B.Sc. STATISTICS - I YEAR

DJS1C : PROBABILITY THEORY

SYLLABUS

Unit - I

Probability: sample space – Events - algebraic operations on events- definition of probability - independent events – conditional probability - addition and multiplication theorems of probability – Bayes Theorem.

Unit - II

Random variables: Discrete and continuous random variables – distribution function - properties – probability mass function and probability density function – discrete and continuous probability distributions.

Unit - III

Multiple random variables: Joint, marginal and conditional distribution functions - independence of random variables – transformation of random variables (one and two dimensional - concepts only) and their distribution functions.

Unit - IV

Mathematical expectation: Expectation – properties – Cauchy - Schwartz inequality, conditional expectation and conditional variance – theorems on expectation and conditional expectation. Moment generating function, cumulant generating function, characteristic function, probability generating function and their properties. Tchebychev's inequality

Unit - V

Limit Theorems: Convergence in probability, weak law of large numbers – Bernoulli's theorem, Khintchine's theorem (statements only) – Simple form of central limit theorem i.i.d random variables.

REFERENCE BOOKS::

1. Goon, A.M., M. K. Gupta and B. Das Gupta (2002) Fundamentals of Statistics- Vol. I., World Press, Ltd, Kolkata.
2. Gupta, S.C. and V.K. Kapoor (2002) Fundamentals of Mathematical Statistics, Sultan Chand & Sons, New Delhi.
3. Bhat, B.R. (2007) Modern Probability Theory, (third edition), New Age International (P) Ltd, New Delhi.
4. Lipschutz, S. (2008) Probability Theory (Second Edition), Schaum's Outline Series, McGraw Hill, New York.
5. Mood, A.M., F.A. Graybill and D.C. Boes (1974) Introduction to Theory of Statistics McGraw Hill Book Co.,
6. Spiegel, M.R. and Ray, M. (1980) Theory and Problems of Probability and Statistics, Schaum's Outline Series, McGraw Hill, New York.

PROBABILITY THEORY

UNIT – I

1.1 Random Experiment

An experiment is an operation whose output cannot be predicted with certainty. If in each trail of an experiment conducted under identical conditions, the outcome is not unique, but may be any one of the possible outcomes, then such an experiment is called Random Experiment.

1.2 Sample Space

A sample space can be defined as the set of all possible outcomes of an experiment and is denoted by S.The set S = $\{E_1, E_2, E_3,..., E_n\}$ is called a sample space of an experiment satisfying the following two conditions

(i) Each element of the set S denotes one of the possible outcomes

(ii) The outcome is one and only one element of the set S whenever the experiment is performed. For example, in a tossing a coin Sample space consists of head and tail S={H,T} and the two coins are tossed then the sample space S ={HH,HT,TH,TT}.

1.3 Trail and Events

Any particular performance of a random experiment is called trail and the outcome or combinations of outcomes are termed as event.

1.4 Exhaustive Events

The total number of possible outcome of a random experiment is known as the exhaustive events. For example, in a tossing a coin head and tail are the two exhaustive cases. In drawing two cards from a pack of cards, the exhaustive number of cases is $^{52}C_2$, since 2 cards can be drawn out of 52 cards in $^{52}C_2$ ways.

1.5 Favourable Events

The number of cases favourable to an event in a trail is the number of outcomes which entail the happening of the event. For example, in throwing of two dice, the number of cases favourable to getting the sum 5 is (2,3),(3,2),(1,4) and (4,1)

1.6 Mutually Exclusive Events

Events are said to be mutually exclusive or incompatible if the happening of any one of them precludes the happening of all the others, i.e., if no two or more of them can happen simultaneously in the same trail. For example, in tossing a coin, both head and tail cannot occur in a single trail.

1.7 Equally Likely Events

Outcomes of a trail are said to be equally likely if taking into consideration all the relevant evidences, there is no reason to expect one in preference to the others. For example, in tossing a coin, getting a head and tail are equally likely events.

1.8 Independent Events

Several events are said to be independent if the happening of an event is not affected by the supplementary knowledge concerning the occurrence of any number of the remaining events. For example, in tossing a unbiased coin, the event of getting a head in the first toss is independent of getting a head in the second, third and subsequent throws.

1.9 Algebraic Operations of Events

For events A, B, C, then

(i) $(A\cup B) = \{\omega\in S: \omega\in A \text{ or } \omega\in B\}$

(ii) $(A\cap B) = \{\omega\in S: \omega\in A \text{ and } \omega\in B\}$

(iii) A^c or $\overline{A}$ (A complement) = $\{\omega\in S: \omega\notin A\}$

(iv) $A-B= \{\omega\in S: \omega\in A \text{ but } \omega\notin B\}$

(v) $A \subset B \Rightarrow$ for every $\omega\in A$, $\omega\in B$

(vi) $B\supset A \Rightarrow A \subset B$

(vii) A=B if and only if A and B have same elements, *i.e.*, $A \subset B$ and $B \subset A$

(viii) $A\cup B$ can be denoted by A+B if A and B are disjoint.

(ix) A and B are disjoint (mutually exclusive) $\Rightarrow A\cap B =\phi$

Notes : **Algebra of Sets**

Commutative law $A\cup B = B\cup A$, $A\cap B = B\cap A$

Associative law $A\cup(B\cup C) = (A\cup B)\cup C$, $A\cap(B\cap C) = (A\cap B)\cap C$

Distributive Laws $A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Complementary law $A \cup \overline{A} = S$, $A \cap \overline{A} = \phi$, $A \cup S = S$, $A \cap S = A$, $A \cup \phi = A$, $A \cap \phi = \phi$

Difference law $A - B = A \cap \overline{B}$. $A - B = A - (A \cap B) = (A \cup B) - B$

$A-(B-C) = (A-B) \cup (A-c)$, $(A \cup B)-C = (A-C) \cup (B-C)$

DeMorgan's Law $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$

1.10 MATHEMATICAL (OR CLASSICAL OR PRIORI) PROBABILITY

If a random experiment or a trail results in 'n' exhaustive, mutually exclusive and equally likely outcomes out of which 'm' are favourable to the occurrence of an event E, then the probability 'P' of occurrence of E , usually denoted by P(E), is given by

$$P(E) = \frac{\text{Number of favourable Cases}}{\text{Total number of exhaustive cases}} = \frac{m}{n}$$

1.11 STATISTICAL (OR EMPIRICAL) PROBABILITY

If an experiment is performed repeatedly under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the event occurs to the numbers of the trails, as the number of trails becomes infinitely large, is called the probability of happening of the event, it begin assumed that the limit is finite and unique. Symbolically, if in N trails an event E happens M times, then the probability of the happening of E, denoted by P(E) is given by

$$P(E) = \lim_{N \to \infty} \frac{M}{N}$$

1.12 AXIOMS OF PROBABILITY

The axioms approach was given by A.N Kolmogrov. With each event E_i in a finite sample space S, associate a real number, say $P(E_i)$ called the probability of an event E_i satisfying the conditions:

(i) ***Nonnegative**:* $0 \leq P(E_i) \leq 1$.

This implies that the probability of an event is always non-negative and can never exceed. If P (A) = 1, the event A is certainly going to happen and if P (A) = 0, the event is certainly not going to happen (impossible event).

(ii) ***Certainty*** **:** The probability of the sample space is 1.P(S) =1,

(iii) **Union** : If $\{A_n\}$ is any finite or infinite sequence of disjoint events in B, then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i)$$

(axioms of additivity)

1.13 Theorems on Probability

Theorem 1.1 Probability of the impossible event is zero, *i.e.*, P(ϕ) =0

Proof:

Impossible event contains no sample point and hence the certain event S and the impossible event ϕ are mutually exclusive.

$\therefore$ S∪ϕ =S ⇒ P(S∪ϕ) =P(S)

⇒ P(S∪ϕ) =P(S)+P(ϕ) using Axiom (iii) of probability

P(S) = P(S)+P(ϕ)

⇒P(ϕ) =P(S)-P(S) = 0

⇒P(ϕ) = 0

Theorem 1.2 Probability of the complementary event $\overline{A}$ of A is given by $P(\overline{A}) = 1 - P(A)$

Proof :

A and $\overline{A}$ are mutually disjoint events, so that A∪$\overline{A}$ = S

⇒ P(A∪$\overline{A}$)=P(S) from axioms (ii) and (iii)

P (A) +P ($\overline{A}$) =P(S)

⇒ P (A) +P ($\overline{A}$) = 1

⇒ P ($\overline{A}$) = 1- P (A)

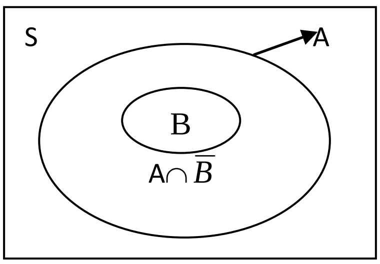
$P(\overline{A}) = 1 - P(A)$

Theorem 1.3 If B ⊂ A, then

(i) P(A∩$\overline{B}$) = P(A) –P(B) (ii) P(B) ≤ P(A)

Proof

(i) When B ⊂ A, B and A∩$\overline{B}$ are mutually exclusive Events so that A = B∪ (A∩$\overline{B}$)



⇒ P (A) = P[B∪ (A∩$\overline{B}$)]

= P(B)+P (A∩$\overline{B}$) by axioms(iii)

⇒ P(A) - P(B)= P(A∩$\overline{B}$)

$\therefore$ $P(A\cap \overline{B}) = P(A) - P(B)$

(ii) $P(A\cap \overline{B}) \geq 0 \Rightarrow P(A) - P(B) \geq 0 \Rightarrow P(A) \geq P(B)$

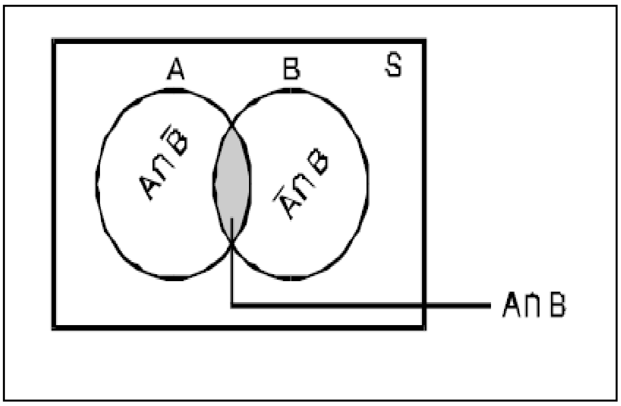
Hence $B \subset A \Rightarrow P(B) \leq P(A)$

Theorem 1.4 If A and B are independent events, Prove that

(i) $\overline{A}$ and B are independent

(ii) A and $\overline{B}$ are independent and

(iii) $\overline{A}$ and $\overline{B}$ are independent



Proof : If A and B are independent events,

then $P(A\cap B) = P(A).P(B)$

(i) From the diagram

$B = (A\cap B)\cup(\overline{A}\cap B)$

Also $(A\cap B)$ and $(\overline{A}\cap B)$ are mutually exclusive

$P(B) = P[(A\cap B)\cup(\overline{A}\cap B)]$

$= P(A\cap B) + P(\overline{A}\cap B)$

$P(B) - P(A\cap B) = P(\overline{A}\cap B)$

$P(\overline{A}\cap B) = P(B) - P(A\cap B) = P(B) - P(A)P(B)$

$= P(B)[1 - P(A)] = P(B)P(\overline{A})$

$\therefore P(\overline{A}\cap B) = P(B)P(\overline{A})$ $\overline{A}$ and B are independent

(ii) From the diagram

$A = (A\cap \overline{B})\cup(A\cup B)$

Also $(A\cap \overline{B})$ and $(A\cup B)$ are mutually exclusive

$P(A) = P[(A\cap \overline{B})\cup(A\cap B)] = P(A\cap \overline{B}) + P(A\cap B)$

$P(A) - P(A\cap B) = P(A\cap \overline{B})$

$P(A\cap \overline{B}) = P(A) - P(A\cap B) = P(A) - P(A)P(B) = P(A)[1-P(B)]$

$P(A\cap \overline{B}) = P(A)P(\overline{B})$

$\therefore$ A and $\overline{B}$ are independent

(iii) $\overline{A}$ and $\overline{B}$ are independent

DeMorgan's Law, $\overline{A} \cap \overline{B} = \overline{A \cup B}$

$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - [P(A)+P(B)-P(A \cap B)]$

$= 1-[P(A)+P(B)-P(A)P(B)]$

$=1-P(A)-P(B)+P(A)P(B)$

$=1-P(A)-P(B)[1-P(A)]$

$=[1-P(A)][1-P(B)] = P(\overline{A})\ P(\overline{B})$

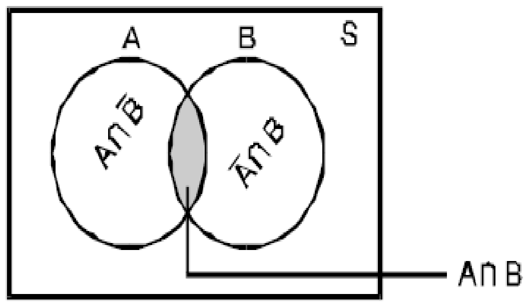
$P(\overline{A} \cap \overline{B}) = P(\overline{A})\ P(\overline{B})$

$\therefore$ $\overline{A}$ and $\overline{B}$ are independent

1.14 Addition Theorem of Probability

If A and B are any two events and are not disjoint, then $P(A \cup B) = P(A)+P(B)-P(A \cap B)$

Proof:



From the Venn diagram

Let $A = [(A \cap \overline{B}) \cup (A \cap B)]$

$P(A) = P[(A \cap \overline{B}) \cup (A \cap B)]$ using axiom (iii)

$\Rightarrow P(A) = P(A \cap \overline{B}) + P(A \cap B)$ ....(1)

Let $B = [(\overline{A} \cap B) \cup (A \cap B)]$

$P(B) = P[(\overline{A} \cap B) \cup (A \cap B)]$ using axiom (iii)

$\Rightarrow P(B) = P(\overline{A} \cap B) + P(A \cap B)$ .....(2)

From (1)+(2) , we get

$$\Rightarrow P(A)+P(B) = P(A \cap \overline{B}) + P(A \cap B) + P(\overline{A} \cap B) + P(A \cap B)$$

$$= P(A \cup B) + P(A \cap B)$$

$$\Rightarrow P(A)+P(B) - P(A \cap B) = P(A \cup B)$$

$\therefore$ **$P(A \cup B) = P(A)+P(B)-P(A \cap B)$**

Similarly for the three events

$$P(A \cup B \cup C) = P(A)+P(B)+PC)-P(A \cap B)-P(A \cap C)-P(B \cap C)+P(A \cap B \cap C)$$

1.15 Multiplication Theorem of Probability for Independent Events

If A and B are the two events with positive probabilities $\{P(A)\neq 0, P(B)\neq 0\}$ then A and B are independent if and only if $P(A\cap B) = P(A) .P(B)$

Proof

If an event happen in $n_1$ways of which a_1 are successful and the event B can happen in n_2 ways of which a_2 are successful, and to combine each successful event in the first with each successful event in the second case. Thus the total number of possible cases in a_1x a_2. Similarly, the total number of possible cases is n_1x n_2. By the definition the

probability occurrence of both events, $\frac{a_1 x\ a_2}{n_1 x\ n_2} = \frac{a_1}{n_1} x \frac{a_2}{n_2}$

But $P(A) = \frac{a_1}{n_1}$ and $P(B) = \frac{a_2}{n_2}$

$$P(A\cap B) = P(A) .P(B)$$

1.15 CONDITIONAL PROBABILITY

The multiplication theorem is not applicable in the case of dependent events. If A and B are the two events are said to be dependent, when B can occur only when A is known to have occurred. The probability attached to such an event is called conditional probability and its denoted by $P(B/A)$

$$P(B/A) = \frac{P(A\cap B)}{P(A)}$$

The general terms of multiplication in its modified form in terms of conditional probability becomes

$$P(A\cap B) = P(B)\ P(A/B)$$
$$P(A\cap B) = P(A)\ P(B/A)$$

1.16 BAYES THEOREM

Statement

If $E_1, E_2, E_3, \ldots, E_n$ are mutually disjoint events with $P(E_i) \neq 0, (i = 1,2,3,\ldots,n)$, then for any arbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A)>0$, we have

$$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)P(A \mid E_i)} = \frac{P(E_i)P(A \mid E_i)}{P(A)}$$

Proof :

Let $E_1, E_2, E_3, \ldots, E_n$ are and mutually disjoint events and A be any another event on the sample space, then $(A \cap E_i) = E_i$ (i =1,2,3,...n) are mutually disjoint events

$$A = (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \ldots \cup (A \cap E_n)$$

$\Rightarrow$ $P(A) = P[(A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \ldots \cup (A \cap E_n)]$ by using axioms of (iii)

$\Rightarrow$ $P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) + \ldots + (A \cap E_n)$

$$P(A) = \sum_{i=1}^{n} P(A \cap E_i)$$

Multiplication theorem of probability

$$\boldsymbol{P(A \cap B) = P(A \mid B)P(B)}$$

$$P(A) = \sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)$$

$$P(E_i \mid A) = \frac{P(A \cap E_i)}{P(A)}$$

$$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{P(A)} = \frac{P(E_i)P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)P(A \mid E_i)}$$

$$\therefore\ P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{\sum P(E_i)P(A \mid E_i)} = \frac{P(E_i)P(A \mid E_i)}{P(A)}$$

Problems

Example 1.1 Two unbiased dice are thrown. Find the probability that (i) both the dice show the same number,(ii) the first die shows 6 ,and (iii) the total of the numbers on the die is greater than 8

Solution : In a random throw of two dice, since each of six faces of one can associated with the each of six faces of the other die, the total number of cases is 6x6 = 36 as follows

$$S = \begin{pmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{pmatrix}$$

Exhaustive number of cases n(s) = 36

(i) The favourable cases that the both dice, shown the same number are (1,1),(2,2),(3,3),(4,4),(5,5), and(6,6) *i.e.*, n(E) = 6

$$P(\text{two dice shown the same number}) = \frac{6}{36} = \frac{1}{6}$$

(ii) The favourable cases that the first die shows 6 are (6,1),(6,2),(6,3),(6,4),(6,5), and (6,6) *i.e.*, n(E) = 6

$$P(\text{first die shows } 6) = \frac{6}{36} = \frac{1}{6}$$

(iii) The cases favourable to getting a total of more than 8 are (3,6),(4,5),(4,6),(5,4),(5,5),(5,6),(6,3),(6,4),(6,5),(6,6) i.e., n(E) = 10

$$P(\text{getting a total of more than } 8) = \frac{10}{36} = \frac{5}{18}$$

Example 1.2 Four cards are drawn at random from a pack of 52 cards, find the probability

(i) they are a king, a queen, a jack and an ace.

(ii) two are kings and two are queens.

(iii) Two are black and two are red

(iv) There are a two cards of hearts and two cards of diamonds

Solution: Four cards can be drawn from a well-shuffled pack of 52 cards in $^{52}C_4$, which gives the exhaustive number of cases n(S) = $^{52}C_4$

(i) 1 king can drawn out of 4 kings in $4C_1$ ways similarly 1 queen,1 jack and an ace can each be drawn in $4C_1$ ways

Favourable numbers of cases n(E) = $4C_1 x 4C_1 x 4C_1 x 4C_1$

Required probability = $\frac{4C_1 x 4C_1 x 4C_1 x 4C_1}{52C_4}$

(ii) 2 king can drawn out of 4 kings in $4C_2$ ways similarly 2 queen be drawn in $4C_2$ ways

Required probability = $\frac{4C_2 x 4C_2}{52C_4}$

(iii) Since there are 26 black cards and 26 red cards in a pack of cards,

Required probability = $\frac{13C_2 x 13C_2}{52C_4}$

Example 1.3 What is the chance that a leap year selected at random will contain 53 Sundays?

Solution: In leap year (which consists of 366 days), there are 52 complete weeks and 2 days over. The possible combinations for these two days are ,

(i) Sunday and Monday (ii) Monday and Tuesday (iii) Tuesday and Wednesday

(iv) Wednesday and Friday (v) Friday and Saturday (vi) Saturday and Sunday

Required probability = $\frac{2}{7}$

Example 1.4 A bag contains 8 white and 4 red balls. 5 balls are drawn at random. What is the probability that 2 of them are red and 3 of them white?

Solutions : The total number of balls in the bag = 8 + 4 = 12

The number of balls drawn = 5

5 balls can be drawn from 12 balls in $12C_5$ ways i.e., $n(S) = 12C_5$

2 red balls can be drawn from 4 red balls in $4C_2$ ways

3 white balls can be drawn from 8 white balls in $8C_3$ ways

$\therefore$ The number of favourable cases = $n(E)= 4C_2 x 8C_3$

$$\text{Required Probability} = \frac{4C_2 x 8C_3}{12C_5}$$

Example 1.5 An urn contains 6 white, 4 red and 9 black balls. If 3 balls are drawn at random, find the probability that: (i) two of the balls drawn are white, (ii) one is of each colour, (iii) none is red, (iv) at least one is white.

Solution

Total number of balls in the urn is 6+4+9 = 19. Since 3 balls can be drawn out of 19 in $19C_3$ ways , the exhaustive number of cases are $19C_3$

(i) two of the balls drawn are white

$$\text{The required probability} = \frac{{}^{6}C_2 x\ {}^{13}C_1}{{}^{19}C_3}$$

(ii) one is of each colour

$$\text{The required probability} = \frac{{}^{6}C_1 x\ {}^{4}C_1 x\ {}^{3}C_1}{{}^{19}C_3}$$

(iii) None is red,

$$\text{The required probability} = \frac{{}^{15}C_3}{{}^{19}C_3}$$

(iii) at least one is white

$$\text{The required probability} = \frac{{}^{13}C_3}{{}^{19}C_3}$$

Example:1.6 In a given race the odds in favour of three horses A,B,C are 1:2, 1:3, 1:4 respectively. Assuming that a dead heat is impossible, find the probability that one of them will win the race.

Solution: If P(A), P(B), P(C) are the probabilities of winning of the horses A, B, C respectively, then

$$P(A) = \frac{1}{1+2} = \frac{1}{3} \qquad P(B) = \frac{1}{1+3} = \frac{1}{4} \qquad P(C) = \frac{1}{1+4} = \frac{1}{5}$$

These events are mutually exclusive, the chance that one of them wins

$$P(AUBUC) = P(A) + P(B) + P(C)$$

$$= \frac{1}{3} + \frac{1}{4} + \frac{1}{5} = \frac{47}{60}$$

The required probability = $\frac{47}{60}$

Example 1.7 An integer is chosen at random from the first two hundred digits. What is the probability, that the integer chosen is divisible by 6 or 8.

Solution: P(divisible by 6 **or** 8) =P(divisible by 6 $\cup$ 8)

A= integer chosen is divisible by 6

B = integer chosen is divisible by 8

$$P(A) = \frac{33}{200} \qquad P(B) = \frac{25}{200} \qquad P(A \cap B) = \frac{8}{200}$$

$$P(A \cup B) = P(A)+P(B)-P(A \cap B) = \frac{33}{200} + \frac{25}{200} - \frac{8}{200} = \frac{58-8}{200} = \frac{1}{4}$$

Required probability = $\frac{1}{4}$

Example :1.8 A is known to hit the target in 2 out of 5 shots whereas B is known to hit the target in 3 out of 4 shots. Find the probability of the target being hit when they both try?

Solution:

Let A be the event that 'A' hits the target and B the event that 'B' hits the target.

$$P(A) = \frac{2}{5} \qquad P(B) = \frac{3}{4}$$

$P(A\cup B) = P(A)+P(B)- P(A\cap B)$

$= P(A)+P(B)-P(A).P(B)$ [since A and B are independent]

$$P(A\cup B) = \frac{2}{5}+\frac{3}{4} - \frac{2}{5}\frac{3}{4} = \frac{8+15-6}{20} = \frac{17}{20}$$

Example 1.9 If the letters of the word "REGULATIONS" be arranged at random, what is this chance that there will be exactly 4 letters between R and E.

Solution:

The word "REGULATIONS" consists of 11 letters. The two letters R and E CAN OCCUPY $11P_2$ that is 11

The number of ways in which there will be exactly 4 letters between R and E are enumerated below

(i) R is in the 1^{st} place and E is in the 6^{th} place
(ii) R is in the 2^{nd} place and E is in the 7^{th} place
(iii) R is in the 3^{rd} place and E is in the 8^{th} place
(iv) R is in the 4^{th} place and E is in the 9^{th} place
(v) R is in the 5^{th} place and E is in the 10^{th} place
(vi) R is in the 6^{th} place and E is in the 11^{th} place

Since R and E interchange their position,

The required number of favourable cases is 2x 6 = 12.

The required probability is = 12/110 = 6/25

Example 1.10 A letter is taken out at random from "ASSISTANT" and another is taken out from "STATISTICS". What is the chance that they are same letters?

Solutions

ASSISTANT → AA I N SSS TT

STATISTICS→ A II C SSS TTT Here N and C are not common

Probability of choosing A = $\frac{^{2}C_1}{^{9}C_1} \times \frac{^{1}C_1}{^{10}C_1} = \frac{1}{45}$

Probability of choosing I = $\frac{1}{9} \times \frac{2}{10} = \frac{1}{45}$

Probability of choosing S = $\frac{3}{9} \times \frac{3}{10} = \frac{1}{10}$

Probability of choosing I = $\frac{2}{9} \times \frac{3}{10} = \frac{1}{15}$

$$\text{Total Probability} = \frac{1}{45}+\frac{1}{45}+\frac{1}{10}+\frac{1}{15}=\frac{19}{90}$$

Example 1.11 From a city population, the problem of selection (i) a male or a smoker is 7/10 (ii) a male smoker is 2/5 (iii) a male, if a smoker is already selected is 2/3 find the probability of selecting (a) non-smokers (b) a male and c) smoker, if a male is first selected.

Solution:

A : a male is selected   B : a smoker is selected

$$\text{Given } P(A\cup B) = \frac{7}{10} \;;\; P(A\cap B) = \frac{2}{5} \;;\; P(A \mid B) = \frac{2}{3}$$

(a) The probability of selecting a non-smoker is

$$P(\overline{B}) = 1 - P(B) = 1 - \frac{P(A\cap B)}{P(A \mid B)} = 1 - \frac{2/5}{2/3} = 1 - \frac{3}{5} = \frac{2}{5}$$

P(B) = 2/5

(b) The probability of selecting a male (by addition theorem)

$$P(A) = P(A\cup B) + P(A\cap B) - P(B) = \frac{7}{10}+\frac{2}{5}-\frac{3}{5}=\frac{1}{2}$$

(c) The probability of selecting a smoker if a male is first selected is

$$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{2/5}{1/2} = \frac{4}{5}$$

Example 1.12 If A and B are events such that $P(A\cup B)= \frac{3}{4}$, $P(A\cap B)=\frac{1}{4}$ and $P(\overline{A}) = \frac{2}{3}$

Find $P(\overline{A}/B)$.

Solution: $$P(\overline{A}/B) = \frac{P(\overline{A}\cap B)}{P(B)} = \frac{P(B) - P(A\cap B)}{P(B)}$$

$$P(A) = 1 - P(\overline{A}) = 1 - \frac{2}{3} = \frac{1}{3} \quad ; \quad P(A\cup B) = 3/4$$

$$\text{ie., } P(A\cup B) = P(A)+P(B)- P(A\cap B) \Rightarrow \frac{3}{4} = \frac{1}{3} + P(B) - \frac{1}{4}$$

$$\Rightarrow P(B)= \frac{3}{4} + \frac{1}{4} - \frac{1}{3} = \frac{9+3-4}{12} = \frac{8}{12} = \frac{2}{3}$$

$$\text{Hence } P(\bar{A}/B) = \frac{\frac{2}{3} - \frac{1}{4}}{\frac{2}{3}} = 0.625$$

Example: 1.13 Two friends A and B apply for two vacancies at the same post. The probabilities of their selection are ¼ and 1/5 respectively. What is the chance that i) One of them will be selected, (ii) both will be selected, (iii) none will be selected, (iv) atleast one will be selected and (v) at most one will be selected.

Solution:

Let A – Event A is selected, B – Event B is selected

$P(A) = \frac{1}{4}$ $\quad P(B) = \frac{1}{5}$

$P(\bar{A}) = 1\text{-}P(A) = 1\text{-}\frac{1}{4} = \frac{3}{4}$; $P(\bar{B}) = 1\text{-}P(B) = 1\text{-}\frac{1}{5} = \frac{4}{5}$

(i) One of them will be selected

i.e.,P[(A is selected but B is not selected)OR(B is selected but A is not selected)

$$P[(A \cap \bar{B}) \cup (\bar{A} \cap B)] = P(A)\,P(\bar{B}) + P(\bar{A})\,P(B) = \frac{1}{4} x \frac{4}{5} + \frac{1}{5} x \frac{3}{4} = \frac{4}{20} + \frac{3}{20} = \frac{7}{20}$$

*P(*One of them will be selected)= $\frac{7}{20}$

(ii) both will be selected $\Rightarrow P(A \cap B) = P(A)P(B) = \frac{1}{4}\,\frac{1}{5} = \frac{1}{20}$

(iii) none will be selected $\Rightarrow P(\bar{A} \cap \bar{B}) = P(\bar{A})P(\bar{B}) = \frac{3}{4}\,\frac{4}{5} = \frac{12}{20} = \frac{3}{5}$

(iv) atleast one will be selected

P(One of them will be selected or both will be selected) = $\frac{7}{20} + \frac{1}{20} = \frac{8}{20} = \frac{2}{5}$

(v) atmost one will be selected

P(none will be selected or One of them will be selected) = $\frac{3}{5} + \frac{7}{20} = \frac{12+7}{20} = \frac{19}{20}$

Example 1.14 A card is drawn from a well-shuffled deck of 52 cards. What is the probability that it is either a spade or a king?

Solution

If A and B denote the events of drawing a 'spade card' and a 'king' respectively, then the event A consists of 13 sample points, whereas the event B consists of 4 sample points. Therefore,

$$P(A) = \frac{13}{52} \quad P(B) = \frac{4}{52}$$

The compound event $(A \cap B)$ consists of only one sample point, viz.; king of spade. So,

$$P(A \cap B) = \frac{1}{52}$$

$$P(A \cup B) = P(A)+P(B)- P(A \cap B) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52}$$

Example: 1.15 A bag contains 5 white and 3 black balls. Two balls are drawn at random one after the other without replacement. Find the probability that both balls drawn are black.

Solution : The probability of drawing a black ball in the first draw.

$$P(A) = \frac{3}{5+3} = \frac{3}{8}$$

The probability of drawing the second black ball given that the first ball drawn is black

$$P(B/A) = \frac{2}{5+2} = \frac{2}{7}$$

$\therefore$ The probability that both balls drawn are black is $P(A \cap B) = P(A)\,P(B/A) = \frac{3}{8} \text{ x } \frac{2}{7} = \frac{3}{28}$

Example: 1.16 A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?

Solutions: Let A be the event that the first tube is good and B be the event that second is also good

$$P(\text{first tube is good}) = P(A) = \frac{6C_1}{10C_1} = \frac{3}{5}$$

P(Both the tubes are good) = P (A∩B) = $\frac{6C_2}{10C_2} = \frac{3}{5}$

$$\therefore\ P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{3/5} = \frac{5}{9}$$

Example 1.17 The contents of urns I,II and III are as follows

(i) 1 white, 2 black and 3 red balls

(ii) 2 white , 1 black and 1 red balls and

(iii) 4 white, 5 black and 3 red balls

One urn is chosen at random and two balls drawn from it. They happen to be white and red. What is the probability that they come from urns I, II and III?

Solutions: Let E_1, E2, and E_3 denote the events that the urn I,II, and III is chosen respectively, and let A be the event that the two balls are taken from the selected urn are white and red. Using Baye's theorem

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$$P(A|E_1) = \frac{1x3}{6C_2} = \frac{1}{5} \qquad P(A|E_2) = \frac{2x1}{4C_2} = \frac{1}{3} \qquad P(A|E_3) = \frac{4x3}{12C_2} = \frac{2}{11}$$

$$P(E_1|A) = \frac{P(E_1)P(A|E_1)}{\sum_{i=1}^{3} P(E_i)P(A|E_i)} = \frac{\frac{1}{3} x \frac{1}{5}}{\frac{1}{3} x \frac{1}{5} + \frac{1}{3} x \frac{1}{3} + \frac{1}{3} x \frac{1}{11}} = \frac{33}{118}$$

$$P(E_2|A) = \frac{P(E_2)P(A|E_2)}{\sum_{i=1}^{3} P(E_i)P(A|E_i)} = \frac{\frac{1}{3} x \frac{1}{3}}{\frac{1}{3} x \frac{1}{5} + \frac{1}{3} x \frac{1}{3} + \frac{1}{3} x \frac{1}{11}} = \frac{55}{118}$$

$$P(E_3|A) = \frac{P(E_3)P(A|E_3)}{\sum_{i=1}^{3} P(E_i)P(A|E_i)} = \frac{\frac{1}{3} x \frac{1}{11}}{\frac{1}{3} x \frac{1}{5} + \frac{1}{3} x \frac{1}{3} + \frac{1}{3} x \frac{1}{11}} = \frac{30}{118}$$

Example 1.18: A factory produces a certain type of outputs by three type of machines. The respectively daily production figures are Machine I: 3,000 units; Machine II: 2,500 units; Machine III: 4,500 units. Past experience shows that 1 percent of the output produced by the machine I is defective. The corresponding fraction of defectives for the other two machines are1.2 percent and 2 percent respectively. An item is drawn at random from the days production run and found is to be defective. What is the probability that it comes from the output of (i) Machine I, (ii) Machine II and (iii) Machine III.

Solution : Let E_1,E_2,and E_3 are the events that the output produced by machines I,II and III respectively and A be the event that the output is defective

Using Baye's theorem

$$P(E_1) = \frac{3000}{10,000} \; ; \quad P(E_2) = \frac{2500}{10,000} \quad ; \quad P(E_3) = \frac{4500}{10,000}$$

$P(A/E_1) = 1\% = 0.01$; $P(A/E_2) = 1.2\% = 0.012$; $P(A/E_3) = 2\% = 0.02$

The probability that an item selected at random from day's production is defective is given by

$$P(A) = \sum_{i=1}^{3} P(E_i)P(A/E_i)$$

$$P(A) = 0.03x0.01 + 0.25x0.012 + 0.45x0.02 = 0.015$$
$$= 0.015$$

(i) $P(E_1/A) = \dfrac{P(E_1)P(A/E_1)}{P(A)} = \dfrac{0.003}{0.015} = \dfrac{1}{5}$

(ii) $P(E_2/A) = \dfrac{P(E_2)P(A/E_2)}{P(A)} = \dfrac{0.003}{0.015} = \dfrac{1}{5}$

(iii) $P(E_3/A) = \dfrac{P(E_3)P(A/E_3)}{P(A)} = \dfrac{0.009}{0.015} = \dfrac{3}{5}$

Aliter : The posterior probabilities can be obtained elegantly in a tabular form as given below

Event E_i	Prior Probabilities $P(E_i)$	Conditional Probabilities $P(A/E_i)$	Joint probabilities $P(E_i)\,P(A/E_i)$	Posterior Probabilities $P(E_i \mid A)$
E_1	$\frac{3000}{10,000}=0.30$	0.01	0.003	$\frac{0.003}{0.015}=\frac{1}{5}$
E_2	$\frac{2500}{10,000}=0.25$	0.012	0.003	$\frac{0.003}{0.015}=\frac{1}{5}$
E_3	$\frac{4500}{10,000}=0.45$	0.020	0.009	$\frac{0.009}{0.015}=\frac{3}{5}$
Total	1.00		P(A) = 0.015	1

Example 1.19 There are three coins, identical in appearance, one of which is ideal and the other two biased with probabilities 1/3 and 2/3 respectively for a head. One coin is taken at random and tossed twice. If a head appears both the times, what is the probability that the ideal coin was chosen.

Solution

Let B is the event of obtaining 2 heads in two tosses of the selected coin and A_1, A_2, A_3 respectively the events of choosing the first (ideal), second and third coins , we have

$$\text{Probability of getting a head in a toss} = \frac{1}{2}$$

$$P(A_1)= P(A_2)=P(A_3) = \frac{1}{3} \quad \text{and } P(B/A_1) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

Also, probabilities of turning head up with second and third coins respectively being 1/3 and 2/3 , we have

$$P(B/A_2) = \left(\frac{1}{3}\right)^2 = \frac{1}{9} \quad \text{and } P(B/A_3) = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$

Using Baye's theorem

$$P\left(A_1/B\right) = \frac{P(A_1)\,P(B/A_1)}{P(A_1)\,P(B/A_1) + P(A_2)\,P(B/A_2) + P(A_3)\,P(B/A3)}$$

$$= \frac{\frac{1}{3} \text{ x } \frac{1}{4}}{\frac{1}{3} \text{ x } \frac{1}{4} + \frac{1}{3} \text{ x } \frac{1}{9} + \frac{1}{3} \text{ x } \frac{4}{9}} = \frac{\frac{1}{12}}{\frac{29}{108}}$$

$$P\left(A_1/B\right) = \frac{9}{29}$$

Unit – II

2. **RANDOM VARIABLE**

Let S be a sample space associated with a given random experiment. A real valued function defined on S and taking values in R(-∞,∞) is called one dimensional random variable.

A random variable X is a rule which associates uniquely a real number with every elementary event $E_i \in S$, *i* =1,2,3,...n i.e, a random variable is a real valued function which maps the sample space on to the real line. Discrete Random Variables and Continuous Random Variables are the two types of a random variable.

2.1 DISCRETE RANDOM VARIABLE

A variable which can assume only a countable number of real values and for which the value which the variable takes depends on chance is called discrete random variable. In other words, a real valued function defined on a discrete sample space is called a discrete random variable. For instance, numbers of members of family, number of students in a class, number of passenger in a bus, tossing a coin and rolling a dice are the example of discrete random variable.

2.1.1 Probability Mass Function

If X is one dimensional discrete random variable taking at most a countable in finite number of values x_1, x_2, x_3,... then it is probabilistic behaviour at each real point described by a function called the probability mass function.

Definition:

If X is a discrete random variable with distinct $x_1, x_2, x_3,...x_n,...$, then the function P(x) defined as $P_X(x) = \begin{cases} P(X = x_i) & if\ x = x_i \\ 0 & if\ x \neq x_i; i = 1,2,3,... \end{cases}$ is called the probability mass function of random variable X

Remarks: The numbers $p(x_i)$; i =1,2,3,... must satisfy the following conditions:

(i) $P(x_i) \geq 0$ and (ii) $\sum_{i=1}^{\infty} P(x_i) = 1$

2.2 CONTINUOUS RANDOM VARIABLE

A random variable which can assume any value from a specified interval of the form [a,b] is known as continuous random variable.

2.2.1 PROBABILITY DENSITY FUNCTION

If X is a continuous random variable, it will have infinite number of values in any interval however small. The probability that this variable lies in the infinitesimal interval ($x,x+dx$) is expressed as $f(x)\,dx$, where the function $f(x)$ is called probability density function *(p.d.f)*, satisfying the following conditions

$$(i)\ f(x) \geq 0 \quad \forall x \quad (ii) \int_{-\infty}^{\infty} f(x)\,dx = 1$$

2.3 DISTRIBUTION FUNCTION

Let X be a random variable, the function F defined for all real x by $F(x) = P(X \leq x)$ is called the distribution function*(d.f)* or cumulative distribution function of the random variable X.

If random variable X is discrete then distribution function is $F(x) = P(X \leq x)$

If X is continuous random variable then distribution function is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$$

2.3.1 Properties of Distribution Function

1. If F is the distribution function of random variable X and if a<b then

   $$P(a < X \leq b) = F(b) - F(a)$$

2. If F is the distribution function of random variable X then

   (i) $0 \leq F(X) \leq 1$(ii)$F(x) \leq F(Y)$ if $x < y$

3. If F is the distribution function of random variable X then

   $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$ and $F(\infty) = \lim_{x \to \infty} F(x) = 1$

4. $\frac{d}{dx}\left(F(x)\right) = f(x)$

Example 2.1 If the random variable X takes the value 1, 2, 3 and 4 such that

2P(X=1)=3P(X=2) = P (X=3)=5P(X=4).Find the probability distribution?

Solution:

$2P(X=1)= k \Rightarrow P(X=1) = k/2$

$3P(X=2) = k \Rightarrow P(X=2) = k/3$

$P(X=3) = k$

$5P(X=4)= k \Rightarrow P(X=4) = k/5$

$$\sum_{x=1}^{4} P(x_i) = 1$$

$$\frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$k = \frac{30}{61}$$

The probability distribution is

x	1	2	3	4
P(X=x)	15/61	10/61	30/61	6/61

Example 2.2 A random variable X has the following probability function

x	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*
P(x)	*0*	*k*	*2k*	*2k*	*3k*	k^2	$2k^2$	$7k^2+k$

(i) Find k, (ii) Evaluate P(X<6),P(X≥6) and P(o<X<5) (iii) Determine the distribution function of X and (iv) P(X≤a)>1/2 find the minimum value of a,

Solution:

$$\sum_{x=0}^{7} P(x_i) = 1$$

$$k+2k+2k+2k+3k+k^2+2k^2+7k^2+k=1$$

$\Rightarrow 10k^2+9k-1=0 \quad \Rightarrow (10k-1)(k+1) = 0 \quad \Rightarrow k = \frac{1}{10}$ or k = -1(negative)

Hence $k = \frac{1}{10}$

(ii) P(X<6) = P(X=0)+ P(X=1)+ P(X=2)+ P(X=3)+ P(X=4)+ P(X=5)

$$= k + 2k + 2k + 2k + 3k + k2$$

$$P(X < 6) = \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100} = \frac{81}{100}$$

$P(X \geq 6) = 1 -$ P(X<6) $= 1 - \frac{81}{100} = \frac{19}{100}$

$P(X \geq 6) = \frac{19}{100}$

P(0<X<5) = P(X=1)+ P(X=2)+ P(X=3)+ P(X=4)

$= k+2k+2k+2k+3k = 8k = \frac{8}{10}$

$P(0<X<5) = \frac{8}{10}$

(iii) *Distribution function of X*

$F(x) = P(X \leq x)$

x	$F(x) = P(X \leq x)$
0	0
1	$k = \frac{1}{10}$
2	$k + 2k = 3k = \frac{3}{10}$
3	$k + 2k + 2k = 5k = \frac{5}{10}$
4	$k + 2k + 2k + 3k = 8k = \frac{8}{10}$
5	$k + 2k + 2k + 3k + k^2 = 8k + k^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$
6	$k + 2k + 2k + 3k + k^2 + 2k^2 = 8k + 3k^2 = \frac{8}{10} + \frac{3}{100} = \frac{83}{100}$
7	$k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 9k + 10k^2 = \frac{9}{10} + \frac{10}{100} = 1$

(iv) *P(X≤a)>1/2 find the minimum value of a*

From the distribution function P(X≤4) = $\frac{8}{10} = \frac{4}{5} > \frac{1}{2}$

a = 4

Example 2.3 A discrete random variable X has the following probability distribution

x :	0	1	2	3	4	5	6	7	8
p(x):	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) Find the value of 'a'

(ii) $P(0 < X < 3)$

(iii) $P(X \geq 3)$

(iv) Find the distribution function of X

Solution: We have $\sum_{i=1}^{n} P(X = x) = 1$

a+3a+ 5a+7a+9a+11a+13a+ 15a+17a =1

$\therefore$ 81a = 1 $\Rightarrow$ a = $\frac{1}{81}$

$\therefore$ The actual probability distribution is

x	0	1	2	3	4	5	6	7	8
P(X=x)	$\frac{1}{81}$	$\frac{3}{81}$	$\frac{5}{81}$	$\frac{7}{81}$	$\frac{9}{81}$	$\frac{11}{81}$	$\frac{13}{81}$	$\frac{15}{81}$	$\frac{17}{81}$

$P(0 < X < 3) = P(X = 1) + P(X = 2) = \frac{3}{81} + \frac{5}{81} = \frac{8}{81}$

$P(0 < X < 3) = \frac{8}{81}$

$P(X \geq 3) = 1 - P(X<3) = 1 - \left\{\frac{1}{81} + \frac{3}{81} + \frac{5}{81}\right\} = \frac{72}{81}$

The distribution function of X is

x	0	1	2	3	4	5	6	7	8
F(x)	0	$1/81$	$4/81$	$9/81$	$16/81$	$25/81$	$36/81$	$49/81$	1

Example 2.4 For the following density function, f(x) = $ae^{-|x|}$, $-\infty < x < \infty$, find the value of 'a'

Solution:

Given f(x) is a pdf.

$$\therefore \int_{-\infty}^{\infty} f(x)dx = 1$$

$$a\int_{-\infty}^{\infty} e^{-|x|}dx = 1$$

$$2a\int_{0}^{\infty} e^{-x}dx = 1$$

$$2a\left(\frac{e^{-x}}{-1}\right)_0^{\infty} = 1$$

$$2a\left(\frac{e^{-\infty}}{-1} - \frac{e^{-0}}{-1}\right) = 1$$

$$2a = 1 \Rightarrow a = \frac{1}{2}$$

Example 2.5 The diameter of an electric cable, say X, is assumed to be a continuous random variable with *p.d.f : f(x) = 6x(1-x) , 0≤x≤1.*

(i)Determine a number b such that P(X<b)=P(X>b).

(ii) Compute $P(X \le \frac{1}{2} / \frac{1}{3} \le X \le \frac{2}{3})$

Solution (i)

$$P(X < b) = P(X > b)$$

$$\Rightarrow \int_0^b f(x)dx = \int_b^1 f(x)\, dx$$

$$\Rightarrow \int_0^b 6x(1-x)dx = \int_b^1 6x(1-x)dx$$

$$\Rightarrow 6\int_0^b (x - x^2)dx = 6\int_b^1 (x - x^2)dx$$

$$\Rightarrow \left(\frac{x^2}{2}+\frac{x^3}{3}\right)_0^b = \left(\frac{x^2}{2}+\frac{x^3}{3}\right)_b^1$$

$$\Rightarrow \left[\left(\frac{b^2}{2}+\frac{b^3}{3}\right)-\left(\frac{0^2}{2}+\frac{0^3}{3}\right)\right] = \left[\left(\frac{1^2}{2}+\frac{1^3}{3}\right)-\left(\frac{b^2}{2}+\frac{b^3}{3}\right)\right]$$

$$\Rightarrow 3b^2 - 2b^3 = (1 - 3b^2 + 2b^3)$$

$$\Rightarrow 4b^3 - 6b^2 + 1 = 0$$

$$\Rightarrow (2b-1)(2b^2 - 2b - 1) = 0$$

$$\therefore 2b - 1 = 0 \quad \Rightarrow \quad b = \frac{1}{2} \, or$$

$$2b^2 - 2b - 1 = 0 \Rightarrow b = \frac{2 \pm \sqrt{4+8}}{4} = \frac{1 \pm \sqrt{3}}{2}$$

Hence $b = \frac{1}{2}$, is the only real value lying between 0 and 1

$$(ii)\ P(X \le \frac{1}{2} / \frac{1}{3} \le X \le \frac{2}{3}) = \frac{P\left(X \le \frac{1}{2} \cap \frac{1}{3} \le X \le \frac{2}{3}\right)}{P\left(\frac{1}{3} \le X \le \frac{2}{3}\right)}$$

$$= \frac{P\left(\frac{1}{3} \le X \le \frac{1}{2}\right)}{P\left(\frac{1}{3} \le X \le \frac{2}{3}\right)} = \frac{\int_{1/3}^{1/2} 6x(1-x)dx}{\int_{1/3}^{2/3} 6x(1-x)dx}$$

$$= \frac{13/54}{13/27} = \frac{11}{26}$$

$$P(X \le \frac{1}{2} / \frac{1}{3} \le X \le \frac{2}{3}) = \frac{11}{26}$$

Example 2.6 Let X be a continuous random variable with *p.d.f* given by

$$f(x) = \begin{cases} kx & , 0 \le x < 1 \\ k & , 1 \le x < 2 \\ -kx + 3k & , 2 \le x < 3 \\ 0 & , otherwise \end{cases}$$

(i) find the value of k (ii)Determine the c.d.f

Solution:

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\int_0^1 kx\,dx + \int_1^2 k\,dx + \int_2^3 (-kx + 3k)\,dx = 1$$

$$k\left(\frac{x^2}{2}\right)_0^1 + k(x)_1^2 + \left(-k\frac{x^2}{2} + 3kx\right)_2^3 = 1$$

$$k\left(\frac{1^2}{2} - \frac{0^2}{2}\right) + k(2-1) + \left(\left(-k\frac{3^2}{2} + 3k3\right) - \left(-k\frac{2^2}{2} + 3k2\right)\right) = 1$$

$$k\left(\frac{1}{2}\right) + k + \left(\left(-k\frac{9}{2} + 9k\right) - \left(-k\frac{4}{2} + 6k\right)\right) = 1$$

$$\frac{k}{2} + k + \left((k)\left(-\frac{9}{2} + 9\right) - (k)(-2+6)\right) = 1$$

$$\frac{k}{2} + k + \left((k)\left(\frac{-9+18}{2} - 4\right)\right) = 1$$

$$\frac{k}{2} + k + \left((k)\left(\frac{-9+18-8}{2}\right)\right) = 1$$

$$\frac{k}{2} + k + \frac{k}{2} = 1$$

$$\Rightarrow \frac{k + 2k + k}{2} = 1$$

$$\Rightarrow \frac{4k}{2} = 1 \quad \Rightarrow 2k = 1 \quad k = \frac{1}{2}$$

(ii) The *c.d.f*

For any *x*, such that $-\infty < x < 0$;

$$F(x) = \int_{-\infty}^{x} f(x)dx = 0$$

For any *x*, where $0 \le x < 1$;

$$F(x) = \int_{-\infty}^{0} 0\,dx + \int_0^x kx\,dx = k\int_0^x x\,dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_0^x = \frac{1}{2}\left(\frac{x^2}{2} - \frac{0}{2}\right) = \frac{x^2}{4}$$

For any *x*, where 1≤x<2;

$$F(x)=\int_{-\infty}^{0} 0\,dx+\int_{0}^{1} kx\,dx+\int_{1}^{x} k\,dx \;=k\int_{0}^{1} x\,dx+k\int_{1}^{x} dx$$

$$=\frac{1}{2}\int_{0}^{1} x\,dx+\frac{1}{2}\int_{1}^{x} dx \;=\frac{1}{2}\left(\frac{x^2}{2}\right)_0^1+\frac{1}{2}(x)_1^x=\frac{1}{2}\left(\frac{1^2}{2}-\frac{0^2}{2}\right)+\frac{1}{2}(x-1)=\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+\frac{1}{2}(x-1)$$

$$=\frac{1}{4}+\frac{x-1}{2}=\frac{1+2(x-1)}{4}=\frac{1+2x-2}{4}$$

$$F(x)=\frac{2x-1}{4}$$

For any *x*, where 2≤x<3;

$$F(x)=\int_{-\infty}^{0} 0\,dx+\int_{0}^{1} kx\,dx+\int_{1}^{2} k\,dx+\int_{2}^{x} -kx+3k\,dx \;=k\int_{0}^{1} x\,dx+k\int_{1}^{2} dx+k\int_{2}^{x} -x+3\,dx$$

$$=\frac{1}{2}\int_{0}^{1} x\,dx+\frac{1}{2}\int_{1}^{2} dx+\frac{1}{2}\int_{2}^{x} -x+3\,dx$$

$$=\frac{1}{2}\left(\frac{x^2}{2}\right)_0^1+\frac{1}{2}(x)_1^2+\frac{1}{2}\left(-\frac{x^2}{2}+3x\right)_2^x$$

$$=\frac{1}{2}\left(\frac{1^2}{2}-\frac{0^2}{2}\right)+\frac{1}{2}(2-1)+\frac{1}{2}\left(\left(-\frac{x^2}{2}+3x\right)-\left(-\frac{2^2}{2}+3(2)\right)\right)$$

$$=\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{2}(1)+\frac{1}{2}\left(\left(-\frac{x^2}{2}+3x\right)-(-2+6)\right)$$

$$=\frac{1}{4}+\frac{1}{2}+\frac{1}{2}\left(-\frac{x^2}{2}+3x-4\right)=\frac{1}{4}+\frac{1}{2}+\left(-\frac{x^2}{4}+\frac{3}{2}x-\frac{4}{2}\right)=\frac{1+2-x^2+6x-8}{4}$$

$$F(x)=\frac{-x^2+6x-5}{4}$$

For any *x*, x≥3;

$$F(x)=\int_{-\infty}^{0} 0\,dx+\int_{0}^{1} kx\,dx+\int_{1}^{2} k\,dx+\int_{2}^{1} -kx+3k\,dx+\int_{3}^{x} 0\,dx=k\int_{0}^{1} x\,dx+k\int_{1}^{2} dx+k\int_{2}^{3} -x+3\,dx$$

$$=\frac{1}{2}\int_{0}^{1} x\,dx+\frac{1}{2}\int_{1}^{2} dx+\frac{1}{2}\int_{2}^{3} -x+3\,dx$$

$$=\frac{1}{2}\left(\frac{x^2}{2}\right)_0^1+\frac{1}{2}(x)_1^2+\frac{1}{2}\left(-\frac{x^2}{2}+3x\right)_2^3$$

$$=\frac{1}{2}\left(\frac{1^2}{2}-\frac{0^2}{2}\right)+\frac{1}{2}(2-1)+\frac{1}{2}\left(\left(-\frac{3^2}{2}+3(3)\right)-\left(-\frac{2^2}{2}+3(2)\right)\right)$$

$$=\frac{1}{2}\left(\frac{1}{2}\right)+\frac{1}{2}(1)+\frac{1}{2}\left(\left(-\frac{9}{2}+9\right)-(-2+6)\right)$$

$$=\frac{1}{4}+\frac{1}{2}+\frac{1}{2}\left(-\frac{9}{2}+9-4\right)=\frac{1}{4}+\frac{1}{2}+\frac{1}{2}\left(\frac{-9}{2}+5\right)=\frac{1}{4}+\frac{1}{2}+\frac{1}{2}\left(\frac{-9+10}{2}\right)=\frac{1}{4}+\frac{1}{2}+\frac{1}{4}=1$$

$$F(x)=1$$

Hence the distribution function F(x) is given by

$$F(x)=\begin{cases} 0 & \text{for } -\infty \le x < 0 \\ \dfrac{x^2}{4} & \text{for } 0 \le x < 1 \\ \dfrac{2x-1}{4} & \text{for } 1 \le x < 2 \\ \dfrac{-x^2+6x-5}{4} & \text{for } 2 \le x < 3 \\ 1 & \text{for } 3 \le x < \infty \end{cases}$$

Example 2.7 The cumulative distribution of continuous random variable X is given by

$$F(x)=\begin{cases} 0, & x<0 \\ x^2, & 0 \le x < 1/2 \\ 1-\dfrac{3}{25}(3-x), & 1/2 \le x < 3 \\ 0, & x \ge 0 \end{cases}$$

Find (i) Probability density function of X (ii) $P(|X| \le 1)$ and (iii) $P(1/3 \le X < 4)$

Solution:

We know that $f(x)=\dfrac{d}{dx}F(x)$

The points x = 0, ½ , 3 are points of continuity

$$\therefore f(x) = \begin{cases} 0, & x < 0 \\ 2x, & 0 \le x < \frac{1}{2} \\ \frac{6}{25}(3-x), & \frac{1}{2} \le x < 3 \\ 0, & x \ge 3 \end{cases}$$

$$P(|X| \le 1) = P(-1 \le X \le 1) = F(1) - F(-1) = \frac{3}{25}$$

$$P(\tfrac{1}{3} \le X < 4) = F(4) - F(\tfrac{1}{3}) = 1 - \frac{1}{9} = \frac{8}{9}$$

2.4 Discrete distributions

i) Binomial ii) Poisson iii) Normal

2.4.1 BINOMIAL DISTRIBUTION:

Binomial distribution is also known as Bernoulli distribution after the Swiss mathematician James Bernoulli (1654-1705) who discovered it in 1700 and was first published in 1712, eight years of his death. The distribution can be used in the following conditions

(i) The outcome of any trial can only take on two possible values, say success and Failure.
(ii) There is a constant probability p of success on each trial;
(iii) The experiment is repeated n times (i.e. n trials are conducted);
(iv) The trials are statistically independent (i.e. the outcome of past trials does not Affect subsequent trials);

Suppose an experiment is repeated 'n' times and each trail is independent. Let us assume that each trail results in two possible mutually exclusive and exhaustive outcomes i.e. success and failure. Let X is random variable represents total no. of successes in 'n' trails. Let the probability of success in each trail is p and the probability of failure is q=1-p and p remains constant from trail to trail. Now, we have to find out the probability of x successes in n trails.

Let us suppose that a particular order of outcomes of x successes in n repetitions be as follows

SSSSSFFFSSFS………FS(x number of successes and n-x failures)

Since, the trails are all independent the probability for the joint occurrence of the event is

pppppqqqppqp……..qp

= (pppppp…..x times)(qqqqqq…… (n-x) times)

= $p^x q^{n-x}$

Further in a series of n trails x successes and n-x failures can occur in ^nc_x ways. So, the required probability of x successes in n trails is

$$P(X=x) = {}^nc_x\, p^x q^{n-x},\quad x = 0,1,2,........,n$$

This is called probability distribution of Binomial random variable X or simply Binomial distribution. Symbolically this can be written as B(X; n, p)

Definition : A random variable X is said to be follow a binomial distribution if its probability function is given by

$$P(X=x) = {}^{n}c_x \; p^x q^{n-x}, \quad x = 0,1,2,........,n$$

and p + q =1

Where n, p is called parameters of the binomial distribution. Mean and variance of the Binomial distribution is np and npq

Example 2.8 Find the binomial distribution for which the mean is 4 and variance is 3

Solution

Mean =np Variance =npq

Given Mean= np=4

Variance =npq =3

$$\frac{npq}{np} = \frac{3}{4}$$

$$q = \frac{3}{4}$$

$$p = 1 - q = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\therefore np = 4$$

$$n\frac{1}{4} = 4$$

$$n = 16$$

The required binomial distribution is $P(X = x) = 16C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{n-x}$

Example 2.9 Find p for a binomial random variable X if n = 6 and if 9P[X=4] = P[X=2]

Solution:

Let X~ B(6,p)

$$\therefore P[X = x] = 6c_x p^x q^{n-x}, n = 0,1,2---6$$

Given that 9P[X=4] = P[X=2]

$$9 \times 6c_4 p^4 q^2 = 6c_2 p^2 q^4$$

$$9p^2 = q^2$$

$$9p2 = (1-p)^2$$

ie., $8p^2+2p-1 = 0$

ie, $p = \frac{1}{4} \; or \; -\frac{1}{2}$

But $p = -\frac{1}{2}$ is impossible

Hence $p = \frac{1}{4}$

Example 2.10 In a binomial distribution consisting of 5 independent trails, probabilities of 1 and 2

Successes are 0.4096 and 0.2048. Find the parameter of 'P' of the distribution

Soln : Let X ~ B(n,p) the probability mass function is

$P(X=x) = \left({}^{n}C_{x}\right)P^{x}q^{(n-x)}$ **; p+q =1 ; x =0,1,2,3,…….**

$P(X = 1) = \binom{5}{1}pq^{4} = 0.4096$ ---------------**(1)**

$P(X = 2) = \binom{5}{2}p^{2}q^{3} = 0.2048$ ---------------**(2)**

From (1) and (2) we get

$$= \frac{\binom{5}{1}pq^{4} = 0.4096}{\binom{5}{2}p^{2}q^{3} = 0.2048} = \frac{5q = 0.4096}{10p = 0.2048} = \frac{5(1-p)}{10p} = 2$$

$$= 5(1-p) = 2x10p$$

$$5 - 5p = 20p$$

$$5 = 20p + 5p$$

$$5 = 25p$$

$$==> 25p = 5$$

$$p = \frac{5}{25} = \frac{1}{5}$$

Example 2.11 Ten coins thrown simultaneously. Find the probability of getting at least 7 heads

Solution: Given n = 10, probability of getting a head = p = ½ , q = 1-p = ½

Probability mass function of binomial distribution is

$P(X = x) = nC_{x}p^{x}q^{n-x}$, x = 0, 1,2,…….10

ie., $P(X = x) = 10C_{x}(1/2)^{x}(1/2)^{10-x} = 10C_{x}(1/2)^{10}$

$P(X \geq 7) = P(X =7)+ P(X =8)+ P(X =9)+ P(X =10)$

$= 10C_7(1/2)^{10} +10C_8(1/2)^{10} + 10C_9(1/2)^{10} + 10C_{10}(1/2)^{10}$

$P(X \geq 7) = 0.172$

2.4.2 POISSON DISTRIBUTION:

Poisson distribution is a discrete probability distribution, which is the limiting case of the binomial distribution under certain conditions.

1. When n is very indefinitely very large
2. Probability of success is very small.
3. $np =\lambda$ is finite

Definition : A discrete random variable X is said to be follow a Poisson distribution if the probability mass function is given by

$$p(X = x) = P(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,2,3.........\infty$$

Where e = 2.7183 and $\lambda > 0$

Here λ is called the parameter of the Poisson distribution.

Example 2.12 The probability of an item to be defective is 0.01. Find the probability that a sample
of 100 items randomly selected will contain not more than one defective item.

Solution: Given p = 0.01, n= 100, mean $\lambda = np = 0.01\times 100 = 1$,

Probability mass function of Poisson distribution is $P(X =x) = \frac{e^{-\lambda}\lambda^x}{x!}$, x=0,1,2….

$$P (X \leq 1) = P(X =0) +P(X =1) = e^{-1}+\frac{e^{-1}.1}{1!} = 2e^{-1}$$

Example 2.13 It is known from the past experience that in a certain plant there are on the average
4 industrial accidents .Find the probability that in a given year there will be less than 4 accidents.

Solution:

Let X denote the number of accidents in a year.

Given $\lambda = 4$

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,2,3,…….$$

P(less than 4 accidents) = P(X < 4) = P(X =0)+ P(X =1) + P(X = 2)+ P(X = 3)

$$= \left(\frac{4^0 e^{-4}}{0!} + \frac{4^1 e^{-4}}{1!} + \frac{4^2 e^{-4}}{2!} + \frac{4^3 e^{-4}}{3!} \right) = 0.4335$$

$$= e^{-4} \left(\frac{4^0}{0!} + \frac{4^1}{1!} + \frac{4^2}{2!} + \frac{4^3}{3!} \right) = 0.4335$$

2.4.3 Normal Distribution:

A continuous random variable X is said to follow a Normal distribution with parameter mean μ and variance σ^2 if its probability density is given by

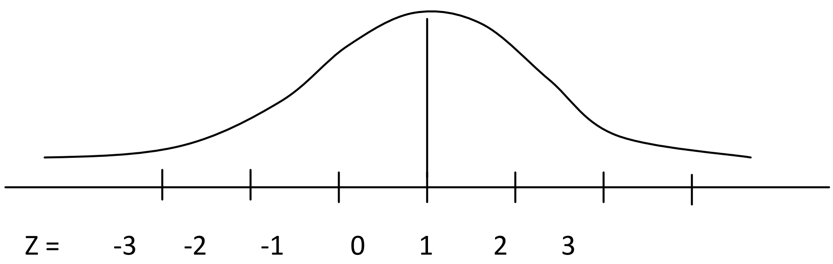
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

Characteristics of a normal probability distribution

1. The normal curve is bell-shaped and has a single peak at the exact center of the distribution.
2. The arithmetic mean, median, and mode of the distribution are equal and located at the peak.
3. Half the area under the curve is above and half is below this center point (peak).
4. The normal probability distribution is symmetrical about its mean.
5. It is asymptotic - the curve gets closer and closer to the x-axis but never actually touches it.

The standard normal probability distribution is a normal distribution with a mean of 0 and a standard deviation of 1 is called the standard normal distribution.

$$\mathbf{z} = \frac{\mathbf{(X - \mu)}}{\boldsymbol{\sigma}}$$



Example 2.14 X is a normal variate with mean 30 and S.D 5. Find the probabilities that (i) 26 ≤ X ≤ 40 (ii) X ≥ 4 5 (iii) |X - 30| > 5

Solution Here mean $\mu = 30$ and $\sigma = 5$

(i)When X = 26 and $Z = \frac{x-\mu}{\sigma} = \frac{26-30}{5} = -0.8$

When X = 40 and $Z = \frac{x-\mu}{\sigma} = \frac{40-30}{5} = 2$

P (26 ≤ X ≤ 40) = P (- 0.8 ≤ X ≤ 2)
= P(- 0.8 ≤ X ≤ 0) +P(0 ≤ X ≤ 2)
= P(0 ≤ X ≤ 0.8) +P(0 ≤ X ≤ 2) (symmetry)
= 0.2881 +0. 4772 = 0.7653

P (26 ≤ X ≤ 40) = 0.7653

(ii) When X =45 $Z = \frac{x-\mu}{\sigma} = \frac{45-30}{5} = 3$

P(X.≥ 45) = P(Z ≥ 3) = 0.5 - P(0 ≤ X ≤ 3) = 0.5 – 0.4986 = 0.0014

(iii) P(|X - 30| ≤ 5) = P(25 ≤ X ≤ 35) = P(-1 ≤ Z ≤ 1) = 2 P(0 ≤ Z ≤ 1) = 2 x 0.3413 =0.6826

P(|X - 30| > 5) = 1- P(|X - 30| ≤ 5) = 1 – 0.6826 = 0.3174

Example 2.15 The weight of adult cocker spaniel are normally distributed with a mean $\mu = 25$ lb and a standard deviations $\sigma = 3$ lb. find the probability that a) cocker's weight is less than 23 lb b) weight is between 20 lb and 27 lb c) weight is more than 29 lb

Solution

a) Find the probability that the cocker's weight is less than 23 lb.

   P(x < 23) = $P\left(z < \frac{23-25}{2}\right)$ = P(z < -.67) = .2514

b) Find the probability that the weight is between 20 lb and 27 lb.

   P(20 < x < 27) = $P\left(\frac{20-25}{3} < z < \frac{27-25}{3}\right)$ =

   P(-1.67 < z <.67) = .7486 - .0475 = .7011

c) Find the probability that the weight is more than 29 lb.

   P(x > 29) = $P\left(z > \frac{29-25}{3}\right)$ = P(z > 1.33)

   = 1 - .9082 = .0918

Example 2.16 In a distribution exactly normal, 10.03% of the items are under 25 kilogram weight and 8.97% of the items are under 70 kilogram weight. What are the mean and variance of the distribution?

Solution

Let x denote the weight (in kilograms) of the item.

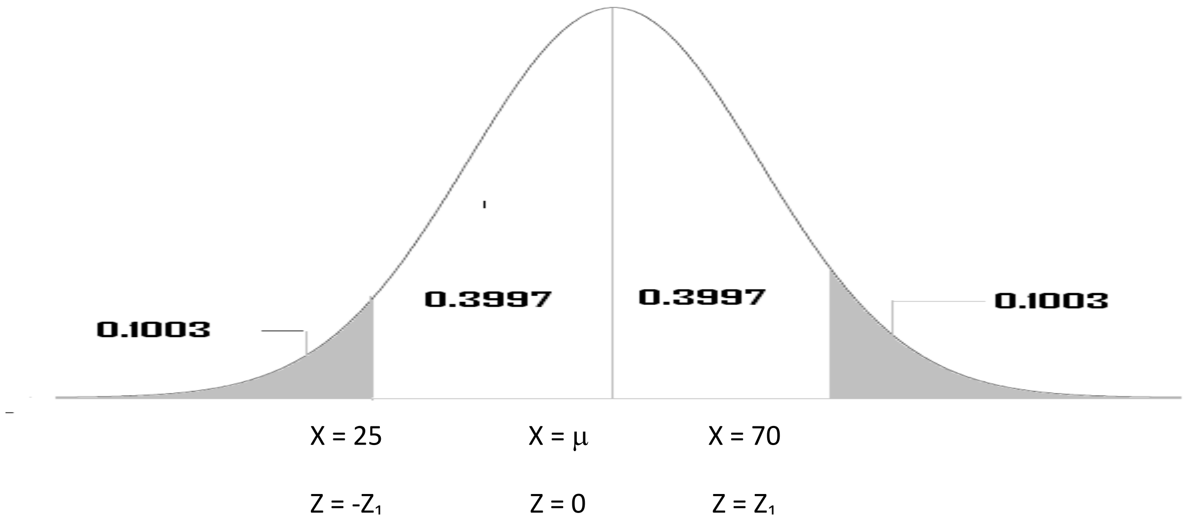
If $X \sim N(\mu, \sigma^2)$ then given are

$P(X < 25) = 0.1003$ and $P(X<70) = 0.8997$

The points x =25 and x = 70 are located as shown below

When X = 25 , $Z = \frac{25-\mu}{\sigma} = -Z_1 (\text{say})$ -----------------(1)

When X = 70 $Z = \frac{70-\mu}{\sigma} = Z_2 (\text{say})$ ----------------------(2)



From the diagram

$P(Z < -Z_1) = 0.1003$

$P(Z < Z_2) = 0.8997$ and now $P(0<Z<Z_2) = 0.3997 \Rightarrow Z_2 = 1.28$ (from normal table)

$P(Z < -Z_1) = 0.1003 \Rightarrow P(Z>Z_1) = 0.1003$

$P(0<Z<Z_1) = 0.5 - 0.1003 = 0.3997 \Rightarrow Z_1 = 1.28$ (from normal table)

Substuting the values of Z_1 and Z_2 in (1) and (2)

$$\frac{25-\mu}{\sigma} = -1.28$$

$$\Rightarrow 25 - \mu = -1.28\sigma \quad \text{---------------(3)}$$

$$\frac{70-\mu}{\sigma}=1.28 \qquad \text{--------------------(4)}$$

$$\Rightarrow 70-\mu=1.28\sigma$$

Solving the equation 3 and 4 we get $\mu = 47.5$ and $\sigma = 17.578$

Unit – III

3. TWO DIMENSIONAL RANDOM VARIABLES

Let S be the sample space of a random experiment and let X and Y be two random variables defined on S. Thus X=X(s) and Y=Y(s) are two functions which assign real numbers x and y to each outcome $s \in S$. Then the pair (X,Y) is called two dimensional random variable.

3.1 JOINT PROBABILITY MASS FUNCTIONS

Let X and Y be the two dimensional discrete random variable. Let us suppose that X can assume 'n' values $x_1, x_2, \ldots .x_n$ and y can assume 'm' values $y_1, y_2, \ldots, y_m$. Let us consider the probability of ordered pair (x_i, y_j) where i = 1, 2, 3,….,n and j = 1, 2, 3,….,m defined by

$$P_{ij} = P(X = x_i, Y = y_j) = P(x_i, y_j)$$

The function P(x,y) defined for any ordered pair (x,y) is called joint probability function of x and y which satisfied the following condition

(i) $P_{ij} \geq 0 \ \forall$ i, j (ii) $\sum_{i=1}^{n}\sum_{j=1}^{m} P_{ij} = 1$ and its represented in a tabular form as follows:

Y \ X	y_1	y_2	y_3		y_m	P(X=x) Total
x_1 x_2 . . . . Xn	P_{11} P_{12} . . . . Pn_1	P_{12} P_{22} . . . . Pn_2	P_{13} P_{23} . . . . Pn_3	… … …	P_1m P_2m . . . . P_{nm}	P_1 P_2 . . . . Pn.
Total P(Y = y)	$P_{.1}$	$P_{.2}$	$P_{.3}$		P.m	$\Sigma P_{.1} = \Sigma .P$

3.2 Marginal Probability Function

Let (X, Y) be a two dimensional discrete random variable then which takes up countable number of values (x_i, y_j). Then the probability distribution of X and is determined as follows.

$$P(X = x_i) = P(X = x_i \cap Y = y_1) + P(X = x_i \cap Y = y_2) + \cdots + P(X = x_i \cap Y = y_m)$$

$$= P_{i1} + P_{i2} + \cdots + P_{im}$$

$$\sum_{j=1}^{m} P_{ij} = P_{i.}$$

is known as the marginal probability mass function of X.

Similarly , the marginal probability mass function of random variable Y

$$P(Y = y) = \sum_{i=1}^{n} P_{ij} = P_{\cdot j}$$

3.3 Joint Probability Density Function

If X and Y are continuous random variable then their joint probability density function *f(x,y)* if $P\left\{x - \frac{dx}{2} \le X \le x + \frac{dx}{2}, y - \frac{dy}{2} \le Y \le y + \frac{dy}{2}\right\} = f(x,y)\,dx\,dy$ and provided *f(x,y)* satisfies the following conditions

$$i)\quad f(x,y) \ge 0$$

$$ii)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

3.4 Marginal density function

When (X, Y) is a two dimensional continuous random variable then the marginal density function of a random variable X and Y is defined as

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad , \quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

3.5 Marginal Distribution Functions

The Marginal distribution function X and Y respectively with respect to joint distribution function F_{xy} (x, y) is

$$F_x(X) = P(X \leq x) = P(X \leq x.\ Y < \infty)$$

$$= \lim_{y \to \infty} F_{xy}(x, y) = F_{XY}(X, \infty)$$

$$F_Y(y) = P(Y \leq y) = P(Y \leq y.\ X < \infty)$$

$$= \lim_{y \to \infty} F_{xy}(x, y)$$

$$= F_{xy}(\infty, y)$$

$F_x(X)$ is termed as marginal distribution function of X corresponding to the joint distribution function [F_{xy} (x, y)] Similarly; $F_y(Y)$ is termed as marginal distribution function of Y corresponding to the joint distribution function [Fxy (x, y)].

If (X,Y) is a continuous random variable then the cumulative distribution function is defined as

$$F(X \leq x, Y \leq y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f(x, y)\ dx.dy$$

3.6 CONDITIONAL PROBABILITY DISTRIBUTION

Let (X, Y) be a discrete two dimensional random variable then the conditional probability mass function in of X given Y = y and conditional probability function of Y given that $X=x_j$

$$P(X / Y) = \frac{P(X = x_i, Y = y_j)}{P(Y = y_j)}$$

$$P(Y = y_j / X = x_i) = \frac{P(X = x_i, Y = y_j)}{P(X = x_i)}$$

If $f(X/Y)$ is a two dimensional continuous random variable then,

$$P(X = x / Y = y) = \frac{f(x, y)}{f_Y(y)}$$

$$P(Y = y / X = x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

Example3.1, From the following Bivariate probability distribution of X and Y. Find (i) P(X ≤ 1, Y = 2), (ii) P(X ≤ 1), (iii) P(Y = 3), (iv) P(Y ≤ 3), (v) P(X ≤ 3),

Solution

X \ Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

Solution

X \ Y	1	2	3	4	5	6	Total
0	0	0	1/32	2/32	2/32	3/32	8/32
1	1/16	1/16	1/8	1/8	1/8	1/8	10/16
2	1/32	1/32	1/64	1/64	0	2/64	8/64
Total	3/32	3/32	11/64	13/64	12/64	16/64	1

(i) P(X ≤ 1, Y = 2) = P(X = 0, Y = 2) + P(X = 1, Y = 2) = 0 + 1/16 =1/16

(ii) P(X ≤ 1) = P(X = 0) + P(X = 1) = 8/32 + 10/16 = 28/32 = 7/8

(iii) P(Y = 3) = 11/64

(iv) P(Y ≤ 3) = P(Y = 1) + P(Y = 2) + P(Y = 3) = 3/32 + 3/16 + 11/64 = 23/64

(v) P(X ≤ 3, Y ≤ 4) = P(X = 0, Y = 1) + P(X = 1, Y = 1) + P(X = 2, Y = 1)

+ P(X = 0, Y = 2) + P(X = 1, Y = 2) + P(X = 2, Y = 2)

+ P(X = 0, Y = 3) + P(X = 1, Y = 3) + P(X = 2, Y = 3)

+ P(X = 0, Y = 4) + P(X = 1, Y = 4) + P(X = 2, Y = 4)

$$= \left(0+\frac{1}{16}+\frac{1}{32}\right)+\left(0+\frac{1}{16}+\frac{1}{32}\right)+\left(\frac{1}{32}+\frac{1}{8}+\frac{1}{64}\right)+\left(\frac{2}{32}+\frac{1}{8}+\frac{1}{64}\right)$$

$$= \frac{3}{32}+\frac{3}{32}+\frac{11}{64}+\frac{13}{64} = \frac{12+11+13}{64} = \frac{36}{64}$$

$$P(X \le 3,\ Y \le 4) = \frac{36}{34}$$

Example 3.2 A two dimensional random variable X and Y have a joint probability function

$P(x, y) = \dfrac{2x+y}{27}$ Where X and Y can assume that value 0, 1 and 2. Find

(i). The marginal probability function of a *x* and *y*

(ii). The conditional distribution of *x/y* and *y/x*

Solution

The joint probability mass function P(x, y) = $\dfrac{2x+y}{27}$; x = 0, 1, 2 ; y = 0, 1,2

X \ Y	0	1	2	Total P(X=*x*)
0	0	1/27	2/27	3/27
1	2/27	3/27	4/27	9/27
2	4/27	5/27	6/27	15/27
Total P(Y = y)	6/27	9/27	12/27	1

(i) The marginal probability function of a x

x	0	1	2
P(X=*x*)	3/27	9/27	15/27

The marginal probability function of a y

y	0	1	2
P(Y= *y*)	6/27	9/27	12/27

(ii) Conditional probability P(x/y)

$$P(y=0/x=0) = \frac{P(x=0/y=0)}{P(x=0)} = 0/37 = 0$$

$$P(y=0/x=1) = \frac{P(x=0/y=1)}{P(x=1)} = \frac{2/27}{9/27} = \frac{2}{9}$$

$$P(y=0/x=2)=\frac{P(x=0/y=2)}{P(x=2)}=\frac{4/27}{15/27}=\frac{4}{15}$$

$$P(y=1/x=0)=\frac{P(x=1/y=0)}{P(x=0)}=\frac{1/27}{3/27}=\frac{1}{3}$$

$$P(y=1/x=1)=\frac{P(x=1/y=1)}{P(x=1)}=\frac{3/27}{9/27}=\frac{1}{3}$$

$$P(y=1/x=2)=\frac{P(x=1/y=2)}{P(x=2)}=\frac{5/27}{15/27}=\frac{1}{3}$$

$$P(y=2/x=0)=\frac{P(x=2/y=0)}{P(x=0)}=\frac{2/27}{3/27}=\frac{2}{3}$$

$$P(y=2/x=1)=\frac{P(x=2/y=1)}{P(x=1)}=\frac{4/27}{9/27}=\frac{4}{9}$$

$$P(y=2/x=2)=\frac{P(x=2/y=2)}{P(x=2)}=\frac{6/27}{15/27}=\frac{2}{5}$$

Conditional probability P(x/y)

$$P(x=0/y=0)=\frac{P(x=0/y=0)}{P(y=0)}=0/{}^{6}\!/_{27}=0$$

$$P(x=0/x=1)=\frac{P(x=0/y=1)}{P(y=1)}=\frac{1/27}{9/27}=\frac{1}{9}$$

$$P(x=0/y=2)=\frac{P(x=0/y=2)}{P(y=2)}=\frac{2/27}{12/27}=\frac{1}{6}$$

$$P(x=1/y=0)=\frac{P(x=1/y=0)}{P(y=0)}=\frac{2/27}{6/27}=\frac{1}{3}$$

$$P(x=1/y=1)=\frac{P(x=1/y=1)}{P(y=1)}=\frac{3/27}{9/27}=\frac{1}{3}$$

$$P(x=1/y=2)=\frac{P(x=1/y=2)}{P(y=2)}=\frac{4/27}{12/27}=\frac{1}{3}$$

$$P(x=2/y=0)=\frac{P(x=2/y=0)}{P(y=0)}=\frac{4/27}{6/27}=\frac{2}{3}$$

$$P(x=2/x=1)=\frac{P(x=2/y=1)}{P(y=1)}=\frac{5/27}{9/27}=\frac{5}{9}$$

$$P(x=2/y=2)=\frac{P(x=2/y=2)}{P(y=2)}=\frac{6/27}{12/27}=\frac{1}{2}$$

Example 3.3 If X and Y are two random variable having a joint density function,

$$f(x,y)=\begin{cases}\dfrac{6-x-y}{8} & ; \quad 0<x<2 \\ & \quad\ 2\le y<4 \\ 0 & ; \quad \text{otherwise}\end{cases}$$

Find (i) P(X < 1 $\cap$ Y < 3), (ii) P(X < 1/Y < 3), (iii) P(X + Y < 3)

Solution

$$P(X<1\cap Y<3)=\int_2^3\int_0^1 f(x,y)\,dx\,dy$$

$$=\int_2^3\left[\int_0^1 \frac{1}{8}(6-x-y)\,dx\right]dy$$

$$=\frac{1}{8}\int_2^3\left[6x-x^2/2-xy\right]_0^1\,dy$$

$$=\frac{1}{8}\int_2^3\left[\left(6(1)-\frac{(1)^2}{2}-(1)y\right)-\left(6(0)-\frac{(0)^2}{2}-(0)y\right)\right]dy$$

$$=1/8\int_2^3(6-\frac{1}{2}-y)\,dy$$

$$=1/8\int_2^3\left(11/2-y\right)\,dy$$

$$=\frac{1}{8}\left[\frac{11y}{2}-\frac{y^2}{2}\right]_2^3$$

$$= \frac{1}{8}\left[\left(\frac{11}{2}(3) - \frac{(3)^2}{2}\right) - \left(\frac{11}{2}(2) - \frac{(2)^2}{2}\right)\right]$$

$$= \frac{1}{8}\left[\left(\frac{3^2}{2} - \frac{9}{2}\right) - \left(\frac{22}{2} - \frac{4}{2}\right)\right] = \frac{1}{8}(12-9) = \frac{3}{8}$$

(ii) P(X < 1/Y < 3)

$$P(X < 1/Y < 3) = \frac{P(X < 1 \cap Y < 3}{P(Y < 3)}$$

$$f_Y(\text{y}) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_0^2 \frac{1}{8}(6 - x - y)\,dx$$

$$= \frac{1}{8}\int_0^2 (6 - x - y)\,dx$$

$$P(X < 1/Y < 3) = \frac{\frac{3}{8}}{\frac{5}{8}} = \frac{3}{5}$$

(iii) P(X + Y < 3)

$$= \int_0^1 \int_2^3 \frac{6 - x - y}{8}\,dy\,dx$$

$$= \frac{1}{8}\int_0^1 \left(6y - xy - \frac{y^2}{2}\right)_2^{3\text{-x}} dx$$

$$= \frac{1}{8}\int_0^1 \left[\left(6(3-x) - x(3-x) - \frac{(3-x)^2}{2}\right) - \left(6(2) - x(2) - \frac{2^2}{2}\right)\right] dx = \frac{5}{24}$$

Example 3.4 The joint probability density function of a two dimensional random variable (*X*, *Y*) is given by

$$f(x) = \begin{cases} 2 & ; \quad 0 < x < 1,\ 0 < y < x \\ 0, & \quad \text{elsewhere} \end{cases}$$

(i) Find the marginal density function of X and Y

(ii) Find the conditional density function of Y given X = x and X given Y = x.

(iii) Check for independence of X and Y

Solution

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_0^x 2\, dy = 2[y]_0^x$$

$$= 2\,(x - 0) = 2x$$

$f_X(x) = 2x \quad 0 < x < 1$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx$$

$$= \int_y^1 2\, dx = 2[x]_y^1 = 2[1 - y]$$

$f_Y(y) = 2\text{-}2y,\ 0 < y < 1$

The conditional density function of X and Y is,

$$f\left(\frac{y}{x}\right) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x}$$

(ii) If X and Y are independent, the $f_{xy}(x,y) = f_X(x)\ f_Y(y)$

R.H.S:

$f_X(x)\, f_Y(y) = 2x \,.\, 2(1\text{-}y) = 4x\,(1\text{-}y) = f(x, y)$

$\therefore$ X and Y are not independent.

Example 3.5 The joint probability density function of two dimensional random variable

variable $f(x, y) = \begin{cases} k\ x\ (x\text{-}y); & 0 < x < 2;\ -x < y < x \\ 0 \quad ; & \text{otherwise} \end{cases}$

(i) Find the constant 'k' and (ii) the marginal density function of a r.v X and Y.

Solution

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{\infty} f(x, y)\, dy \quad dx = 1$$

$$\int_{0}^{2} \int_{-x}^{x} kx(x - y)\, dy \quad dx = 1$$

$$k\int_{0}^{2} \int_{-x}^{x} (x^2 - xy)\, dy \quad dx = 1$$

$$k\int_{0}^{2} \left[x^2 y - \frac{xy^2}{2} \right]_{-x}^{x} \quad dx = 1$$

$$k\int_{0}^{2} \left[\left(x^2(x) - x.\frac{x^2}{2} \right) - \left(x^2(-x) - \frac{x(-x)^2}{2} \right) \right] dx = 1$$

$$k\int_{0}^{2} \left[\left(x^3 - \frac{x^3}{2} \right) - \left((-x^3) - \frac{x^2}{2} \right) \right] dx = 1$$

$$k\int_{0}^{3} \left[(x^3 - \frac{x^3}{2} - x^3 + \frac{x^3}{2} \right] dx = 1$$

$$k\int_{0}^{3} 2x^3 dx = 1$$

$$k\left[\frac{2x^4}{4} \right]_{0}^{2}$$

$$k\left[\frac{2(2)^4}{4} - \frac{0^4}{4} \right] = 1$$

$$k\left[\frac{32}{4} - 1 \right] = 1$$

$$8k = 1$$

$$k = \frac{1}{8}$$

ii) Marginal density function of X

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$\int_{-x}^{x} kx(x-y)\, dy \ = \frac{1}{8}\int_{-x}^{x} (x^2 - xy)\, dy$$

$$= \frac{1}{8}\left[x^2 y - \frac{xy^2}{2}\right]_{-x}^{x} = \frac{1}{8}\left[\left(x^3 - \frac{x^3}{2}\right) - \left(x^2(-x) - \frac{x(-x)^2}{2}\right)\right]$$

$$= \frac{1}{8}\left[x^3 - \frac{x^3}{2} + x^3 + \frac{x^3}{2}\right] = \frac{2x^3}{8}$$

$$f_X(x) = \frac{x^3}{4}$$

Marginal density function of Y

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

$$= \int_{-y}^{2} kx(x-y)dx = \ = \frac{1}{8}\int_{-y}^{2} (x^2 - xy)dx = \frac{1}{8}\left[\frac{x^3}{3} - \frac{x^2 y}{2}\right]_{-y}^{2}$$

$$= \frac{1}{8}\left\{\left(\frac{2^3}{3} - \frac{2^2 y}{2}\right) - \left(\frac{(-y)^3}{3} - \frac{(-y)^2 y}{2}\right)\right\}$$

$$= \frac{1}{8}\left\{\left(\frac{8}{3} - \frac{4y}{2}\right) - \left(\frac{-y^3}{3} - \frac{y^3}{2}\right)\right\}$$

$$= \frac{1}{8}\left\{\frac{8}{3} - \frac{4y}{2} + \frac{y^3}{3} + \frac{y^3}{2}\right\}$$

$$= \frac{1}{8}\left(\frac{16 - 12y + 2y^3 + 3y^3}{6}\right)$$

$$f_Y(y) = \frac{1}{8}\left(\frac{5y^3 - 12y + 16}{6}\right)$$

$$f_Y(y) \ = \ \frac{5y^3 - 12y + 16}{48}, -2 < y < 0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

$$= \int_{y}^{2} kx(x-y)dx$$

$$= \frac{1}{8}\int_{y}^{2} (x^2 - xy)dx$$

$$= \frac{1}{8}\left[\frac{x^3}{3} - \frac{x^2 y}{2}\right]_y^2 = \frac{1}{8}\left[\left(\frac{2^3}{3} - \frac{2^2 y}{2}\right) - \left(\frac{y^3}{3} - \frac{y^2.y}{2}\right)\right]$$

$$= \frac{1}{8}\left[\left(\frac{8}{3} - \frac{4y}{2}\right) - \left(\frac{y^3}{3} - \frac{y^3}{2}\right)\right]$$

$$= \frac{1}{8}\left[\frac{8}{3} - \frac{4y}{2} - \frac{y^3}{3} - \frac{y^3}{2}\right]$$

$$= \frac{1}{8}\left[\frac{16 - 12y - 2y^3 + 3y^3}{6}\right]$$

$$= \frac{1}{8}\left[\frac{16 - 12y + y^3}{6}\right]$$

$$f_Y(y) = \frac{y^3 - 12y + 16}{48}, 0 < y < 2$$

Example 3.6 The joint distribution of X and Y is given by *f*(*x*,*y*) = $f(x,y) = 4xye^{-(x^2+y^2)}$ x ≥ 0, y ≥ 0. Test whether x and y are independent are not. Find (i) conditional density of *f*(*x*/*y*).

Solution

Marginal probability density function of X is given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$= \int_{0}^{\infty} 4xy \ e^{-(x^2+y^2)}dy$$

$$= \int_0^\infty 4xy\ e^{-x^2} . e^{-y^2} dy \qquad Put \quad y^2 = t$$

$$2y \ \ dy = dt$$

$$= 4xe^{-x^2} \int_0^\infty y\ e^{-y^2} dy$$

$$= 4xe^{-x^2} \int_0^\infty y\ e^{-t}\ \frac{dt}{2y}$$

$$= 2xe^{-x^2} \int_0^\infty e^{-t} dt$$

$$= 2xe^{-x^2} \left[\frac{e^{-t}}{-1} \right]_0^\infty$$

$$= 2xe^{-x^2} \left[\left(\frac{e^{-\infty}}{-1} \right) - \left(\frac{e^{-0}}{-1} \right) \right]$$

$$= f_X(x) = 2x\ e^{-x^2}\ x \ \geq \ 0$$

Marginal density function of Y,

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y) dx$$

$$= \int_0^\infty 4xy \ \ e^{-(x^2+y^2)} dy$$

$$= \int_0^\infty 4xy\ e^{-x^2} . e^{-y^2} dy \qquad Put \quad x^2 = t$$

$$2x \ \ dx = dt$$

$$= 4ye^{-y^2} \int_0^\infty x\ e^{-x^2} dx$$

$$= 4ye^{-y^2} \int_0^\infty x\ e^{-t}\ \frac{dt}{2x}$$

$$= 2ye^{-y^2} \int_0^\infty e^{-t} dt$$

$$= 2xe^{-x^2} \left[\frac{e^{-t}}{-1} \right]_0^\infty$$

$$= 2xe^{-x^2} \left[\left(\frac{e^{-\infty}}{-1} \right) - \left(\frac{e^{-0}}{-1} \right) \right]$$

$$f_y(y) = 2ye^{-y^2},\ y \geq 0$$

If X and Y are independent then.

f_{XY} (x, y) = f_x(x). f_y(y)

$$= 2ye^{-y^2},\ 2xe^{-x^2} = 4xy\ e^{-x^2},\ e^{-y^2} = f_{XY}(x, y)$$

$\therefore$ X and Y are independent.

(ii) **Conditional density function**

$$f(x/y) = \frac{f(x,y)}{f_y(y)} = \frac{4xy\ e^{-(x^2+y^2)}}{2y\ e^{-y^2}} = 2x\ e^{-x^2}$$

Example 3.7 The joint distribution function of two random variable of (x, y) is given by,

$$F(x, y) = \begin{cases} 1 - e^{-x} - e^{-y} + e^{-(x+y)}, x \geq 0, y \geq 0 \\ 0 \qquad\qquad\qquad\qquad\text{, otherwise} \end{cases}$$

Find (i) Marginal density function of r.v. X and Y

(ii) Check X and Y are independent

Solution

To find the joint density function,

$$f(x, y) = \frac{\partial^2}{\partial x\ \partial y} F(x, y)$$

$$= \frac{\partial^2}{\partial x\ \partial y}\left[1 - e^{-x} - e^{-y} + e^{-(x+y)}\right]$$

$$= \frac{\partial^2}{\partial x\ \partial y}\left[1 - e^{-x} - e^{-y} + e^{-x}.e^{-y}\right]$$

$$= \frac{\partial}{\partial x}\left[0 - 0 - (-1)e^{-y} + e^{-x}e^{-y}(-1)\right]$$

$$= \frac{\partial}{\partial x}\left[0 - 0 - (-1)e^{-y} - e^{-x}.e^{-y}\right]$$

$$= \frac{\partial}{\partial x}\left[e^{-y} - e^{-x}.e^{-y}\right] = (0 - (-1)e^{-x}.e^{-y}) = e^{-x}.e^{-y} = e^{-(x+y)}$$

$$f(x,y) = e^{-(x+y)}$$

Marginal density function of X and Y

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$= \int_0^{\infty} e^{-(x+y)}dy = \int_0^{\infty} e^{-x}\ e^{-y}dy = e^{-x}\int_0^{\infty} e^{-y}dy = e^{-x}\left[\frac{e^{-y}}{-1}\right]_0^{\infty}$$

$$= e^{-x}\left[\left(\frac{e^{-\infty}}{-1}\right) - \left(\frac{e^{-0}}{-1}\right)\right]$$

$$= e^{-x}\left[0-(-1)\right] = e^{-x}$$

$$f_X(x) = e^{-x},\ x \geq 0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

$$= \int_0^{\infty} e^{-(x+y)}dx = \int_0^{\infty} e^{-x}\ e^{-y}dx = e^{-y}\int_0^{\infty} e^{-x}dx = e^{-y}\left[\frac{e^{-x}}{-1}\right]_0^{\infty}$$

$$= e^{-y}\left[\left(\frac{e^{-\infty}}{-1}\right) - \left(\frac{e^{-0}}{-1}\right)\right]$$

$f_Y(y) = e^{-y},\ y \geq 0$

If X and Y are independent, then

$f_{XY}(xy) = f_x(x).\ f_y(y)$

$= e^{-x}\ .\ e^{-y}$

$= e^{-(x+y)} = f_{XY}(xy)$

$\therefore$ X and Y are independent.

Example 3.8 The J.P.d.f of two dimensional random variable x and y is given by,

$$f(x,y) = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4} \quad ; \quad 0 < x < \infty \quad ; \quad 0 \leq y < \infty$$

Find (i) Marginal density of *x* and *y* and conditional distribution x/y and y/x.

Solution

$$fx(x) = \int_{-\infty}^{\infty} f(x,y)dy$$

$$=\int_0^\infty \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}dy$$

$$=\frac{9}{2}\int_0^\infty \frac{(1+x+y)}{(1+x)^4(1+y)^4}dy$$

$$=\frac{9}{2(1+x)^4}\int_0^\infty \frac{(1+x+y)}{(1+y)^4}dy$$

$$=\frac{9}{2(1+x)^4}\int_0^\infty \left[\frac{x}{(1+y)^4}+\frac{(1+y)}{(1+y)^4}\right]dy$$

$$=\frac{9}{2(1+x)^4}\int_0^\infty \left[x(1+y)^{-4}+(1+y)^{-3}\right]dy$$

$$=\frac{9}{2(1+x)^4}\left[\frac{x(1+y)^{-3}}{-3}+\frac{(1+y)^{-2}}{-2}\right]_0^\infty$$

$$=\frac{9}{2(1+x)^4}\left[\left(\frac{x(1+\infty)^{-3}}{-3}+\frac{(1+\infty)^{-2}}{-2}\right)-\left(\frac{x(1+0)^{-3}}{-3}+\frac{x(1+0)^{-2}}{-2}\right)\right]$$

$$=\frac{9}{2(1+x)^4}\left[\left(0+0\right)-\left(\frac{x}{-3}+\frac{y}{-2}\right)\right]$$

$$=\frac{9}{2(1+x)^4}\left[\frac{x}{3}+\frac{1}{2}\right]$$

$$=\frac{9}{2(1+x)^4}\left[\frac{2x+3}{6}\right]$$

$$f_X(x)=\frac{3(2x+3)}{4(1+x)^4},\quad 0\le x<\infty$$

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)dx$$

$$=\int_0^\infty \frac{9(1+x+y)}{2(1+x)^4(1+y)^4}dx$$

$$=\frac{9}{2(1+y)^4}\int_0^\infty \frac{(1+x+y)}{(1+x)^4}dx$$

$$=\frac{9}{2(1+y)^4}\int_0^\infty \left[\frac{1+x}{(1+x)^4}+\frac{y}{(1+x)^4}\right]dx$$

$$= \frac{9}{2(1+y)^4} \int_0^{\infty} \left[(1+x)^{-3} + y(1+x)^{-4} \right] dy$$

$$= \frac{9}{2(1+y)^4} \left[\frac{(1+x)^{-3}}{-2} + \frac{y(1+x)^{-3}}{-3} \right]_0^{\infty}$$

$$= \frac{9}{2(1+y)^4} \left[\left(\frac{(1+\infty)^{-2}}{-2} + \frac{y(1+\infty)^{-3}}{-3} \right) - \left(\frac{(1+0)^{-2}}{-2} + \frac{y(1+0)^{-3}}{-3} \right) \right]$$

$$= \frac{9}{2(1+y)^4} \left[(0+0) - \left(\frac{1}{-2} + \frac{y}{-3} \right) \right]$$

$$= \frac{9}{2(1+y)^4} \left[\frac{1}{2} + \frac{y}{3} \right]$$

$$= \frac{9}{2(1+y)^4} \left[\frac{3+2y}{6} \right]$$

$$f_Y(y) = \frac{3(2y+3)}{4(1+y)^4}, \quad 0 \le y < \infty$$

(ii) Conditional probability

$$f(x/y) = \frac{f(x,y)}{f_y(y)}$$

$$= \frac{\dfrac{9(1+x+y)}{2(1+x)^4(1+y)^4}}{\dfrac{3(2y+3)}{4(1+y)^4}} = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4} \frac{4(1+y)^4}{3(2y+3)}$$

$$= \frac{6(1+x+y)\ (1+y)^4}{(1+x)^4(1+y)^4(2y+3)} = \frac{6(1+x+y)}{(1+x)^4(2y+3)}$$

$$f(y/x) = \frac{f(x,y)}{f_X(x)}$$

$$= \frac{\dfrac{9(1+x+y)}{2(1+x)^4(1+y)^4}}{\dfrac{3(2x+3)}{4(1+x)^4}} = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4} \frac{4(1+x)^4}{3(2x+3)}$$

$$f(y/x) = \frac{6(1+x+y)\ (1+x)^4}{(1+x)^4(1+y)^4(2x+3)}$$

3.7 Transformation of One dimensional Random Variable

Let X be a random variable defined one sample space S and Y = g(X) be a strictly monotonic increasing or decreasing function such that Y is also random variable defined on S. Let $f_Y(y)$ be the probability density function of Y. Then

$$f_Y(y) = \frac{d}{dy}\left(F_Y(y)\right) \text{ Where}$$

$$\begin{aligned} F_Y(y) &= P[Y \le y] \\ &= P[g(X) \le y] \\ &= P[X \le g^{-1}(y)] \\ &= P[X \le x] \\ &= F_X(x) \ \ where\ x = g^{-1}(y) \end{aligned}$$

Differentiating both sides with respect to y, we have

$$\frac{d}{dy}F_Y(y) = \frac{d}{dy}F_X(x)$$

$$f_Y(y) = \frac{d}{dy}F_X(x)\left|\frac{dx}{dy}\right|$$

$$\therefore f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

3.8 Transformation of Two dimensional Random Variable

Let (X,Y) be a continuous random variable with joint probability density function *f(x,y)*. Let U and V be the transformations such that U = *u(x,y),* V=*v(x,y)*. Then the joint probability density function of (U,V) is

$$g(u,v) = f(x,y)\left|J\right|$$

Where J is the Jacobian transformation

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial x}{\partial u} & \frac{\partial x}{\partial u} \end{vmatrix}$$

Unit –IV

4.1 MATHEMATICAL EXPECTATON

The 'average' value of a random phenomenon is also termed as its mathematical expectation or expected value. Once we have constructed the probability distribution for a random variable, to compute a mean or expected value of the random variables, where the weights are probabilities associated with the corresponding values. The mathematical expression for computing the expected value of a discrete random variable X with the probability mass function and computing the expected value of a continuous as random variable X with the probability density function are denoted byE(X)

$$E(X)=\begin{cases}\sum_{i=1}^{n} x_i\, P(X=x_i) & \textit{for discrete random variable}\\[2ex] \int_{-\infty}^{\infty} x\, f(x)\, dx & \textit{for continuous random variable}\end{cases}$$

4.1.1 Properties of Expectation

Property 1. Addition Theorem of Expectation

If X and Y are random variables then E(X + Y) = E (X) + E(Y), provided all the expectation exists.

Proof

Let X and Y be a continuous random variables with joint p.d.f $f_{XY}(x, y)$ and marginal probability density functions of $f_X(x)$ and $f_Y(y)$ respectively.

$$E(X)=\int_{-\infty}^{\infty} x\, f(x)\, dx \qquad E(Y)=\int_{-\infty}^{\infty} y\, f(y)\, dy$$

$$E(X+Y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x+y)\, f_{XY}(x,y)\, dx\, dy$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{XY}(x,y)dx.dy+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{XY}(x,y)dx\, dy$$

$$=\int_{-\infty}^{\infty} x\left[\int_{-\infty}^{\infty} f_{XY}(x,y)dy\right]dx+\int_{-\infty}^{\infty} y\left[\int_{-\infty}^{\infty} f_{XY}(x,y)dx\right]dy=\int_{-\infty}^{\infty} x\, f_X(x)dx+\int_{-\infty}^{\infty} y\, f_Y(y)dy$$

E (X + Y) = E (X) + E (Y)

Property 2: Multiplication theorem of Expectation

If X and Y are independent random variables, then E(XY) = E(X), E(Y).

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \; f_{XY}(x,y)dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_X(x) f_Y(y)\; dx.\, dy$$ X, Y are independent

$$= \int_{-\infty}^{\infty} x\, f_X(x)dx \int_{-\infty}^{\infty} y\, f_Y(y)dy$$

E(XY)= E (X) . E (Y)

Property 3 If X is a random variable and 'a' is constant.

(i) E[a ψ(X)] = a E[ψ(X)] (ii) E[ψ(X) + a] = E[ψ(X)] + a

Where ψ (X) is a function of X, is a r.v and all the expectation are exists.

Proof (i)

$$E[a\ \psi(X)] = \int_{-\infty}^{\infty} a\ \psi(x) f(x)\ dx = a\int_{-\infty}^{\infty} \psi(x) f(x)\ dx$$

$$E[a\ \psi(X)] = a\ E[\psi(X)]$$

(ii)

(i)

$$E[\psi(X)+a] = \int_{-\infty}^{\infty} [\psi(x)+a] f(x)\ dx = \int_{-\infty}^{\infty} \psi(x) f(x)\ dx + \int_{-\infty}^{\infty} a\, f(x)\ dx$$

$$= E[\psi(X)] + a\int_{-\infty}^{\infty} f(x)\ dx \qquad \left(\therefore \int_{-\infty}^{\infty} f(x)\ dx = 1\right)$$

$$= E[\psi(X)] + a$$

Property 4. If X is a random variable and a and b are constants then E(aX + b) = a E(X) + b provided all the Expectations exists.

Proof

$$E(aX+b) = \int_{-\infty}^{\infty} (ax+b) f(x)dx = \int_{-\infty}^{\infty} axf(x)dx + \int_{-\infty}^{\infty} bf(x)dx$$

$$= a\int_{-\infty}^{\infty} xf(x)dx + b\int_{-\infty}^{\infty} f(x)dx \qquad \left(\therefore \int_{-\infty}^{\infty} f(x)dx = 1\right)$$

$E(aX+b)$ = a E (X) + b

Property 5 If $X \geq 0$ then $E(X) \geq 0$.

Proof

If x is continuous random variable such that $X \geq 0$ then

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} xf(x) > 0$$

[If $X \geq 0$ $f(X) = 0$ for $n < 0$] provided the expectation exists.

Property 6

If X and Y are two random variables such that $Y \leq X$, then $E(Y) \leq E(X)$, provided all expectations exists.

Proof:

Since $Y \leq X$

We have r.v $Y - X \leq 0 \Rightarrow X - Y \geq 0$.

Hence $E(X-Y) \geq 0$

$E(X) - E(Y) \geq 0$

$E(X) \geq E(Y)$

$\Rightarrow$ $E(Y) \leq E(X)$.

4.2 Variance

The variance of a random variable X is defines as

$$Var(X) = E(X^2) - (E(X))^2$$

4.2.1 Property

Let X is a random variable then $V(aX+b) = a^2V(X)$ where a and b are constants

If $Y=aX+b$ then

$E[Y] = E(aX+b) = aE[X]+b$

$\mathbf{Y}\text{-}E[Y] = Y-(aE[X]+b)$

$= (aX+b)-(aE[X]+b)$

$= (aX+b-aE[X]-b)$

$= aX-aE[X]+b-b$

$=aX-aE[X]$

$\mathbf{Y}\text{-}E(Y) = a(X-E[X])$

Taking expectation and squaring on both sides we get

E[Y-E(Y)]^2 =E[a(X-E(X))]^2

$= a^2 [E[X-E[X]]^2]$

$= a^2 [E[X^2-2XE[X]+(E[X])^2]$

$= a^2 [E[X^2]-2E[X]E[X]+(E[X])^2]$

$= a^2 [E[X^2]-2(E[X])^2+(E[X])^2]$

$= a^2 [E[X^2]-(E[X])^2]$

$V(aX+b)=a^2 V(X)$

Example: 4.1 Find the expectation and variance of the number on a die when thrown

Solution

Let X be a random variable representing the number on a die when thrown. Then X can take any one of the values 1,2,3,4,5,6 each with equal probability 1/6

x	1	2	3	4	5	6
P(X=*x*)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$$E(X)=\sum_{i=1}^{6} x_i P(X=x_i)=1\frac{1}{6}+2\frac{1}{6}+3\frac{1}{6}+4\frac{1}{6}+5\frac{1}{6}+6\frac{1}{6}$$

$$=\frac{1+2+3+4+5+6}{6}$$

$$E(X)=\frac{21}{6}$$

Example 4.2 If a pair of fair dice is tossed and X denotes the sum of the numbers on them, find the expectation of X.

Solution: Clearly X may be at least 2 and at most 12

X	2	3	4	5	6	7	8	9	10	11	12
P(X)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$E(X)=\sum_{i=2}^{12} x_i P(X=x_i)=2\frac{1}{36}+3\frac{2}{36}+4\frac{3}{36}+5\frac{4}{36}+6\frac{5}{36}+7\frac{6}{36}+8\frac{5}{36}$$

$$+9\frac{4}{36}+10\frac{3}{36}+11\frac{2}{36}+12\frac{1}{36}$$

$$=\frac{1}{36}[2+6+12+20+30+42+48+36+30+22+12]$$

$$E(X)=\frac{252}{36}=7$$

Example 4.3 If X be a random variable with the following probability distribution

X	-3	6	9
P(x)	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

Find E(X),E(X^2) and E$(2X+1)^2$

Solution

$$E(X)=\sum x_i P(X=x_i)=-3\frac{1}{6}+6\frac{1}{2}+x\frac{1}{3}=\frac{-3+18+18}{6}=\frac{33}{6}=\frac{11}{2}$$

$$E(X)=\frac{11}{2}$$

$$E(X^2)=\sum x_i^2 P(X=x_i)=(-3)^2\frac{1}{6}+6^2\frac{1}{2}+9^2\frac{1}{3}=\frac{93}{2}$$

$$E(X^2)=\frac{93}{2}$$

$$\mathrm{E}(2\mathrm{X}+1)^2=\mathrm{E}\left[4\mathrm{X}^2+4\mathrm{X}+1\right]=\mathrm{E}\left[4\mathrm{X}^2\right]+\mathrm{E}[4\mathrm{X}]+\mathrm{E}[1]$$

$$=4\mathrm{E}\left[\mathrm{X}^2\right]+4\mathrm{E}[\mathrm{X}]+1$$

$$=4.\frac{93}{2}+4.\frac{11}{2}+1=209$$

$$\mathrm{E}(2\mathrm{X}+1)^2=209$$

Example: 4.4 In a continuous distribution the probability density function of X is $f(x)=\begin{cases}\frac{3}{4}x(2-x) & ,0<x<2\\ 0 & ,otherwise\end{cases}$ Find the expectation of the distribution.

Solution.

$$E(X) = \int_0^2 x\ f(x)\,dx = \int_0^2 x\,.\frac{3}{4}x(2-x)\,dx$$

$$= \frac{3}{4}\int_0^2 x^2(2-x)\,dx\ = \frac{3}{4}\int_0^2 2x^2 - x^3\ dx$$

$$= \frac{3}{4}\left[2\frac{x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left[\left(2\frac{2^3}{3} - \frac{2^4}{4}\right) - \left(2\frac{0^3}{3} - \frac{0^4}{4}\right)\right]$$

$$= \frac{3}{4}\left[2\frac{8}{3} - \frac{16}{4}\right] = \frac{3}{4}\left[\frac{16}{3} - \frac{16}{4}\right]$$

$$= \frac{3}{4}\left[\frac{16}{3} - 4\right] = \frac{3}{4}\left[\frac{16-12}{3}\right] = \frac{3}{4}\left[\frac{4}{3}\right] = 1$$

$$E(X) = 1$$

4.3 Cauchy-Schwartz Inequality

If X and Y are random variables taking real values, then $[E(XY)]^2 \leq E(X^2)\,E(Y^2)$

Proof

Consider the expression $(X+tY)^2$ which is a function of real variable t. Since it is always non-negative for all real values of X,Y and t, it follows that

$$E(X+tY)^2 \geq 0\ \forall t$$

$E(X^2+2XYt+t^2Y^2) \geq 0\ \forall t$

$E(X^2)+2t\,E(XY)+t^2\,E(Y^2) \geq 0\ \forall t$

$$\text{i.e., } \varphi(t) = At^2+Bt+C \geq 0\ \forall t$$

Treating as a quadratic in t , its roots will be real i.e., $t \geq 0$

$$\text{where } A= E(Y^2),\ B=2\,E(XY)\ C= E(X^2) \geq 0\ \forall t$$

Now $\varphi(t) \geq 0$ implies $B^2-4AC \leq 0$

$$\therefore 4E[(XY)] - 4E(X^2)\,E(Y^2) \leq 0$$

$$\Rightarrow [E(XY)]^2 \leq E(X^2)\,E(Y^2)$$

4.4. Conditional Expectation and Conditional Variance

Discrete Case: The conditional expectation of mean value of a continuous function g(X,Y) is given that Y = y_j is defined by,

$$E\left\{g(X,Y)/Y=y_j\right\}=\sum_{i=1}^{\infty}\ \sum g(x_i,y_j)P(X=x_i/Y=y_j)$$

$$=\sum\frac{g(x_i,y_j)P(X=x_i\cap Y=y_j)}{P(Y=y_j)}$$

(ie) $=E\left\{g(X,Y)/Y=y_j\right\}$ is nothing but the expectation of function $g(x_i, y_j)$ of X with respect to the conditional distribution of X when y = y_j. In particular, the conditional expectation of a discrete random variable X is given Y = y_j

$$E\left\{X/Y=y_j\right\}=\sum x_i\quad P(X=x_i/Y=y_j)$$

The conditional variance of X given y = y_j is given by

$$V\left\{X/Y=y_j\right\}=E\left\{X-E(X/Y=y_j)^2/Y=y_j\right\}$$

Continuous case

The conditional expectation of g(X, Y) on hypothesis Y = y is given by

$$E\left\{g(X,Y)/Y=y)\right\}=\int_{-\infty}^{\infty}g(x,y)\,f_{X/Y}(x/y)dx$$

$$=\int_{-\infty}^{\infty}g(x,y)\frac{f(x,y)}{f_Y(y)}dx$$

In particular, the conditional mean of x given y = y is defined as

$$E\left\{X/Y=y)\right\}=\int_{-\infty}^{\infty}x\frac{f(x,y)}{f_Y(y)}dx$$

Similarly,

$$E\left\{Y/X=x)\right\}=\int_{-\infty}^{\infty}y\frac{f(x,y)}{f_X(x)}dy$$

The conditional variance of X defined as

$$V(X/Y=y)=E\left[\left(X-E(X/Y=Y)\right)^2/Y=y\right]$$

$$V(Y/X=x)=E\left[\left(Y-E(Y/X=x)\right)^2/X=x\right]$$

Theorem 4.1 The expected value of X is equal to the expectation of the conditional expectation of X given that is symbolically,

$$E(X) = E\left\{E(X/Y)\right\}$$

$$E\left\{E(X/Y)\right\} = E\left\{\sum_i x_i \quad P(X = x_i / Y = y_j)\right\}$$

$$= E\left\{\sum_i x_i \quad \frac{P(X = x_i \cap Y = y_j)}{P(y = y_j)}\right\}$$

$$= \sum_j \quad \left\{\sum_i x_i \quad \frac{P(X = x_i \cap Y = y_j)}{P(\mathrm{Y} = y_j)}\right\} P(\mathrm{Y} = y_j)$$

$$= \sum_i x_i \sum_j \quad P(X = x_i \cap Y = y_j)$$

$$= \sum_j x_i \sum_i \quad P(X = x_i \cap y = y_j)$$

$$= \sum_j x_i \quad P(X = x_i) = E\left(X\right) \quad = \mathrm{E(X)}$$

⇨ E{E (X/Y)} = E(X)

Hence proved.

Theorem 4.2

The variance of X can be regarded as consisting of two parts the expectation of conditional variance and variance of conditional expectation symbolically

Var(X) = E[V(X/Y)] + V [E(X/Y)]

= E[V(X/Y)] + V [E(X/Y)]

$$= E\left\{E(\mathrm{X}^2/\mathrm{Y}) - \left[E(\mathrm{X}/\mathrm{Y})\right]^2\right\} + \left[\left\{E(\mathrm{X}/\mathrm{Y})\right\}^2\right] - \left[E\left\{E(\mathrm{X}/\mathrm{Y})\right\}\right]^2$$

$$= E\left\{E(\mathrm{X}^2/\mathrm{Y})\right\} - E\left\{E(\mathrm{X}/\mathrm{Y})\right\}^2 + E\left\{E(\mathrm{X}/\mathrm{Y})\right\}^2 - \left[E\left\{E(\mathrm{X}/\mathrm{Y})\right\}\right]^2$$

$$= E\left\{E(\mathrm{X}^2/\mathrm{Y})\right\} - \left[E\left\{E(\mathrm{X}/\mathrm{Y})\right\}\right]^2$$

$$= E\left\{E(\mathrm{X}^2/\mathrm{Y})\right\} - \left[E(\mathrm{Y})\right]^2$$

$$= E\left\{\sum_i x_i^2 P(\mathrm{X} = x_i / \mathrm{Y} = y_j)\right\} - [E(\mathrm{X})]^2$$

$$= E\left\{\sum_i x_i^2 \frac{P(\mathrm{X} = x_i \cap Y = y_j)}{P(\mathrm{Y} = y_j)}\right\} - [E(\mathrm{X})]^2$$

$$= \sum_j \left[\left\{ \sum_i x_i^2 \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)} \right\} P(Y = y_j) \right] - [E(X)]^2$$

$$= \sum_j x_i^2 \sum_j P(X = x_i \cap Y = y_j) - [E(X)]^2$$

$$= \sum_j x_i^2 P(X = x_i) - [E(X)]^2$$

$$= E(X^2) - [E(X)]^2$$

= Var (X) =

$\Rightarrow$ Var (X) = E[V(X/Y)] + V [E(X/Y)]

Hence the theorem

EXAMPLE : 4.5 Let X and y be a two random variable each taking three values -1, 0, 1 having joint probability function of x and y

X \ Y	-1	0	1
-1	0	0.1	0.1
0	0.2	0.2	0.2
1	0	0.1	0.14

(i) Show that X and Y having different expectation.

(ii) Find the Variance of X and Y

(iii) Given that Y = 0 what is the conditional probability distribution of X.

(iv) Find the Var (Y/X = -1)

Solution

X \ Y	-1	0	1	P(Y=y)
-1	0	0.1	0.1	0.2
0	0.2	0.2	0.2	0.6
1	0	0.1	0.14	0.2
P(X = x)	0.2	0.4	0.4	1

(i) Expectation of X and Y are

$$E(X) = \sum x_i p_i = (-1)(0.2) + (0)(0.4) + (1)(0.4) = 0.2$$

$$E(Y) = \sum y_j p_i = (-1)(0.2) + (0)(0.6) + (1)(0.2) = 0$$

$$E(X) \neq E(Y)$$

$\therefore$ X and Y are having different expectation.

(ii) Variance of X and Y

$$Var(X) = E(X^2) - \left(E(X)\right)^2$$

$$E(X^2) = \sum x_i^2 P(X = x_i) = (-1)^2(0.2) + (0)^2(0.4) + (1)^2(0.4)$$

$$= 0.2 + 0 + 0.4 = 0.6$$

$$E(X^2) = \backslash 0.6$$

$$Var(X) = E(X^2) - \left(E(X)\right)^2 = 0.6 - (0.2)^2 = 0.6 - 0.04 = 0.56$$

$$Var(X) = 0.56$$

$$Var(Y) = E(Y^2) - \left(E(Y)\right)^2$$

$$E(Y^2) = \sum y_j^2 P(Y = y_j) = (-1)^2(0.2) + (0)^2(0.6) + (1)^2(0.2)$$

$$= 0.2 + 0 + 0.2 = 0.4$$

$$E(Y^2) = \backslash 0.4$$

$$Var(Y) = E(Y^2) - \left(E(Y)\right)^2 = 0.4 - (0)^2 = 0.4 - 0 = 0.4$$

$$Var(Y) = 0.4$$

(iii) Conditional probability of X when Y = 0

$$\text{P(X} = -1/\text{ Y} = 0) = \frac{P(\text{X} = -1 \cap Y = 0)}{P(\text{Y} = 0)} = \frac{0.2}{0.6} = \frac{1}{3}$$

$$\text{P(X} = 0/\text{ Y} = 0) = \frac{P(\text{X} = 0 \cap Y = 0)}{P(\text{Y} = 0)} = \frac{0.2}{0.6} = \frac{1}{3}$$

$$\text{P(X} = 1/\text{ Y} = 0) = \frac{P(\text{X} = 1 \cap Y = 0)}{P(\text{Y} = 0)} = \frac{0.2}{0.6} = \frac{1}{3}$$

(iv) V (Y|X = -1)

$$Var\left(Y/X=-1\right)=E\left(Y/X=-1\right)^2-\left[E\left(Y/X=-1\right)\right]^2$$

$$E\left(Y/X=-1\right)=\sum_y y\ P\left(Y=y/X=-1\right)$$

$$=(-1)(0)+(0)(0.2)+(1)(0)$$

$$E\left(Y/X=-1\right)\ =0$$

$$E\left(Y/X=-1\right)^2=\sum_y y^2\ P\left(Y=y/X=-1\right)$$

$$=(-1)^2(0)+(0)^2(0.2)+(1)^2(0)$$

$$E\left(Y/X=-1\right)^2\ =0$$

$$\therefore\ Var\left(Y/X=-1\right)=E\left(Y/X=-1\right)^2-\left[E\left(Y/X=-1\right)\right]^2$$

$$Var\left(Y/X=-1\right)=0-0=0$$

Example 4.6 Let $f\left(x,y\right)\ =\ \begin{cases}8xy,\ 0<x<y<1\\ 0,\quad elsewhere\end{cases}$.

Find (a) E(Y|X= x) Var (Y|X=x)

Solution : (a)

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\int_x^1 8xy\ dy=8x\int_x^1 y\ dy=8x\left[\frac{y^2}{2}\right]_x^1$$

$$=8x\left[\frac{1^2}{2}-\frac{x^2}{2}\right]=8x\left[\frac{1^2-x^2}{2}\right]$$

$$f_X(x)=4x(1-x^2),\ o<x<1$$

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\int_0^y 8xy\ dx=8y\int_0^y x\ dx=8y\left[\frac{x^2}{2}\right]_0^y$$

$$=8y\left[\frac{y^2}{2}-\frac{0^2}{2}\right]=8y\left[\frac{y^2}{2}\right]$$

$$f_Y(y)=4y^3,\ o<y<1$$

$$f_{X/Y}(x/y)=\frac{f(x,y)}{f_Y(y)}=\frac{8xy}{4y^3}$$

$$f_{X/Y}(x/y)=\frac{2x}{y^2}$$

$$f_{Y/X}(y/x)=\frac{f(x,y)}{f_X(x)}=\frac{8xy}{4x(1-x^2)}$$

$$f_{Y/X}(y/x)=\frac{2y}{(1-x^2)}$$

(b) Var(Y/X=x) = E(Y^2/X=x)-{E(Y/X=x}^{2}

$$E\left(Y\mid X=x\right)=\int_x^1 y\, f_{Y/X}(y/x)dy=\int_x^1 y\ \frac{2y}{(1-x^2)}dy$$

$$=\frac{2}{(1-x^2)}\int_x^1 y^2\ dy=\frac{2}{(1-x^2)}\left[\frac{y^3}{3}\right]_x^1$$

$$=\frac{2}{(1-x^2)}\left[\frac{1^3}{3}-\frac{x^3}{3}\right]=\frac{2}{(1-x^2)}\left[\frac{1^3-x^3}{3}\right]$$

$$E\left(Y\mid X=x\right)=\frac{2}{3}\left[\frac{1-x^3}{1-x^2}\right]$$

$$E\left(Y^2\mid X=x\right)=\int_x^1 y^2\ f_{Y/X}(y/x)dy=\int_x^1 y^2\ \frac{2y}{(1-x^2)}dy$$

$$=\frac{2}{(1-x^2)}\int_x^1 y^3\ dy=\frac{2}{(1-x^2)}\left[\frac{y^4}{4}\right]_x^1$$

$$=\frac{2}{(1-x^2)}\left[\frac{1^4}{4}-\frac{x^4}{4}\right]=\frac{2}{(1-x^2)}\left[\frac{1^4-x^4}{4}\right]$$

$$E\left(Y^2\mid X=x\right)=\frac{1+x^2}{2}$$

$$Var(Y/X=x)=E\left(Y^2\mid X=x\right)-\left(E\left(Y\mid X=x\right)\right)^2$$

$$=\left[\frac{1+x^2}{2}\right]-\left(\frac{2}{3}\left[\frac{1-x^3}{1-x^2}\right]\right)^2$$

$$Var(Y/X=x)=\frac{1+x^2}{2}-9\left(\frac{1-x^3}{1-x^2}\right)^2$$

4.5 MOMENT GENERATING FUNCTION

The Moment Generating Function (M.G.F) of a random variable X defined as

$$M_X(t) = E(e^{tX}) = \begin{cases} \int e^{tx} f(x)dx & \text{for continuous probability distributions} \\ \sum_x e^{tx} p(x = x) & \text{for discrete probability distributions} \end{cases}$$

$$M_X(t) = E(e^{tX}) = \int e^{tx} f(x)dx$$

$$\therefore M_X(t) = E(e^{tX}) = E\left(1 + tX + \frac{t^2X^2}{2!} + ... + \frac{t^rX^r}{r!} + ...\right)$$

$$= 1 + t\,E(X) = \frac{t^2}{2!}E(X^2) + \cdots + \frac{t^r}{r!}E(X^r) + ...$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + ...$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r^1$$

Where $\mu_r' = E(X^r) = \begin{cases} \int x^r\ f(x)\ dx & \textit{for continuous distribution} \\ \sum_x x^r\, p(x) & \textit{for discrete distribution} \end{cases}$

is the rth moment of X about origin. Thus the coefficient of $\frac{t^r}{r!}$ in $M_X(t)$ gives μ_r' (about origin). Since $M_X(t)$ generates moments, it is known as moment generating function. Differentiating moment generating function w.r. to 't' 'r' time and put t = 0 we get.

$$\left[\frac{d^r}{dt^r}M_X(t)\right]_{t=0} = \mu_r'$$

put r = 1

$$\mu_1' = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = E(X) = Mean$$

put r = 2

$$\mu_2' = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = E(X^2)$$

$$Variance = \mu_2' - (\mu_1')^2 = E(X^2) - (E(X))^2$$

4.5.1 Properties of Moment generating function:

Property 1

$M_{cX}(t) = E[e^{tcX}]$, c is a constant.

By definition

L.H.S. $M_{cx}(t) = E[e^{tcx}]$

R.H.S. $M_x(ct) = E[e^{ctx}]$ = L.H.S

$\therefore M_{cX}(t) = E[e^{tcX}]$

Property 2

The moment generating function of the sum of a number of random variables is equal to the product of their respective moment generating function.

$$M_{(X_1+X_2+X_3+X_4+\cdots+X_n)}(t) = M_{X_1}(t)M_{X_2}(t)M_{X_3}(t)\cdots M_{X_n}(t)$$

Proof

$$M_{(X_1+X_2+X_3+X_4+\cdots X_n)}(t) = E\left[e^{t(x_1+x_2+\ldots.x_n)}\right]$$

$$= E\left[e^{tx_1}.e^{tx_2}\ldots.e^{tx_n}\right]$$

$$= E\left[e^{t\,x_1}\right]E\left[e^{t\,x_2}\right]\ldots.E\left[e^{t\,x_n}\right]$$

$$= M_{X_1}(t)M_{X_2}(t)M_{X_3}(t)\ldots.M_{X_n}(t)$$

Property 3 Effect of change of origin and scale on MGF.

Let us transform X to the new variable U by changing both the origin and scale in X as follows $U = \frac{X-a}{h}$ where a and h are constants

Moment generating function about U about origin is given by

$$M_U(t) = E(e^{tU}) = E\left(e^{t\left(\frac{X-a}{h}\right)}\right)$$

$$= E\left(e^{\left(\frac{tX\text{-}at}{h}\right)}\right) = E\left(e^{\left(\frac{tX}{h}-\frac{at}{h}\right)}\right)$$

$$= E\left(e^{\frac{tX}{h}}.e^{\frac{-at}{h}}\right) = e^{\frac{-at}{h}}E\left(e^{\frac{tX}{h}}\right)$$

$$M_U(t) = e^{\frac{-at}{h}}M_X(t/h)$$

Where $M_X(t)$ is the M.G.F of X about orgin.

4.5.2 Limitations of Moment Generating Function

1. ***A random variable X may not have moments although its moment generating function exists.***

Consider a discrete random variable X with probability density function is

$$f(x)=\frac{1}{x(x+1)} \text{ for } x=1,2,3,\ldots \text{ and '0' otherwise}$$

$$E(X)=\sum_{x=1}^{\infty} x\,f(x) = \sum_{x=1}^{\infty} \frac{x}{x(x+1)}$$

$$=\sum_{x=1}^{\infty} \frac{1}{(x+1)} \quad =\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots$$

$$=\left\{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots\right\}-1$$

$$E(X) = \sum_{x=1}^{\infty} \frac{1}{x}-1$$

Since $\Sigma\sum_{1}^{\infty}\frac{1}{x}$ is divergent series, E(X) does not exists and consequently no moment of X exists,how ever , the mgf of X is given by

$$M_X(t)=\sum_{x=1}^{\infty} e^{tx} f(x)=\sum_{x=1}^{\infty} e^{tx} \frac{1}{x(x+1)}$$

Let z = e^t

$$M_X(t)=\sum_{x=1}^{\infty} \frac{z^x}{x(x+1)} \quad =\frac{z^1}{1.2}+\frac{z^2}{2.3}+\frac{z^3}{3.4}+\cdots$$

$$= z^1\left(1-\frac{1}{2}\right)+z^2\left(\frac{1}{2}-\frac{1}{3}\right)+z^3\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots$$

$$=\left(z-\frac{z}{2}\right)+\left(\frac{z^2}{2}-\frac{z^2}{3}\right)+\left(\frac{z^3}{3}-\frac{z^3}{4}\right)+\cdots$$

$$=\left(z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots\right)-\left(\frac{z}{2}+\frac{z^2}{3}+\frac{z^3}{4}+\cdots\right)$$

$$= \left(z+\frac{z^2}{2}+\frac{z^3}{3}+\cdots\right)-\left(\left(1+\frac{z}{2}+\frac{z^2}{3}+\frac{z^3}{4}+\cdots\right)-1\right)$$

$$=-\log\ (1-z)-\left(\left(1+\frac{z}{2}+\frac{z^2}{3}+\frac{z^3}{4}+\cdots\right)-1\right)$$

$$= -log\ (1\text{-}z)\text{-}\left(\frac{z}{z}\left(1+\frac{z}{2}+\frac{z^2}{3}+\frac{z^3}{4}+\cdots\right)-1\right)$$

$$= -log\ (1\text{-}z)\text{-}\left(\frac{1}{z}\left(z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\cdots\right)-1\right)$$

$$= -log\ (1\text{-}z)\text{-}\frac{1}{z}\left(z+\frac{z^2}{2}+\frac{z^3}{3}+\frac{z^4}{4}+\cdots\right)+1$$

$$= -log\ (1\text{-}z)\text{-}\frac{1}{z}\left(-log(1-z)\right)+1$$

$$= -log\ (1\text{-}z)+\frac{1}{z}log(1-z)+1$$

$$for\ |z| < 1 \ \Rightarrow |e^t| < 1 \Rightarrow t < 0$$

$$= 1+\left(\frac{1}{z}-1\right)log(1-z)$$

$$= 1+\left(\frac{1}{e^t}-1\right)log(1-e^t) = 1+\left(e^{-t}-1\right)log(1-e^t), t<0$$

So that $M_X(t) = 1$ for t=0 , Hence $M_X(t)$ exists for $t \leq 0$.

2. ***A random variable X can have moment generating function along with some or all moments, yet the but m.g.f does not generate the moments.***

Let consider a discrete random variable X with probability functions

$$P(X = 2^x) = \frac{e^{-1}}{x!} \text{ for } x = 0,1,2,\ldots \quad \text{Then}$$

$$E(X^r) = \sum_{x=0}^{\infty}\left(2^x\right)^r P(X = 2^x) = \sum_{x=0}^{\infty}\left(2^x\right)^r \frac{e^{-1}}{x!}$$

$$= e^{-1}\sum_{x=0}^{\infty}\frac{\left(2^r\right)^x}{x!} = e^{-1}\left[1+\frac{2^r}{1!}+\frac{\left(2^r\right)^2}{2!}+\cdots\right] = e^{-1}e^{2^r}$$

$$E(X^r) = e^{2^r-1}$$

Hence all the moments of X exists. The m.g.f of X, if it exists, is given by

$$M_X(t) = \sum_{x=0}^{\infty} e^{(t\ 2^x)}\left(\frac{e^{-1}}{x!}\right) = e^{-1}\sum_{x=0}^{\infty} e^{t\ 2^x}\left(\frac{1}{x!}\right)$$

By D' Alembert's ratio test the series on the RHS is convergent for $t \leq 0$ and diverges for $t > 0$. Hence $M_X(t)$ cannot be differentiated at t=0 and has no Maclurin's expansion and consequently it does not generate moments.

3. ***A random variable X can have some or all moments, but m.g.f does not exist except perhaps at one point.***

Let consider X be a random variable with probability function

$$P(X = \pm 2^r) = \begin{cases} \frac{e^{-1}}{2x!}; x = 0,1,2,\ldots \\ 0 \quad , otherwise \end{cases}$$

The distribution being symmetric, moments of odd order about origin vanish

i.e., $\mu_{2r+1} = 0 \Rightarrow E(X^{2r+1}) = 0$

Now, $E(X^{2r}) = \sum_{x=0}^{\infty} (\pm 2^x)^{2r} \frac{e^{-1}}{2x!} = e^{-1} \sum_{x=0}^{\infty} \frac{(2^x)^{2r}}{x!} = e^{(2^{2r}-1)}$

Thus all the moments of X exists. The m.g.f of X, if it exists, is given by

$$M_X(t) = \sum_{x=0}^{\infty} \left\{ \left(e^{t.2^x} + e^{-t.2^x} \right) \frac{1}{2ex!} \right\} = e^{-1} \sum_{x=0}^{\infty} \left\{ \frac{Cosh(t 2^x)}{x!} \right\}$$

Which is only convergent for t = 0. Hence m.g.f of X does not exists at t=0.

Example 4.7 Let the random variable X assume the value of r with probability law $P(X = r) = q^{r-1}.p$. r = 1, 2, 3. Find the moment generating function and hence find its mean and variance.

Solution

$M_X(t) = E(e^{tr})$

$$= \sum_{r=1}^{\infty} e^{tr} \; p(x = r)$$

$$= \sum_{r=1}^{\infty} e^{tr} \; q^{r-1}.p$$

$$= \sum_{r=1}^{\infty} e^{t\,r} \; q^r.q^{-1}.p$$

$$= \frac{p}{q} \sum_{r=1}^{\infty} (qe^t)^r$$

$$= \frac{p}{q} \sum_{r=1}^{\infty} (qe^t)^r$$

$$= \frac{p}{q} (qe^t)\left[1 + (qe^t) + (qe^t)^2 + \ldots\right]$$

$$= p\; e^t (1 - qe^t)^{-1}$$

$$M_X(t) = \frac{Pe^t}{(1-qe^t)}$$

Mean $E(\mathrm{X}) = \left[\frac{d}{dt} M_X{}^{(t)}\right]_{t=0}$

$$\frac{d}{dt} M_X(t) = \frac{d}{dt} \quad \frac{pe^t}{(1-qe^t)}$$

$$= p\frac{d}{dt} e^t (1-qe^t)^{-1}$$

$$= p\left[e^t(-1)(1-qe^t)^{-2}(-qe^t) + e^t(1-qe^t)^{-1}\right]$$

$$= p\left[\frac{qe^{2t}}{(1-qe^t)^2} + \frac{e^t}{(1-qe^t)}\right]$$

$$= p\left[\frac{qe^{2t} + e^t(1-qe^t)}{(1-qe^t)^2}\right]$$

$$= p\left[\frac{qe^{2t} + e^t - qe^t e^t}{(1-qe^t)^2}\right] = p\left[\frac{qe^{2t} + e^t - qe^{2t}}{(1-qe^t)^2}\right]$$

$$= \frac{pe^t}{(1-qe^t)^2}$$

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \frac{pe^0}{(1-qe^0)^2} = \frac{p}{(1-q)^2} \frac{p}{p^2}$$

$$E(X) = \frac{1}{p}$$

$$E(X^2) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0}$$

$$\left[\frac{d^2}{dt^2} M_X(t)\right] = \frac{d^2}{dt^2}\left(\frac{pe^t}{(1-qe^t)}\right)$$

$$= \frac{d}{dt}\left[\frac{pe^t}{(1-qe^t)^2}\right]$$

$$= p\frac{d}{dt} \quad e^t(1-qe^t)^{-2}$$

$$= p\left[e^t(-2)(1-qe^t)^{-2-1}(-qe^t) + e^t(1-qe^t)^{-2}\right]$$

$$= p\left[2qe^t.e^t(1-qe^t)^{-3} + e^t(1-qe^t)^{-2}\right]$$

$$= P\left[\frac{2qe^{2t}}{(1-qe^t)^3}+\frac{e^t}{(1-qe^t)^2}\right]$$

$$= P\left[\frac{2qe^{2t}+e^t(1-qe^t)}{(1-qe^t)^3}\right]$$

$$E(X^2)=\left[\frac{d^2}{dt^2}M_x(t)\right]_{t=0} \quad P\left[\frac{2qe^0+e^0(1-qe^0)}{(1-qe^0)^3}\right]$$

$$= P\left[\frac{2q+1-q}{(1-q)^3}\right]$$

$$= P\left[\frac{q+1}{(1-q)^3}\right]=P\left[\frac{q+1}{p^3}\right]$$

$$E(X^2) = \frac{(q+1)}{p^2}$$

Var (X) = E (X^2) – (E(X))^2

$$=\left(\frac{(q+1)}{p^2}\right)-\left(\frac{1}{p}\right)^2=\frac{q+1}{p^2}-\frac{1}{p^2}=\frac{q+1-1}{p^2}$$

$$\text{Var (x)} = \frac{q}{p^2}$$

Example 4.8 A random variable X has probability function $p(x)=\frac{1}{2^x}\ x=1,2,3,\ldots$ Find the moment generating function, mean and variance.

Solution:

$$M_X(t)=E(e^{tx})=\sum_{x=1}^{\infty}e^{tx}p(x)=\sum_{x=1}^{\infty}e^{tx}\frac{1}{2^x}=\sum_{x=1}^{\infty}\frac{e^{tx}}{2^x}=\sum_{x=1}^{\infty}\left(\frac{e^t}{2}\right)^x$$

$$=\left(\frac{e^t}{2}\right)^1+\left(\frac{e^t}{2}\right)^2+\left(\frac{e^t}{2}\right)^3+\left(\frac{e^t}{2}\right)^4+\cdots$$

$$=\frac{e^t}{2}\left[1+\left(\frac{e^t}{2}\right)+\left(\frac{e^t}{2}\right)^2+\left(\frac{e^t}{2}\right)^3+\left(\frac{e^t}{2}\right)^4+\cdots\right]$$

$$= \frac{e^t}{2}\left[1+\left(\frac{e^t}{2}\right)+\left(\frac{e^t}{2}\right)^2+\left(\frac{e^t}{2}\right)^3+\left(\frac{e^t}{2}\right)^4+\cdots\right]$$

$$= \frac{e^t}{2}\left[1-\frac{e^t}{2}\right]^{-1} = \frac{e^t}{2}\left[\frac{2-e^t}{2}\right]^{-1} = \frac{e^t}{2}\left[\frac{2}{2-e^t}\right]$$

$$M_X(t) = \left[\frac{e^t}{2-e^t}\right]$$

Mean

$$E(X) = \left[\frac{d}{dt}M_X(t)\right]_{t=0}$$

$$\frac{d}{dt}M_X(t) = \frac{d}{dt}\frac{e^t}{\left(2-e^t\right)} = \left[\frac{\left(2-e^t\right)e^t - e^t\left(-e^t\right)}{\left(2-e^t\right)^2}\right] = \left[\frac{2e^t - e^te^t + e^te^t}{\left(2-e^t\right)^2}\right] = \left[\frac{2e^t}{\left(2-e^t\right)^2}\right]$$

$$E(X) = \left[\frac{d}{dt}M_X(t)\right]_{t=0} = \left[\frac{2e^t}{\left(2-e^t\right)^2}\right]_{t=0} = \left[\frac{2e^0}{\left(2-e^0\right)^2}\right] = \left[\frac{2}{(2-1)}\right] = 2$$

$$E(X) = 2$$

$$Variance \;=\; E(X^2) - \left(E(X)\right)^2$$

$$E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0}$$

$$\left[\frac{d^2}{dt^2}M_X(t)\right] = \left[\frac{d^2}{dt^2}\frac{e^t}{2-e^t}\right] = \left[\frac{d}{dt}\frac{2e^t}{\left(2-e^t\right)^2}\right] = \left[\frac{\left(2-e^t\right)^2(2e^t) - 4e^t\left(2-e^t\right)\left(-e^t\right)}{\left(2-e^t\right)^4}\right]$$

$$E(X^2) = \left[\frac{d^2}{dt^2}M_X(t)\right]_{t=0} = \left[\frac{\left(2-e^t\right)^2(2e^t) - 4e^t\left(2-e^t\right)\left(-e^t\right)}{\left(2-e^t\right)^4}\right]_{t=0}$$

$$= \left[\frac{\left(2-e^0\right)^2(2e^0) - 4e^0\left(2-e^0\right)\left(-e^0\right)}{\left(2-e^0\right)^4}\right] = \left[\frac{(2-1)2+4(2-1)(1)}{(2-1)^4}\right] = \frac{2+4}{1}$$

$$E(X^2) = 6$$

$$Variance \;=\; E(X^2) - \left(E(X)\right)^2 = 6-(2)^2 = 6-4 = 2$$

Example 4.9 Find the m.g.f of the random variable X having p.d.f is defined as

$$f(x)=\begin{cases} x & \text{for } 0\le x\le 1 \\ 2-x & \text{for } 1\le x\le 2 \\ 0 & \text{otherwise} \end{cases}$$

Solution:

$$M_X(t)=\int_{-\infty}^{\infty} e^{tx} f(x)\,dx=\int_0^1 e^{tx}\,x\,dx+\int_1^2 e^{tx}\,(2-x)\,dx$$

$$=\int_0^1 x\,e^{tx}\,dx+\int_1^2 (2-x)\,e^{tx}dx$$

$$=\left[x\left(\frac{e^{tx}}{t}\right)-\left(\frac{e^{tx}}{t^2}\right)\right]_0^1+\left[(2-x)\left(\frac{e^{tx}}{t}\right)-(-1)\left(\frac{e^{tx}}{t^2}\right)\right]_1^2$$

$$=\left[\left((1)\left(\frac{e^{t(1)}}{t}\right)-\left(\frac{e^{t(1)}}{t^2}\right)\right)-\left((0)\left(\frac{e^{t(0)}}{t}\right)-\left(\frac{e^{t(0)}}{t^2}\right)\right)\right]$$

$$+\left[\left((2-2)\left(\frac{e^{t(2)}}{t}\right)-(-1)\left(\frac{e^{t(2)}}{t^2}\right)\right)-\left((2-1)\left(\frac{e^{t(1)}}{t}\right)-(-1)\left(\frac{e^{t(1)}}{t^2}\right)\right)\right]$$

$$=\left[\left(\frac{e^t}{t}-\frac{e^t}{t^2}\right)+\left(\frac{e^0}{t^2}\right)\right]+\left[\left(0+\frac{e^{2t}}{t^2}\right)-\left(\frac{e^t}{t}+\frac{e^t}{t^2}\right)\right]=\left[\frac{e^t}{t}-\frac{e^t}{t^2}+\frac{1}{t^2}\right]+\left[\frac{e^{2t}}{t^2}-\frac{e^t}{t}-\frac{e^t}{t^2}\right]$$

$$=\frac{e^t}{t}-\frac{e^t}{t^2}+\frac{1}{t^2}+\frac{e^{2t}}{t^2}-\frac{e^t}{t}-\frac{e^t}{t^2}$$

$$=\frac{e^{2t}}{t^2}-2\frac{e^t}{t^2}+\frac{1}{t^2}=\frac{1-2e^t+e^{2t}}{t^2}=\frac{\left(1-e^t\right)^2}{t^2}=\left(\frac{1-e^t}{t}\right)^2$$

$$M_X(t)=\left(\frac{1-e^t}{t}\right)^2$$

4.6 CUMULANTS

Cummlants generating function K(t) is defined as $K_X(t) = log_e M_X(t)$

Provided the right hand side can be exoanded as a convergent series in power of t or If the logarithm of the m.g.f of a distribution can be expanded as a convergent series in powers of t viz.,

$$K_X(t) = k_1 t + k_2 \frac{t^2}{2!} + k_3 \frac{t^3}{3!} + \cdots + k_r \frac{t^r}{r!} + \cdots = log M_X(t)$$

$$= log\left(1 + t\mu_1' + \frac{t^2}{2!}\mu_2' + \cdots + \frac{t^r}{r!}\mu_r' + \ldots\right)$$

Then the coefficients $k_1, k_2, \ldots$. Are called the first, second cumulant of the distribution and $K_X(t)$ is called the cumulative function.

Differentiating r times both sides with respect to t and putting t = 0 and we have

$$k_r = \left[\frac{d^r}{dt^r} log\, M_X(t)\right]_{t=0} = \left[\frac{d^r}{dt^r} K_X(t)\right]_{t=0}$$

4.6.1 Properties of Cumulants

Property 1 : Additive Property

The r^{th} cumulant of the sum of the independent random variables is equal to the sum of the r^{th} cumulants of the individual variables. Symbolically

$$k_r(X_1+X_2+X_3+\ldots+X_n) = k_r(X_1) + kr(X_2) + kr(X3) + \ldots + kr(Xn)$$

where X_i, i=1,2,…,n are independent random variables.

Proof

Since X_i, i=1,2,…,n are independent,

$$M_{X_1+X_2+X_3+\cdots+X_n}(t) = M_{X_1}(t) M_{X_2}(t) M_{X_3}(t) \cdots M_{X_n}(t)$$

Taking logarithm of each side

$$K_{X_1+X_2+X_3+\cdots+X_n}(t) = K_{X_1}(t) + K_{X_2}(t) + K_{X_3}(t) + \cdots + K_{X_n}(t)$$

Differentiating with respect to 'r' times and put t= 0 we get

$$\left[\frac{d^r}{dt^r} K_{X_1+X_2+X_3+\cdots+X_n}(t)\right]_{t=0} = \left[\frac{d^r}{dt^r} K_{X_1}(t)\right]_{t=0} + \left[\frac{d^r}{dt^r} K_{X_2}(t)\right]_{t=0} + \cdots + \left[\frac{d^r}{dt^r} K_{X_n}(t)\right]_{t=0}$$

$$\therefore k_r(X_1+X_2+X_3+\ldots+X_n) = k_r(X_1) + kr(X_2) + kr(X_3) + \ldots + kr(X_n)$$

Property 2: Effect of change of Origin and scale on Cumulants

Let $U = \frac{X-a}{h}$ then

$$M_U(t) = e^{\frac{-at}{h}} M_X(t/h)$$

Taking logarithm on both sides

$$log[M_U(t)] = log\left[e^{\frac{-at}{h}} M_X(t/h) \right]$$

$$K_U(t) = log\, M_U(t) = \frac{-at}{h} + K_X(t/h)$$

$$k_1' t + k_2' \frac{t^2}{2!} + k_3' \frac{t^3}{3!} + \cdots + k_r' \frac{t^r}{r!} + \cdots = \frac{-at}{h} + k_1(t/h) + k_2 \frac{(t/h)^2}{2!} + \cdots + k_r \frac{(t/h)^r}{r!}$$

Where k_r' and k_r are the r^{th} cumulants of U and X respectively. Comparing coefficients,

we get $k_1' = \frac{k_1 - a}{h}$ *and* $k_r' = \frac{k_r}{h^r}; r = 2,3,.....$

Thus except the first cumulant, all the cumulants are independent of change of origin. But the cumulants are not invariant of change of scale as the r^{th} cumulant of U is $(1/h^r)$ times the r^{th} cumulant of the distribution of X.

4.7 CHARACTERISTIC FUNCTION

In some case moment generating function does not exists. The characteristic function defined as

$$\phi_X(t) = E(e^{itX}) = \begin{cases} \int e^{itx} \; f(x) \; dx & \text{for continuous probability distribution} \\ \sum_x e^{itx} \; p(x) & \text{for discrete probability distribution} \end{cases}$$

4.7.1 Properties of characteristic function

Property 1

For all real t, we have

(i) $\phi(0) = \int_{-\infty}^{\infty} dF(x) = 1$

(ii) $| \phi(t) | \le | = \phi(0)$

Property 2

ϕ (t) is continuous everywhere, i.e., ϕ (t) is continuous function of 't' in $(-\infty,\infty)$. Rather ϕ (t) is uniformly continuous in 't'.

Proof

$$\text{For } h \neq 0 \,|\, \phi_X(t+h) = \phi_X(t) =| \int_{-\infty}^{\infty} [e^{i(t+h)} - e^{itx}] dF(x) |$$

$$\leq \int_{-\infty}^{\infty} | e^{itx}(e^{ihx} - 1) | dF(x) = \int_{-\infty}^{\infty} | e^{ihx} - 1 | dF(x)$$

The last integral does not depend on 't'. If it tends to zero as h → 0 then ϕ_x (t) is uniformly continuous in 't'

Now $| e^{ihx} - 1 | \leq | e^{ihx} | + | 1 | \leq 1 + 1 = 2$

$$\therefore \int_{-\infty}^{\infty} | e^{ihx} - 1 | dF(x) \leq 2 \int_{-\infty}^{\infty} | dF(x) = 2$$

Hence by Dominated convergence theorem (D.C.T) taking the limit inside the integral sign.

$$\lim_{h \to 0} | \phi_X(t+h) - \phi_X(t) | \leq \int_{-\infty}^{\infty} \lim_{h \to 0} | e^{ihx} - 1 | dF(x) = 0$$

$$\Rightarrow \lim_{h \to 0} \phi_X(t+h) = \phi_X(t), \forall t$$

Hence $\phi_x(t)$ is uniformly continuous in 't'.

Property 3

$\phi_X(-t)$ and $\phi_x(t)$ are conjugate functions.

$\phi_X(-t) = \overline{\phi_X(t)}$, where a is the complex conjugate of 'a'.

Proof

$\phi_x(t) = E(e^{itx}) = E\,[\text{Cos } t_x + i \text{ Sin} t_x]$

$\overline{\phi_X(t)} = E(\text{Cos } tX - i \text{ Sin} t X)$

$= E\{\text{Cos } (-t) X + i \text{ Sin } (-t) X \}$

$= E(e^{-itx}) = \phi_x(-t)$

Property 4

If the distribution function of a r.v.x is symmetrical about zero, ie if

1 – F(x) = F(-x)

⇨ F(-x) = f(x)

Proof

By the definition the $\phi_x(t)$ is real valued and even function of t

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)dx \qquad \text{put } x=-y$$

$$= \int_{-\infty}^{\infty} e^{-ity} f(-y)dy$$

$$= \int_{-\infty}^{\infty} e^{-ity} f(-y)dy \qquad (f(-y) = f(y)$$

= ϕ_x (-t)

⇨ ϕ_x (-t) is an even function of 't'

Property 5

If X is some r.v with characteristic function ϕ_x (t) and u $\mu_r' = E(X^r)$ exists.

$$\mu_r' = (-i)^r \left| \frac{\partial^r}{\partial t^r} \phi_x(t) \right|_{t=0}$$

Proof

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)dx$$

Differentiating (under the integral sign)'r' times w.r. to t, we get

$$\frac{\partial^r}{\partial t^r} \phi(t) = \int_{-\infty}^{\infty} (ix)^r \; e^{itx} f(x)dx$$

$$= \int_{-\infty}^{\infty} i^r x^r \; e^{itx} f(x)dx$$

$$= (i)^r \int_{-\infty}^{\infty} x^r \; e^{itx} f(x)dx$$

$$\therefore \left| \frac{\partial^r}{\partial t^r} \phi_x(t) \right|_{t=0} = (i)^r \left| \int_{-\infty}^{\infty} x^r \; e^{itx} f(x)dx \right|_{t=0}$$

$$= (i)^r \int_{-\infty}^{\infty} x^r f(x)dx$$

$$= (i)^r E(x^r) = i^r \mu_r'$$

Hence

$$\mu_r' = \left(\frac{1}{i}\right)^r \left|\frac{\partial^r}{\partial t^r}\phi_X(t)\right|_{t=0} = (-i)^r \left|\frac{\partial^r}{\partial t^r}\phi(t)\right|_{t=0}$$

Property 6

$\phi_{cx}(t) = \phi_x(ct)$ c is constant.

Property 7

If X_1 and X_2 are independent random variables, then,

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t) + \phi_{X_2}(t)$$

Property 8 Effect of change of origin and scale on characteristic Function.

If $U = \frac{x-a}{h}$, a and h being constants, then

$$\phi_u(t) = e^{-iat/h}\ \phi_X\left(t/h\right)$$

In particular we take a = E(x) = μ(say) and h = σ_x = σ, then the characteristic function of the standard variate.

$$Z = \frac{X - E(\mathrm{X})}{\delta_X} = \frac{X-\mu}{\delta}$$ is given by

$$\phi_Z(t) = e^{-i\mu t/\sigma}\phi(t/\sigma)$$

Example: Find the characteristic function of the Poisson distribution

Solution:

The probability mass function of a Poisson distribution is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}; x = 0,1,2,3,.....$$

$$\phi_X(t) = \sum_{x=0}^{\infty} e^{itx} P(X = x) = \sum_{x=0}^{\infty} e^{itx} \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty} e^{-\lambda} \frac{e^{itx}\lambda^x}{x!} =$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\left(\lambda e^{it}\right)^x}{x!} = e^{-\lambda}\left[\frac{\left(\lambda e^{it}\right)^0}{0!} + \frac{\left(\lambda e^{it}\right)^1}{1!} + \frac{\left(\lambda e^{it}\right)^2}{2!} + \cdots\right]$$

$$= e^{-\lambda}\left[1 + \frac{\left(\lambda e^{it}\right)^1}{1!} + \frac{\left(\lambda e^{it}\right)^2}{2!} + \cdots\right]$$

$$\phi_X(t) = e^{-\lambda} e^{\lambda e^{it}} = e^{-\lambda + \lambda e^{it}} = e^{-\lambda(1 - e^{it})}$$

$$\phi_X(t) = e^{-\lambda(1 - e^{it})}$$

Example 4.10 Find the characteristic function of a pdf $f(x) = \frac{\alpha}{2} e^{-\alpha|x|}, -\infty < x < \infty$

Solution Let

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx = \int_{-\infty}^{\infty} e^{itx} \frac{\alpha}{2} e^{-\alpha|x|}\, dx$$

$$= \frac{\alpha}{2} \int_{-\infty}^{\infty} e^{itx} e^{-\alpha|x|}\, dx = \frac{\alpha}{2}\left[\int_{-\infty}^{0} e^{itx} e^{-\alpha(-x)}\, dx + \int_{0}^{\infty} e^{itx} e^{-\alpha(x)}\, dx\right]$$

$$= \frac{\alpha}{2}\left[\int_{-\infty}^{0} e^{itx} e^{\alpha x}\, dx + \int_{0}^{\infty} e^{itx} e^{-\alpha x}\, dx\right] = \frac{\alpha}{2}\left[\int_{-\infty}^{0} e^{itx+\alpha x}\, dx + \int_{0}^{\infty} e^{itx-\alpha x}\, dx\right]$$

$$= \frac{\alpha}{2}\left[\int_{-\infty}^{0} e^{(\alpha+it)x}\, dx + \int_{0}^{\infty} e^{-(\alpha-it)x}\, dx\right] = \frac{\alpha}{2}\left[\left(\frac{e^{(\alpha+it)x}}{(\alpha+it)}\right)_{-\infty}^{0} + \left(\frac{e^{-(\alpha-it)x}}{-(\alpha-it)}\right)_{0}^{\infty}\right]$$

$$= \frac{\alpha}{2}\left[\left(\left(\frac{e^{(\alpha+it)(0)}}{(\alpha+it)}\right) - \left(\frac{e^{(\alpha+it)(-\infty)}}{(\alpha+it)}\right)\right) + \left(\left(\frac{e^{-(\alpha-it)\infty}}{-(\alpha-it)}\right) - \left(\frac{e^{-(\alpha-it)0}}{-(\alpha-it)}\right)\right)\right]$$

$$= \frac{\alpha}{2}\left[\int_{-\infty}^{0} e^{(\alpha+it)x}\, dx + \int_{0}^{\infty} e^{-(\alpha-it)x}\, dx\right] = \frac{\alpha}{2}\left[\left(\frac{e^{(\alpha+it)x}}{(\alpha+it)}\right)_{-\infty}^{0} + \left(\frac{e^{-(\alpha-it)x}}{-(\alpha-it)}\right)_{0}^{\infty}\right]$$

$$= \frac{\alpha}{2}\left[\left(\left(\frac{e^{(\alpha+it)(0)}}{(\alpha+it)}\right) - \left(\frac{e^{(\alpha+it)(-\infty)}}{(\alpha+it)}\right)\right) + \left(\left(\frac{e^{-(\alpha-it)\infty}}{-(\alpha-it)}\right) - \left(\frac{e^{-(\alpha-it)0}}{-(\alpha-it)}\right)\right)\right]$$

$$=\frac{\alpha}{2}\left[\left(\left(\frac{e^{0}}{(\alpha+it)}\right)-(0)\right)+\left((0)-\left(\frac{e^{0}}{-(\alpha-it)}\right)\right)\right]=\frac{\alpha}{2}\left[\frac{1}{(\alpha+it)}+\frac{1}{(\alpha-it)}\right]$$

$$=\frac{\alpha}{2}\left[\frac{(\alpha-it)+(\alpha+it)}{(\alpha+it)(\alpha-it)}\right]=\frac{\alpha}{2}\left[\frac{\alpha-it+\alpha+it}{(\alpha^{2}-(it)^{2})}\right]=\frac{\alpha}{2}\left[\frac{2\alpha}{\left(\alpha^{2}-\left(i^{2}t^{2}\right)\right)}\right]=\frac{\alpha}{2}\left[\frac{2\alpha}{\left(\alpha^{2}-(-1)t^{2}\right)}\right]$$

$$\varphi_X(t)=\left[\frac{\alpha^{2}}{\left(\alpha^{2}+t^{2}\right)}\right]$$

Example 4.11 Show that the distribution which the characteristic function $\mathbf{e^{-|t|}}$ has the density function is $f(x)=\frac{1}{\pi}\frac{dx}{1+x^{2}}\quad -\infty\le x\le\infty$

Solution

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\phi_X(t)e^{-itx}dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-|t|}\,e^{-itx}dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-|t|}(\cos tx-i\sin tx)dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-|t|}(\cos tx)dt-i\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-|t|}(\sin tx)dt$$

$$=\frac{1}{2\pi}\int_{0}^{\infty}e^{-|t|}(\cos tx)dt=\frac{1}{2\pi}\int_{0}^{\infty}e^{-(t)}(\cos tx)dt=\frac{1}{2\pi}\int_{0}^{\infty}e^{-t}(\cos tx)dt$$

$$=\frac{2}{2\pi}\int_{0}^{\infty}e^{-t}(\cos tx)dt\;=\frac{1}{\pi}\left[\frac{e^{-t}}{1+x^{2}}(-\cos xt+x\sin xt)\right]_{0}^{\infty}$$

$$f(x)=\frac{1}{\pi}\frac{1}{1+x^{2}}$$

4.8 Probability Generating Function

For a random variable X which takes integral values 0,1,2,3,…only, we define the probability generating function P(s) by

$$P(s) = p_0 + p_1 s + p_1 s^2 + \cdots = \sum_{n=0}^{\infty} p_n s^n = E(s^X)$$

The coefficient of s^n in the expansion of P(s) in powers of s gives P(X=n). This explains P(s) is called the probability generating function (p.g.f).

4.9 TCHEBYCHEV'S INEQUALTIY

The role of standard deviation as a parameter to characterize variance precisely interpreted by means of the well known Chebychev's inequality. The theorem discovered in 1853 was later an discreased in 1856 by Bienayme.

Definition

If X is a random variable with mean μ and variance σ^2, then for any positive number k, we have

$$P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

$$P\{|X-\mu| < k\sigma\} \leq 1-\frac{1}{k^2}$$

Proof: Case (i) x is continuous γ. ν. By the definition.

$$\sigma^2 = \sigma_x^2 = E[X - E(X)]^2$$

$$= E[X - \mu]^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx. \qquad f(x) \text{ is p.d.f of } x$$

$$= \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu-k\sigma}^{\mu+k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)dx$$

$$\geq \int_{-\infty}^{\mu-k\sigma} (x-\mu)^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} (x-\mu)^2 f(x)dx$$

We know that

$x \leq \mu - k\sigma$ and $x \geq \mu + k\sigma \Leftrightarrow |x - \mu| \geq k\sigma$

substituting $(x - \mu) = k\sigma$ we get.

$$\sigma^2 \geq \int_{-\infty}^{\mu-k\sigma} (k\sigma)^2 f(x)dx + \int_{\mu+k\sigma}^{\infty} (k\sigma)^2 f(x)dx$$

$$\geq (k\sigma)^2 \left[\int_{-\infty}^{\mu-k\sigma} f(x)dx + \int_{\mu+k\sigma}^{\infty} f(x)dx \right] \geq (k\sigma)^2 \left[\int_{-\infty}^{\mu-k\sigma} f(x)dx + \int_{\mu+k\sigma}^{\infty} f(x)dx \right]$$

$$\geq k^2\sigma^2 \left[P(\mathrm{X} \leq \mu - k\sigma)\right] + \left[P(\mathrm{X} \geq \mu - k\sigma)\right]$$

$$\sigma^2 \geq k^2\sigma^2 P[|x-\mu| \geq k\sigma]$$

$$P\left(|x-\mu| \ \geq k\sigma\right) \leq \sigma^2 \frac{1}{\sigma^2 k^2}$$

Also since

$$P\left\{|X-\mu| \geq k\sigma\right\} + P\left\{|X-\mu| < k\sigma\right\} = 1$$

We get

$$P\left\{|X-\mu| < k\sigma\right\} = 1 - P\left\{|X-\mu| \geq k\sigma\right\} = 1 - \{ 1/k^2$$

Case (ii)

In the case of discrete random variable, the proof exactly similarly on replacing integration by summation

We late kσ = C > 0

$$P\left\{|X-\mu| \geq c\right\} \leq \frac{\sigma^2}{c^2} \text{ and} - P\left\{|X-\mu| < c\right\} \geq 1 - \frac{\sigma^2}{c^2}$$

$$\Rightarrow P\left\{|X - E(\mathrm{X})| \geq c\right\} \leq \frac{Var(\mathrm{X})}{c^2} \text{ and } P\left\{|X - E(\mathrm{X})| < c\right\} \geq 1 - \frac{Var(\mathrm{X})}{c^2}$$

Example 4.12 If x is the number scored in a throw of a fair die, show that the Tchebychev's inequality gives P{|x-μ| > 2.5σ} < 0.47, where μ is the mean of X, while the actual probability is zero.

Solution

Here X is a random variable which takes the values 1,2,3,4,5,6 with probability 1/6. Hence

$$E(\mathrm{X}) = \sum_{i=1}^{6} x_i \ p(x = x_i)$$

$$= \left(1 \text{ x } \frac{1}{6}\right) + \left(2 \text{ x } \frac{1}{6}\right) + \left(3 \text{ x } \frac{1}{6}\right) + \left(4 \text{ x } \frac{1}{6}\right) + \left(5 \text{ x } \frac{1}{6}\right) + \left(6 \text{ x } \frac{1}{6}\right)$$

$$= \ \frac{1}{6}\left(1+2+3+4+5+6\right) = \ \frac{21}{6} = \frac{7}{2}$$

$$E(X^2) = \left(1^2 x \frac{1}{6}\right) + \left(2^2 x \frac{1}{6}\right) + \left(3^2 x \frac{1}{6}\right) + \left(4^2 x \frac{1}{6}\right) + \left(5^2 x \frac{1}{6}\right) + \left(6^2 + \frac{1}{6}\right)$$

$$E(X^2) = \frac{1}{6}(1+4+9+25+36) = \frac{91}{6}$$

$$Var(X) = E(X^2) - (E(X))^2 = \left(\frac{91}{6}\right) - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12} = 2.9167$$

For k > 0, Tchebychev's in equality gives $P\{|X - E(X)| > k\} < \frac{Var\ X}{k^2}$

Where Choosing k = 2.5

$$P\{|X - \mu| > 2.5\} < \frac{2.9167}{(2.5)^2} = 0.47$$

The actual probability P is given by

$$P\{|X - \mu| \geq 0.25\} = P\{|X - E(X)| \geq 0.25\}$$
$$= P\{|X - 3.5| \geq 0.25\}$$
$$= P[X \text{ lies out side the limits 3.5-2.5 and } 3.5+2.5 \text{ i.e., 1 and 6}]$$

= 0 (Since X being number on a die cannot lie outside the limit 1 and 6)

Example 4.13 A fair die is tossed 720 times. Use Chebyshev's inequality to find a lower bound for getting 100 to 140 sixes.

Solution: Let X be the number of sixes obtained when a die is thrown 720 times.

p = probability of success in a single throw = $\frac{1}{6}$

q = 1- $\frac{1}{6}$ = $\frac{5}{6}$ and here n = 720

Thus X follows Binomial distribution with

Mean =μ = np ie., σ = 120

Variance σ^2 = npqie., σ^2= 100 and σ = 10.

We have to find lower bound for the probability P(100 < X < 140)

Now, by Chebyshev's inequality

P(|X- μ| <kσ) ≥ 1- $\frac{1}{k2}$

ie., P(μ- kσ< X < μ+ kσ) ≥ 1- $\frac{1}{k2}$

Comparing P(100 < X < 140) with LHS we get,

$\mu - k\sigma = 100$ ie. $120-10k = 100$

and $\mu + k\sigma = 140$ ie. $120+10k = 140$

Subtracting, $-20k = -40$ ie., $k=2$

$P(100 < X < 140) \geq \frac{3}{4}$

Hence the lower bound is $\frac{3}{4} = 0.75$

UNIT –V

5.1 CONVERGENCE IN PROBABILITY

Definition

A sequence of random variables X_1, X_2, …., X_n is said to convergence in probability to a constant a, if for any $\in > 0$

$$\lim_{n\to\infty} P(|X_n - a| < \in) = 1$$

or its equivalent. $\lim_{n\to\infty} P(|X_n - a| \geq \varepsilon) = 0$ and

we write $X_n \xrightarrow{P} a$ as $n \to \infty$.

If there exists a random variable X such that

$X_n - X \xrightarrow{P} a$ as $n \to \infty$ then we says that the given sequence $\{X_n\}$ of random variables converges in probability to the random variable X.

5.2 Weak Law of Large Numbers (WLLN)

Statement:

Let $X_1, X_2, X_3, \ldots.. X_n$ be a sequence of random variables and $\mu_1, \mu_2, \ldots.., \mu_n$ be their respective expectation and Let

$$B_n = Var(X_1 + X_2 + \ldots + X_n) < \infty$$

when $P\left\{\left|\frac{X_1 + X_2 + \ldots. + X_n}{n} - \frac{\mu_1 + \mu_2 + \ldots.\mu_n}{n}\right| < \varepsilon\right\} \geq 1 - \eta$

for all $n > n_0$ where $\in$ and η are arbitrary small positive numbers, provided.

$\lim_{n\to\infty} \frac{B_n}{n^2} \to 0$

Proof

Using Chebychev's inequality to the random variable $\frac{X_1 + X_2 + \ldots. + X_n}{n}$ we get any $\in > 0$

$$P\left\{\left|\frac{X_1 + X_2 + \ldots. + X_n}{n} - E\left(\frac{X_1 + X_2 + \ldots. + X_n}{n}\right)\right| < \varepsilon\right\} \geq 1 - \frac{B_n}{n^2 \varepsilon^2}$$

$$\left[\text{Since Var}\left(\frac{X_1+X_2+....+X_n}{n}\right)=\frac{1}{n^2}Var(X_1+X_2+....+X_n)\right]$$

$$P\left\{\left|\frac{X_1+X_2+....+X_n}{n}-\frac{\mu_1+\mu_2+....\mu_n}{n}\right|<\varepsilon\right\}\geq 1-\frac{B_n}{n^2\varepsilon^2}$$

So, far, nothing is assumed about the behaviour of B_n for indefinitely increasing values of n. Since $\in$ is arbitrary, we assume $\frac{B_n}{n^2\varepsilon^2}\rightarrow 0$ as n becomes indefinitely large.Thus, having chosen two arbitrary small positive numbers $\in$ andη, number no can found so that in equality $\frac{B_n}{n^2\in^2}<\eta$ will hold for n > n_0 consequently, we shall have

$$P\left\{\left|\frac{X_1+X_2+....+X_n}{n}-\frac{\mu_1+\mu_2+....\mu_n}{n}\right|<\varepsilon\right\}\geq 1-\eta \text{ for all } n > n_0\ (\varepsilon,\eta)$$

This conclusion leads to the following important results, known as W.L.L.N,With the probability approaching unity or certainty as near as we please, we may expect that the arithmetic mean of values actually assumed by n random variables will differ from the arithmetic mean of their expectations by less than any given number, however small, provided the number of variables can be taken sufficiently large and provided the condition.

$$\frac{B_n}{n^2}\rightarrow 0 \text{ as } n \rightarrow\infty \text{ is fulfilled}$$

Remarks

1. Weak law of large numbers can also be stated as,

$$\overline{X_n}\xrightarrow{P}\overline{\mu_n} \text{ Provided}$$

$$\frac{B_n}{n^2}\rightarrow 0 \text{ as } n \rightarrow\infty$$

2. For the existence of the law, we assume the following conditions
   (i) E (X_i) exists for all i.
   (ii) B_n = Var(X_1+X_2+….+X_n) exists and
   (iii) $\frac{B_n}{n^2}\rightarrow 0$ as n $\rightarrow\infty$

Condition (i) is necessary, without it the law itself cannot be stated. But the condition (ii) & (iii) are not necessary (iii) however a sufficient condition.

5.3 Bernoulli's Law of Large Numbers

Statement: Let there be n trails of an event, each trail resulting in a success or failure. If X is the number of successes in n trails with constant probability p of success for each trail, then E(X) = n p and Var(X) = npq, q = 1-p. The variable X/n represents the proportion of success or the relative frequency of success and

$$E\left(\frac{X}{n}\right) = p \; and \; Var\left(\frac{X}{n}\right) = \frac{1}{n^2} Var(X) \text{ then.}$$

$$P\left\{\left|\frac{X}{n} - p\right| < \varepsilon\right\} \to 1 \text{ as n } \to \infty$$

$$\Rightarrow P\left\{\left|\frac{X}{n} - p\right| \geq \varepsilon\right\} \to 0 \text{ as n } \to \infty$$

for any assigned $\varepsilon > 0$. This implies that (x/n) converges in probability to p as n → ∞.

5.4 Khinchin's theorem

Statement: If X_i's are identically and independently distributed random variables, the only condition necessary for the law of large number to hold is that $E(X_i)$; i = 1,2,3,…..,n should exists.

5.5 Central limit theorem

If X_i (i = 1, 2, …,n) be independent random variables such that $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$ then under certain very general conditions, the random variables $S_n = X_1 + X_2 + \ldots + X_n$ is asymptotically normal with mean μ and standard deviation σ where,

$$\mu = \sum_{i=1}^{n} \mu_i \quad \text{and} \quad \sigma^2 = \sum_{i=1}^{n} \mu_i^2$$

Central Limit theorem for (independent and identically distributed) variables was proved by Linderberg and Levy. If $X_1, X_2, \ldots Xn$ are independent and i.i.d random variables with $E(X_i) = \mu_i$ $Var(X_i) = \sigma_i^2$ i = 1,2,3 …..n then the sum $S_n = X_1 + X_2 + \ldots + Xn$ is asymptotically normal with mean $\mu = n\mu_1$ and variance $\sigma_i^2 = n\sigma_i^2$

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