B.Sc. PHYSICS - I YEAR

DJK1A : PROPERTIES OF MATTER

SYLLABUS

UNIT I : ELASTICITY

Introduction - Modulus of elasticity - Poisson's ratio- Relation between elastic constants and Poisson's ratio - Energy stored - Twisting couple on a wire - Torsional pendulum (with and without weights) – determination of rigidity modulus of a rod by static torsion method.

UNIT II : BENDING OF BEAMS

Expression for Bending moment- Cantilever – expression for depression – Experiment to find youngs' modulus – Cantilever oscillation – expression for period – uniform bending – I form girders.

UNIT III : SURFACE TENSION

Molecular interpretation - surface energy- Pressure difference across a curved surface- Excess pressure in liquid drops and air bubbles - Molecular forces - Variation of surface tension with temperature - Capillary rise and energy consideration - Jaeger's method.

UNIT IV : VISCOSITY

Streamlined motion – turbulent motion – coefficient of viscosity – rate of flow of liquid in a capillary tube - Poiseuilles formula - Stoke's fall- analogy between liquid flow and current flow – equation of continuity of flow of liquid – energy possessed by a flowing liquid.

UNIT V : GRAVITATION

Laws of gravitation, gravitational field and potential, acceleration due to gravity and its variation, escape velocity, Kepler's laws and planetary motion, motion of satellites, Geostationary orbit.

Books for study

1. Properties of Matter - Brijlal & Subramaniam.
2. Properties of Matter - D.S. Mathur
3. Properties of Matter - Murugesan

Books for Reference

1. Physics, Robert Resnick, David Halliday, Jearl Walker Wiley and Sons Inc., Sixth Edition.
2. H.R Gulati- fundamental of general properties of matter- R.Chand and co- fifth edition

UNIT - I

ELASTICITY

1.1 Introduction :

Elasticity is the property by virtue of which a body offers .resistance to any deforming force. A material body makes use of this property to regain its original condition when the deforming forces are removed. All bodies can be deformed by the action of external forces. Bodies which can completely regain their original condition of shape and size on removal of deforming forces are said to be **perfectly elastic**. Bodies which retain their deformed nature even after the removal of the deforming forces are said to be **perfectly plastic**. If external forces fail to produce any deformation or relative displacements of the particles of the body, the body is said to be perfectly rigid. In general there are no bodies which are perfectly elastic or perfectly plastic. Even a quartz fiber which is the nearest approach to a perfectly elastic body does not regain its original size and shape from very large deformations. Similarly putty which is the nearest approach to a perfectly plastic body tends to regain from small deformations. Thus a body is said to be more elastic or plastic when compared to another. Bodies which are homogeneous and isotropic are considered here.

1.2 Stress and Strain:

A deforming force or load is the combination of external forces acting on a body and its effect is to change the form or the dimensions of the body. When there is a load on the body, the forces of reaction come into play internally in it, tending to restore it to its original condition. This restoring or recovering force per unit area set up inside the body is called the **stress.** It is equal and opposite to the load within elastic limit. If the internal force developed is perpendicular to the surface it is called **normal stress.** The normal stress may be compressive or expansive (tensile) according as a decrease or increase in volume is involved.

Stress is measured in terms of deforming force acting per unit area of the surface. The Unit of stress is Pascal (and its dimension is $ML^{-1}T^{-2}$).

The change produced in the body due to change in dimension of a body under a system of forces in equilibrium is called **strain.** It is the amount of deformation suffered by a body under applied internal forces. Strain is measured by the change in dimension for unit dimension and hence it has no unit. The nature of the strain depends on the nature of the deforming forces.

The ratio of the change in length per unit length is known as **linear strain** or **longitudinal strain** which is created by longitudinal stress.

When equal inward or outward forces are applied normal to all the faces of a cube, a change in volume is produced. The ratio of the change in volume per unit volume is known as volume strain.

When equal and opposite forces act tangentially along two opposite faces of a cube, a change in shape is produced. Such a strain is called **shearing strain** or **shear** and is measured by the angle through which a line on the body normal to the force is turned.

1.3 Hooke's Law:

The maximum value of the stress within which a body completely regains its original conditions of shape and size when the deforming forces are removed is known as the **elastic limit.**

Hooke's law states that within elastic limits, the stress is directly proportional to strain. i.e., the ratio of the stress to the strain is a constant. This constant is called the **modulus of elasticity** of the material of body.

i.e. $$\frac{stress}{strain} = modulus of elasticity$$

Elasticity of a substance is due to the intermolecular forces. When this force is great as in a solid, the modulus of elasticity is high. That is even for a great amount of stress developed, strain will be very small. But when the intermolecular forces are small as in a gas, the modulus of elasticity is very small, (i.e) even for a small amount of stress, the corresponding strain will be more.

1.4 Elastic behavior of a material:

The elastic behavior of a material of wire can be studied by plotting a curve between the stress along the y axis and the corresponding strain on the x axis. The curve is called **stress – strain curve.** Let a wire be clamped at one end loaded at the other end gradually from zero value until the wire breaks down. The nature of the stress- strain curve for low carbon steel wire is shown in figure 1.1

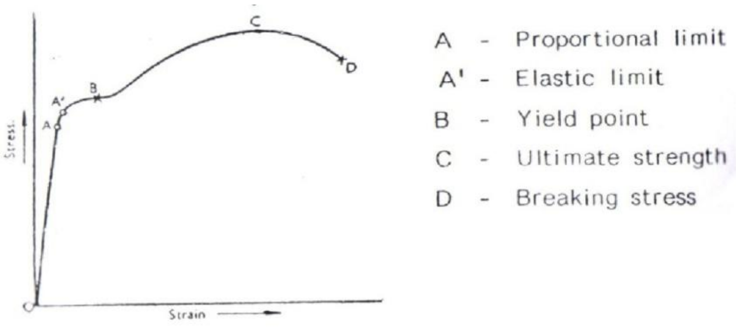


Fig 1.1 The Stress – strain curve for low carbon steelwire

The Part OA of the curve is a straight line which shows that upto the point A, stress is proportional to strain i.e. Hooke's law is obeyed. The point A is called the proportional limit which is measured by the maximum stress that can be developed in the given material without causing a deviation from Hooke's law. In the vicinity of the point A there lies another point A' called the elastic limit. i.e.upto the point A' the wire behaves as a perfectly elastic body. It should however be remembered here that it is not necessary that for the part AA' of the curve, the stress should be proportional to the strain. These two points A and A' are very nearer to each other and may coincide for some materials. If the wire is loaded beyond the elastic limit A' the wire gets stretched and attains a permanent set. i.e. there is a permanent deformation in the body after the removal of the deforming forces.

On increasing the load still further a point B, called yield point at which extension of the wire increase rapidly without an increase in the load. For a given material, the yield point is usually determined by the minimum value of stress for which the material begin to deform appreciably without an increase of load. The value of the stress at the yield point is called **yield strength** of that material. The elongation without addition in load is called **creeping** and this behavior of the metal is called **yielding.** If the wire if further loaded, a point represented by C is reached after which the wire begins to neck down or flow locally so that its cross-sectional area no longer remains uniform. At this point C the wire begins to thin down at some point where it finally breaks. At the point C the value of the developed stress is maximum and is called the ultimate tensile strength (or) tensile strength of the given material. The tensile strength is defined as the maximum value of tensile stress withstand by the material before fracture under a steady load.

$$\text{Tensile Strength} = \frac{\text{Max .tensile load}}{\text{Original cross} - \text{sectional area}}$$

Usually tensile strength of metals and alloys increases on cooling and decreases on heating. The stress corresponding to the point D where the wire actually breaks down is called the **breaking stress.** The value of breaking stress is of no practical importance whereas the position of point C is very useful in knowing the ultimate strength of the material. The nominal value of the breaking stress is found to be less than that of the ultimate strength.

If a body is subjected to a constant stress, it loses its elastic property even within its elastic limit. It will regain its elastic property if it is allowed to rest sufficiently. Similarly a wire is loaded repeatedly or subjected to a large number of cycles of stresses it gets tired or ruptured due to gradual fracture of the material and hence loses its strength apparently. Thus the **elastic fatigue** may be defined as the apparent loss of strength of material or as the progressive fracture of the material caused by repeated stress in it.

Substances like quartz, phosphor bronze and silver fibers are regain their original condition immediately on removal of the deforming forces. That is why they are frequently employed as the suspensions in galvanometers and electrometers, etc. But some other materials, like glass fibers take hours to recover from the strain. This delay in regaining the normal condition is called **elastic after effect.**

Normally the working stress on a body is kept far below the ultimate tensile stress and is never allowed to cross the elastic limit. The above fact is practiced by all design Engineers to get higher stability and reliability of the structures. The ratio between the ultimate tensile stress and the working stress is called the **safety factor.**

$$\text{i.e. Safety Factor} = \frac{\text{Ultim ate tensile Stress}}{\text{Working Stress}}$$

Working load or working stress is determined by the designer on the basis of his experience and knowledge. Thus the safety factor depends upon the engineering material and the standard at workmanship.

1.5 Factors affecting Elasticity:

Effect of Stress: We have seen that the action of large constant stress or the repeated number of cycles of stresses acting in a body affects the elasticity of the body gradually. Taking these into account, the working stress on an engineering pieces is kept for below its ultimate tensile strength.

Effect of temperature: Normally the elasticity decreases with the increase of temperature. A **carbon** filament which is highly elastic at normal temperature becomes plastic when it a at high temperature. **Lead** is not a very good elastic material. But at low temperature it becomes a very good elastic material. **Inward** is a special alloy used for making pendulums and its elasticity is not affected by temperature changes. **Creep resistance** is a property by which the material can withstand its elastic property without fracture at high temperatures and during quick loading. Dispersion hardened materials and coarse grained materials have better creep resistance at high temperatures and hence they can withstand their elastic properties even at high temperatures.

Effect of impurities : The elastic property of a material may increase or decrease due to the addition of impurities. If we add carbon in minute quantities to molten iron, the elastic properties of iron is increased enormously. But if the carbon content is more than 1% in iron, then the strength of iron decreases. Similar the addition of Potassium in gold increases the elastic properties of gold. If any addition of impurity atoms distorts the lattice structure of base metal, then elastic property of the base metal decreases. These kind of impurity atoms generally have different atoms and therefore act as centers of distortion which decrease the elastic properties of the base metal.

Effect of heat treatment and metal processing: A grain consists of many small interlocking crystals. The various heat treatment processes are adopted to get the desired physical and mechanical properties through the changes in micro constituents of the material. Annealing (heating and then slow cooling) is one of them which is adopted to increase softness and ductility in the materials. But it decreases the elastic properties of the material by decreasing the tensile strength and yield point of the material. This is due to formation of large crystal grains. Hammering and rolling are the metal processing techniques to make thin plates and sheets. These break up the grains into smaller units or fine grains resulting an increase of elastic properties. So metals with fine grains are stronger than the metals with large or coarse grains. However for high temperature applications, we are using materials with large grains because they have high creep resistance.

Effect of crystalline nature: For a given metal, the modulus of elasticity is more when it is in single crystal form and in the poly crystalline state, its modulus of elasticity is comparatively small. However for most of the engineering uses, we are using poly crystalline form of metals due to its increased mechanical properties like ductility, malleability, etc.

1.6 Three Modulii of elasticity :

Corresponding to the three types of strain, there are three kinds of modulii of elasticity. They are Young's modulus (E), Bulk modulus (K), and Rigidity modulus (N).

a) **Young's Modulus (E):** When the deforming force or load is applied to the body only along a particular direction, the change per unit length produced in that direction is called longitudinal or linear or elongation strain. The force applied per unit area of cross-section is called longitudinal (or) linear stress.

Within the elastic limit, the ratio of the linear stress to linear strain is called the Young's Modulus.

$$\text{Therefore, } E = \frac{\text{Linear Stress}}{\text{linear Strain}} = \frac{F/a}{\ell/L} \frac{F/\ell}{aL} \text{ Pascals}$$

Where F is the force applied normal to the area of cross-section 'a' and ℓ is the change in length produced in an original length 'L'.

b) **Bulk Modulus (K) :** The uniform applied force acting normally on the whole surface of the body produces a change in volume and there is no change of shape. The force applied per unit area or pressure gives the bulk stress. The change in volume per unit volume gives the bulk strain. Thin the bulk modulus 'K' is defined as the ratio of the bulk stress to the bulk strain.

$$\text{Therefore, } K = \frac{\text{bulk stress}}{\text{bulk strain}} = \frac{F/a}{v/V} = \frac{F/V}{aV} = \frac{PV}{V} \text{ Pascals}$$

Where F/a= P, is the normal stress or bulk stress or pressure acting on a surface area ‘a’ and ‘v’ is the change in volume produced in an original volume ‘V’

The reciprocal of the bulk modulus of a substance is called its compressibility.

c) **Rigidity Modulus (N):** Here the applied force changes the shape of the body without causing any change in its volume. Let us consider a solid cube ABCDPQRS whose lower face DCQP is fixed. A tangential force is applied on the upper face ABRS as shown in the figure 1.2. Due to the application of force F on upper face ABRS an equal and opposite force comes into play on the lower fixed face DCQP. These two forces form a couple which makes the layers, parallel to the two faces to move one over the other. Thus the point A shifted to A', B to B', R to R' and S to S'. That is, the lines joining the two faces turn through an angle ‘Ө’. The face ABCD is then said to be sheared through an angle ‘Ө’.

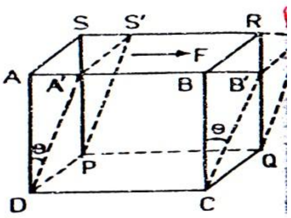


Fig. 3.2 Application of shearing force on a solid cube

Thus the angle of shear or shearing strain or (simply) shear ‘Ө’ is defined as the angle through which a line which was originally perpendicular to the tangential force has turned. The shearing stress or tangential stress is the tangential force per unit area of the face ABRS. Thus the rigidity modulus 'N' is defined as the ratio of the shearing stress to the angle of shear.

$$N = \frac{\text{Shearing Stress}}{\text{angle of Shear}} = \frac{F/a}{\theta} = \frac{F}{a\theta} = \text{Pascal}$$

All solids have three modulii of elasticity, and fluids (gases and liquids) have only bulk modulus of elasticity.

Poisson's Ratio (σ) : When a wire is stretched by means of a force it is elongated. It is observed that along with an increase in the length of wire, a corresponding contraction in its diameter also takes place. The ratio of the change in diameter to the initial diameter is known as lateral strain. Within the elastic limit, the ratio of the lateral strain to the longitudinal strain is called Poisson's ratio **σ**.

$$\text{Thus } \boldsymbol{\sigma} = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

$$\therefore \boldsymbol{\sigma} = \frac{decreaseindiameter/originaldiameter}{increaseinlength/originallength} = \frac{\Delta D/D}{\Delta l/l}$$

where ΔD and Δl are change in diameter and length respectively and D and l are original diameter and length respectively. If α and β and the longitudinal strain per unit stress and the lateral strain per unit stress respectively, then

$$\sigma = {}^{\beta}/_{\alpha}$$

1.6 Work done per unit volume in deforming a body:

Work is done by the deforming forces when the body is strained or deformed. The energy so spent in doing work is stored within the body in the form of elastic potential energy or strain energy which appears as heat when the stress in it is relieved.

(a) Workdone per unit volume in stretching a wire :

Let F be the force applied to a wire fixed at the upper end.

Workdone in producing a small increase in length $'dl$ = F.dl.

The young's modulus of elasticity $E = \frac{FL}{al}$

$$\therefore F = Eal/L$$

Workdone during the stretch of the wire from 0 to l is given by

$$w = \int_0^l \frac{Eal}{L}.dl$$

$$= \frac{Ea}{L}\frac{l^2}{2}$$

$$= \frac{1}{2}\frac{Eal}{L}l$$

$$\text{Since} F = \frac{Eal}{L} \;,$$

$$w = \frac{1}{2}\,F.l$$

$$= \frac{1}{2} \text{ x Strecthing force x elongation produced}$$

$$\text{Workdone per unit volume} = \frac{1}{2} \text{ x } \frac{\text{Streching force}}{a} \text{ x } \frac{\text{elongation produced}}{L}$$

$$= \frac{1}{2} \text{ x Stress x strain}$$

b) Workdone per unit volume in changing the volume of a solid :

The workdone in changing the volume from o to v is given by

$$w = \int_0^V P\,.dv$$

The bulk modulus of elasticity $K = \frac{PV}{v}$

$$/\ W = \int_0^V \frac{Kv}{V}\, dv = \frac{1}{2}\frac{K}{V}v^2$$

$$= \frac{1}{2}\rho.V = \frac{1}{2} \text{ x bulk stress x change in volume}$$

$$\text{Workdone per unit volume} = \frac{1}{2} \text{ x bulk stress x bulk strain.}$$

C) Workdone per unit volume during shearing strain :

In the case of shear, we can prove that

Workdone $W = \int_0^l F.\, d\,l = \int_0^l NL\, l\, dl = \frac{1}{2}\, NL\, l^2$

$$= \frac{1}{2}\, F\, l \quad [\text{Since } N = \frac{F}{a\theta} = \frac{FL}{L^2 l} = \frac{F}{Ll}]$$

Therefore, workdone per unit volume $= \frac{1}{2}$ x shearing stress x angle of shear

Thus we find that in any kind of strain, workdone per unit volume is equal to $\frac{1}{2}$ x stress x strain.

1.8 Relation between three modulii of Elasticity:

First Part: To derive the relation between E and N:

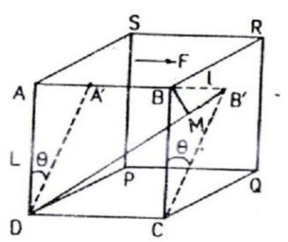


Fig 1.4 Application of a Shearing force on a cube.

Let a tangential force 'F' be applied to the upper face ABRS of a solid cube ABCD PQRS whose lower face DCQP is fixed. Further we assume that the solid cube is a homogenous and isotropic elastic medium. Let the length of the cube be equal to L meters. Now consider the, vertical section ABCD of the cube. The face ABCD is displaced to the position A 'B' C D. The diagonal DB increases to DB' whereas the diagonal AC decreases A'C.

$$\text{Shearing Stress} = \frac{F}{\text{area ABRS}} = \frac{F}{L^2} = T \ldots 1.1$$

Let α and β be the longitudinal and lateral strains per unit stress. We know that a shearing stress along AB is equivalent to an equal tensile stress along DB and an equal compression stress along AC right angles. Hence, extension along diagonal DB due to tensile stress, along DB = DB .T.α

Extension along diagonal DB due to compression stress along DC = DB .T.β

Therefore, total extension along DB = DB .T. (α + β)

$= \sqrt{2LT}\,((\alpha + \beta) \ldots \quad 1.2$

Draw a perpendicular BM on DB'. Then Practically DB= DM and the increase in the length of diagonal DB is equal to B'M.

Since θ is very small, $A\hat{B}'C = 90^\circ$

Hence $B\hat{B}'M = 45^\circ$

Therefore B'M=BB' $\cos 45^\circ = \frac{BB'}{\sqrt{2}} = \frac{l}{\sqrt{2}} \ldots \quad 1.3$

Where l = BB'. Equations 1.2 and 1.3 are giving the value of B'M.

Therefore, B'M = $\frac{l}{\sqrt{2}} = \sqrt{2}\ L\,T\,(\alpha + \beta)$

Rearranging this equation, we get

$$\frac{T.L}{l} = \frac{1}{2(\alpha + \beta)}$$

$$\frac{T}{l/L} = \frac{T}{\theta} = \frac{1}{2\,\alpha(1+\beta/\alpha)}$$

Since $\frac{T}{\theta} = N$,

$$\frac{1}{\alpha} = E \text{ and } \sigma = \beta/\alpha$$

$$N = \frac{E}{2(1+\sigma)}$$

(or) $E = 2N\,(1+\sigma)$ ... 1.4

Second Part:

To derive the relation between E and K:

Consider an unit cube ABCDPQRS, Let the stresses T_X act perpendicular to faces. ASPD and BRQC, the stresses T_y act perpendicular to faces SRQP and ABCD and the stresses T_Z act perpendicular to faces ABRS and AQPD respectively. If α is the elongation per unit length per unit stress along the direction of the applied stress then elongation produced in the edges AB, BR and BC will be $T_x\alpha$, $T_y\alpha$ and $T_z\alpha$ respectively.

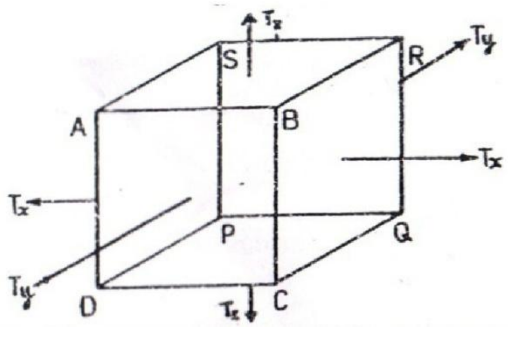


Fig 1.5 Application of bulk stress on a cube.

If β is the contraction produced per unit length per unit stress in a direction perpendicular to the applied stress, then contraction produced perpendicular to the edges AB, BR and BC will be $T_x\beta$, $T_y\beta$ and $T_z\beta$ respectively. Hence the resultant lengths of AB, BR and BC are as follows:

$AB' = 1 + T_x\alpha - T_y\beta - T_z\beta$

$BR' = 1 + T_y\alpha - T_x\beta - T_z\beta$

$BC' = 1 + T_z\alpha - T_x\beta - T_y\beta$

∴ New volume of the cube = $AB' \times BR' \times BC' = V'$

$$= \left(1 + T_x\alpha - T_y\beta - T_z\beta\right) \cdot \left(1 + T_y\alpha - T_z\beta - T_x\beta\right) \cdot \left(1 + T_z\alpha - T_x\beta - T_y\beta\right)$$

Since α and β are very small, terms containing their squares and higher powers can be neglected.

$V' = 1 + \alpha\left(T_x + T_y + T_z\right) - 2\beta\left(T_x + T_y + T_z\right)$

$$= 1 + (\alpha - 2\beta)\left(T_x + T_y + T_z\right)$$

Assume that the stresses acting on all the faces are equal.

i.e. $T_x = T_y = T_z = T$

Therefore $V' = 1 + 3T(\alpha - 2\beta)$

Hence increase in volume $= V' - V = [1 + 3T(\alpha - 2\beta)] - 1$

$$= 3T(\alpha - 2\beta)$$

Instead of applying the stretching force outwardly, let a pressure P be applied on all the faces to compress the cube. Then the contraction in volume is also equal to $3P(\alpha - 2\beta)$.

Here the compressive stress is represented by pressure 'P'

$$\text{Bulk Strain} = \frac{\text{Change in Volume}}{\text{Original Volume}} = \frac{3P(\alpha-2\beta)}{1}$$

$$\text{Now Bulk Modulus K} = \frac{\text{Bulk Stress}}{\text{Bulk Strain}} = \frac{P}{3P(\alpha - 2\beta)} = \frac{1}{3(\alpha - 2\beta)}$$

$$= \frac{1}{3\alpha(\alpha - 2\beta/\alpha)} = \frac{E}{3(1-2\sigma)}$$

$$E = 3K(1-2\sigma) \qquad \ldots\ldots 1.5$$

Third Part:

To connect the equations connecting E and N & E and K. From equations (1.4) and (1.5), we get,

$$E = 2N(1+\sigma) = 3K\,(1-2\sigma) \qquad \ldots\ldots\ 1.6$$

From equation (1.4), we get

$$2 + 2\sigma = \frac{E}{N}$$

From equation 1.5 we get

$$1 - 2\sigma = \frac{E}{3K} \ldots \qquad 1.7$$

Adding the equations (1.6) and (1.7), we get

$$3 = \frac{E}{N} + \frac{E}{3K} = E\left(\frac{1}{N} + \frac{1}{3K}\right)$$

Rearranging this equation, we get

$$\frac{3}{E} = \frac{1}{N} + \frac{1}{3K}$$

Multiplying both sides of this equation by 3, we get

$$\frac{9}{E} = \frac{3}{N} + \frac{1}{K} \ldots\ldots \qquad 1.8$$

Poisson's Ratio $'\sigma'$in terms of K and N:

From equation (1.6), we, get

$$2N\,(1+\sigma) = 3\,K\,(1-2\sigma)$$

$$2N\sigma + 6\,K\sigma = 3K - 2N$$

Therefore $\sigma = \frac{3K-2N}{6K+2N} \ldots \qquad 1.9$

Limiting values of $'\sigma'$:

From equation (1.6), We get

3K (1 – 2 $\sigma) = 2N\ (1+\sigma)$

We know that K and N are always positive quantities. Further the least value of N and K are equal to zero.

When N= 0, the maximum value of Poisson ratio is given by

$$\sigma = \frac{3K}{6K} = \frac{1}{2}$$

On the other hand if N = $\infty.,$ then

the minimum value value of Poisson's ratio is given by

$$\sigma = \frac{\frac{3K}{N} - 2}{\frac{6K}{N} + 2} = \frac{\frac{3K}{\infty} - 2}{\frac{6K}{\infty} + 2} = -1$$

Hence -1 <$\sigma < \frac{1}{2}$

Alternative Method:

i) If $\sigma$is to be positive ,the right hand side expression must be positive, Hence left hand side expression is also positive. This possible if 2 $\sigma < 1\ (or) < \frac{1}{2}$

ii) If σ is a negative quantity, then the left hand side expression is positive. Hence the right hand side expression is also positive. This ic possible only when (1 + σ) is positive or σ>-1.

Therefore, the limiting values of $\sigma$are -1 and 0.5 (or) -1 <σ< 0.5

In practice, however σcannot be negative, Since a negative value of σ implies a lateral extension instead of lateral contraction in a direction normal to that along which extension takes place. There is no such material so far. In actual practice, the value of σ lies between 0.2 to 0.4.

1.9 Twisting couple on a Cylinder:

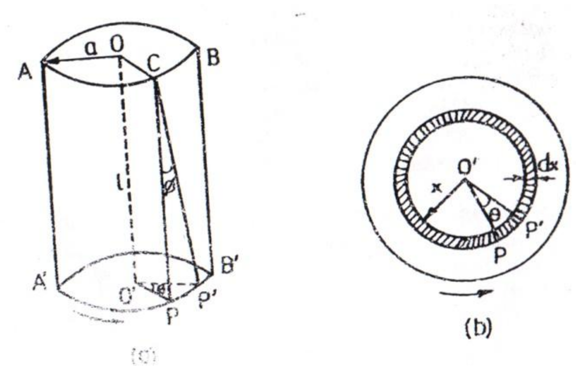


Fig 1.6

a) Twisting couple on a cylinder
b) Base view of the cylinder with coaxial shell.

Consider a short cylinder (or wire) of length ℓ and radius a, clamped at the upper end AB. Let a twisting couple be applied to the face A'B' as shown by the arrow head in a direction perpendicular to the length of the cylinder (Fig. 1.6 a).

As a result of this external twisting couple 'C', the radius of each circular cross-section of the cylinder is turned about the axis of cylinder through an angle θ called the angle of twist. Hence the radius O`P is twisted through an angle θ to the position O'P' as shown in figure 1.6 a, This is called pure shear since there is no change in length or radius and only the shape of the cylinder is changed. Due to elasticity of the material, a restoring couple is set up inside the cylinder which is equal and opposite to the twisting couple under equilibrium.

A line CP on the rim of the cylinder parallel to oo' is displaced to cp' through an angle an Øcalled angle of shear, due to twisting couple. The displacement PP’ is maximum for the points lying on the rim and goes on decreasing as we move towards o', the centre of the cylinder. Let us calculate the value of the twisting couple on this cylinder. Imagine this solid cylinder consisting of coaxial cylindrical shells.

Consider one such cylindrical shell of radius x and thickness dx (Fig. 1.6 b).

The angle of shear ∅will have the maximum value when x = a and least at 0'. But the angle of twist θ will be the same for all shells. Since ∅is small, PP' = $l\emptyset$

Similarly PP' = x θ (Refer Fig.-1.6b)

Therefore, $l\emptyset$ = xθ

$$\emptyset = \frac{X\theta}{l}$$

Rigidity Modulus N $= \frac{T}{\emptyset} = \frac{T}{\frac{X\theta}{l}}$ Where T is the shearing stress acting on the cylinder.

(or) $$T = \frac{Nx\theta}{l}$$

The base area of the hollow cylindrical shell of thickness

dx = 2 $\pi x dx$

Therefore, the shearing force acting on this area = 2 $\pi x dx . \frac{Nx\theta}{l}$

$$= \frac{2\pi N\theta}{l} x^2 \ dx$$

Moment of this force about oo' (axis of cylinder) = $\frac{2\pi N\theta}{l} x^3 \ dx$

This expression gives the magnitude of the couple required to twist an infinitesimally thin cylindrical shell of radius x through an angle θ. Hence the total couple 'C', required to twist the whole cylinder of radius 'a' about its own axis oo', may be obtained by integrating the above expression between the limits x = 0 to x = a.

Thus C = $\int_0^a \frac{2\pi N\theta}{l} x^3 dx$

$$= \frac{2\pi N\theta}{l}\left[\frac{X^4}{4}\right]_0^a = \frac{\pi N\theta a^4}{2l}$$

In the above expression if $\theta = 1$ radian, then we get,

Twisting couple per unit twist = $\frac{\pi N a^4}{2l}$

This twisting couple required to produce a twist of unit radian in the cylinder is called the torsional rigidity or modulus of torsion for the material of the cylinder.

1.10 Shafts:

A Shaft is thick rod of high rigidity modulus of elasticity that can rotate on bearings about its own axis with an arrangement for the application of a couple at one end and with an attachment to a load at the other end. A good shaft should transmit the couple applied at one end to the other end without any appreciable twist to itself. Even for large couples applied, the twist in the shaft should be very small.

Thus the efficiency of a shaft varies as $\frac{C}{\theta}$ or $\frac{\pi N a^4}{2l}$.

For good shafts with high transmission efficiency, we prefer thick rods of material of high rigidity modulus of elasticity.

1.1 Torsion pendulum:

A torsion pendulum is used to determine the rigidity modulus ‘N’ of the materials of wire and the moment of inertia of a given disc or cylinder about its axis of suspension by the method of torsinal oscillations.

Principle:

A torsion pendulum consists of a metal wire clamped to a rigid support at one end and carries a heavy circular disc at the other end. When the disc is subjected to a slight rotation and left free, it starts oscillating periodically about the wire as axis.

When a wire or cylinder of length $'\ell'$ and radius 'a' is subjected to an external couple or torque, it is twisted and a restoring couple proportional to the twist is developed in it due to elastic reaction. This restoring couple produces an angular acceleration in the wire in a direction opposite to that of the twist. During untwisting itself, it rotates beyond its equilibrium position. Hence it is twisted again and the produced angular acceleration is now in the opposite direction. This process is repeated and thus the system executes torsional oscillations. Consider an intermediate state when the wire is under twist $'\theta'$ and the disc is moving with an angular acceleration,

$$\alpha = \frac{d^2\theta}{dt^2}. \quad At this stage,$$

Potential energy confined to wire, equal to the work done in twisting it through θ

$$= \int_0^\theta Moment of the couple . d\theta$$

$$= \int_0^\theta c\theta d\theta = \frac{1}{2}c\theta^2$$

Where C is the couple per unit twist.

Kinetic energy confined to the rotating suspended mass

$$= \frac{1}{2}\sum m_i v_i^2 = \frac{1}{2}\omega^2 \sum m\, iri2$$

Where m_i's and v_i's are the mass and velocity of the particles ,constituting the suspended mass and $v_i = r_i\ \omega$. Where r_i is the radius vector of the i^{th} particle from the axis of suspension and (ω) is its angular velocity.

Therefore the total energy of the system $= \frac{1}{2}c\theta^2 + \frac{1}{2}I\omega^2$

According to the law of conservation of energy, this total energy is a constant at every instant.

$$\text{Therefore } \frac{1}{2}c\theta^2 + \frac{1}{2}I\omega^2 = \text{Constant}$$

$$\frac{1}{2}c\theta^2 + \frac{1}{2}I\left(\frac{d\theta}{dt}\right)^2 = \text{Constant}$$

Differentiating this with respect to time , we get

$$\frac{1}{2}c\ 2\theta\frac{d\theta}{dt} + \frac{1}{2}I\ \ 2\ \ \frac{d\theta}{dt}\ .\frac{d^2\theta}{dt^2} = o$$

$$\therefore I\frac{d^2\theta}{dt^2} + C\theta = 0$$

$$\text{i.e.}\frac{d^2\theta}{dt^2} + \frac{C}{I}\theta = 0 \qquad \text{(or)}$$

$$\frac{d^2\theta}{dt^2} = -\frac{C}{I}\theta$$

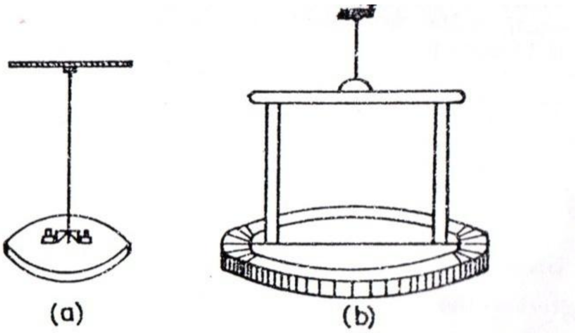
This equation represents a simple har monic motion of period.

$$T = 2\pi\sqrt{1/C}$$

Thus the torsional oscillations made by the torsion pendulum are simple harmonic and the period of the oscillation is controlled by moment of Inertia of the suspended mass about the axis of suspension and couple per unit twist produced in the wire, carrying the suspended mass.

Experiments:

(i) Determination of Rigidity modulus of a wire and moment of inertia of a circular disc about the axis of its suspension:



The torsion pendulum is formed by the given circular disc suspended whose rigidity modulus is to be determined. The experiment consists of three parts:

First, the disc is set into oscillations without any cylindrical masses on the disc. The mean period of oscillation 'T_0' is found out.

$$\text{Now } T_0 = 2\pi\sqrt{I_o/c}$$

Where I_o is the moment of inertia of the disc about the axis of suspension.

A symmetrical line (diameter) is drawn on the suspended disc passing through the point of suspension. Now two equal cylindrical masses (= 500 gms) are placed symmetrically on this line such that they are very nearer to the axis of wire as shown in figure 1.7a. The distance d_1 of the centre of gravity of each mass from the axis of wire is measured. The mean period of oscillations 'T_1' is found out by making torsional oscillation with masses on the disc,

$$\text{Therefore} \quad T_1 = 2\pi\sqrt{I_1/c}$$

where $I_1 = i_o + 2\, i_o + 2\, md_1^2$ from the parallel axis theorem. Here i_o is the moment of inertia of each mass about an axis passing through its centre and perpendicular to its plane and m is the mass of each cylindrical mass.

Therefore $T_2 = 2\pi\sqrt{I_2/c}$

Where $I_2 = i_O + 2\, i_o + 2\, md_2^2$

Now $I_2 - I_1 = 2\, m\, (d_2^2 - d_1^2)$

$T_2^2 - T_1^2 = \frac{4\pi^2}{C}(I_2 - I_1)$

Therefore $\frac{T_0^2}{(T_2^2 - T_1^2)} = \frac{I_o}{(I_2 - I_1)}$

Substituting the value of $(I_2 - I_1)$ in this equation, we get

$$\frac{I_0}{2m(d_2^2 - d_1^2)} = \frac{T_0^2}{(T_2^2 - T_1^2)}$$

$I_o = 2m(d_2^2 - d_1^2).\frac{T_0^2}{(T_2^2 - T_1^2)}$ kg m^2

Thus the moment of inertia of the disc about the axis of rotation is calculated substituting the values of T_0, T_1, T_2, d_1 and d_2 in the above formula.

The rigidity modulus of the material of the wire can be calculated as follows:

We know that,

$$C = \frac{\pi N a^4}{2\ell}$$

Where a is the radius wire and ℓ is the length of wire.

Since $T_0^2 = 4\pi^2 \frac{I_o}{C}$ and using the above equation for I_0, we get

$$C = \frac{\pi N a^4}{2\ell} = 4\pi^2 \frac{I_0}{T_0^2} = \frac{4\pi^2}{T_0^2}\, 2m(d_2^2 - d_1^2).\frac{T_0^2}{(T_2^2 - T_1^2)}$$

Therefore $N = \frac{16\pi m \ell (d_2^2 - d_1^2)}{a^4 (T_2^2 - T_1^2)}$ Pascals

Determination of the moment of inertia of an irregular body:

Here the torsion pendulum is formed by a cradle suspended by a steel wire (Refer figure 1.7b). The cradle is in the form of a horizontal circular disc fixed to a rectangular metallic frame as in figure. At the centre of disc, there is a concentric circular grove to place the body. First the mean period of oscillation 'T_0' is found out for any mass on the cradle.

$$\text{Therefore,} \quad T_0 = 2\pi\sqrt{I_o/c}$$

where I_0 is the moment of inertia of the cradle about the axis of rotation. Now place a regular body on the cradle such that the axis of the wire passes through the centre of gravity of the body placed in the cradle. Find the mean period of oscillation 'T_1'.

$$\text{Therefore,} \quad T_1 = 2\pi\sqrt{(I_o + I_1)/C}$$

Where I_1 is moment of inertia of the regular body which can determined with the help of the dimensions of the body.

Replace the regular body by the given irregular body. Find the mean period of oscillation 'T_2'.

$$\text{Therefore,} \quad T_2 = 2\pi\sqrt{(I_o + I_2)/C}$$

Where I_2 is the moment of inertia of the irregular body at the axis passing through its centre of gravity and perpendicular to the plane.

$$T_1^2 - T_0^2 = \frac{4\pi^2}{C} I_1$$

$$T_2^2 - T_0^2 = \frac{4\pi^2}{C} I_2$$

$$\frac{(T_1^2 - T_0^2)}{(T_2^2 - T_0^2)} = \frac{I_1}{I_2} \text{ (or) } I_2 = I_1\left(\frac{T_2^2 - T_0^2}{T_1^2 - T_0^2}\right)$$

Substituting the values of I_1, To, T_1 and T_2 in the above equation the value of ' I' the moment of inertia of the given irregular body can be determined.

UNIT - II

BENDING OF BEAMS

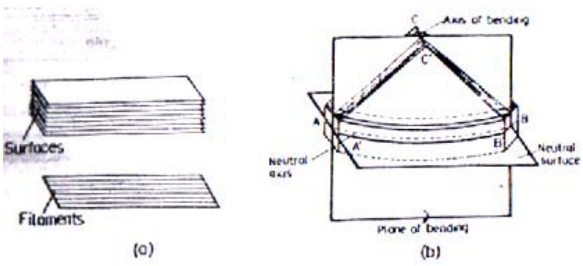
A beam is a rod or a bar of uniform cross section of homogeneous and isotropic elastic material whose length is large compared to its thickness.

2.1 Basic Assumptions involved in the simple theory of bending:

- The cross section of the beam remains unaltered during best so that that shearing stresses over any section are negligibly small.
- The radius of curvature of bent beam is large compared with its thickness.
- The minimum deflection of the beam is small compared with its length and
- The Young's modulus of the beam is not changed during bending. Thus we are going to see the simple and pure bending only.

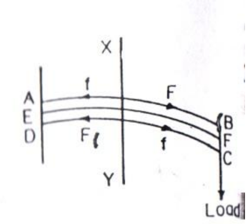
2.2 Plane of bending and Neutral axis of a bent beam:

A beam may be considered as consisting of a number of thin plane horizontal layers called **surfaces** placed one above the other. Now each plane layer or surface consists of a number of parallel longitudinal metallic fibres placed side by side and are called longitudinal filaments lying on convex side of the bent beam are elongated and those lying on concave side are shortened. However some of the filaments lying in the median plane of the beam remain unaltered in length and are called **neutral filaments** and the median plane containing these is known as **neutral surface**.



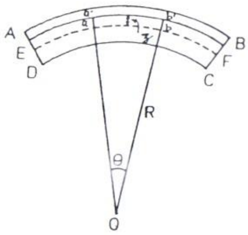
The plane in which bending takes place is known as **plane of bending** and obviously it is the vertical plane when the beam is placed horizontally. The line attained by the intersection of neutral surface and plane of bending is called **neutral axis**. A line perpendicular to the plane of bending on which centre of curvature of all the bent filaments lie is called **axis of bending.**

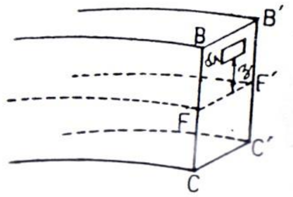
2.3 Bending moment:



In the bent beam let EF be the neutral surface (in the above fig). The plane XY represents a transverse section of the beam normal to EF. The filament AB, shown above the neutral surface gets elongated and thus it is under a stretching force F. Similarly the filament CD, shown below the neutral surface, gets compressed and thus it is under a compressive force 'F'. These two forces constitute a clockwise couple. This couple is called **external couple** or **bending couple** and it has atendency to rotate the beam clockwise. Again as the beam is at rest, the moment of this couple must be balanced by an internal couple tending to rotate the beam anticlockwise. Thus when the filament above EF is stretched by F, an equal but opposite restoring force f arises in it. Similarly when the filament below EF is compressed by F, an equal and opposite restoring force f arises in it. These forces f and f constitute an anticlockwise internal couple which is called **balancing couple** or **restoring couple** which has a tendency to rotate the beam anticlockwise. If the moments of all external couples acting over all filaments in the cross section XY are added we get the moment of the external couple which bends the beam. If the moments of all the internal couples acting over all filaments in the section XY are added we get the moment of the internal restoring couple which balance the external couple. Thus in equilibrium position, moment of bending couple = moment of restoring couple.

The moment of the internal restoring couple is called bending moment or internal bending moment of the beam.





Let a beam ABCD having rectangular cross section be bent in the form of an arc of a circle of radius R with the centre at O. Consider a small portion ab of neutral axis of the beam subtending an angle θ at the centre O. a'b' is another small portion of a filament at a distance 'z' above the neutral filament ab. Before bending a'b' = ab. After bending, a'b' > ab since a'b' is above the neutral surface.

When, θ is small

$$a'b' = (R + z)\theta$$

$$ab = R\theta$$

Therefore increase in length of small element a’b’ = a’b’ – ab.

$$= (R + z)\theta - R\theta$$

= zθ

Strain in a’b’= $\frac{increaseinlength}{originallength} = \frac{Z\theta}{R\theta} = \frac{Z}{R}$

Let BB′C′C be the cross section of the beam perpendicular to plane of bending (refer figure 2.). The line FF' lies in the neutral surface. Let us consider an area of cross section δA of a′b' at a distance z above the neutral line FF' on this cross section BB'C'C.

The Young's Modulus of the material of the beam 'E' $= \frac{Stress}{Strain}$

Therefore, stress on this area $\delta A = E.Strain = E\frac{Z}{R}$

Total internal force on the area

$$\delta A = E\frac{Z}{R}\delta A$$

Moment of this force about the neutral line FF'

$$= E\frac{z}{R}\delta A.z = \frac{E}{R}\delta Az^2$$

So the total moment of these internal forces acting above and below the neutral line

FF' $= \frac{E}{R}\sum\delta Az^2$

where $\sum\delta Az^2 = I_g$ the geometrical moment of inertia of the cross section area of the beam about a horizontal axis through its centroid. I_g is also equal to AK^2 where A is the cross sectional area of the beam and K, the radius of gyration of this cross sectional area about a horizontal axis through its centroid.

Thus the moment of the restoring .couple or the bending moment

$$= \frac{E}{R}I_g$$

As discussed above, $\frac{Elg}{R}$, the moment of all the internal forces balances the external couple. The quantity $EI_g = EAK^2$ is called the flexural rigidity of the beam. Geometrical moment of inertia of the beam is also equal to the (mechanical) moment of inertia 'I' if the beam has an unit mass per unit area.

Note:

(i) If the cross section of the beam is rectangular then A = b x d where b is the breadth of the face BB'C'C and d the thickness of the beam. The moment of inertia of the rectangle BB'C'C about the axis FF$'$ parallel to the side BB$'$= $MK^2 = M\frac{d^2}{12}$

(i.e) $K^2 = d^2/12$

Therefore, geometrical moment of inertia of the beam

$$l_g = AK^2 = \frac{bdd^2}{12} = \frac{bd^3}{12}$$

(ii) If the cross section of the beam is circular and has a radius 'r', then A = πr^2.

Moment of inertia about $FF^1 = MK^2 = M\frac{r^2}{4}$

Therefore, $K^2 = \frac{r^2}{4}$

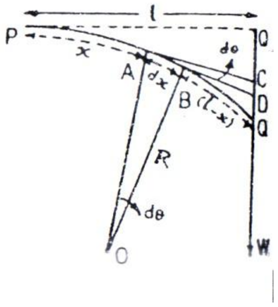
Hence geometrical moment of inertia of the beam about the neutral surface

$$'I_g' = \pi r^2 \frac{r^2}{4} = \frac{\pi r^4}{4}$$

2.4 Depression of a cantilever:

A cantilever is a beam fixed horizontally at one end and loaded at the other end. The Young's modulus of the material of the cantilever can be determined using the value of depression produced in that cantilever.

Let PQ be the neutral axis of a cantilever fixed at P. Let ℓbe its length and the weight of the cantilever be negligible. It is loaded at Q with a weight and so the end Q is depressed to Q$^{'}$.



Consider a point A at a distance x from the fixed end 'P' as shown in the above figure

The moment of the couple due to the load 'W' = bending couple

$$= W.AQ = W(\ell - x)$$

This must be equal to the moment of restoring couple (or) bending moment, $\frac{EI_g}{R}$, under equilibrium conditions.

$$\text{Therefore, } W(\ell - x) = \frac{EI_g}{R}$$

Where R is the radius of curvature of the neutral axis at A. Let B be another point at a distance dx from A and AB subtending an angle 'dθ' at O. When θ is small, dx = Rθ dx

$$\text{Hence, } R = \frac{dx}{D\theta}$$

Substituting the value of R in the above equation, we get

$$W(\ell - x) = EI_g \frac{d\theta}{dx}$$

$$W(\ell - x)dx = EI_g d\theta$$

Draw tangents at A and B meeting the vertical line OQ' at C and D respectively.

Then the depression of B below A is evidently

$$CD = dy = (\ell - x)d\theta$$

$$\left(\frac{dy}{\ell - x}\right) = d\theta$$

Substituting the value of dθ in the above equation, we get

$$W(\ell - x)dx = EI_g\left(\frac{dy}{\ell - x}\right)$$

$$i.e. W(\ell - x)^2 dx = EI_g dy$$

$$\text{Thus dy} = \frac{W(\ell - x)^2}{EI_g} dx$$

Therefore total depression of the cantilever ‘y’

$$= \frac{W}{EI_g}\int_0^{\ell}(\ell - x)^2 \, dx = \frac{W}{EI_g}\int_0^{\ell}(\ell^2 - 2\ell x + x^2)\, dx$$

$$= \frac{W}{EI_g}\left[\ell^2 x - \frac{2\ell \; x^2}{2} + \frac{x^3}{3}\right]_0^{\ell} = \frac{W}{EI_g}\left(\ell^3 - \ell^3 + \frac{\ell^3}{3}\right)$$

$$y = \frac{W\ell^3}{3EI_g}$$

Hence the Young's modulus of the material of the cantilever

$$E = \frac{W\ell^3}{3yI_g}$$

Note: When the weight of the cantilever is effective, then in addition to the weight W at Q the weight of the portion $(\ell - x)$ of the cantilever is also acting at the mid point or the centre of gravity of this portion. If w be the weight per unit length of the cantilever, a weight of

w $(\ell - x)$ is acting at a distance $(\ell - x)/2$ from the section AB.

Under equilibrium conditions, the total bending moment,

$$W(\ell - x) + w(\ell - x)\frac{(\ell - x)}{2} = \frac{EI_g}{R} = EI_g \frac{d\theta}{dx}$$

$$\frac{w(\ell - x)dx + \frac{w}{2}(\ell - x)^2 dx}{EI_g} = d\theta$$

Therefore dy = $(\ell - x)d\theta = \frac{W}{EI_g}(\ell - x)dx + \frac{w}{2EI_g}(\ell - x)^3\, dx$

$$\therefore Total\ deepression\quad 'y' = \frac{W}{EI_g}\int_o^{\ell}(\ell - x)^2\ dx + \frac{w}{2EI_g}\int_o^{\ell}(\ell - x)^3\, dx$$

Put $(\ell - x) = u$

Therefore –dx = du and $\int_o^{\ell}\ dx = \int_o^{\ell} du$

Hence y = $\frac{W}{EI_g}\int_o^{\ell} u^2 du + \frac{w}{2EI_g}\int_o^{\ell} u^3 du$

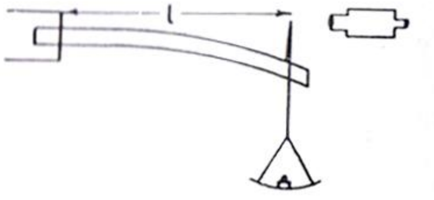
$$= \frac{W\ell^3}{3EI_g} + \frac{w\ell^4}{8EI_g}$$

If $W_1 = w\ell = weight\ of\ the\ cantilever, then$

$$y = \left(W + \frac{3}{8}W_1\right)\frac{\ell^3}{3EI_g}$$

When $W_1 \ll W \quad then\ y = \frac{W\ell^3}{3EI_g}$

2.5 **Young's modulus by Cantilever (Statical Method):**



A given rod (or bar) whose Young's modulus is required is clamped firmly and horizontally at one end. A weight hanger is suspended from the free end of it as shown in the above figure . A vertical pin is also fixed firmly to the free end of the rod using wax. A travelling microscope is focussed on the pin and the image of the tip of the pin is made to coincide with the point of intersection of the cross wires.

The microscope reading (R_0) is noted. (We can take the reading by coinciding the image of the tip of the pin with the horizontal cross wire also). A small mass($\simeq$ 50gms)is placed in the weight hanger. The loaded end of the rod then gets depressed and the top of the pin also gets lowered by the same amount. The microscope is adjusted to get the image of the tip at the point of intersection of the cross wires and the microscope reading is again noted. The load is increased in equal steps and the corresponding microscope readings are noted. These observations are repeated while loads are removed from the weight hanger in equal steps. The results are tabulated as under:

Load kg	Microscope readings			Depression for M kg. wt. M
	While loading m	While Unloading M	Mean m	

Mean value of y = …… m

The distance from the fixed end of the rod to the point where, the weight hanger is suspended is measured and it is taken as the length $'\ell'$ of the cantilever. If the cantilever has circular cross section, its radius is measured using a screw gauge; if it has rectangular cross section then its breadth and thickness are measured.

We know that,

W= $\frac{y\ell^3}{3EI_g}$ (assuming the weight of the rod is negligible)

Therefore, $E = \frac{W\ell^3}{3\,y\,I_g} = \frac{M_g\ell^3}{3\,y\,I_g}$

If the cantilever has circular cross section, $I_g = \frac{\pi r^4}{4}$

If the cantilever has rectangular cross section $l_g = \frac{bd^3}{12}$

2.6 Oscillations of a Cantilever:

Let a rod be clamped rigidly at one end and a load is attached at the other end. Assume that the mass of the rod is negligible. Let the load be depressed a little and released. The rod begins to oscillate simple harmonically due to its bending and unbending.

The bending force on the cantilever = Mg

Where M is the mass of the load attached at the free end and g is the acceleration produced in it.

The restoring force produced = $-K_y$ = $\frac{-3EI_g \quad y}{\ell^3}$

Where y is the displacement of the cantilever from its equilibrium position and

Kis the force constant (or) force per unit displacement of the cantilever.

In equilibrium,

Bending force = Restoring force

Therefore

$$Mg = -\frac{3EI_g y}{\ell}$$

$$M\frac{d^2y}{dt^2} = \frac{-3EI_g y}{\ell^3}$$

Therefore, $\frac{d^2y}{dt^2} = \frac{-3EI_g y}{M\,\ell^3} = -\omega^2 y$

This is the equation of a simple harmonic motion with angular frequency

$$\omega^2 = \frac{3EI_g}{M\ell^3}$$

Therefore the load executes SHM with a period 'T' given by

$$T = 2\pi\sqrt{\frac{M\ell^3}{3EI_g}} \quad \text{(or)} \quad T^2 = \frac{4\pi^2 M\,\ell^3}{3\,EI_g}$$

2.7 Young's modulus by Cantilever (Dynamical method):

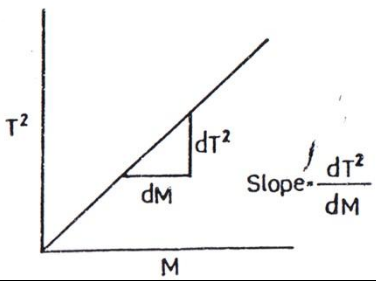
Let a cantilever be loaded with a mass 'M' kg as shown in the above figure. Let the loaded end of the cantilever be depressed from its equilibrium position and released. Find the period of oscillation of the cantilever. Then add another 50gms and once again find the period. The load is increased in equal steps and the corresponding periods are noted. Then draw a graph between the square of the period and the corresponding mass in the weight hanger. It should be a straight line. Find the slope of the line.

$$Slope = \frac{dT^2}{dM} = \frac{4\,\pi^2\,\ell^3}{3\,EI_g}$$

$$Therefore, E = \frac{4\pi^2\ell^3}{3\,I_g} x \frac{1}{slope}$$

Note: By means of finding the mean value of $\frac{M}{T^2}$ of different loads, one can also find the value of E. In that case,

$$E = \frac{4\pi^2\ell^3}{3\,I_g}\frac{M}{T^2}$$



2.8 Workdone in bending a Cantilever:

We know that ,

Work done = Force x displacement of the cantilever

Bending force = Mg = $3EI_g\, {}^{y}/_{\ell^3}$

because, in equilibrium conditions, the bending force is equal and opposite to the restoring force.

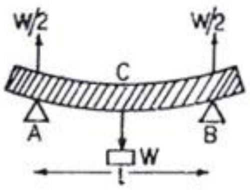
/Work done $=\frac{3EI_g}{\ell^3}\, y.y = \frac{3EI_g}{\ell^3}y^2$

This is stored in the form of elastic potential energy of the cantilever.

2.9 Uniform Bending and Non uniform Bending:

When an uniform load is acting on the beam, the envelope of the bent beam forms an arc of a circle and the bending is called **uniform bending**. When we load the beam only at a point of the beam the envelope of the bent would not form an arc of a circle and the bending is called **non uniform bending**. Therefore cantilever bending is a non uniform bending.

Non uniform bending (A beam supported symmetrically on two knife edges and loaded In the middle):



Consider a light beam supported symmetrically on two knife edges A and B at a distance 'ℓ' apart with a load W at the middle point 'C' of the beam. The reaction at each knife edge is equal to $\frac{W}{2}$ in the upward direction.

Since the middle part of the beam is practically horizontal, it may be equal to two inverted cantilevers fixed at C and being loaded at A and B with a load

$\frac{W}{2}$acting in the upward direction. The length of each inverted cantilever is equal to $\ell/2$.

By considering the above facts, the 'elevation of the beam at A or B above C or the depression of the beam at the middle is given by

$$y = \frac{\frac{W}{2} \cdot (\frac{\ell}{2})^3}{3\, EI_g} = \frac{W\ell^3}{48\, EI_g}$$

Experiment:

The given beam is supported on two knife edges in the same horizontal level, equal lengths projecting beyond the supports. A vertical pin is fixed at the centre of beam by means of wax. The weight hanger is attached at the middle using thread. In the microscope, the image of the tip of the pin is made to coincide with the horizontal cross-wire,, The loads are added to the hanger in steps of 50 gms and the microscope, is adjusted so that the tip of the image of the pin just coincides with the horizontal crosswire in each case and the microscope readings are noted. The observations are repeated while unloading the hanger in same steps and the readings are tabulated as under:

Load kg.	Microscope Readings			Depression for M k.g.wt 'y' M
	While loading m	While unloading M	Mean m	

Mean value of y= ……..m

using the formula $E = \frac{Mg\ell^3}{48\, yI_g}$, E can be determined

Uniform Bending:

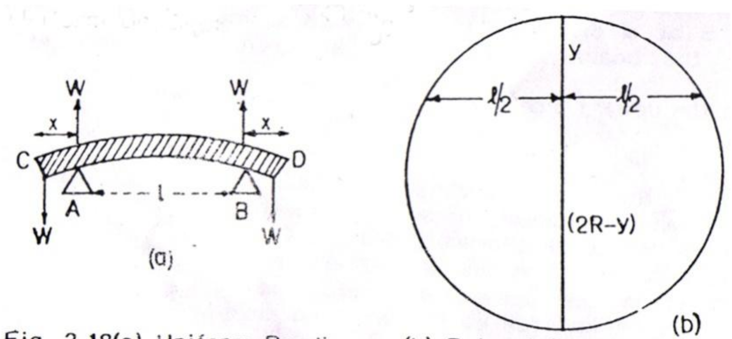


Fig. 3.18(a) Uniform Bending. (b) Rule of Sagitha.

Consider a light beam CD supported symmetrically on two knife edges at A and B and loaded with equal weights W at each end as shown in the above figure. Hence the reaction at each knife edge is equal to W. Now the beam bends uniformly and forms an arc of a circle of radius R.

The bending moment =W.x =$\frac{EI_g}{R}$

where x is the distance from loaded end and knife edge.

If the centre of the beam is elevated through a distance y, then by property of circles (Rule of Sagitha)

$$\frac{\ell}{2}.\frac{\ell}{2} = (2R - y).y$$

$$ie.\frac{\ell^2}{4} = 2Ry - y^2$$

Since y is very small, y^2 *can be negiected*.

Therefore, $\frac{\ell^2}{4} = 2\,Ry$

$$\frac{1}{R} = \frac{8y}{\ell^2}$$

Substituting the value of $\frac{1}{R}$ in the above equation, we get

$$Wx = \frac{EI_g}{2}.8y$$

$$y = \frac{Wx\ell^2}{8\,EI_g}$$

$$E = \frac{Wx\ell^2}{8\,yI_g}$$

Experiment:

The given beam is supported on two knife edges in the same horizontal level, equal lengths projecting beyond the supports. A vertical pin is fixed at the centre of the beam by means of wax. Two weight hangers are attached at a distance x from the knife edges. In the microscope the image of the tip of the pin is made to coincide with the horizontal cross-wire. The loads are added to the hangers in steps of 50 gms simultaneously and the microscope is adjusted so that the tip of the image of the pin just coincides with the horizontal cross-wire in each case and the microscope readings are noted. The observations are repeated while unloading the hanger in same steps and the readings are tabulated as under:

Load k.g.	Microscope readings			Depression 'y' for M kg. wt. M
	while loading m	while unloading m	Mean m	

Mean value of y = ………m

Then using the formula, E= $\frac{Mg\ x\ \ell^2}{8\ yI_g}$, E can be determined.

2.10 Application to girders – I form girders:

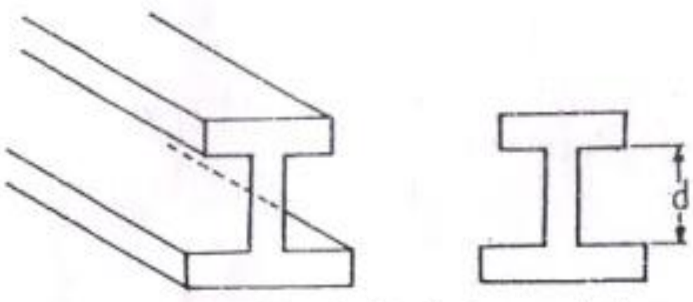


Fig. 3.19 I shape girders.

When a heavy girder is supported at its ends, it is bent non uniformly under its own weight into the form of an inverted double cantilever. We know that the depression of its mid-point is given by

$$y = \frac{Mg\ \ell^3}{48\ EI_g}$$

If it has rectangular cross section of breadth b and thickness 'd'

$$I_g = \frac{bd^3}{12}$$

$$\text{Therefore,}\quad y = \frac{Mg\ \ell^3}{48\ E\frac{bd^3}{12}} = \frac{Mg\ \ell^3}{4\ E\ bd^3}$$

When a beam is used as a girder, it should have minimum depression under its own weight. Further depression of the girder should be small for a given load also. This can be achieved by decreasing its length or span, increasing E and increasing b or d. When we decrease the length of the girder, the depression is reduced. But it is found that the decreasing ℓ to get minimum depression is not economical inso many respects. By selecting the girder material with high Young's modulus (like steel) one can get the small depression. Since d occurs in the equation in the form of d^3, therefore a smaller change in d produces the same effect as a larger change in b. The corresponding increase in volume of the girder will be much smaller when d is increased than when b is increased so as to have the same value of depression. it is therefore more economical to have a large depth end(small breadth). For purposes of stability the upper and lower parts of the cross section will be broader so that the section will have the shape o**f I.** This can be explained in another way. When a girder is supported at its two ends, its middle part is depressed and the surfaces above and below its neutral surface are compressed and extended respectively. Compression is maximum at the upper face and extension is maximum at the lower face since stresses are maximum there. Stresses are decreasing as we proceed towards the neutral surface from either side. It follows therefore that the upper and lower faces of the girder should be much stronger that its middle portions. In other words, the middle portion of the girder may be made of a much smaller breadth than the upper and the lower faces, thus saving a good amount of material with no loss in its strength. That is why the girders have the shape of **I.**

UNIT - III

SURFACE TENSION

3.1 Introduction

Any liquid in small quantity, so that gravity influence is negligibly small, will always assume the form of a spherical drop. e.g., rain drops, small quantities of mercury placed on a clean glass plate etc. So a liquid must experience some kind of force, so as to occupy a minimum surface area. This contracting tendency of a liquid surface is known as surface tension of liquid. This is a fundamental property of every liquid.

Surface tension is that property of liquids owing to which they tend to acquire minimum surface area.

Small liquid drops acquire spherical shape due to surface tension. Big drops flatten due to weight.

The following experiment illustrates the tendency of a liquid to decrease its surface area.

When a camel hair brush is dipped into water, the bristles spread out [Fig. 3.1 (a)]. When the brush is taken out, the bristles cling together on account of the films of water between them contracts [Fig. 3.1 (b)]. This experiment clearly shows that the surface of a liquid behaves like an elastic membrane under tension with a tendency to contract. This tension or pull in the surface of a liquid is called its surface tension.

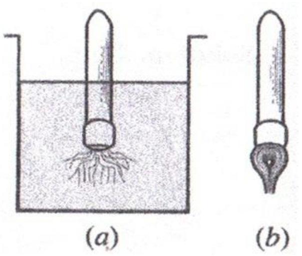


Fig. 3.1

Definition: Surface tension is defined as the force per unit length of a line drawn in the liquid surface, acting perpendicular to it at every point and tending to pull the surface apart along the line.

Unit of Surface Tension: Surface tension is force per unit length. So its SI unit is Newton per meter (Nm^{-1})

Dimensions of Surface Tension: Surface tension is the ratio of a force to a length.

Surface tension = force/length

Dimensions of force = MLT^{-2}

Dimensions of length = L

$\therefore$ Dimensions of surface tension = $\frac{MLT^{-2}}{L} = MT^{-2}$

The dimensional formula for surface tension is $[MT^{-2}]$

3.2. Molecular interpretation:

Consider three molecules *A*, *B* and *C* of a liquid (Fig. 3.2). The circles around them indicate their respective spheres of influence.

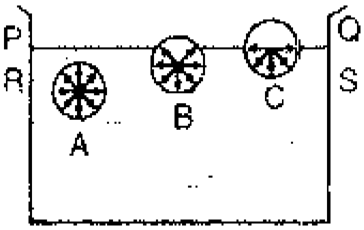


Fig.3.2

(i) The molecule A is well within the liquid. It is attracted equally in all directions by the other molecules lying within its sphere of influence. Therefore, it does not experience any resultant force in any direction. This happens only as long as the sphere of influence is well within the liquid.

(ii) The sphere of influence of molecule *B* lies partly outside the liquid. The upper half of the sphere contains fewer molecules attaching the molecule *B* upwards, than the lower half attracts it downwards. Hence, there is resultant downward force acting on *B*.

(iii) The molecule C lies on the surface of the liquid. Half of its sphere of influence lies above the surface of the liquid and contains only a few vapor molecules. But there are many liquid molecules in its entire lower half. Thus the resultant downward force in this case is the maximum.

Draw a plane *RS* parallel to the free surface *PQ* of the liquid at a distance equal to the molecular range. The layer of the liquid between the planes *PQ* and *RS* is called the *surface film.* Hence all the molecules in the surface film are pulled downward due to the cohesive force between molecules.

If a molecule is to be brought from the interior of the liquid to the surface of the liquid, work has to be done against the downward cohesive force acting upon it. Hence, molecules in the surface film have greater potential energy than the molecules inside the liquid. The potential energy of a system tends towards a minimum. Hence the surface film tends to contract, so as to contain minimum number of molecules in it. Thus the surface of the liquid is under tension and behaves like a stretched elastic membrane.

Surface Energy: The potential energy per unit area of the surface film is called its surface energy.

3.3 Pressure Difference across a Liquid Surface

(a) If the free surface of the liquid is plane [Fig. 3.3 (a)], the resultant force due to Surface Tension

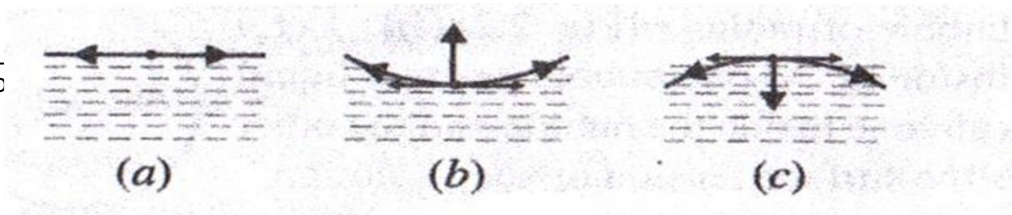


Fig.3.3

on a molecule on the surface is zero.

(b) If the free surface of the liquid is concave [Fig. 3.3 (b)],the resultant force due to Surface Tension on a molecule on the surface acts vertically upwards.

(c) If the free surface of the liquid is convex [Fig. 3.3(c)], the resultant force due to Surface Tension on a molecule on the surface acts vertically downwards (into the liquid).

3.4 Excess pressure Inside a Liquid Drop

A spherical liquid drop has a convex surface [Fig. 3.4 (i)]. The molecules near the surface of the drop experience a resultant force, acting inwards due to surface tension. Therefore the pressure inside the drop must be greater than the pressure outside it. Let this excess pressure inside the liquid drop over the pressure outside it he *p*.

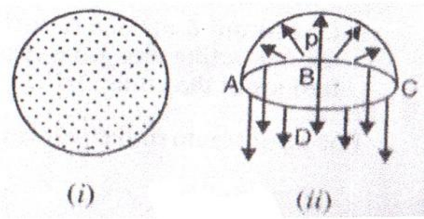


Fig.3.4

Imagine the drop to be divided into two exactly equal halves. Consider the equilibrium of the upper half of the drop [Fig. 3.4 (ii)]. r is the radius of the drop and σ it's Surface Tension.

Upward force on the plane face ABCD due to the excess pressure p $= p\,\pi\,r^2$

Downward force due to surface tension acting along the circumference of the circle ABCD $= \sigma\, 2\pi\, r$

$$p\,\pi\,r^2 = \sigma\,2\,\pi\,r$$

$$p = \frac{2\sigma}{r}$$

Example. What would be the pressure inside a small air bubble of 10^{-4} m radius, situated just below the surface of water. Surface Tension of water may be taken to be $70\text{x}10^{-3}\ \text{Nm}^{-1}$ and the atmospheric pressure to be $1.012\text{x}10^{5}\ \text{Nm}^{-2}$.

Solution.

Excess of pressure inside the spherical air bubble over that of the atmosphere $= p = \frac{2\sigma}{r}$

Here, $\sigma = 70 \times 10^{-3}\ Nm^{-1}$, $r = 10^{-4}$ m

Excess pressure $= \frac{2\sigma}{r} = \frac{2X(70X10^{-3})}{10^{-4}} = 1400\ Nm^{-2}$

Total pressure inside the air bubble = Atmospheric pressure + Excess pressure = $1.012 \times 10^{5} + 1400 = 1.026 \times 10^{5}\ Nm^{-2}$.

3.5 Excess Pressure inside a Soap Bubble

Consider a soap bubble of radius r [Fig. 3.5 (i)]. Let p be the excess pressure inside it. A soap bubble has two liquid surfaces in contact with air, one inside the bubble and the other outside the bubble. σ is the surface tension of soap solution.

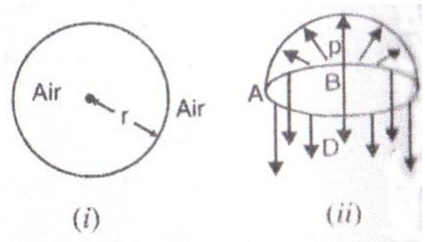


Fig.3.5

Consider the equilibrium of the upper half (or the upper hemisphere of the bubble [Fig. 3.5 (ii)].

Upward force on the plane face ABCD due to the excess pressure p $= p\,\pi\, r^2$

Downward force due to surface tension acting along the circumference of the circle ABCD $= 2 \times \sigma\, 2\pi r = 4\pi r \sigma$

For equilibrium of the hemisphere,

$$p\,\pi\, r^2 = 4\,\pi\, r\sigma$$

$$p = \frac{4\sigma}{r}$$

Example: The Pressure of air in a bubble of 7×10^{-3}m diameter is 8×10^{-3} m of water above the atmosphere pressure. Calculate the Surface Tension of the soap. **Solution.**

Excess of pressure inside a soap bubble over that outside it $= p = \frac{4\sigma}{r}$

Here, $p = 8 \times 10^{-3}$ m of water $= (8 \times 10^{-3}) \times 1000 \times 9.81\ Nm^{-2}$

$= 78.48\ Nm^{-2}$

r = (7x10^{-3}) /2=3.5x10^{-3} m.

$$\sigma = \frac{pr}{4} = \frac{78.48x93.5x10^{-3}}{4}$$

= 68.67x10^{-3} Nm^{-1}

3.6 Molecular forces:

There are two kinds of molecular forces:

(i) adhesive forces (ii) cohesive forces.

(i) Forces of attraction between molecules of different substances are known as adhesive forces. For example, the force of attraction between the glass molecules of a beaker and molecules of water contained in it is an adhesive force. Adhesive force is different for different pairs of substances.

(ii) Force of attraction between molecules of the same substance is called cohesive force. This force varies inversely probably as the eighth power of the distance between two molecules. Hence, it is very appreciable when the distance between two molecules is small. It is the greatest in solids, less in liquids and the least in gases. Therefore, a solid has a definite shape, a liquid has a definite free surface and a gas has neither.

The maximum distance up to which a molecule exerts a force of attraction on another is called the range of molecular attraction and is generally of the order of 10^{-9} m. A sphere with the molecule as centre and the range of molecular attraction as radius is called the sphere of influence of the molecule. The molecule attracts and is, in turn, attracted by the molecules present inside this sphere.

3.7 Variation of Surface Tension with Temperature

Liquids are of two types, *viz., (i)* unassociated liquid and *(ii)* associated liquid. An unassociated liquid contains the individual molecules of that liquid. Example: Benzene and carbon tetrachloride. An associated liquid contains groups of molecules of quite another type. These groups, however, tend to break up into single molecules with a rise in temperature. At the ordinary temperatures, water is known to consist of groups, consisting of two H_2O molecules, in addition to ordinary single H_2Omolecules. Thus water is an associated liquid at these temperatures.

The S.T. of an unassociated liquid is found to decrease with rise of temperature, according to the simple formula $\sigma_t = \sigma_o(1 - \alpha t)$where σ_t is the S.T. at $t^o\ C, \sigma_o at\ 0^o\ C\ and\ \alpha$ is the temperature coefficient of S.T. for the liquid. Van der Waals and Ferguson suggested other relations from which could be easily deduced that the S.T. is zero at the critical temperature. The best relation connecting S.T. and temperature, for both associated and unassociated liquids, is due to Eotvos. This formula was later modified by Ramsay and Shields.

This is represented by

$\sigma\ (Mvx)^{3/2}\ = k\ (\theta_c - \theta - d) where\ \sigma = Surface\ tension\ at\ \theta\ K, \theta_c$ = Critical temperature, d = a constant, varying from 6 to 8 for most of the liquids, k = another `constant having the value 2.12 for associated liquids and 2.22 for unassociated liquids.

x = Coefficient of association

$$= \frac{effective\ molecular\ weight\ of\ associated\ liquid}{mol. wt. of\ the\ unassociated\ liquid\ with\ the\ same\ molecules}$$

M= molecular weight of the unassociated liquid and *v* its specific volume. This shows that the S.T. is zero, when $\theta = (\theta_c - d)$ i.e., at a temperature a little below the critical temperature.

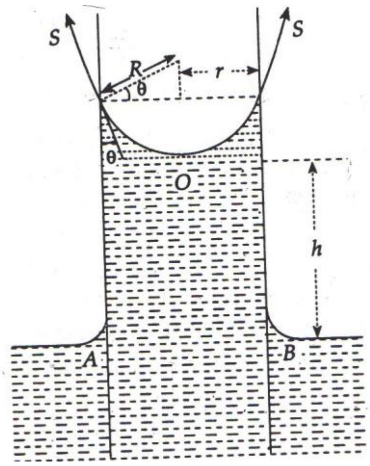


Fig 3.6

3.8 Capillary rise and energy consideration

If a tube of very narrow bore (called capillary) is dipped in a liquid, it is found that the liquid in the capillary either ascends or descends relative to the surrounding liquid. This phenomenon is called capillarity. The root cause of capillarity is the difference in pressures on two sides of (concave and convex) curved surface of liquid.

When a uniform capillary tube, open at both ends, is partially dipped vertically (fig 3.6) in a liquid that wets the tube, the surface of the liquid inside the tube is concave upward. The pressure in the liquid just below the meniscus is less than the atmospheric pressure above by 2S/R, where S is the surface tension of the liquid and R is the radius of curvature of meniscus. Hence the liquid rises in the capillary tube till the weight of the volume of liquid lifted in it is balanced by the above difference in pressure.

Let *h* be the height of the liquid column in the capillary above the free surface of the liquid outside. If $h \gg r$, the radius of the tube, the meniscus at the top may be considered hemispherical of radius of curvature R≈ r.

Volume of the liquid lifted, V=volume of a liquid cylinder of height *h*+ volume of liquid in the meniscus.

$$V = \pi r^2 h + (\pi r^2 \times r - \frac{2}{3}\pi r^2)$$

$$= \pi r^2 h + \frac{1}{3}\pi r^3$$

$$= \pi r^2 \left(h + \frac{1}{3}r\right)$$

Weight of the liquid lifted $= \pi r^2 \left(h + \frac{1}{3}r\right) \rho g,$ where ρ is the density of the liquid.

The liquid makes contact with the tube along a line $2\pi r$. If S be the tension, acting tangentially to the liquid surface, the vertical component of it is S$Scos\theta$. so that the total upward force due to it is $2\pi rcos\theta$.

At equilibrium, therefore, we have

$$2\pi rScos\theta = \pi r^2(h + \frac{1}{3}r)\rho g$$

$S = \frac{r(h+1/3\ r)\rho g}{2cos\theta}$ (3.1)

If $\theta = 0$, as is the case of pure water in clean glass, $cos\theta = 1$ and so we obtain

$S = \frac{1}{2}r\left(h + \frac{1}{3}r\right)\rho g$ from (3.1)

$$= \frac{1}{2}rH\rho g$$

Where $H = h + \frac{1}{3}r,$ the effective height.

When a capillary tube is dipped vertically into a liquid which wets the walls of the tube, there is a rise of the liquid inside the tube. The rise, obviously, takes place against the action of gravity and the liquid, therefore, must gain in potential energy. The question, therefore, arises as to where does it get this increasing potential from. For , according to the law of conservation of energy, energy can only be converted from one form into another, but cannot be created. the explanation is, however, simple.

We have three surfaces of separation to consider when a capillary tube is immersed in a liquid, viz., i) an air-liquid surface ii) an air-glass surface and iii) an glass-liquid surface, each having its own surface tension, different from the others, and equal to its free surface energy per unit area.

Now, as the plane liquid surface in the tube acquires a curvature, the air liquid surface increases and, as the liquid rises in the tube, the glass liquid surface increases, the air glass surface decreasing by an equal amount. Thus, the surface energy of the air liquid and the glass liquid surfaces increases while that of the air glass surface decreases by the same amount. In other words, the energy required to raise the liquid in the capillary tube is obtained from the surface energy of the air glass surface.

On the other hand, a liquid, which does not wet the walls of the tube, gets depressed inside it, below its level outside the tube. In this case, obviously, the glass liquid surface decreases, whereas the air glass surface increases by an equal amount, resulting in an net increase in the surface energy of the whole system .This energy is derived from the depression of the liquid inside the tube, those gravitational potential energy is thus decreased by an equal amount.

3.9Jaegar's Method

Principle: The experiment is based on the principle that the pressure inside an air bubble in a liquid is greater than the pressure outside it by 2σ/*r*. Here σ is the S.T. of the liquid and *r* the radius of the air bubble. This excess pressure can be directly found and hence σ can be calculated.

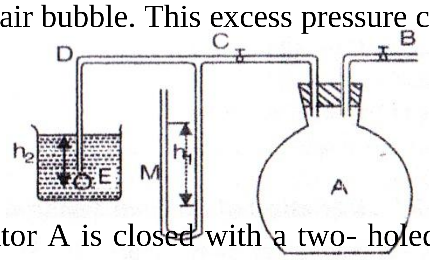


Fig 3.7

Apparatus: An aspirator A is closed with a two- holed stopper through which pass two glass tubes (fig 3.7). One of these is connected to a water reservoir through a stopcock B and the other is joined through a tap C to a manometer M and a vertical tube DE. The tube DE ends in a narrow orifice at E and dips into the experimental liquid contained in a beaker.

Experimental Details: If the stopcock B is opened, water flows into the aspirator and the air in the aspirator is displaced. The displaced air forces its way through the tube DE and forms air bubbles at E. The size of each air bubble gradually grows. When its radius becomes equal to the radius of the tube at E, it becomes unstable and breaks away. During the growth of the bubble, the pressure inside increases and reaches a maximum value at the instant of detachment. The difference in manometer levels h_1 is noted just when the bubble detaches itself. At the moment of detachment,

the pressure inside the bubble = $p_1 = H + h_1 \rho_1 g$, where

H= atmospheric pressure, h_1 = the difference in manometer levels, and

ρ_1 = density of the manometric liquid.

The pressure outside the bubble at the same time } $= p_2 = H + h_2 \rho_2 g$

Where h_2 = Length of the tube dipping in the experimental liquid and

ρ_2 =Density of the experimental liquid.

Excess pressure inside the bubble } $= p = (H + h_1\rho_1 g) - (H + h_2 \rho_2 g)$

$$= (h_1 \rho_1 - h_2 \rho_2) g$$

But the excess pressure inside the bubble = $2\sigma/r$

Hence $2\sigma/r = (h_1\rho_1 - h_2\rho_2) g$

or $\sigma = \frac{1}{2} rg (h_1\rho_1 - h_2\rho_2)$

Advantages:

1. The angle of contact need not be known

2. The continual renewal of the liquid air interface helps in avoiding contamination

3. The experiment does not require a large quantity of liquid.

4. The liquid in the beaker may be heated to various temperatures. Hence the S.T. of a liquid can be determined at various temperatures.

Drawbacks:

1. The exact value of the radius of the bubble when it breaks away cannot be ascertained.

2. The drop may not be hemispherical and of quite the same radius as the aperture at *E*.

3. The calculations are based on the assumption of static conditions but the phenomenon is not entirely statical.

For these reasons, this method does not give very accurate results for the surface tension.

UNIT IV
VISCOSITY

4.1 Introduction

Consider two parallel planes A and B separated by a distance [Fig.4.1]. Let the plane B be at rest and the plane A be moving with a uniform velocity. Let the space between the two plans be filled with a gas or liquid. So the layer of liquid in contact with B will be at rest while the layer in contact with A will move the maximum velocity. The layers in between will move the different velocities, decreasing from A to B. So a velocity gradient is set up. Because of this, the liquid will exert a force at A in a direction opposite to its motion tending to reduce its velocity. Similarly at B, a force will be exerted urging it to move in the direction of motion of A. As a net result, the relative velocity between the layers A and B will gradually decreases. This property of the liquid by which it resists the relative motion between its different layers is known as viscosity or internal friction of the liquid.

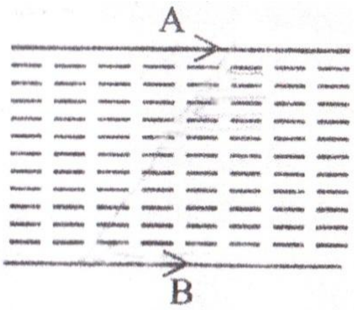


Fig. 4.1

It is to be noted that an external tangential force has to be applied to maintain this relative motion between the layers of the liquid; otherwise the liquid will not flow. According to Newton's third law of motion, internal forces from within the liquid will be brought into play opposing the flow of the liquid. These internal forces are called viscous forces or viscous drag. These forces are similar to forces of friction in solids.

4.2 Stream - lined, and turbulent motion

Consider the flow of liquid through a narrow tube. The velocity of flow is greatest along the axis of the tube. The layers in contact with the walls of the tube will be at rest. An external pressure - head is to be applied for the liquid to flow. If this pressure - head is constant, the liquid settles down into steady motion. Each particle will move parallel to the axis of the liquid, with a constant velocity gradient along the radius of the tube. This orderly motion is possible only if the tube is narrow enough and the pressure head is not large. Such a smooth and orderly motion of the liquid is called **stream - lined motion.**

If however, the pressure - head is large enough, the liquid particles are accelerated axially. Now the resultant motion of the liquid is not orderly, but violent. Such a motion is called **turbulent motion**.

The velocity of the fluid at which orderly motion ceases and **turbulent** motion sets in is known as the **Critical Velocity** (V_c) For orderly motion, external pressure head $P \, \alpha \, V_c^2$. It has been found that

$$V_c = \frac{K\eta}{\rho r}$$

Where ρ is the density of the liquid, η is the viscosity of the liquid and r is the radius of the tube. K is a constant called Reynolds' number. For narrow tubes, K is approximately 1000.

4.3 Coefficient of Viscosity

Consider a liquid flowing over a horizontal surface. The layer in contact with the surface is at rest. The velocities of other layers increase uniformly from layer to layer. The velocity is maximum for the top layer [Fig. 4.2].

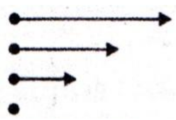


Fig. 4.2

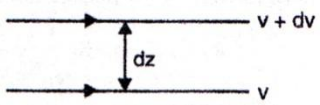


Fig. 4.3

Consider two layers of liquid separated by a distance dz [Fig. 4.3]. Let υ and υ+dυ be the velocities of two layers. So the velocity gradient is dυ/dz. Let A be surface area of the layer. The viscous force is directly proportional to the surface area A and velocity gradient dυ/dz.

$$F = A\frac{d\upsilon}{dz} or \; F = \eta \, A\frac{d\upsilon}{dz}$$

Definition: The coefficient of viscosity is defined as the tangential force per unit area required to maintain a unit velocity gradient.

Unit of η is $N \, s \, m^{-2}$. It is called the Pascal second.

$$Dimensions \; of \; (\eta) = \frac{(F)}{(A)[(d\upsilon ldz)]} = \frac{MLT^{-2}}{L^2\left(\frac{LT^{-1}}{L}\right)} = [ML^{-1}T^{-1}]$$

4.4 Rate of Flow of Liquid in a Capillary Tube - Poiseuille's formula

Consider horizontal capillary tube to length l and radius a through which a liquid flows [Fig. 4.4] is the coefficient of viscosity of the liquid. p is the pressure difference between the ends of the tube. The velocity of the liquid is maximum along and is zero at the walls. (dv/dr) is the velocity gradient.

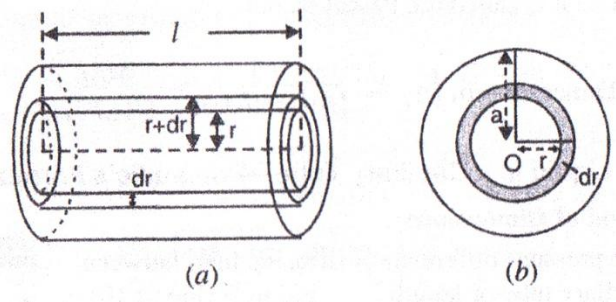


Fig. 4.4

Consider a cylindrical shell of the liquid of inner radius r an outer radius r+dr (Fig. 4.4 (b)]

The surface area of the shell = A = 2πrl.

The backward dragging viscous force acting on this layer is

$$F_1 = -\eta A \frac{dv}{dr} = -\eta 2\pi r l \frac{dv}{dr}$$

The driving force on the liquid shell, accelerating it forward

$$F_2 = px\pi r^2$$

$\pi r^2 = Area\ of\ cross -$ section of the innercylinder.

Here,

When the motion is steady,

backward dragging force $(F_{1)} =$ driving force (F_2)

$$-\eta 2\pi r l \frac{dv}{dr} = p\pi r^2$$

$$dv = \frac{-p}{2\eta l} r\, dr$$

Integrating, $v = \frac{-p}{2\eta l}\frac{r^2}{2} + C$ (C is constant)

When $r = a, v = 0$

$$0 = \frac{-p}{2\eta l}\frac{a^2}{2} + C.$$

or $C = \frac{pa^2}{4\eta l}$

$$v = \frac{p}{4\eta l}\left(a^{2-r^2}\right).$$

Volume of liquid flowing per second through this shell

dV = (Area of cross –section of the shell) x Velocity

$=[2\pi \text{ r dr}] \left[\frac{p}{4\eta l}(a^2 - r^2\right]$

$\frac{\pi p}{2\eta l}(a^2 r - r^{3)}\, dr$

The Volume of the liquid flowing out per second is obtained by integrating the expression for dv between the limits r = 0 to r=a

$$V = \int_0^a \frac{\pi p}{2\eta l}(a^2 r - r^3)dr = \frac{\pi p}{2\eta l}\left[a^2\frac{r^2}{2} - \frac{r^4}{4}\right]\begin{matrix} a \\ 0 \end{matrix}$$

$$= \frac{\pi\ p}{2\eta l}\frac{a^4}{4}$$

$$\text{V} = \frac{\pi p a}{8\eta l}^4$$

This is Poiseuille’s formula for the rate of flow of liquid through a capillary tube.

4.5. Stokes’ Law

Suppose a small metallic sphere is dropped into a highly viscous liquid. The viscous force F experienced by a falling sphere depends on

(i) the terminal velocity $\upsilon$of the ball

(ii) the radius r of the ball and

(iii) the coefficient of viscosity (η) of the liquid.

F= $k\upsilon^a r^b\ \eta^c$

Here, k is a dimensionless constant.

The dimensions of these quantities are: $F = MLT^2$; $\upsilon = LT^{-1}; r = L;$

$\eta = ML^{-1}T^{-1}$ **(**k is a number; it has no dimension)

$$MLT^2 = (LT^{-1})^a L^b (ML^{-1}T^{-1})^c$$

$$MLT^2 = M^c\, L^{a+b-c}\, T^{-a-c}$$

Equating the powers of M, L and Ton either side,

$$c = 1;\ a + b - c = 1\ and - a - c = -2$$

Solving, $a = 1; b = 1 and\ c = 1$

∴ $F = k\upsilon r\eta$

Stokes experimentally found the value of k to be 6π

∴ $F = 6\pi\upsilon r\eta$**.**

Expression for terminal velocity

Let ρ be the density of the ball and ρ' the density of the liquid.

The weight of the ball = $\frac{4}{3}\pi r^3\,\rho g$

The weight of the displaced liquid =$\frac{4}{3}\pi r^3\,\rho g$

The apparent its terminal velocity υ weight of the ball = viscous force F.

$$6\pi \upsilon r\eta = \frac{4}{3}\pi r^3\,(\rho - \rho')g$$

or $$\upsilon = \frac{2}{9}\frac{r^2}{\eta}\,(\rho - \rho')g$$

∴ $$\eta = \frac{2}{9}\frac{r^2}{\upsilon}\,(\rho - \rho')g$$

4.6. Determination of η of a Highly Viscous Liquid (Stokes' Method)

Stokes' method is suitable for highly viscous liquids like castor oil and glycerin. The experimental liquid is taken in a tall and wide jar [Fig.4.5]. Four or five marks A,B,C,D… are drawn in the outside of the jar at intervals of 5 cm. A steel ball is gently dropped centrally into the jar. The time taken by the ball to move through the distances. AB, BC, CD, … are noted. When the times for two consecutive transits are equal, the ball has reached terminal velocity.

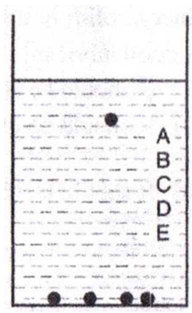


Fig. 4.5

Now another ball is gently dropped into the jar. When the ball just reaches a mark below the terminal stage, the time (1) taken by the ball to move through a definite distance (x) is noted.

∴ Terminal velocity = υ = x/t.

The experiment is repeated for varying distances. The mean value of $\boldsymbol{\upsilon}$ is found.

The radius of the ball is measured accurately with a screw gauge. The density of the ball ρ and the density of the liquid ρ' are found by the principle of Archimedes.

η is calculated using the formula,

$$\eta = \frac{2}{9}\frac{r^2}{\upsilon}\,(\rho - \rho')\text{ g}$$

4.7 Analogy between liquid flow and current flow

1. As the liquid flows through a tube, mass of the liquid gets displaced. The corresponding volume of liquid flowing per sec is given by

$V = \frac{\pi P r^4}{8\eta l}$ (1)

In a current carrying conductor charges get displaced. The

electric current is given by I= $Q/t = \frac{\text{charge}}{\text{time}}$ (2)

Thus V is analogous to I.

Volume of liquid flow / sec. is analogous to charge flow / see

or current.

2) V = $\frac{P}{\left(\frac{8\eta l}{\pi r^4}\right)}$ (3)

Also we know that,

Electric current (I) = $\frac{\text{potenti al difference or EMF}}{\text{Resistance}} = \frac{E}{R}$ (4)

Comparing (3) & (4) we see that the pressure – head P is analogous to the EMF or potential difference an

$\frac{8\eta l}{\pi r^4}$ corresponds to R, the resistance.

So $\frac{8\eta l}{\pi r^4}$ is called the Viscous resistance.

Thus **electrical resistance is analogous to viscous resistance.**

3) P corresponds to E

Flow of liquid depends on the pressure difference between the ends of the tube. Similarly flow of current depends on the potential difference between the ends of a wire. Thus potential difference is analogous to pressure difference or pressure head.

4.8 Equation of Continuity

Consider a liquid of density p flowing through a non –uniform tube AB (Fig. 4.6). Let a_1

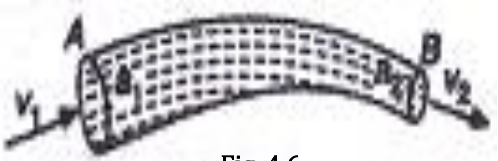


Fig. 4.6

and a_2 be the cross- sectional areas of the tube at the points A and B. Let the velocity of the liquid at A and B is v_1 and v_2 respectively.

The mass of liquid crossing each section of the tube per unit time must be by the same.

$$\therefore a_1 v_1 \;\; p = a_2 v_2 p \;\; or \;\; a_1 v_1 = a_2 v_2.$$

This is the 'equation of continuity'.

4.9 Energy of a Liquid in flow

We have the three types of energy possessed by a liquid in flow, viz, (i) kinetic energy, (ii) potential energy and (iii) pressure energy.

(i) Kinetic Energy.

Clearly, the kinetic energy of a mass m of a liquid, flowing with velocity v, is given by $\frac{1}{2}mv^2$.If we consider unit volume of the liquid, m=ρ, the density of the liquid, and, therefore, we have

Kinetic energy per unit volume of the liquid $=\frac{1}{2}\rho v^2$

And, if we consider unit mass of the liquid, m=1, and, therefore.

Kinetic energy per unit mass of the liquid $=\frac{1}{2}v^2$

(ii) Potential Energy.

The potential energy of a liquid of mass m at a height h above the earth's surface is equal to mgh. Again if we consider unit volume of the liquid, m= ρ, the density of the liquid, and, therefore.

Potential Energy per unit volume of the liquid = ρgh

But, if we consider unit mass of the liquid, m=1 and we have

Potential Energy per unit mass of the liquid = gh.

(iii) Pressure Energy.

Consider a tank A, Containing a liquid of density ρ provided with narrow side tube T, cross-sectional area

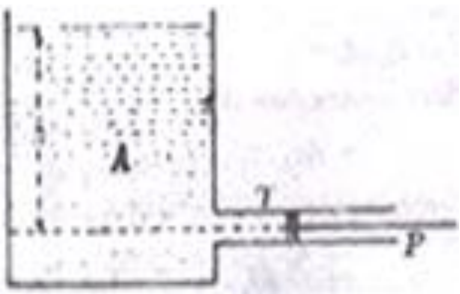


Fig. 4.7

a, properly fitted with a piston P that can be smoothly moved in and out, Let the hydrostatic pressure due to the liquid, at level of the axis of the side tube, be p, so that the force on the piston is = p.a. If, therefore, more liquid is to be introduced into the tank, this much force has to be applied to the piston in moving it inwards. Let the piston be moving slowly inwards through a distance x, so that the velocity of the liquid be very small and there may be not kinetic energy acquired by it. Then, clearly, a volume of liquid a. x., or a mass a. x. ρ of it, is forced into the tank, and an amount of work p.a.x. is performed to do so. This work, (or energy), p.a.x, required to make the liquid move against pressure p, without imparting any velocity to it, thus becomes the energy of the mass a. x. ρ of the liquid in the tank, for it can do the same amount of work in pushing the piston back, when escaping from the tank. It is referred to a s the pressure energy of the liquid.

Thus pressure of a mass a.x.ρ of the liquid is equal to p.a.x and, therefore,

Pressure energy per unit mass of the liquid = $\frac{p.a.x.}{a.x.\rho} = \frac{p}{\rho} = \frac{pressure}{density}$

Now, if we consider unit volume of the liquid, we have pressure energy of volume a.x of the liquid = p.a.x

Pressure energy per unit volume of the liquid = $\frac{\mathrm{p.a.x}}{\mathrm{a.x}}$ = p, the pressure of the liquid.

Total energy of the liquid in motion = Pressure energy + Kinetic energy + Potential energy

∴ Total energy per unit mass of the flowing liquid = $\frac{\mathrm{p}}{\rho} + \frac{\mathrm{v}^2}{2} + \mathrm{gh}$

UNIT V

GRAVITATION

5.1. Introduction

Humanity has speculated about the origin of the universe since the dawn of man. The real renaissance of astronomy began with Nicholas Copernicus, who made the first fully predictive mathematical model of a heliocentric system which is against Ptolemy’s geocentric system. Galileo discovered that Nicholas Copernicus was right and that the earth was not the centre of the solar system. In the following century, this model was elaborated and expanded by Kepler and supporting observations made using a telescope were presented by Galileo. He had developed three laws governing the motion of the five then-known planets. He did not have a theoretical model for the principles governing this movement, but rather achieved them through trial and error over the course of his studies. Newton's work, nearly a century later, was to take the laws of motion he had developed and apply them to planetary motion to develop a rigorous mathematical framework.

Newton realized that all motion, whether it was the orbit of the moon around the earth or an apple falling from a tree, followed the same basic principles. By his dynamical and gravitational theories, he explained Kepler’s laws and established the modern quantitative science of gravitation. Newton's law of gravity defines the attractive force between all objects that possess mass. Understanding the law of gravity, one of the fundamental forces of physics, offers profound insights into the way our universe functions. This universal force would also act between the planets and the Sun, providing a common explanation for both terrestrial and astronomical phenomena.

5.2. Newton's Law of Gravitation

Thus Newton succeeded in reducing the three laws of Kepler into a single law known as the Newton’s law of gravitation. It describes the attraction between two points of mass in space separated from some distance, *r*. The forces of attraction depend on the mass of each object and the magnitude of *r*.

The law states that every particle of matter in this universe attracts every other particle with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

If m_1 and m_2 are the masses of two particles situated at a distance *r* apart, the force of attraction between them is given by

$F \alpha \frac{m_1 m_2}{r^2}$

$$F = \frac{G m_1 m_2}{r^2}$$

Where, F is called the gravitational force and G is a proportionality constant. G is known as universal gravitational constant. It is termed a "universal constant" because it is thought to be the same at all places and all times. Its unit and dimensions are $Nm^2 kg^{-2}$ and $M^{-1}L^3T^{-2}$ respectively and has a value of $6.673 \times 10^{-11}\ Nm^2\ kg^{-2}$. The gravitational constant is numerically equal to the force exerted by a mass of 1 kg on another equal mass situated at a distance of 1 metre from it.

Newton's Law is called the Universal Law of Gravitation in the sense that it hold good everywhere, right from huge interplanetary distances to the smallest terrestrial ones.

5.3. Gravitational potential and gravitational field intensity

A region of space around a body within which a gravitational force of attraction can be experienced is called its gravitational field.

The gravitational potential *V* at a point in a gravitational field is the amount of work done in moving an unit mass from this point to infinity against the gravitational force of attraction. The gravitational potential difference between two points in a gravitational field is the amount of work done in taking an unit mass from one point to the other point against the gravitational force of attraction. Thus the gravitational potential *V* is defined as the gravitational potential energy per unit mass of a body in a gravitational field.

i.e. $V = \frac{-GMm}{r}\frac{1}{m} = \frac{-GM}{r}$

where $\frac{-GMm}{r}$ is the gravitational potential energy of a particle of mass *m* and the earth having mass *M*.

The negative potential energy of the system containing the particle of mass *m* and earth indicates that the particle is bound to the earth by earth's attractive force on the particle.

The gravitational field intensity 'g' at a point is the gravitational force experienced by an unit mass placed at that point.

Thus $g = \frac{F}{m} = \frac{-GMm}{mr^2} = \frac{-GM}{r^2}$

The negative sign indicates the attractive nature of the gravitational force.

The gravitational field is a vector field. Each point in this field has a vector associated with it.

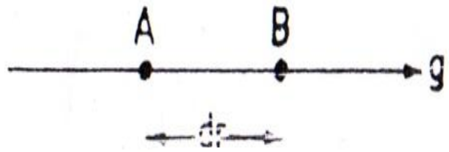


Fig.5.1

Consider two points A and B lying very near to each other at a distance dr in a gravitational field ‘g’ of a particle acting in the direction indicated in figure 5.1.

The work done in taking an unit mass from B to $A = g.dr$

This work also represents the difference of potential dV between A and B. Hence

$dV = -gdr$ (or) $g = \frac{-dV}{dr}$

Where, the negative sign shows only that the intensity decreases as distance increases. Hence the intensity of gravitational field at a point can also be defined as the gravitational potential gradient at that point.

Gravitational potential due to a point mass

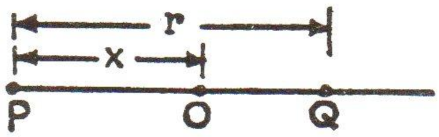


Fig 5.2

Let a point mass *m* be situated at P (Fig.5.2). The attraction due to it at a point O, distant *x* from P, is directed towards P and is of magnitude $\frac{Gm}{x^2}$. The work done on the system in moving a unit mass by a small amount *dx* is $\frac{Gm}{x^2}\ dx$.

This is equal to the potential difference *dV* between the points *dx* apart.

So

$$dV = \frac{Gm}{x^2}dx$$

Hence Potential at Q, distant *r* from P, is

$V = \int_{\infty}^{r} G\frac{m}{x^2}dx = Gm\int_{\infty}^{r}\frac{dx}{x^2} = -Gm[\frac{1}{x}]_{\infty}^{r}$

$= -\frac{Gm}{r}$

5.4. Gravitational Potential and field intensity due to a spherical shell

Consider an uniform spherical shell of radius R and centre O. Let σ be its surface density.

Therefore Mass of the shell $M = 4\pi R^2\sigma$

Case 1: Potential at external points

Let us find an expression for the potential at a point P outside the shell. Join PO and let AOA' be the diameter which passes through P. Consider a thin slice of the shell between two planes BC and $B'C'$ at right angles to $AOA^{'}$. Join PB, OB and $OB^{'}$.

Let ∟AOB = θ and ∟AOB' = θ+ dθ

OP = r PB = a

Hence the arc $BB^{'} = Rd\theta$

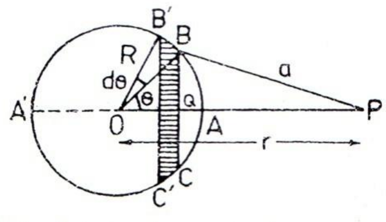


Fig.5.3. Potential due to a spherical shell

Radius of the slice $B^{'}BCC^{'} = BQ = R\ Sin\theta$

Area of the slice = $2\pi R\ Sin\theta\ Rd\theta$

Mass of the slice = $2\pi R\ Sin\theta\ Rd\theta\ \sigma$

Each point of the slice is at the same distance a from P

Potential at P due to the slice = $dV = -G\frac{2\pi R^2\ \sigma Sin\theta d\theta}{a}$

In the triangle $OPB, a^2 = R^2 + r^2 - 2Rrcos\theta$

Differentiating this, we get

$$2a\ da = 2Rrsin\theta \quad d\theta R \text{ and } r \text{ are constants}$$

$$a = \frac{Rr \quad sin\theta \quad d\theta}{da}$$

Hence $$dV = \frac{-G\ 2\pi R\ ^2\sigma\ sin\theta \ \ d\theta\ \ da}{Rrsin\theta d\theta}$$

$$\frac{-G\ 2\ \pi R\ \sigma\ da}{r} \qquad (5.1)$$

Potential V at P due to the whole spherical shell is obtained by integrating the above equation between the limits $a = r - R$ and $a = r + R$

$$V = -\int_{r-R}^{r+R} \frac{G\ 2\pi R\ \sigma\ da}{r} = \frac{-G\ 2\pi R\ \sigma}{r}[a]_{r-R}^{r+R}$$

$$\frac{-G\ 2\pi R\ \sigma\ .2R}{r} = \frac{-G\ 4\pi R^2 \sigma}{r}$$

Thus $V = \frac{-GM}{r}$

Where $M = 4\pi\ R^2\ \sigma$, the mass of the shell.

Thus for the external points, this spherical shell behaves as if the whole mass of the shell is concentrated at its centre.

Case 2: Potential for points on surface of the shell

Let us consider a point which lies on the surface of the shell itself. The limits for the value of a will be 0 and 2R. Hence

Potential at a point on the surface of the shell $V = \int_0^{2R} \frac{-G2\pi R\sigma da}{r}$

$$= \frac{-G2\pi R\sigma}{r} [a]_0^{2R}$$

$$= \frac{-G2\pi R\sigma}{r} 2R$$

$$= \frac{-G4\pi R^2\sigma}{r}$$

$$= \frac{-GM}{r} = \frac{-GM}{R} \qquad [r=R]$$

$$V = \frac{-GM}{R}$$

Case 3: Potential for internal points

If the point P is inside the shell, the potential at that point is obtained by integrating equation (5.1) between the limits $a = R - r$ and $a = R + r$.

$$V = -\int_{R-r}^{R+r} G \frac{2\pi R\sigma\ da}{r} = -G\ \frac{2\pi R\sigma}{r} [R + r - R + r]$$

$$= -G\ 4\pi\sigma R = -G\ \frac{4\pi\ R^2\sigma}{R} = \frac{-GM}{R}$$

Thus the gravitational potential is constant for all internal points and is equal to the value of the potential on the surface of the shell.

The gravitational field intensity due to a spherical shell:

Case 1: For external points

We know that the gravitational potential for external points,

$$V = \frac{-GM}{r}$$

$$g = \frac{-dV}{dr} = \frac{-GM}{r^2} \qquad (5.2)$$

Thus the intensity is inversely proportional to the square of the distance of the point from the centre of the shell.

Case 2: For points on the surface of the shell

Putting *r*=*R* in the expression (5.2) we get the gravitational field at the point on the surface of the shall

$$g = \frac{GM}{R^2}$$

Case 3: For Internal points

For internal points $V = \frac{-GM}{R}$

$$g = \frac{-dV}{dr} = 0$$

Hence the intensity is zero for all internal points due to constant potential inside.

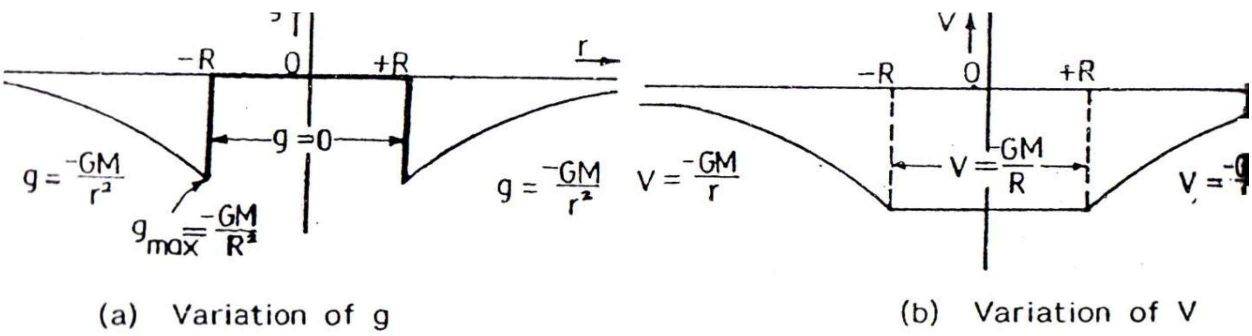


Fig. 5.4. Graphical representations of the variation of the field intensity and the potential of a spherical shell

5.5. Gravitational potential and field intensity due to a solid sphere

Case 1: For External points

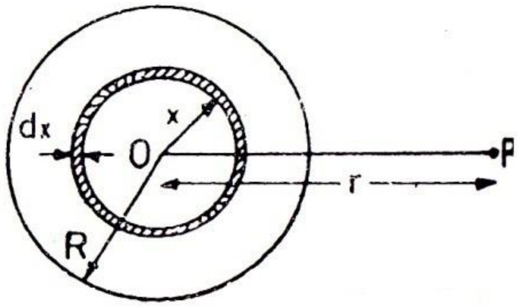


Fig 5.5

Let the radius of the sphere be R and let P be a point outside the sphere at a distance r from its centre O. If ρ is the mass per unit volume of the sphere, the mass of the sphere is given by

$$M = \frac{4}{3}\pi R^3 \rho$$

The sphere may be imagined to be made up of a large number of concentric spherical shells. Consider a shell of radius x and thickness dx.

Surface area of the shell $= 4\pi\, x^2$

Volume of the shell $= 4\pi\, x^2 \,.\, dx$

Mass of the shell $= 4\pi\, x^2\, dx.\rho$

Each shell will produce a potential at the point P as if its mass is concentrated at O.

Therefore Potential due to shell at $P = -\frac{G\,.Mass\ of\ shell}{r}$

$$= -G\,\frac{4\pi\,\rho\, x^2\, dx}{r}$$

Potential due to the solid sphere $= -\int_0^R G\,\frac{4\pi\,\rho\, x^2\, dx}{r}$

$$= -G\,\frac{4\pi\rho}{r}\frac{R^3}{3}$$

$= -\frac{GM}{r}$ Since $M = \frac{4}{3}\pi R^3\rho$, Mass of sphere (5.3)

Further gravitational intensity at P is given by

$$g = \frac{-dV}{dr} = \frac{-GM}{r^2} \qquad (5.4)$$

Thus for external points, this solid sphere behaves as if the whole mass of the sphere is concentrated at its centre.

Case 2: For points on the surface of the shell

If the point P lies on the surface of the solid sphere, we have *r*=*R*

Putting *r*=*R* in eqn (5.3)we get

The potential at point on surface= $\frac{-GM}{R}$

and the gravitational intensity

$g = \frac{dV}{dr} = \frac{-GM}{R^2}$ putting *r*=*R* in equation (5.4)

Case 3: For internal points

Let P be a point inside the sphere at a distance r from its centre. Then this point P is considered to be an external point for all shells having radii $x < r$ and an internal point for the shells having radii $x > r$.

Therefore Potential at P = $-\int_0^R \frac{Gdm}{r} + \int_r^R \frac{-Gdm}{x}$

where dm is the mass of the shell of radius *x* and thickness *dx*.

Therefore $dm = 4\pi\rho x^2\, dx$

Therefore $V = \frac{-G}{r}\int_0^r 4\pi\rho x^2 dx - G\int_r^R \frac{4\pi\rho x^2 dx}{x}$

$$= -\frac{G}{r}M' - G\,4\pi\rho\left[\frac{x^2}{2}\right]_r^R$$

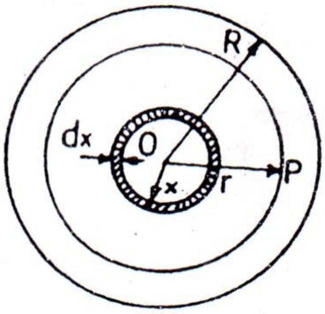


Fig 5.6

Where $M' = \frac{4}{3}\pi r^3\rho$ = mass of the sphere with radius r.

Hence $V = -\frac{GM'}{r} - G\,4\pi\rho\,[\frac{R^2}{2} - \frac{r^2}{2}]$

$$= -\frac{GM'}{r} - G\frac{4\pi\rho R^2}{2} + G\frac{4\pi\rho r^2}{2}$$

Since $\frac{4}{3}\pi R^3\rho = M$ = mass of the whole solid sphere and

$$M' = \frac{4}{3}\pi r^3 \rho$$

$$M' = M\frac{r^3}{R^3}$$

$$V = -\frac{GM'}{r} - \frac{3GM}{2R} + \frac{3GM'}{2r}$$

$$= \frac{GM'}{2r} - \frac{3GM}{2R}$$

$$-\frac{3GM}{2R}\left[1 - \frac{r^2}{3R^2}\right]$$

Further gravitational intensity at P is given by

$$g = -\frac{dV}{dr} = -\frac{G\,M\,r}{R^3}$$

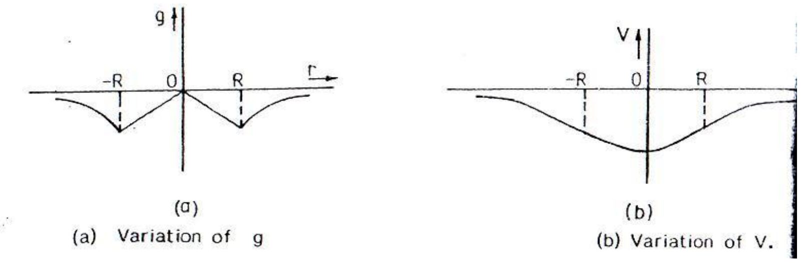


Fig. 5.7 Graphical representations of variation of field intensity and potential of a solid sphere

From Fig. 5.7., we see that for points outside the sphere, the field intensity and potential are the same as that due to a particle of mass M at the centre of the sphere. For internal points, the field intensity linearly increases with r in its magnitude and is maximum at the surface of the sphere and the potential is maximum at the centre and then gradually decreases.

5.5. Acceleration due to Gravity

The force of attraction exerted by the earth on a body is called gravitational pull or gravity. We know that when force acts on a body, it produces acceleration. Therefore, a body under the effect of gravitational pull must accelerate. The acceleration produced in a freely falling body under the effect of gravity is called acceleration due to gravity, it is denoted by g. Using Newton's second law of motion ($F = m\,g$) and Newton's law of gravitation, we get

$$F = mg = \frac{GMm}{R^2}$$

$$g = \frac{GM}{R^2}$$

From the expression, it is clear that the value of 'g' is independent of mass, shape and size of the body but depends upon mass and radius of the earth. i.e. earth produces same acceleration in a light as well as heavy body.

Acceleration due to gravity is a vector quantity and its direction is always towards the centre of the earth. The S.I. Unit of 'g' is m/s^2 or N/Kg. the dimensional formula of acceleration due to gravity is $[M^0LT^{-2}]$. Its average value is taken to be 9.8 m/s^2 on the surface of the earth at mean sea level. It is constant at a given place. However it slightly differs from place to place on the surface of the earth.

5.6. Variation of the acceleration due to gravity

The value of acceleration due to gravity varies due to the following factors:
(a) Shape of the earth (b) Axial rotation of the earth (c) Depth below the earth surface and (d) Height above the earth surface

a) Variation of g with shape of earth

The earth is not a perfect sphere, but bulges at equator and flattened at the poles. Its equatorial radius is about 21km more than the polar radius. Therefore if a body is taken from pole to equator its distance from the centre of the earth will change. Consequently, the gravitational force also varies. As g is inversely proportional to square of radius of earth, the value of g is minimum at the equator and maximum at the poles.

b) Variation of g with latitude (rotation of the earth)

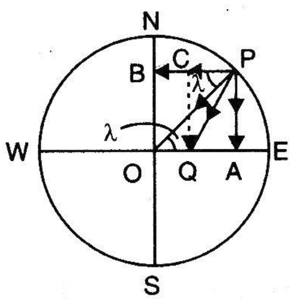


Fig. 5.8

Consider the earth to be a perfect sphere of radius R with centre at O. We know that the earth rotates about its own axis with a certain angular velocity ω. During rotation, each particle lying on the surface of the earth must rotate in a horizontal circle with same angular velocity ω about the rotation axis.

Now, let a particle of mass m be situated on the surface of the earth at a point p of latitude λ. If the earth were atrest, the particle at p experiences a force *mg* along the radius PO towards O. As the earth is rotating about its polar axis NS, the body at p describes a horizontal circle with centre at B and radius BP=Rcosλ. During rotation the primary particle at p experiences centrifugal force which acts along BP, away from B=$mBP\omega^2$

=m(Rcos λ)ω^2

=$mR\omega^2 cos\lambda$

Force *mg* acts along PO. Resolve *mg* into two rectangular components (i) *mg* sin λ along PA and (ii) *mg* cos λ along PB. Out of the resolved component along PB, a portion m R $\omega^2 \cos\lambda$ is used in overcoming centrifugal force.

Let the net force be represented by PC. Then

$$PC = mg\cos\lambda - mR\,\omega^2\cos\lambda \text{ and } PA = mg\sin\lambda$$

The resultant force (*mg'*) experienced by P is along PQ, such that

$$(PQ)^2 = (PC)^2 + (PA)^2 or\ PQ = [(PC)^2 + (PA)^2]^{1/2}$$

i.e.,

$$mg' = [(mg\cos\lambda - mR\,\omega^2\cos\lambda)^2 + (mg\sin\lambda)^2]^{1/2}$$

$$= mg\,[1 + \frac{R^2\omega^4}{g^2}\cos^2\lambda - \frac{2\,R\omega^2}{g}\cos^2\lambda]^{1/2}$$

∴

$$mg' = mg\,[1 - \frac{2R\omega^2}{g}\cos^2\lambda]^{1/2} \quad \text{[neglecting } \frac{R^2\omega^4}{g^2}\cos^2\lambda]$$

$$= mg[1 - \frac{R\omega^2\cos^2\lambda}{g}]$$

$$(\because \frac{R\omega^2}{g}\ is\ small, its\ higher\ powers\ can\ be\ neglected\ )$$

$$g' = g\,[1 - \frac{R\omega^2\cos^2\lambda}{g}]$$

Or $g' = g - R\omega^2\cos^2\lambda$

This is the required expression. As cos λ and ω are positive, therefore $g' < g$

This shows the value of g decreases due to earth's rotation.

at the equator $\lambda = 0^o$ therefore $g' = g - R\omega^2$ [minimum]

at the poles $\lambda = 90^o$ therefore $g' = g$ [maximum]

c) Variation of *g* with altitude (height)

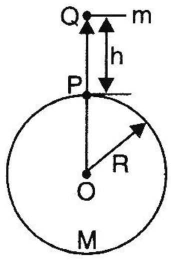


Fig 5.9

Assuming the earth to be an uniform solid sphere of mass M and radius R with centre O. If a body of mass m is initially placed on the surface of the earth at P (fig 5.9) then we know the acceleration due to gravity is

$$g = \frac{G\,M}{R^2} \qquad (5.5)$$

If the body is raised to a height h, above the surface of the earth at a point Q, then its distance from the centre of the earth is $(R + h)$. Now the acceleration due to gravity exerted by the earth on the body is

$$g' = \frac{G\,M}{(R+h)^2} \qquad (5.6)$$

Dividing (5.6) by (5.5), we get

$$g'/g = \frac{GM}{(R+h)^2}\frac{R^2}{GM} = \frac{R^2}{(R+h)^2}$$

$$= \frac{R^2}{R^2[1 + \frac{h}{R}]^2}$$

$$g' = \frac{g}{[1 + h/R]^2}$$

$$g' = g[1 + {}^{h}/_{R}]^{-2}$$

Expanding this equation, by using binomial theorem. We have

$g' = g[1 - {}^{2h}/_{R}]$ Since $h << R$, Neglecting the higher powers of h/R

This expression shows that acceleration due to gravity decreases with the increase of height or altitude from the surface of earth.

d) Variation of g with depth

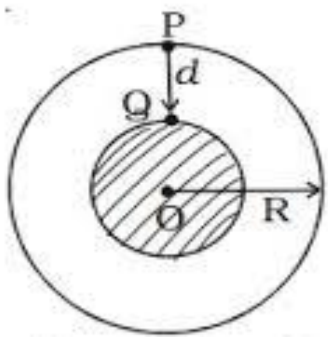


Fig.5.10

Consider that earth is a homogeneous sphere of mass M and radius R with centre at O**.** If a body of mass *m* is placed at point P on the surface of the earth, the value of acceleration due to gravity on the surface of the earth at this point P is given by

$$g = \frac{GM}{R^2}$$

Let ρ be the uniform density of material of the earth

$$M = \frac{4}{3}\pi R^3 \rho$$

$$g = \frac{G[\frac{4}{3}\pi R^3 \rho]}{R^2}$$

$g = \frac{4}{3}\pi \rho G R$ (5.7)

Now, Let the body be placed at Q, a distance d below the surface of the earth. Its distance from the centre O of the earth is (R-d). A sphere of radius (R-d) is drawn from O. The body at Q is situated at the surface of inner solid sphere and lies inside the outer spherical shell. The gravitational force on a body inside a shell is always zero. Therefore, the gravitational force of attraction acting on a body is only due to inner solid sphere.

Acceleration due to gravity on the surface of the earth at the point Q is

$$g' = \frac{GM'}{(R-d)^2}$$

$$M' = \frac{4}{3}\pi\,(R-d)^3\rho$$

$$g' = \frac{G\frac{4}{3}\pi\,(R-d)^3\rho}{(R-d)^2}$$

$$= \frac{4}{3}\pi\ \mathrm{G}\ (\mathrm{R}-\mathrm{d})\ \rho \qquad (5.8)$$

Dividing (5.8) by (5.7), we get

$$\frac{g'}{g} = \frac{R-d}{R} = 1 - \frac{d}{R}$$

$g' = g\,[1 - \frac{d}{R}]$

Therefore, the acceleration due to gravity decreases with increase of depth.

5.7. Escape velocity

When an object is thrown vertically upwards, it reaches a certain height and returns back to the earth. While throwing upwards the height it reaches will vary with the initial velocity. So when it is thrown up with a certain minimum initial velocity, the object overcome the gravitational pull and goes beyond the earth's gravitational field and escapes from earth. The initial velocity needed to achieve that condition is called escape velocity. So, escape velocity is defined as the minimum initial velocity that will take a body away above the surface of a planet when it's projected vertically upwards.

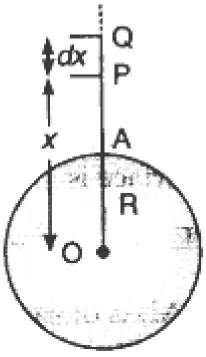


Fig. 5.11

Let earth be a perfect sphere of mass M, radius R with centre at O. Let a body of mass *m* to be projected from a point A on the surface of earth (planet). Join OA and produce it further. Take two points P and Q at a distance *x* and (*x* + *dx*) from the centre O of the earth.

To calculate the escape velocity of the earth, let the minimum velocity to escape from the earth's surface be v_e. Then, kinetic energy of the object of mass ***m*** is

K.E = $\frac{1}{2}mv_e^2$

When the projected object is at point **P** which is at a distance ***x*** from the center of the earth, the force of gravity between the object and earth is

$$F = GMm/x^2$$

Work done in taking the body against gravitational attraction from **P** to **Q** is given by

$$dW = Fdx = \frac{GMm}{x^2}\mathrm{d}x$$

The total amount of work done in taking the body against gravitational attraction from surface of the earth to infinity can be calculated by integrating the above equation within the limits ***x*= R** to***x***= ∞. Hence, total work done is

$$W = \int_R^\infty dW = \int_R^\infty \frac{GMm}{x^2}dx$$

$$= GMm\int_R^\infty x^{-2}dx = GMm\left[\frac{x^{-1}}{-1}\right]_R^\infty$$

$$or, W = \frac{GMm}{R}$$

For the object to escape from the earth's surface, kinetic energy given must be equal to the work done against gravity going from the earth's surface to infinity, hence

$$\frac{1}{2}mv_e^2 = \frac{GMm}{R}$$

$$v_e = \sqrt{\frac{2GM}{R}}$$

Since,

$$g = \frac{GM}{R^2}$$

$$V_e = \sqrt{2gR}$$

The relation shows that the escape velocity of an object does not depend on the mass of the projected object but only on the mass and radius of the planet from which it is projected. The escape velocity at the Earth's surface is about 11.2 kilometers per second (25,000 miles per hour) and the escape velocity on the Moon's surface is 2.4 kilometers per second (5,300 miles per hour).

5.8. Kepler’s laws and planetary motion

German astronomer Johannes Kepler after a life time study work out three empirical laws which accurately describe the revolutions of the planets around the sun and are known as Kepler’s laws of planetary motion. It opened the way for the development of celestial mechanics. These laws are

1. **The law of Orbits:** Every planet moves around the sun in an elliptical orbit with sun at one of the foci.
2. **The law of Area:** The line joining the sun to the planet sweeps out equal areas in equal interval of time. *i.e.* areal velocity is constant. According to this law planet will move slowly when it is farthest from sun and more rapidly when it is nearest to sun. It is similar to law of conservation of angular momentum.

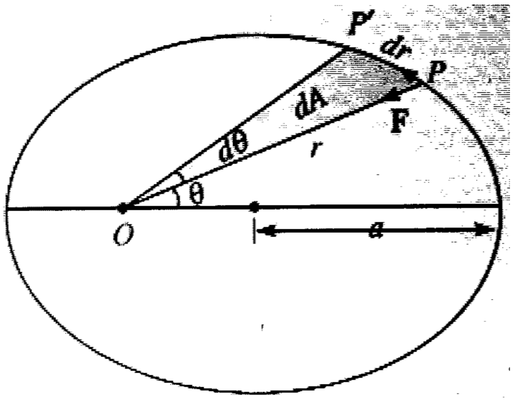


Fig 5.12

Let the particle move from position P to position P' in an infinitesimal time interval *dt* as shown in fig.5.12. The area *dA*, which the radius vector *r* sweeps, is given by

$$dA = \frac{1}{2} r dr = \frac{1}{2} r (r d\theta)$$

where d$\theta$is the angle swept by the radius vector. Here we have assumed that PP' is a straight line since *dr* is infinitesimally small. The areal velocity is given by

$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \omega$ (5.9)

where ω is the angular velocity of the particle. Now if *m* is the mass of the particle, then its momentum *L* is

$L = mr^2 \omega$ (5.10)

Combining eqn (5.9) and (5.10) we get

$$\frac{dA}{dt} = \frac{L}{2m} = Consant$$

3. The law of periods: The Square of period of revolution (*T*) of any planet around sun is directly proportional to the cube of the semi-major axis of the orbit.

If T be the periodic time of describing the ellipse by the planet, we have

T=$\frac{area\ \ of\ ellipse}{areal\ \ velocity}$

The area of ellipse πab and areal velocity= $\frac{L}{2m}$

$$T = \frac{\pi ab}{L/2m} = \frac{2\pi abm}{L}\ or\ T^2 = \frac{4\pi^2 m^2 a^2 b^2}{L^2}$$

The latus rectum of ellipse is, *l* =$\frac{b^2}{a}$

$\therefore T^2 = \frac{4\pi^2 m^2 a^3 l}{L^2} = or\ T^2 \propto a^3$

5.9. Satellite Motion

Satellites are natural or artificial bodies describing orbit around a planet under its gravitational attraction. Moon is a natural satellite while INSAT-1B is an artificial satellite of earth. Artificial satellites can be launched from the surface of earth with the help of rockets. A multistage rocket carries the satellite up to the required height of the orbit. Its last stage tilts the satellite into its orbit and gives a final push to acquire the required velocity. The velocity required to put the satellite into its orbit around the earth is orbital velocity v_0. These satellites can be made to revolve around the earth in circular orbits.

We will consider a satellite in a circular orbit of a distance (R+h) from the centre of the earth, where R = radius of the earth. If m is the mass of the satellite and v_0 its speed, the centripetal force required for this orbit is

$\frac{m v_0^2}{(R+h)}$ (5.11)

This centripetal force is provided by the gravitational force, which is

$= \frac{GMm}{(R+h)^2}$ (5.12)

For revolution of satellite around the earth, the gravitational pull provides the required centripetal force. Equating equations (5.11) and (5.12) and

$$\frac{mv_0^2}{(R+h)} = \frac{GMm}{(R+h)^2}$$

Since $g = \frac{GM}{R^2}$

$$\frac{mv_0^2}{(R+h)} = \frac{m\ g\ R^2}{(R+h)^2}$$

(or) $v_0^2 = \frac{gR^2}{(R+h)}$

The orbital velocity of the satellite $v_0 = \sqrt{gR}$, if $R >>$ h.

If T is the period of revolution of the satellite.

Then $v_0 = \frac{2\pi\ (R+h)}{T}$

Therefore $\frac{4\pi^2\ (R+h)^2}{T^2} = \frac{g\ R^2}{(R+h)}$

(or) $T^2 = \frac{4\pi^2\ (R+h)^3}{gR^2}$

$$T = 2\pi \sqrt{\frac{(R+h)^3}{gR^2}}$$

[Note : When $h < R,\ T = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{R^3}{GM}}$.

The period of revolution of satellite is found to change with the height of the satellite above earth.

5.10. Geostationary orbit

If the value of (*R*+ *h*) is arranged in such a way that T takes for the satellite to complete one revolution about the earth is exactly 24 hours. This means that if the orbit of the satellite lies on the equatorial plane, the satellite's angular velocity will be same as that of the earth's angular velocity, and the satellite will appear to be situated at the same position when viewed from a point on earth. This orbit is called as geostationary orbit and a satellite placed in such orbit is called a geostationary or geosynchronous satellite, communication satellite.Since the period of revolution of the satellite around the earth be same as that of earth about its own axis, relative velocity of the satellite with respect to earth is zero. So that, the satellite appear stationary from any point on earth.

To have the period of the revolution of the satellite should be 24 hours, it should remain at a height h given by

$$R + h = (\frac{T^2 g R^2}{4\pi^2})^{1/3}$$

$$= \frac{[(24 \times 60 \times 60)^2 \times 9.8 \times (6371 \times 10^3)^2]}{4\pi^2}$$

R+*h*= 42,207 km

$$h = (42{,}207 - 6371)km = 35836\ km$$

The orbital velocity of such a satellite in the geostationary orbit will be about 3.07 km/s. For any other orbit the velocity and period will be different.

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