24x^2 + x + 3 = 0$.

On dividing the equation by $x^2$, it can be written as,

$\Rightarrow 3\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 24 = 0$ ... (1)

Put $\left(x + \frac{1}{x}\right) = t \Rightarrow x^2 + \frac{1}{x^2} = t^2 - 2$.

(1) $\Rightarrow 3(t^2 - 2) + t - 24 = 0$

$\Rightarrow 3t^2 + t - 30 = 0$

$\Rightarrow 3t^2 - 9t + 10t - 30 = 0$

$\Rightarrow 3t(t - 3) + 10(t - 3) = 0$

$\Rightarrow (3t + 10)(t - 3) = 0$

$$\Rightarrow t = \frac{-10}{3} \ (or) \ 3$$

$$\therefore x + \frac{1}{x} = \frac{-10}{3} \quad (or) \ x + \frac{1}{x} = 3$$

$\Rightarrow 3x^2 + 10x + 3 = 0 \ (or) x^2 - 3x + 1 = 0$

$\Rightarrow x = \frac{-1}{3} (or) \ -3 \ (or) \ \frac{3 \pm \sqrt{5}}{2}$.

Therefore, the roots of the equation are: $\pm 1, \frac{-1}{3}, -3, \frac{3 \pm \sqrt{5}}{2}$ .

Horner's Method:

By this method, we can conveniently find both the commensurable and incommensurable roots of a given equation. The procedure is to determine first the integral part and then the decimal part. If the root be commensurable, then we can find it exactly and in case of incommensurable roots, the value can be determined to any number of places of decimals.

The method of finding the positive roots of $f(x) = 0$ is given below. The negative roots of $f(x) = 0$ is obtained by finding the positive roots of $f(-x) = 0$.

Problem:

1. Find the real positive root of the equation,$x^3 + 24x = 50$ to three places of decimals.

Solution:

The given equation is, $f(x) = x^3 + 24x - 50$.

$\Rightarrow f(0) = 0 + 0 - 50 = -50; \quad f(1) = 1 + 24 - 50 = -25; \quad f(2) = 8 + 48 - 50 = 6.$

We can easily see that for all positive values of $x$ greater than 2, $f(x)$ is positive. Hence, the positive real root lies between 1 and 2.

Therefore, the integral part of the root is 1.

Diminish the roots of the equation by 1.

| | | | |
|---|---|---|---|
| 1 | 0 | 24 | -50 |
| | 1 | 1 | 25 |
| | 1 | 25 | -25 |
| | 2 | 2 | |
| | 2 | 27 | |
| | 1 | | |
| | 3 | | |

Therefore, the transformed equation is $x^3 + 3x^2 + 27x - 25 = 0$.

$\Rightarrow f(0) = -25; \quad f(1) = 6.$

$\Rightarrow$This equation has a root between 0 and 1.

Multiply the roots of the equation by 10.

Then, the transformed equation is,$x^3 + 30x^2 + 2700x - 25000 = 0$

By trail, the root of this equation is found to lie between 8 and 9.

Diminish the roots of the equation by 8. We have

| | | | |
|---|---|---|---|
| 1 | 30 | 2700 | -25000 |
| | 8 | 304 | 24032 |
| | 38 | 3004 | -968 |
| | 8 | 368 | |
| | 46 | 3372 | |
| | 8 | | |
| | 54 | | |

The transformed equation is, $x^3 + 54x^2 + 3372x - 968 = 0$

This equation hasa root between 0 and 1.

Multiply the roots by 10. Then the transformed equation is,

$$x^3 + 540x^2 + 337200x - 968000 = 0.$$

Here, the root lies between 2 and 3.

Diminish the roots by 2. We have

| | | | |
|---|---|---|---|
| 1 | 540 | 337200 | -968000 |
| | 2 | 1084 | 676568 |
| | 542 | 338284 | -291432 |
| | 2 | 1088 | |
| | 544 | 339372 | |
| | 2 | | |
| | 546 | | |

Hence, the transformed equation is,

$x^3 + 546x^2 + 339372x - 291432 = 0.$

Multiply the roots of the equation by 10.

The transformed equation is

$x^3 + 5460x^2 + 33937200x - 291432000 = 0$

This equation has a root between 8 and 9.

Hence, diminish the roots by 8.

| | | | |
|---|---|---|---|
| 1 | 5460 | 3372000 | -291432000 |
| | 8 | 43744 | 271847552 |
| | 5468 | 33980944 | -19584448 |
| | 8 | 43808 | |
| | 5476 | 34024752 | |
| | 2 | | |
| | 5484 | | |

Hence, the transformed equation is, $x^3 + 5484x^2 + 34024752x - 1958448 = 0.$

Multiply the roots by 10.The new transformed equation is,

$$x^3 + 54840x^2 + 3402475200x - 1958448000 \;=\; 0.$$

Here, the root of the equations lies between 1 and 2.It is obtained as 1.8285.

Hence, the root correct to three decimal places is 1.829.

The series of arithmetical operations is represented as follows:

| | | | |
|---|---|---|---|
| 1 | 0 | 24 | -50 |
| | 1 | 1 | 25 |
| | 1 | 25 | -25000 |
| | 1 | 2 | 24032 |
| | 2 | 2700 | |
| | 1 | | -968000 |
| | | 304 | 676568 |
| | 30 | 3004 | |
| | 8 | 368 | -291432000 |
| | 38 | | 271847552 |
| | 8 | 337200 | |
| | | 1084 | -19584448000 |
| | 46 | 338284 | |
| | 8 | 1088 | |
| | 540 | | |
| | 2 | 3372000 | |
| | 542 | 43744 | |
| | 2 | 33980944 | |
| | 544 | 43808 | |
| | 2 | | |
| | 5460 | 3402475200 | |
| | 8 | | |
| | 5468 | | |
| | 8 | | |
| | 5476 | | |
| | 8 | | |
| | 54840 | | |

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