DJM3B - REAL AND COMPLEX ANALYSIS

Unit I: Metric spaces – open sets – Interior of a set – closed sets – closure – completeness – Cantor's intersections theorem – Baire – Category Theorem.

Unit II: Continuity of functions – Continuity of compositions of functions – Equivalent conditions for continuity – Algebra of continuous functions – hemeomorphism – uniform continuity – discontinuities connectednon – connected subsets of R – Connectedness and continuity – continuous image of a connected set is connected – intermediate value theorem.

Unit III: Compactness – open cover – compact metric spaces – HerniBorel theorem. Compactness and continuity – continuous image of compact metric space is compact – Continuous function on a compact metric space in uniformly continuous – Equivalent forms of compactness – Every compact metric space is totally bounded – Bolano – Weierstrass property – sequentially compact metric space.

Unit IV: Algebra of complex numbers – circles and straight lines – regions in the complex plane – Analytic functions Cauchy – Rienann equations – Harmonic functions – Bilinear transformation translation, rotation, inversion – Cross – ratio- Fixed points – Special bilinear transformations.

Unit V: Complex Integration – Cauchy's integral theorem – Its extension – Cauchy's integral formula – Morera's theorem – Liouville's theorem – fundamental theorem of algebra – Taylor's series – Laurent's series – Singularities. Residues – Residue Theorem – Evaluation of definite integrals of the following types.$\int_0^{2\pi}$ F (Cos x, sin x) dx and 2 $\int_{-\infty}^{\infty} \frac{f(x)}{g(x)}$ dx

Books for reference:

1. Modern Analysis – Arumugam and Issac.
2. Real Analysis – Vol. III – K. ChandrasekharaRao and K.S. Narayanan, S. Viswanathan Publisher.
3. Complex Analysis – Narayanan &ManicavachagamPillai
4. Complex Analysis – S. Arumugam&Issac.
5. Complex Analysis – P. DuraiPandian
6. Complex Analysis – Karunakaran, Narosa Publishers.

Unit - I

Metric Spaces

Introduction

A Metric Space is a set equipped with a distance function, also called a metric, which enables us to measure the distance between two elements in the set.

1.1 Definition And Examples

Definition 1.1.1 A **Metric Space** is a non empty set **M** together with a function

d : M × M → R satisfying the following conditions.

(i) $d(x , y) \geq 0$ for all $x , y \in M$
(ii) $d(x , y) = 0$ if and only if $x = y$
(iii) $d(x , y) = d(y , x)$ for all $x , y \in M$
(iv) $d(x , z) \leq d(x , y) + d(y , z)$ for all $x , y , z \in M$ **[Triangle Inequality]**

d is called a **metric** or **distance function** on M and **d(x , y)** is called the distance between x and y in M. The metric space M with the metric d is denoted by (M , d) or simply by M when the underlying metric is clear from the context.

Example 1.1.2 Let **R** be the set of all real numbers. Define a function $d : M \times M \rightarrow R$ by $d(x , y) = | x - y |$. Then d is a metric on **R** called the usual metric on **R.**

Proof.

Let $x , y \in$ **R.**

Clearly $d(x , y) = | x - y | \geq 0$.

Moreover, $d(x , y) = 0 \Leftrightarrow |x - y | = 0$.

$$\Leftrightarrow x - y = 0.$$

$$\Leftrightarrow x = y$$

$d(x , y) = | x - y |$

$= | y - x |$

$= d(y ,x)$.

$\therefore d(x , y) = d(y , x)$.

Let $x , y , z \varepsilon$ **R.**

$d(x , z) = | x – z |$

$= | x – y + y - z |$

$\leq | x – y |+| y – z |$

$= d(x , y) + d(y , z).$

$\therefore d(x , z) \leq d(x , y) + d(y , z).$

Hence d is a metric on **R.**

Note. When R is considered as a metric space without specifying its metric, it is the usual metric.

Example 1.1.2

Let M be any non-empty set. Define a function d : M x M →**R** by $d(x , y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$

Then d is a metric on M called the **discrete metric** or **trivial metric** on M.

Proof.

Let $x , y \varepsilon M$.

Clearly $d(x , y) \geq 0$ and $d(x , y) = 0 \Leftrightarrow x = y$.

Also, $d(x , y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$

$= d(y , x)$.

Let $x , y , z \varepsilon M$.

We shall prove that $d(x , z) \leq d(x , y) + d(y , z)$.

Case (i) Suppose $x = y = z$.

Then $d(x , z) = 0$, $d(x , y) = 0$, $d(y , z) = 0$.

$\therefore d(x , z) \leq d(x , y) + d(y , z).$

Case (ii) Suppose $x = y$ and z distinct.

Then $d(x , z) = 1$, $d(x , y) = 0$, $d(y , z) = 1$.

$\therefore d(x , z) \leq d(x , y) + d(y , z).$

Case (iii) Suppose x = z and y distinct.

Then d(x , z) = 0 , d(x , y) = 1 , d(y , z) = 1 .

∴ $d(x , z) \leq d(x , y) + d(y , z)$.

Case (iv) Suppose y = z and x distinct.

Then d(x , z) = 1 , d(x , y) = 1 , d(y , z) = 0.

∴ $d(x , z) \leq d(x , y) + d(y , z)$.

Case (v) Suppose $x \neq y \neq z$.

Then d(x , z) = 1 , d(x , y) = 1 , d(y , z) = 1.

∴ $d(x , z) \leq d(x , y) + d(y , z)$.

In all the cases, $d(x , z) \leq d(x , y) + d(y , z)$.

Hence d is a metric on M.

1.2 OPEN SETS IN A METRIC SPACE

Definition 1.2.1 Let (M , d) be a metric space. Let $a \in M$ and r be a positive real number. The open ball or the open sphere with center a and radius r is denoted by $B_d(a , r)$ and is the subset of M defined by $B_d(a , r) = \{x \in M / d(a , x) < r\}$. We write B(a , r) for $B_d(a , r)$ if the metric d under consideration is clear.

Note. Since d(a, a) = 0 < r, $a \in B_d(a , r)$.

Examples 1.2.2

1. In **R** with usual metric B(a , r) = (a - r , a + r).
2. In $\mathbf{R}^2$ with usual metric B(a , r) is the interior of the circle with center a and radius r.
3. In a discrete metric space M, $B(a , r) = \begin{cases} M & \text{if } r>1 \\ \{ a \} & \text{if } r\leq 1 \end{cases}$

Definition 1.2.3 Let (M , d) be a metric space. A subset A of M is said to be open in M if for each $x \in A$ there exists a real number r > 0 such that $B(x , r) \subseteq A$.

Note. By the definition of open set, it is clear that ∅ and M are open sets.

Examples 1.2.3

1. Any open interval (a , b) is an open set in **R** with usual metric.
   For,
   Let $x \in (a , b)$.

Choose a real number r such that $0 < r \leq \min \{ x-a , b-x \}$.

Then $B(x , r) \subseteq (a , b)$.

$\therefore$ (a , b) is open in R.

2. Every subset of a discrete metric space M is open.

   For,

   Let A be a subset of M.

   If $A = \emptyset$, then A is open.

   Otherwise, let $x \in A$.

   Choose a real number r such that $0 < r \leq 1$.

   Then $B(x , r) = \{ x \} \subseteq A$ and hence A is open.

3. Set of all rational numbers **Q** is not open in **R**.

   For,

   Let $x \in$ **Q**.

   For any real number $r > 0$, $B(x , r) = (x - r , x + r)$ contains both rational and irrational numbers.

   $\therefore B(x , r) \not\subseteq$ **Q** and hence **Q** is not open.

Theorem 1.2.4 Let (M , d) be a metric space. Then each open ball in M is an open set.

Proof.

Let B(a ,r) be an open ball in M.

Let $x \in B(a , r)$.

Then $d(a , x) < r$.

Take $r_1 = r - d(a , x)$. Then $r_1 > 0$.

We claim that $B(x , r_1) \subseteq B(a , r)$.

Let $y \in B(x , r_1)$. Then $d(x , y) < r_1$.

Now, $d(a , y) \leq d(a , x) + d(x , y)$

$< d(a , x) + r_1$

$= d(a , x) + r - d(a , x)$

$= r$.

$\therefore d(a , y) < r$.

$\therefore y \in B(a , r)$.

$\therefore B(x , r_1) \subseteq B(a , r)$.

Hence B(a , r) is an open ball.

Theorem1.2.5 In any metric space M, the union of open sets is open.

Proof.

Let $\{A_\alpha\}$ be a family of open sets in M.

We have to prove $A = \cup A_\alpha$ is open in M.

Let $x \in A$.

Then $x \in A_\alpha$ for some α.

Since A_α is open, there exists an open ball B(x , r) such that $B(x , r) \subseteq A_\alpha$.

$\therefore B(x , r) \subseteq A$.

Hence A is open in M.

Theorem 1.2.6 In any metric space M, the intersection of a finite number of open sets is open.

Proof.

Let $A_1, A_2, \ldots, A_n$ be open sets in M.

We have to prove $A = A_1 \cap A_2 \cap \ldots \cap A_n$ is open in M.

Let $x \in A$.

Then $x \in A_i \forall i = 1, 2, \ldots , n$.

Since each A_i is open, there exists an open ball $B(x , r_i)$ such that $B(x , r_i) \subseteq A_i$.

Take $r = \min \{ r_1 , r_2 , \ldots , r_n \}$.

Clearly $r > 0$ and $B(x , r) \subseteq B(x , r_i) \ \forall i = 1, 2, \ldots , n$.

Hence $B(x , r) \subseteq A_i \forall i = 1, 2, \ldots , n$.

$\therefore B(x , r) \subseteq A$.

$\therefore$ A is open in M.

Theorem 1.2.7 Let (M , d) be a metric space and $A \subseteq M$. Then A is open in M if and only if A can be expressed as union of open balls.

Proof.

Suppose that A is open in M.

Then for each $x \in A$ there exists an open ball $B(x, r_x)$ such that $B(x, r_x) \subseteq A$.

$\therefore A = \bigcup_{x \in A} B(x, r_x)$.

Thus A is expressed as union of open balls.

Conversely, assume that A can be expressed as union of open balls.

Since open balls are open and union of open sets is open, A is open.

1.3 Interior of a set

Definition1.3.1 Let (M , d) be a metric space and $A \subseteq M$. A point $x \in A$ is said to be an interior point of A if there exists a real number $r > 0$ such that $B(x, r) \subseteq A$. The set of all interior points is called as interior of A and is denoted by **Int** A.

Note1.3.2 **Int** $A \subseteq A$.

Example1.3.3 In **R** with usual metric, let A = [1 , 2]. 1 is not an interior points of A, since for any real number $r > 0$, $B(1, r) = (1 - r, 1 + r)$ contains real numbers less than 1. Similarly, 2 is also not an interior point of A. In fact every point of (1 , 2) is a limit point of A. Hence **Int**A = (1 , 2).

Note1.3.4(1)**Int** $\emptyset = \emptyset$ **and Int** M = M.

(2) A is open $\Leftrightarrow$**Int** A = A.

(3) $A \subseteq B \Rightarrow$**Int** A$\subseteq$ **Int** B

Theorem1.3.5 Let (M , d) be a metric space and $A \subseteq M$. Then **Int** A = Union of all open sets contained in A.

Proof.

Let $G = \cup\{ B / B$ is an open set contained in A $\}$

We have to prove **Int** A = G.

Let $x \in$ **Int** A .

Then x is an interior point of A.

$\therefore$ there exists a real number $r > 0$ such that $B(x, r) \subseteq A$.

Since open balls are open, B(x , r) is an open set contained in A.

$\therefore B(x, r) \subseteq G$.

$\therefore x \in G$.

$\therefore$**Int** $A \subseteq G$ …………………………….. (1)

Let $x \in G$.

Then there exists an open se B such that $B \subseteq A$ and $x \in B$.

Since B is open and $x \in B$, there exists a real number $r > 0$ such that $B(x , r) \subseteq B \subseteq A$.

$\therefore$ x is an interior point of A.

$\therefore$ $x \in$ **Int** A .

$\therefore$ $G \subseteq$ **Int** A …………………………... (2)

From (1) and (2), we get **Int** A = G.

Note1.3.6 Int A is an open set and it is the largest open set contained in A.

Theorem1.3.7 Let M be a metric space and $A , B \subseteq M$. Then

(1) **Int** $(A \cap B)$ = (**Int** A) $\cap$ (**Int** A)
(2) **Int** $(A \cup B) \supseteq$ (**Int** A) $\cup$ (**Int** A)

Proof.

(1) $A \cap B \subseteq A \Rightarrow$**Int** $(A \cap B) \subseteq$ **Int** A .

Similarly, **Int** $(A \cap B) \subseteq$**Int** B .

$\therefore$**Int** $(A \cap B) \subseteq$ (**Int** A) $\cap$ (**Int** A) …………………………… (a)

Int$A \subseteq A$ and **Int** $B \subseteq B$.

$\therefore$ (**Int** A) $\cap$ (**Int** A) $\subseteq A \cap B$

Now, (**Int** A) $\cap$ (**Int** A) is an open set contained in $A \cap B$.

But, **Int** $(A \cap B)$ is the largest open set contained in $A \cap B$.

$\therefore$(**Int** A) $\cap$ (**Int** A) $\subseteq$ **Int** $(A \cap B)$ …………………………….. (b)

From (a) and (b) , we get **Int** $(A \cap B)$ = (**Int** A) $\cap$ (**Int** A)

(2) $A \subseteq A \cup B \Rightarrow$**Int** $A \subseteq$ **Int** $(A \cup B)$
Similarly, **Int B**$\subseteq$ **Int** $(A \cup B)$
$\therefore$**Int** $(A \cup B) \supseteq$ (**Int** A) $\cup$ (**Int** A)

Note1.3.8 Int (A ∪ B)need not be equal to(**Int** A) ∪ (**Int** A)

For,

In **R** with usual metric, let A = (0 , 1] and B = (1 , 2).

A ∪ B = (0 , 2).

∴**Int** (A ∪ B) = (0 , 2)

Now, **Int** A (0 , 1) and **Int** B = (1 , 2) and hence (**Int** A) ∪ (**Int** A) = (0 , 2) – { 2 }.

∴**Int** (A ∪ B)≠(**Int** A) ∪ (**Int** A)

1.4 Subspace

Definition1.4.1 Let (M , d) be a metric space. Let M_1 be a nonempty subset of M. Then M_1 is also a metric space under the same metric d. We call (M_1 , d) is a subspace of (M, d).

Theorem1.4.2 Let M be a metric space and M_1 a subspace of M. Let $A \subseteq M_1$. Then A is open in M_1 if and only if $A = G \cap M_1$ where G is open in M.

Proof.

Let $B_1(a , r)$ be the open ball in M_1 with center a and radius r.

Then $B_1(a , r) = B(a , r) \cap M_1$where B(a , r) is the open ball in M with center a and radius r.

Let A be an open set in M_1.

Then $A = \bigcup_{x \in A} B_1(x , r(x))$

$=\bigcup_{x \in A} [B(x , r(x)) \cap M_1)]$

$= [\bigcup_{x \in A} B(x , r(x))] \cap M_1$

$= G \cap M_1$ where $G = \bigcup_{x \in A} B(x , r(x))$ which is open in M.

Conversely, let $A = G \cap M_1$ where G is open in M.

We shall prove that A is open in M_1.

Let $x \in A$.

Then $x \in G$ and $x \in M_1$.

Since G is open in M, there exists an open ball B(x , r) such that $B(x , r) \subseteq G$.

$\therefore B(x , r) \cap M_1 \subseteq G \cap M_1$.

i.e. $B_1(a , r) \subseteq A$.

$\therefore$ A is open in M_1.

Example1.4.3 Consider the subspace $M_1 = [0 , 1] \cup [2 , 3]$ of **R**.

A = [0 , 1] is open in M_1 since $A = (-\frac{1}{2} , \frac{3}{2}) \subseteq M_1$ where $(-\frac{1}{2} , \frac{3}{2})$is open in **R**.

Similarly, B = [2 , 3], C = $[0 , \frac{1}{2}]$, D = $(\frac{1}{2} , 1]$ are open in M_1.

Note that A, B, C, D are not open in **R**.

1.5 Closed Sets.

Definition1.5.1A subset A of a metric space M is said to be closed in M if its complement is open in M.

Examples 1.5.2

1. In **R** with usual metric any closed interval [a , b] is closed.

For,

$[a , b]^c = \mathbf{R} - [a , b] = (- \infty , a) \cup (b , \infty)$.

$(- \infty , a)$ and(b , ∞) are open sets in R and hence $(- \infty , a) \cup (b , \infty)$ is open in **R**.

i.e. $[a , b]^c$ is open in **R**.

$\therefore$ [a , b] is open in **R**.

2. Any subset A of a metric space M is closed since A^c is open as every subset of M is open.

Note. In any metric space M, $\emptyset$ and M are closed sets since $\emptyset^c = M$ and $M^c = \emptyset$ which are open in M. Thus $\emptyset$ and M are both open and closed in M.

Theorem 1.5.3 In any metric space M, the union of a finite number of closed sets is closed.

Proof.

Let $A_1, A_2, \ldots , A_n$ be closed sets in a metric space M.

Let $A = A_1 \cup A_2 \cup \ldots \cup A_n$.

We have to prove A is open in M.

Now, $A^c = [A_1 \cup A_2 \cup \ldots \cup A_n]^c$

$= A_1^c \cap A_2^c \cap \ldots \cap A_n^c$ [By De Morgan's law.]

Since A_i is closed in M, A_i^c is open in M.

Since finite intersection of open sets is open, $A_1^c \cap A_2^c \cap \ldots \cap A_n^c$ is open in M.

i.e. A^c is open in M.

∴ A is closed in M.

Theorem 1.5.4 In any metric space M, the intersection of closed sets is closed.

Proof.

Let $\{A_\alpha\}$ be a family of closed sets in M.

We have to prove $A = \cap A_\alpha$ is open in M.

Now, $A^c = (\cap A_\alpha)^c$

$= \cup A_\alpha^c$ [ByDe Morgan's law.]

Since A_α is closed in M, A_α^c is open in M.

Since union of open sets is open, $\cup A_\alpha^c$ is open.

i.e. A^c is open in M.

∴ A is closed in M.

Theorem 1.5.5 Let M_1 be a subspace of a metric space M. Let $F_1 \subseteq M_1$. Then F_1 is closed in M_1 if and only if $F_1 = F \cap M_1$ where F is a closed set in M.

Proof.

Suppose that F_1 is closed in M_1.

Then $M_1 - F_1$ is open in M_1.

∴ $M_1 - F_1 = A \cap M_1$ where A is open in M.

Now, $F_1 = A^c \cap M_1$.

Since A is open in M, A^c is closed in M.

Thus, $F_1 = F \cap M_1$ where $F = A^c$ is closed in M.

Conversely, assume that $F_1 = F \cap M_1$ where F is closed in M.

Since F is closed in M, F^c is open in M.

∴ $F^c \cap M_1$ is open in M_1.

Now, $M_1 - F_1 = F^c \cap M_1$ which is open in M_1.

∴ F_1 is closed in M_1.

1.6 Closure.

Definition1.6.1 Let A be a subset of a metric space (M , d). The closure of A, denoted by $\overline{A}$, is defined as the intersection of all closed sets which contain A.

i.e. $\overline{A} = \cap \{B \mid B \text{ is closed in M and } B \supseteq A\}$

Note 1.6.2

(1) Since intersection of closed sets is closed, $\overline{A}$ is a closed set.
(2) $\overline{A} \supseteq A$.
(3) $\overline{A}$ is the smallest closed set containing A.
(4) A is closed ⇔ $A = \overline{A}$.
(5) $\overline{\overline{A}} = \overline{A}$.

Theorem 1.6.3 Let (M , d) be a metric space. Let A , B ⊆ M. Then

(1) $A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}$
(2) $\overline{A \cup B} = \overline{A} \cup \overline{B}$
(3) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

Proof.

(1) Let $A \subseteq B$.
$\overline{B} \supseteq B \supseteq A$.
Thus $\overline{B}$ is a closed set containing A.
But $\overline{A}$ is the smallest closed set containing A.
∴ $\overline{A} \subseteq \overline{B}$.

(2) $A \subseteq A \cup B$.
∴by (1), $\overline{A} \subseteq \overline{A \cup B}$.
Similarly , $\overline{B} \subseteq \overline{A \cup B}$.
∴ $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$ ………………………….. (a)
$\overline{A}$ is a closed set containing A and $\overline{B}$ is a closed set containing B.
∴ $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$.
But $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$.
∴ $\overline{A \cup B} \subseteq \overline{A \cup B}$ ………………………….. (b)
From (a) and (b) we get $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(3) $A \cap B \subseteq A$.

$\therefore \overline{A \cap B} \subseteq \overline{A}$.

Similarly, $\overline{A \cap B} \subseteq \overline{B}$.

$\therefore \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$

Note1.6.4 $\overline{A \cap B}$ need not be equal to $\overline{A} \cap \overline{B}$.

For example, in **R** with usual metric take A = (0 , 1) and B = (1 , 2) .

$A \cap B = \emptyset \Rightarrow \overline{A \cap B} = \emptyset$.

But $\overline{A} \cap \overline{B} = [0 , 1] \cap [1 , 2] = \{ 1 \}$.

$\therefore \overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

1.7 Limit Point.

Definition 1.7.1 Let (M , d) be a metric space and $A \subseteq M$. A point $x \in M$ is said to be a limit point of A if every open ball with center x contains a point of A other than x.

i.e. $B(x , r) \cap (A - \{ x \}) \neq \emptyset$ for all $r > 0$.

The set of all limit points of A is denoted by A^{l}.

Example 1.7.2 In **R** with usual metric let A = (0 , 1).

Every open ball with center 0, B(0 , r) = (-r , r) contains points of (0 , 1) other than 0.

$\therefore$ 0 is a limit point of A.

Similarly, 1 is a limit point of A and in fact every point of A is also a limit Point of A.

For each real number $x < 0$, if we choose r such that $0 < r \leq -\frac{x}{2}$, then B(x , r) contains no point of (0 , 1) , and hence x is not a limit point of limit point of A.

Similarly, every real number $x > 0$ is not a limit point of A.

Hence $A^{l} = [0 , 1]$.

Example 1.7.3 In **R** with usual metric, **Z** has no limit point.

For,

Let x be any real number.

If x is an integer, then $B(x , \frac{1}{2}) = (x - \frac{1}{2} , x + \frac{1}{2})$ has no integer other than x.

$\therefore$ x is not a limit point of **Z** .

If x is not an integer, choose r such that $0 < r < |x-n|$ where n is the integer closest to x.

Then $B(x , r) = (x - r , x + r)$ contains no integer.

Hence x is not a limit point of **Z.**

Thus no real number x is a limit point of **Z.**

$\therefore$ $\mathbf{Z}' = \emptyset$.

Example 1.7.4 In **R** with usual metric, every real number is a limit point of **Q** .

For,

Let x be any real number.

Every open ball $B(x , r) = (x - r , x + r)$ contains infinite number of rational numbers.

$\therefore$ x is a limit point of **Q**.

$\therefore$ $\mathbf{Q}' = \mathbf{R}$.

Theorem 1.7.5 Let (M , d) be a metric space and $A \subseteq M$. Then x is a limit point of A if and only if every open ball with center x contains infinite number of points of A.

Proof.

Let x be a limit point of A.

We have to prove every open ball with center x contains infinite number of points of A.

Suppose not.

Then there exists an open ball B(x , r) contains only a finite number of points of A and hence of $(A - \{ x \})$.

Let $B(x , r) \cap (A - \{ x \}) = \{x_1, x_2, \ldots , x_n\}$.

Let $r_1 = \min \{ d(x , x_i) / i = 1 , 2 , \ldots , n \}$.

Since $x \neq x_i$, $d(x , x_i) > 0 \ \forall\ i = 1 , 2 , \ldots , n$ and hence $r_1 > 0$.

Moreover, $B(x , r_1) \cap (A - \{ x \}) = \emptyset$.

$\therefore$ x is not a limit point of A.

This is a contradiction.

$\therefore$ every open ball with center x contains infinite number of points of A.

Conversely, assume that every open ball with center x contains infinite number of points of A.

Then, every open ball with center x contains infinite number of points of

$A - \{x\}$.

Hence x is a limit point of A.

Note 1.7.6 Any finite subset of a metric space has no limit points.

Theorem 1.7.7 Let M be a metric space and $A \subseteq M$. Then $\overline{A} = A \cup A'$.

Proof.

Let $x \in A \cup A'$.

We claim that $x \in \overline{A}$.

Suppose $x \notin \overline{A}$.

Then, $x \in M - \overline{A}$.

Since $\overline{A}$ is closed, $M - \overline{A}$ is open.

$\therefore$ there exists an open ball $B(x, r)$ such that $B(x, r) \subseteq M - \overline{A}$.

$\therefore B(x, r) \cap \overline{A} = \emptyset$.

$\therefore B(x, r) \cap A = \emptyset$. [$\because A \subseteq \overline{A}$].

$\therefore x \notin A \cup A'$, which is a contradiction.

$\therefore x \in \overline{A}$.

$\therefore A \cup A' \subseteq \overline{A}$ ………………………… (1)

Let $x \in \overline{A}$.

We have to prove $x \in A \cup A'$.

If $x \in A$, then $x \in A \cup A'$.

Suppose $x \notin A$.

We claim that $x \in A'$.

Suppose $x \notin A^{\prime}$.

Then there exists an open ball B(x , r) such that $B(x, r) \cap (A - \{x\}) = \emptyset$.

$\therefore B(x, r) \cap A = \emptyset$. [$\because x \notin A$]

$\therefore A \subseteq B(x, r)^c$.

Since B(x , r) is open, $B(x, r)^c$ is closed.

Thus $B(x, r)^c$ is a closed set containing A.

But, $\overline{A}$ is the smallest closed set containing A.

Hence $\overline{A} \subseteq B(x, r)^c$.

Now, $x \notin B(x, r)^c$.

$\therefore x \notin \overline{A}$, which is a contradiction.

$\therefore x \in A^{\prime}$ and hence $x \in A \cup A^{\prime}$.

$\overline{A} \subseteq A \cup A^{\prime}$ ……………………………… (2)

From (1) and (2), we get $\overline{A} = A \cup A^{\prime}$.

Corollary1.7.8 A is closed if and only if A contains all its limit points.

Proof.

A is closed $\Leftrightarrow A = \overline{A}$.

$\Leftrightarrow A = A \cup A^{\prime}$.

$\Leftrightarrow A \subseteq A^{\prime}$.

Corollary 1.7.9 $x \in \overline{A} \Leftrightarrow B(x, r) \cap A \neq \emptyset \ \forall\ r > 0$.

Proof.

$x \in \overline{A} \Rightarrow x \in A \cup A^{\prime}$.

$\therefore x \in A$ or $x \in A^{\prime}$.

If $x \in A$, then $x \in B(x, r) \cap A$.

If $x \in A^{\prime}$, then $B(x, r) \cap (A - \{x\}) \neq \emptyset \ \forall\ r > 0$.

Thus $B(x, r) \cap A \neq \emptyset \ \forall\ r > 0$.

Conversely, let $B(x, r) \cap A \neq \emptyset \ \forall \ r > 0$.

We have to prove $x \in \overline{A}$.

If $x \in A$, then $x \in \overline{A}$.

If $x \notin A$, then $A = A - \{x\}$.

$\therefore B(x, r) \cap (A - \{x\}) \neq \emptyset \ \forall \ r > 0$.

$\therefore$ x is a limit point of A.

$\therefore x \in A'$.

$\therefore x \in \overline{A}$.

Corollary 1.7.10 $x \in \overline{A} \Leftrightarrow G \cap A \neq \emptyset$ for all open set G containing x.

Proof.

Let $x \in \overline{A}$.

We have to prove $G \cap A \neq \emptyset$ for all open set G containing x.

Let G be an open set containing x.

Then there exists an open ball $B(x, r)$ such that $B(x, r) \subseteq G$.

Since $x \in \overline{A}$, $B(x, r) \cap A \neq \emptyset$ and hence $G \cap A \neq \emptyset$.

Conversely, assume that $G \cap A \neq \emptyset$ for every open set containing x.

Then $B(x, r) \cap A \neq \emptyset \ \forall \ r > 0$.

$\therefore x \in \overline{A}$.

1.8 Bounded Sets in a Metric space.

Definition 1.8.1 Let (M , d) be a metric space. A subset A of M is said to be bounded if there exists a positive real number k such that $d(x, y) \leq k \ \forall \ x, y \in A$.

Example 1.8.2 Any finite subset A of a metric space (M , d) is bounded.

For,

Let A be any finite subset of M.

If $A = \emptyset$ then A is obviously bounded.

Let $A \neq \varnothing$ .Then $\{d(x , y)/x , y \in A\}$ is a finite set of real numbers.

Let $k = \max \{d(x , y)/x , y \in A\}$.

Clearly $d(x , y) \leq k$ for all $x , y \in A$.

$\therefore$ A is bounded.

Example 1.8.3 [0,1] is a bounded subset of **R** with usual metric since $d(x , y) \leq 1$ for all $x , y \in [0,1]$.

Example 1.8.4 $(0 , \infty)$ is an unbounded subset of **R.**

Example 1.8.5 Any subset A of a discrete metric space M is bounded since

$d(x , y) \leq 1$ for all $x , y \in A$.

Note 1.8.6 Every open ball $B(x , r)$ in a metric space (M , d) is bounded.

For,

Let $s , t \in B(x , r)$.

$d(s , t) \leq d(s , x) + d(x , t) < r + r$.

$\therefore d(s , t) < 2r$.

Hence $B(x , r)$ is bounded.

Definition 1.8.7 Let (M , d) be a metric space and $A \subseteq M$. The diameter of A, denoted by $d(A)$, is defined by $d(A)= \text{l.u.b} \{d(x , y)/x , y \in A\}$.

Example 1.8.8 In R with usual metric the diameter of any interval is equal to the length of the interval. The diameter of [0 , 1] is 1.

1.9 Complete Metric Spaces.

Definition 1.9.1 Let (M , d) be a metric space. Let (x_n) be a sequence in M. Let $x \in M$. We say that (x_n) converges to x if for every $\varepsilon > 0$ there exists a positive integer N such that $d(x_n , x) < \varepsilon$ for all $n \geq N$. If (x_n) converges to x , then x is called a limit of (x_n) and we write $\lim_{n \to \infty} x_n = x$ or $x_n \to x$.

Note 1.9.2 (1) $x_n \to x$ if and only if for every $\varepsilon > 0$ there exists a positive integer N such that $x_n \in B(x , \varepsilon)$ $\forall$ $n \geq N$. Thus, the open ball $B(x , r)$ contains all but a finite number of terms of the sequence.

(2) $x_n \to x$ if and only if $(d(x_n , x)) \to 0$.

Theorem 1.9.3 The limit of a convergent sequence in a metric space is unique.

Proof.

Let (M , d) be a metric space and let (x_n) be a sequence in M.

Suppose that (x_n) has two limits say x and y.

Let $\varepsilon > 0$ be given.

Since $x_n \to x$, there exists a positive integer N_1 such that $d(x_n , x) < \varepsilon/2$ for all $n \geq N_1$.

Since $x_n \to y$, there exists a positive integer N_2 such that $d(x_n , x) < \varepsilon/2$ for all $n \geq N_2$.

Let $N = \max \{ N_1 , N_2 \}$.

Then, $d(x , y) \leq d(x , x_N) + d(x_N , y)$

$< \varepsilon/2 + \varepsilon/2$

$\therefore d(x , y) < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary , $d(x , y) = 0$.

$\therefore x = y$.

Theorem1.9.4 Let (M, d) be a metric space and $A \subseteq B$. Then

(i) X is a limit point of A $\Leftrightarrow$ there exists a sequence (x_n) of distinct points in A such that $x_n \to x$.

(ii) $X \in \overline{A} \Leftrightarrow$ there exists a sequence (x_n) in A such that $x_n \to x$.

Proof.

(i) Let x be a limit point of A.
Then every open ball B(x , r) contains infinite number of points of A.
Thus, for each natural number n , we can choose $x_n \in B(x , \frac{1}{n})$ such that $x_n \neq x_1, x_2, x_3, \ldots , x_{n-1}$.
Now, (x_n) is a sequence of distinct points in A and $d(x_n , x) < \frac{1}{n}\ \forall n$.
$\therefore (d(x_n , x)) \to 0$.
$\therefore x_n \to x$.

Conversely, assume that there exists a sequence (x_n) of distinct points in A such that $x_n \to x$.

We have to prove x is a limit point of A.

Let it be given an open ball $B(x , \varepsilon)$.

Since $x_n \to x$, there exists a positive integer N such that

$d(x_n , x) < \varepsilon \ \forall\ n \geq N$.

$\therefore\ x_n \in B(x , \varepsilon)\ \forall\ n \geq N$.

Since x_n are distinct points of A, $B(x , \varepsilon)$ contains infinite number of points of A.

Thus, every open ball with center x contains infinite number of points of A.

Hence x is a limit point of A.

(ii) Let $x \in \overline{A}$.
Then $x \in A \cup A^{l}$.
If $x \in A$ then the constant sequence x, x, x, ….. is a sequence in A converges to x.
If $x \notin A$, then $x \in A^{l}$.
$\therefore$ x is a limit point of A.
$\therefore$ by (i), there exists a sequence (x_n) in A converges to x.
Conversely, assume that there exists a sequence (x_n) in A such that
$x_n \to x$.
Then every open ball $B(x , \varepsilon)$ contains points in the sequence and hence points of A.
$\therefore x \in \overline{A}$.

Definition 1.9.5 Let (M , d) be a metric space. Let (x_n) be a sequence in M. Then (x_n) is said to be a Cauchy sequence in M if for every $\varepsilon > 0$ there exists a positive integer N such that $d(x_n , x_m) < \varepsilon$ for all $n , m \geq N$.

Theorem 1.9.6 Every convergent sequence in a metric space (M , d) is a Cauchy sequence.

Proof. Let (x_n) be a convergent sequence in M converges to $x \in M$.

We have to prove (x_n) is Cauchy.

Let $\varepsilon > 0$ be given.

Since $x_n \to x$, there exists a positive integer N such that $d(x_n , x) < \varepsilon/2$ for all $n \geq N$.

$\therefore d(x_n , x_m) \leq d(x_n , x) + d(x , x_m)$

$< \varepsilon/2 + \varepsilon/2$ for all $n , m \geq N$.

$\therefore d(x_n , x_m) < \varepsilon$ for all $n , m \geq N$.

Hence (x_n) is a Cauchy sequence.

Definition1.9.7 A metric space M is said to be complete if every Cauchy sequence in M converges to a point in M.

Example 1.9.8 R with usual metric is complete.

Theorem 1.9.9 A subset A of a complete metric space M is complete if and only if A is closed.

Proof.

Suppose that A is complete.

We have to prove A is closed.

For that it is enough to prove A contains all its limit points.

Let x be a limit point of A.

Then there exists a sequence (x_n) in A such that $x_n \to x$.

Since A is complete $x \in A$.

$\therefore$ A contains all its limit points.

Hence A is closed.

Conversely, assume that A is a closed subset of M.

Let (x_n) be a Cauchy sequence in A.

Then (x_n) be a Cauchy sequence in M.

Since M is complete, there exists $x \in M$ such that $x_n \to x$.

Thus (x_n) is a sequence in A such that $x_n \to x$.

$\therefore x \in \overline{A}$.

Since A is closed A = $\overline{A}$ and hence x ∈ A.

Thus every Cauchy sequence (x_n) in A converges to a point in A.

∴ A is complete.

Note 1.9.10 Every closed interval [a , b] with usual metric is complete since it is a closed subset of the complete metric space **R**.

Theorem 1.9.11 [Cantor's Intersection Theorem]

Let M be a metric space. Then M is complete if and only if for every sequence (F_n) of nonempty closed subsets of M such that $F_1 \supseteq F2 \supseteq \ldots F_n \supseteq \ldots$ and ($d(F_n)$) → 0 , $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Proof.

Let M be a complete metric space.

Let (F_n) be a sequence of nonempty closed subsets of M such that

$F_1 \supseteq F_2 \supseteq \ldots F_n \supseteq \ldots$ ……………………… (1)

and ($d(F_n)$) → 0 , …………………… (2)

We have to prove $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

For each natural number n , we choose a point x_n in F_n.

By (1), $x_n, x_{n+1}, x_{n+2}, \ldots$ all lie in F_n.

i.e. $x_m \in F_n \; \forall \; m \geq n$. …………………… (3)

We claim that (x_n) is a Cauchy sequence in M.

Let ε > 0 be given.

Since ($d(F_n)$) → 0 , there exists a positive integer N such that

$d(F_n) < \varepsilon \; \forall \; n \geq N$.

In particular, $d(F_N) < \varepsilon$. ……………………… (4)

Now, let m , n ≥ N.

Then by (3), $x_m , x_n \in F_N$.

$\therefore d(x_m , x_n) < \varepsilon$. [By (4)]

Thus $d(x_m , x_n) < \varepsilon \ \forall \ m , n \geq N$.

$\therefore (x_n)$ is a Cauchy sequence in M.

Since M is complete, there exists $x \in M$ such that $x_n \to x$.

We show that $x \in \bigcap_{n=1}^{\infty} F_n$.

For any natural number n, x_n, x_{n+1} , x_{n+2} is a sequence in F_n converges to x.

$\therefore x \in \overline{F_n}$.

Since F_n is closed, $\overline{F_n} = F_n$.

$\therefore x \in F_n$.

$\therefore x \in \bigcap_{n=1}^{\infty} F_n$.

Hence $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Conversely, assume that for every sequence (F_n) of nonempty closed subsets of M such that $F_1 \supseteq F_2 \supseteq \ldots F_n \supseteq \ldots$. and ($d(F_n)$) $\to 0$, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

We have to prove M is complete.

Let (x_n) be a Cauchy sequence in M.

We claim that $x_n \to x$ for some $x \in M$.

Define a decreasing sequence of sets $F_1 \supseteq F_2 \supseteq \ldots\ldots \supseteq F_n \supseteq \ldots\ldots$ as follows

$F_1 = \{x_1 ,x_2 , \ldots.., x_n , \ldots\ldots.\}$

$F_2 = \{x_2 ,x_3 , \ldots.., x_n , \ldots\ldots.\}$

….. …… …….. …..

….. …… ……. ……

$F_n = \{x_n ,x_{n+1} , \ldots..,\ldots\ldots.. ..\}$

…… ….. ……. …….

$\therefore \overline{F_1} \supseteq \overline{F_2} \supseteq \ldots\ldots.\supseteq \overline{F_n} \supseteq \ldots\ldots$

Thus $(\overline{F_n})$ is a decreasing sequence of closed sets.

Since (x_n) is a Cauchy sequence, for given $\varepsilon > 0$ there exists a positive integer N such that $d(x_n , x_m) < \varepsilon \ \forall$ n,m $\geq$ N.

$\therefore d(F_N) < \varepsilon$.

Now, $F_n \subseteq F_N \ \forall\ n \geq N \Rightarrow d(F_n) < \varepsilon \ \forall\ n \geq N$.

But $d(F_n) = d(\overline{F_n})$.

$\therefore d(\overline{F_n}) < \varepsilon \ \forall\ n \geq N$ .... (5)

$\therefore (d(\overline{F_n})) \to 0$.

Hence by hypothesis, $\bigcap_{n=1}^{\infty} \overline{F_n} \neq \emptyset$.

Let $x \in \bigcap_{n=1}^{\infty} \overline{F_n}$.

Then $x , x_n \in \overline{F_n}$.

$\therefore d(x_n , x) \leq d(\overline{F_n})$.

$\therefore d(x_n , x) < \varepsilon \ \forall\ n \geq N$ [By (5)]

$\therefore x_n \to x$.

$\therefore$ M is complete.

Note 1.9.12 In the above theorem $\bigcap_{n=1}^{\infty} \overline{F_n}$ contains exactly one point, since if it contains distinct points x and y, then $d(F_n) \geq d(x , y)$ for all n and hence $(d(\overline{F_n}))$ does not converge to 0.

1.10 Baire's Category Theorem.

Definition 1.10.1 A subset A of a metric space M is said to be nowhere dense in M if

Int $\overline{A} = \emptyset$.

Definition 1.10.2 A subset A of a metric space M is said to be of first category in M if A can be expressed as a countable union of nowhere dense sets.

If A is not of first category, then we say it is of second category.

Example1.10.3 In **R** with usual metric, every finite subset A is nowhere dense.

Example 1.10.4 In **R** with usual metric, the subset **Q** is of first category.

For,

Since **Q** is countable it can be expressed as countable union of singleton sets and each singleton set is nowhere dense in **R.**. Thus, **Q** is countable union of nowhere dense sets. Hence **Q** is of first category.

Example 1.10.5 If M is a discrete metric space, then any nonempty subset A of M is not nowhere dense set. Also A is of second category.

Theorem 1.10.6 Let M be a metric space and $A \subseteq M$. Then A is nowhere dense if and only if each nonempty open set contains an open ball disjoint from A.

Proof.

Suppose that A is nowhere dense.

Let G be a nonempty open set.

Since A is nowhere dense, **Int** $\overline{A} = \emptyset$.

$\therefore$ $\overline{A}$ does not contain G.

$\therefore$ there exists $x \in G$ such that $x \notin \overline{A}$.

$x \notin \overline{A} \Rightarrow$ there exists an open ball $B(x , r_1)$ such that $B(x , r_1) \cap A = \emptyset$.

G is open $\Rightarrow$ there exists an open ball $B(x , r_2)$ such that $B(x , r_2) \subseteq G$.

Let $r = \min \{ r_1 , r_2 \}$.

Then G contains $B(x , r)$ and disjoint from A.

Conversely, assume every nonempty open set contains an open ball disjoint from A.

We claim that **Int** $\overline{A} = \emptyset$.

Let $x \in \overline{A}$.

We claim that x is not an interior point of $\overline{A}$.

Suppose x is an interior point.

Then there exists an open ball B(x , r) such that B(x , r) ⊆ $\overline{A}$.

Now, every open ball in B(x , r) intersects with A, which is a contradiction.

Hence x is not an interior point of $\overline{A}$.

∴ **Int** $\overline{A} = \emptyset$.

∴ A is nowhere dense set.

Theorem 1.10.7 [Baire's Category Theorem]

Any complete metric space is of second category.

Proof.

Let M be a complete metric space.

We claim that M is not of first category.

Let (A_n) be a countable collection of nowhere dense sets in M.

We shall prove that $\bigcup_{n=1}^{\infty} A_n \neq M$.

Since M is open and A_1 is nowhere dense, there exists an open ball B_1 of radius less than 1 such that $B_1 \cap A_1 = \emptyset$.

Let F_1 be the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_1.

Now, **Int** F_1 is open and A_2 is nowhere dense.

∴ **Int** F_1 contains an open ball B_2 of radius less than $\frac{1}{2}$ such that $B_2 \cap A_2 = \emptyset$.

Let F_2 be the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_2.

Now, **Int** F_2 is open and A_3 is nowhere dense.

∴ **Int** F_2 contains an open ball B_3 of radius less than $\frac{1}{4}$ such that $B_3 \cap A_3 = \emptyset$.

Let F_3 be the concentric closed ball whose radius is $\frac{1}{2}$ times that of B_3.

Proceeding like this we get a sequence of nonempty closed balls F_n such that

$F_1 \supseteq F_2 \supseteq \ldots\ldots F_n \supseteq \ldots.$ and $d(Fn) < \frac{1}{2^n}$.

$\therefore (d(F_n)) \to 0$ as $n \to \infty$.

Since M is complete, By Cantor's intersection theorem, there exists a point $x \in M$

Such that $x \in \bigcap_{n=1}^{\infty} F_n$.

Moreover, $F_n \cap A_n = \emptyset \ \forall n$.

$\therefore x \notin A_n \quad \forall n$.

$\therefore x \notin \bigcup_{n=1}^{\infty} A_n$.

$\therefore \bigcup_{n=1}^{\infty} A_n \neq M$.

Hence M is of second category.

Corollary 1.10.8 R is of second category.

Proof.

R is a complete metric space. Hence, **R** is of second category.

Unit II

CONTINUITY

2.1 Continuity of functions.

Definition 2.1.1 Let (M_1 , d_1) and (M_2 , d_2) be two metric spaces. Let $a \in M_1$. A function $f : M_1 \to M_2$ is said to be **continuous at a** if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < d_1(x , a) < \delta \Rightarrow d_2(f(x) , f(a)) < \varepsilon$. The function f is said to be continuous if it is continuous at every point of M_1.

Note 2.1.2 $d_1(x, a) < \delta \Rightarrow d_2(f(x) , f(a)) < \varepsilon \Leftrightarrow x \in B(a , \delta) \Rightarrow f(x) \in B(f(a), \varepsilon)$.

$$\Leftrightarrow f\big(B(a, \delta)\big) \subseteq B(f(a), \varepsilon).$$

Theorem 2.1.3 Let (M_1, d_1) and (M_2, d_2) be two metric spaces. A function $f : M_1 \to M_2$ is continuous if and only if $f^{-1}(V)$ is open in M_1 whenever V is open in M_2.

Proof. Assume that f is continuous.

Let V be open in M_1.

We have to prove $f^{-1}(V)$ is open in M_1.

If $f^{-1}(V) = \varphi$, then it is open.

Let $f^{-1}(V) \neq \varphi$.

We shall prove that for each $x \in f^{-1}(V)$ there exists an open ball $B(x , \delta)$

such that $B(x , \delta) \subseteq f^{-1}(V)$.

Let $x \in f^{-1}(V)$. Then $f(x) \in V$.

Since V is open, there exists an open ball $B(f(x) , \varepsilon)$ such that $B(f(x) , \varepsilon) \subseteq V$. ……..(1)

Now, since f is continuous, there exists an open ball $B(x , \delta)$ such that

$f(B(x , \delta)) \subseteq B(f(x) , \varepsilon)$.

By (1), $f(B(x , \delta)) \subseteq V$ and hence $B(x , \delta) \subseteq f^{-1}(V)$.

$\therefore f^{-1}(V)$ is open.

Conversely, assume that $f^{-1}(V)$ is open in M_1 whenever V is open in M_2.

To prove f is continuous, we shall prove that f is continuous at every point of M_1.

Let $x \in M_1$ and let $\varepsilon > 0$ be given.

We know that, $B(f(x), \varepsilon)$ is an open set in M_2.

By hypothesis, $f^{-1}(B(f(x), \varepsilon))$ is open in M_1.

Also, $x \in f^{-1}(B(f(x), \varepsilon))$.

$\therefore$ there exists $\delta > 0$ such that $B(x, \delta) \subseteq f^{-1}(B(f(x), \varepsilon))$.

$\therefore f(B(x, \delta)) \subseteq B(f(x), \varepsilon)$.

$\therefore$ f is continuous at x.

Since $x \in M_1$ is arbitrary, f is continuous on M_1.

Note 2.1.4 f is continuous if and only if inverse image of every open set is open.

Theorem 2.1.5 Let (M_1, d_1) and (M_2, d_2) be two metric spaces. A function $f : M_1 \to M_2$ is continuous if and only if $f^{-1}(W)$ is closed in M_1 whenever W is closed in M_2.

Proof. Assume that f is continuous.

Let W be a closed set in M_2.

Then W^C is an open set in M_2.

By hypothesis, $f^{-1}(W^C)$ is open in M_1.

But $f^{-1}(W^C) = \left[f^{-1}(W)\right]^C$.

$\therefore \left[f^{-1}(W)\right]^C$ is open in M_1.

$\therefore f^{-1}(W)$ is closed in M_1.

Conversely, assume that $f^{-1}(W)$ is closed in M_1 whenever W is closed in M_2.

To prove f is continuous, we shall prove that $f^{-1}(V)$ is open in M_1 whenever V is open in M_2.

Let V be an open set in M_2.

$\therefore V^C$ is a closed set in M_2.

By hypothesis, $f^{-1}(V^C)$ is a closed set in M_1.

(i.e) $\left[f^{-1}(V)\right]^C$ is a closed set in M_1.

$\therefore$ $f^{-1}(V)$ is an open set in M_1.

Thus, inverse image of every open set is open under f.

$\therefore$ f is continuous.

Note 2.1.6 f is continuous if and only if inverse image of every closed set is closed.

Theorem 2.1.7 Let (M_1, d_1) and (M_2, d_2) be two metric spaces. Then $f : M_1 \rightarrow M_2$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

Proof. Assume that f is continuous.

We have to prove $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

Let $A \subseteq M_1$. Then $f(A) \subseteq M_2$.

$\overline{f(A)}$ is a closed set in M_2.

Since f is continuous, $f^{-1}(\overline{f(A)})$ is closed in M_1.

Since $\overline{f(A)} \supseteq f(A)$, $f^{-1}(\overline{f(A)}) \supseteq A$.

But $\overline{A}$ is the smallest closed set containing A.

$\therefore$ $\overline{A} \subseteq f^{-1}(\overline{f(A)})$.

$\therefore f(\overline{A}) \subseteq \overline{f(A)}$.

Conversely, let $f(\overline{A}) \subseteq \overline{f(A)}$ for all $A \subseteq M_1$.

To prove f is continuous, we shall prove that $f^{-1}(W)$ is closed in M_1 whenever W is closed in M_2.

Let W be a closed set in M_2.

By hypothesis, $f(\overline{f^{-1}(W)}) \subseteq \overline{ff^{-1}(W)}$.

$\subseteq \overline{W}$

$= W$ (Since W is closed.).

Thus, $f(\overline{f^{-1}(W)}) \subseteq W$.

$\therefore \overline{f^{-1}(W)} \subseteq f^{-1}(W)$.

Also, $f^{-1}(W) \subseteq \overline{f^{-1}(W)}$.

$\therefore f^{-1}(W) = \overline{f^{-1}(W)}$.

Hence $f^{-1}(W)$ is closed.

$\therefore$ f is continuous.

Theorem 2.1.8 Let (M_1 , d_1) and (M_2 , d_2) be two metric spaces. Let $x \in M_1$. A function $f : M_1 \to M_2$ is continuous at x if and only if $x_n \to x$ in $M_1 \Rightarrow f(x_n) \to f(x)$ in M_2.

Proof.

Suppose that f is continuous at x.

Let (x_n) be a sequence in M_1 such that $x_n \to x$.

We shall prove that $f(x_n) \to f(x)$.

Let $\varepsilon > 0$ be given.

Since f is continuous at x, there exists $\delta > 0$ such that

$d_1 (y , x) < \delta \Rightarrow d_2 (f(y) , f(x)) < \varepsilon$ ………… (1).

Since $x_n \to x$, there exists positive integer N such that

$d_1 (x_n , x) < \delta \ \forall \ n \geq N$.

$\therefore d_2 (f(x_n) , f(x)) < \varepsilon \ \forall \ n \geq N$. [By (1)]

$\therefore f(x_n) \to f(x)$.

Conversely, assume that $x_n \to x \Rightarrow f(x_n) \to f(x)$.

We have to prove f is continuous at x.

Suppose not. Then there exists $\varepsilon > 0$ such that for all $\delta > 0$

$f(B(x , \delta)) \not\subseteq B(f(x) , \varepsilon)$.

Thus for each natural number n, $f(B(x , \frac{1}{n})) \not\subseteq B(f(x) , \varepsilon)$.

Choose x_n such that $x_n \in B(x_n , \delta)$ but $f(x_n) \not\subseteq B(f(x) , \varepsilon)$.

$\therefore$ $d_1 (x_n , x) < \frac{1}{n}$ for all n and $d_2 (f(x_n) , f(x)) \geq \varepsilon$ for all n.

$\therefore$ $x_n \to x$ and $f(x_n)$ does not converge to f(x).

This is a contradiction.

$\therefore$ f is continuous at x.

Problem 2.1.9 Let (M_1 , d_1) and (M_2 , d_2) be two metric spaces. Then prove that any constant function $f : M_1 \to M_2$ is continuous.

Solution.

Let $f : M_1 \to M_2$ be given by f(x) = c where $c \in M_2$ is a constant.

We have to show that f is continuous.

Let V be an open set in M_2.

Now, $f^{-1}(V) = \begin{cases} \emptyset & \text{if } x \notin V \\ M_1 & \text{if } x \in V \end{cases}$.

In both cases , $f^{-1}(V)$ is an open set.

Thus, inverse image of every open set is open under f.

$\therefore$ f is continuous.

Problem 2.1.10 Let M_1, M_2, M_3 be metric spaces. If $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are continuous, then prove that $g \circ f : M_1 \to M_2$ is also continuous.

i.e. composition of two continuous functions is continuous.

Solution.

Let W be an open set in M_3 .

Since g is continuous, $g^{-1}(W)$ is open in M_2.

Since f is continuous, $f^{-1}(g^{-1}(W))$ is open in M_1.

Now, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$.

$\therefore$ $(g \circ f)^{-1}(W)$ is open in M_1.

Hence $g \circ f$ is continuous.

Problem 2.1.11 Let f be a continuous real valued function defined on a metric space M. Let $A = \{ x \in M | f(x) \geq a \text{ where } a \in \mathbf{R} \}$. Prove that A is closed.

Solution.

$A = \{ x \in M \mid f(x) \geq a \text{ where } a \in \mathbf{R} \}$

$= \{ x \in M \mid f(x) \in [a, \infty) \}$

$= f^{-1}([a, \infty))$.

Now, $[a, \infty)$ is a closed subset of **R.**

Since f is continuous, $f^{-1}([a, \infty))$ is a closed subset of M.

$\therefore$ A is closed.

Problem 2.1.12 Let $f : M \to \mathbf{R}$ and $f : M \to \mathbf{R}$ be continuous functions. Prove that

$f+g : M \to \mathbf{R}$ is continuous.

Solution.

Let $x \in M$.

We show that f + g is continuous at x.

Let x_n be a sequence in M such that $x_n \to x$.

Since f and g are continuous, $f(x_n) \to f(x)$ and $g(x_n) \to g(x)$.

$\therefore$ $f(x_n) + g(x_n) \to f(x) + g(x)$.

i.e. $(f+g)(x_n) \to (f+g)(x)$.

$\therefore$ f+g is continuous at x.

Note 2.1.13 In a similar way, we can prove that $f - g$, fg, cf if $c \in \mathbf{R}$ and $\frac{f}{g}$ if $g(x) \neq 0 \ \forall \ x \in M$ are continuous.

2.2 Homeomorphism.

Definition 2.2.1 Let (M_1 , d_1) and (M_2 , d_2) be two metric spaces. A function $f : M_1 \to M_2$ is said to be a homeomorphism if the following holds.

(1) f is a bijection.
(2) f is continuous.
(3) f^{-1} is continuous.

M_1 and M_2 are said to be homeomorphic if there exists a homeomorphism between them.

Definition 2.2.2 A function $f : M_1 \to M_2$ is said to be an open mapping if for every open set G in M_1, f(G) is open in M_2.

i.e. image of every open set in M_1 under f is open in M_2.

Definition 2.2.3 A function $f : M_1 \to M_2$ is said to be a closed mapping if for every closed set F in M_1, f(F) is closed in M_2.

i.e. image of every closed set in M_1 under f is closed in M_2.

Theorem 2.2.4 Let $f : M_1 \to M_2$ be a bijection. Then the following are equivalent.

(1) f is a homeomorphism
(2) f is a continuous open map
(3) f is a continuous closed map

Proof.

We shall prove that (1) $\Leftrightarrow$ (2) and (1) $\Leftrightarrow$ (3) .

Suppose that f is a homeomorphism.

Then f and f^{-1} are continuous.

We have to prove f is an open mapping.

Let G be an open set in M_1.

Since $f^{-1} : M_2 \to M_1$ is continuous, $(f^{-1})^{-1}(G)$ is open in M_1.

i.e. f(G) is open in M_2.

$\therefore$ f is an open map.

Conversely, assume that f is a continuous open map.

We prove that f^{-1} is continuous.

Let G be an open set in M_1.

Since f is an open mapping, f(G) is open in M_2.

i.e. $(f^{-1})^{-1}(G)$ is open in M_2.

$\therefore f^{-1}$ is continuous.

The proof of (1) $\Leftrightarrow$ (3) is similar.

Note 2.2.5 Let $f : M_1 \to M_2$ be a homeomorphism. Then a subset G of M_1 is open in M_1 if and only if f(G) is open in M_2.

For,

Since f is a homeomorphism, f is a continuous open mapping.

Since f is open mapping, G is open in $M_1 \Rightarrow$ f(G) is open in M_2.

Since f is continuous, f(G) is open in $M_2 \Rightarrow f^{-1}(f(G)) = G$ is open in M_1.

$\therefore$ G is open in $M_1 \Leftrightarrow$ f(G) is open in M_2.

Thus a homeomorphism $f : M_1 \to M_2$ gives not only a 1 – 1 correspondence between the elements of the two spaces but also a 1 – 1 correspondence between their open sets.

Note 2.2.6 Let $f : M_1 \to M_2$ be a homeomorphism. Then a subset F of M_1 is closed in M_1 if and only if f(F) is closed in M_2.

Example 2.2.7 The metric spaces (0 ,1) and (0 , ∞) with usual metric are homeomorphic.

For,

Define $f : (0,1) \to (0, \infty)$ by $f(x) = \frac{x}{1-x}$.

We show that f is 1 – 1 and on to.

Let x , y $\in$ (0 , 1).

$f(x) = f(y) \Rightarrow \frac{x}{1-x} = \frac{y}{1-y}$

$\Rightarrow x(1-y) = y(1-x)$

$\Rightarrow x - x y = y - x y$

$\Rightarrow x = y$.

Hence f is 1 – 1.

Let $y \in (0 , \infty)$.

Now, $f(x) = y \Rightarrow \frac{x}{1-x} = y$

$\Rightarrow x = y (1 - x)$

$\Rightarrow x = y - xy$

$\Rightarrow x + xy = y$

$\Rightarrow x (1 + y) = y$

$\Rightarrow x = \frac{y}{1+y}$

$\therefore \frac{y}{1+y} \in (0 , 1)$ is the pre image of y under f.

$\therefore$ f is on to.

Thus f is a bijection and hence $f^{-1}: (0 , \infty) \to (0 , 1)$ by $f(x) = \frac{x}{1 + x}$ is a bijection.

Also, f and f^{-1} are continuous.

$\therefore$ f is a homeomorphism.

2.3 Uniform Continuity.

Definition 2.3.1 Let (M_1 , d_1) and (M_2 , d_2) be a metric space. A function $f : M_1 \to M_2$ is said to be uniformly continuous on M_1, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d_1(x , y) < \delta \Rightarrow d_2(f(x) , f(y)) < \varepsilon$.

Note 2.3.2 Every uniformly continuous function is continuous but the converse need not be true.

Example 2.3.3 The function $f : [0 , 1] \to \mathbf{R}$ given by $f(x) = x^2$ is uniformly continuous on [0 , 1].

For,

Let $\varepsilon > 0$ be given.

Let x , y ∈ [0 , 1].

Now, $|f(x)- f(y)| = |x^2- y^2|$

$= |x + y|\ |x - y|$

$\leq 2\ |x - y|$

Choose $\delta = \frac{\varepsilon}{2}$.

Then, $|x - y| < \delta \Rightarrow |f(x)- f(y)| < \varepsilon$.

∴ f is uniformly continuous on [0 , 1] .

2.4 Discontinuities of R

Definition 2.4.1

A function f: R→R is said to approach to a limit ℓ as x tends to a if given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x-a| < \delta \Longrightarrow |f(x) - \ell\,| < 0$ and we write $\lim_{x \to a} = \ell$.

Definition 2.4.2

A function f is that to have ℓ as the right limit at x=a if given $\varepsilon > 0$ there exists $\delta > 0$ such that $a < x < a + \delta \Longrightarrow |f(x) - \ell\,| < \varepsilon$ and we write $\lim_{x \to a+} = \ell$

Also we denote the right limit ℓ by f(a+)

A function f is that to have ℓ as the right limit at x=a if given $\varepsilon > 0$ there exists $\delta > 0$ such that $a < x < a - \delta \Longrightarrow |f(x) - \ell\,| < \varepsilon$ and we write $\lim_{x \to a-} = \ell$

Also we denote the right limit ℓ by f(a-)

Note 1

$\lim_{x \to a} f(x) = \ell$ if and only if $\lim_{x \to a+} = f(x) = \lim_{x \to a-} f(x) = \ell$.

i.e.

$\lim_{x \to a} f(x) = \ell$ if and only if the left and right limits of f(x) at x = a exist and are equal.

Note 2

The definition of continuity of f at x=a can be formulated as follows.

f is continuous at a if and only if f(a+) = f(a-)=f(a) .

Note 3

If $\lim_{x \to a} f(x)$ does not exist then one of the following happens.

1. $\lim_{x \to a+} f(x)$ does not exists.

2. $\lim_{x \to a-} f(x)$ does not exists.

3. $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ exists and are not equal.

Definition 2.4.3

If a function f is discontinuous at *a* then *a* is called a point of discontinuity for the function.

If *a* is a point of discontinuity of a function then any one of the following cases arises.

i. $\lim_{x \to a} f(x)$ exists but is not equal to f(*a*).

ii. $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ exists and are not equal.

iii. Either $\lim_{x \to a-} f(x)$ or $\lim_{x \to a+} f(x)$ does not exists.

Definition 2.4.4

Let *a* be a point of discontinuity for f(x). *a* is said to be a point of discontinuity of the first kind if $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ exists and both of them are finite and not equal. *a* is said to be a point of discontinuity of the second kind if either $\lim_{x \to a+} f(x)$ or $\lim_{x \to a-} f(x)$ does not exist.

Definition 2.4.5

Let $A \subseteq R$. A function $f : A \to R$ is called monotonic increasing if x , $y \in A$ and $x < y \Rightarrow f(x) \leq f(y)$.

f is called monotonic decreasing if x, $y \in A$ and $x > y \Rightarrow f(x) \geq f(y)$.

f is called monotonic if it is either monotonic increasing or monotonic decreasing.

Theorem 2.4.6

Let $f:[a, b] \to R$ be a monotonic increasing function. Then f has a left limit and a right limit at every point of (a, b). Also f has a right limit at *a* and f has a left limit at *b*. Further

$x < y \Rightarrow f(x+) \leq f(y-)$

Similar result is true for monotonic decreasing functions.

Proof

Let $f : [a, b] \to R$ be monotonic increasing.

Let $x \in [a, b]$. Then $\{f(t) \mid a \leq t < x\}$ is bounded above by f(x).

We claim that $f(x-) = \ell$

Let $\varepsilon > 0$ be given. By definition of l.u.b there exists t such that $a \le t < x$ and $\ell - \varepsilon < f(t) \le \ell$.

$\therefore t < u < x \Rightarrow \ell - \varepsilon < f(t) \le f(u) \le \ell$

($\because$ f is monotonic increasing)

$\Rightarrow \ell - \varepsilon < f(u) \le \ell$

$\therefore x - \delta < u < x \Rightarrow \ell - \varepsilon < f(u) \le \ell$ where $\delta = x - t$

$\therefore f(x-) = \ell$

Similarly we can prove that $f(x+) =$ g. l. b. $\{f(t) \mid x < t \le b\}$.

Now we shall prove that $x < y \Rightarrow f(x+) \le f(y-)$

Let $x < y$.

Now, $f(x+) =$ g.l.b $\{f(t)/x < t \le b\}$

$= $ g.l.b $\{f(t)/x < t \le y\}$ (1)

($\because$ f is monotonic increasing)

Also $f(y-) =$ l.u.b $\{f(t)/a \le t < y\}$

$=$ l.u.b $\{f(t)/x \le t < y\}$ (2)

$\therefore$ $f(x+) \le f(y-)$ [by (1) and (2)]

The proof for monotonic decreasing functions is similar.

Theorem 2.4.7

Let $f:[a, b] \to R$ be a monotonic function. Then the set of points of [a, b] at which f is discontinuous is countable.

Proof

We shall prove the theorem for a monotonic increasing function.

Let $E = \{x \mid x \in [a, b]$ and f is discontinuous at x$\}$.

Let $x \in E$. Then $f(x+)$ and $f(x-)$ exists and $f(x-) \le f(x) \le f(x+)$

If $f(x-) = f(x+)$ then $f(x-) = f(x) = f(x+)$

$\therefore$ f is continuous at x, which is a contradiction.

$\therefore f(x-) \ne f(x+)$

$\therefore f(x-) < f(x+)$

Now choose a rational number r(x) such that $f(x-) < r(x) < f(x+)$

This defines a map r from E to Q which maps x to r(x).

We claim that r is 1-l.

Let $x_1 < x_2$.

$\therefore f(x_1+) < f(x_2-)$.

Also $f(x_1-) < r(x_1) < f(x_1+)$

And $f(x_2-) < r(x_2) < f(x_2+)$

$\therefore r(x_1) < f(x_1+) < f(x_2-) < r(x_2)$

Thus $x_1 < x_2 \Longrightarrow r(x_1) < r(x_2)$.

$\therefore r : E \rightarrow Q$ is 1 - l

$\therefore$ E is countable.

2.5 Connectedness

Definition 2.5.1 A separation of a metric space M is a pair A, B of nonempty disjoint open subsets of M whose union is M.

M is said to be a connected metric space if there is no separation for M.

Example 2.5.2 Any discrete metric space with more than one element is connected.

For,

Let M be a metric space with more than two elements.

Choose an element $a \in M$ and let A = { a }.

Then A^c is a proper subset of M.

Now, A and A^c forms a separation of M.

$\therefore$ M is not connected.

Theorem 2.5.3 Let (M, d) be a metric space. Then M is connected if and only if $\emptyset$ and M are the only sets which are both open and closed in M.

Proof.

Suppose that M is connected.

We have to prove $\emptyset$ and M are the only sets which are both open and closed in M.

Suppose not.

Then there exists a proper subset A of M which is both open and closed in M.

Now, A and A^c forms a separation of M, which is a contradiction.

Conversely, assume that ∅ and M are the only sets which are both open and closed in M.

We have to prove M is connected.

Suppose not.

Then there exists a separation A, B of M.

A is a proper subset of M which is both open and closed in M, a contradiction.

∴ M is connected.

Theorem 2.5.4 Let (M , d) be a metric space. Then the following are equivalent.

(i) The sets A and B form a separation of M.

(ii) A and B are nonempty disjoint closed sets in M whose union is M.

(iii) A and B are nonempty disjoint sets in M whose union is M and $A \cap \overline{B} = \overline{A} \cap B = \emptyset$.

Proof.

We shall prove that **(i) ⇔ (ii) and (ii) ⇔ (iii)**

(i) ⇒ (ii).

Suppose that A and B forms a separation of M.

Then A and B are nonempty disjoint sets in M whose union is M.

We have to prove A and B are closed in M.

Now, $A = B^c$ and $B = A^c$.

Since A and B are open in M, A^c and B^c are closed in M.

i.e., A and B are closed in M.

∴ **(i) ⇒ (ii).**

The proof of **(ii) ⇒ (i)** is similar.

(ii) ⇒ (iii).

Suppose that A and B are nonempty disjoint closed sets in M whose union is M.

We have to prove $A \cap \overline{B} = \overline{A} \cap B = \emptyset$.

Since B is closed, $B = \overline{B}$.

$\therefore A \cap \overline{B} = A \cap B = \emptyset$.

Similarly, $\overline{A} \cap B = \emptyset$.

(iii) ⇒ (i).

Suppose that A and B are nonempty disjoint sets in M whose union is M and

$A \cap \overline{B} = \overline{A} \cap B = \emptyset$.

We have to prove A and B are closed in M.

Let $x \in \overline{A}$.

Since $\overline{A} \cap B = \emptyset$, $x \notin B$.

Since $A \cup B = M$, $x \in A$.

$\therefore \overline{A} \subseteq A$.

But $A \subseteq \overline{A}$.

$\therefore A = \overline{A}$ and hence A is closed.

Similarly, B is closed.

Theorem 2.5.5 Let M be a connected metric space. Let A be a connected subset of M. If B is a subset of M such that $A \subseteq B \subseteq \overline{A}$ then B is connected. In particular, $\overline{A}$ is connected.

Proof.

Suppose B is not connected.

Then there exists a separation B_1 , B_2 of B.

Since B_1 and B_2 are open in B, $B_1 = G_1 \cap B$ and $B_2 = G_2 \cap B$, where G_1 and G_2 are open in M.

Now, $B = B_1 \cup B_2 = (G_1 \cap B) \cup (G_2 \cap B) = (G_1 \cup G_2) \cap B$.

$\therefore B \subseteq G_1 \cup G_2$ and hence $A \subseteq G_1 \cup G_2$.

Take $A_1 = G_1 \cap A$ and $A_2 = G_2 \cap A$.

Then A_1 and A_2 are open in A.

Also, $A_1 \cup A_2 = (G_1 \cap A) \cup (G_2 \cap A)$

$= (G_1 \cup G_2) \cap A$

$= A$ [Since $A \subseteq G_1 \cup G_2$]

$A_1 \cap A_2 = (G_1 \cap A) \cap (G_2 \cap A)$

$= (G_1 \cap G_2) \cap A$

$\subseteq (G_1 \cap G_2) \cap B$ [Since $A \subseteq B$]

$= (G_1 \cap B) \cap (G_2 \cap B)$

$= B_1 \cap B_2$

$= \emptyset$.

Since A is connected, either $A_1 = \emptyset$ or $A_2 = \emptyset$.

Without loss of generality , assume that $A_1 = \emptyset$.

i.e. $G_1 \cap A = \emptyset$.

Since G_1 is open, $G_1 \cap \overline{A} = \emptyset$.

$\therefore G_1 \cap B = \emptyset$. [Since $B \subseteq \overline{A}$]

i.e. $B_1 = \emptyset$, which is a contradiction.

$\therefore$ B is connected .

2.6 Connected subsets of R.

Theorem 2.6.1 A subspace of **R** is connected if and only if it is an interval.

Proof.

Suppose that A is a connected subset of **R** .

We have to prove A is an interval.

Suppose not .

Then, there exists a , b , c $\in$ **R** such that $a < b < c$ and a , c $\in$ A but $b \notin A$.

Define $A_1 = (- \infty , b) \cap A$ and $A_2 = (b , \infty) \cap A$.

Since (- ∞ , b) and (b , ∞) are open in **R** , A_1 and A_2 are open in A.

Moreover, $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = A$.

Clearly $a \in A_1$ and $c \in A_2$.

∴ $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$.

Thus, A is the union of a pair of nonempty disjoint open sets A_1 and A_2 .

∴ A is not connected, which is a contradiction.

Hence A is an interval.

Conversely, assume that A is an interval.

We have to prove A is connected.

Suppose not.

Then, there exists nonempty disjoint closed sets $A_1$and A_2 in A such that $A = A_1 \cup A_2$.

Choose $x \in A_1$ and $z \in A_2$.

Since $A_1 \cap A_2 = \emptyset$, $x \neq z$.

∴ $x < z$ or $z < x$.

Without loss of generality we assume that $x < z$.

Now, $x , z \in A$ and A is an interval.

∴ $[x , z] \subseteq A \subseteq A_1 \cup A_2$.

Hence every element of $[x , z]$ is either in A_1 or in A_2.

Let y = l.u.b. $\{ [x , z] \cap A_1 \}$.

Clearly $x \leq y \leq z$.

By the definition of l.u.b. , for each $\varepsilon > 0$ there exists $t \in [x , z] \cap A_1$ such that

$y - \varepsilon < t \leq y$.

∴ $(y - \ , y + \varepsilon) \cap ([x , z] \cap A_1) \neq \emptyset \ \ \forall \varepsilon > 0$.

.∴ $y \in \overline{[x , z] \cap A_1}$.

Since $[x , z] \cap A_1$ is closed in A , $y \in [x , z] \cap A_1$

$\therefore y \in A_1$.   ……………….. (1)

Again, by the definition of y, for each $\varepsilon > 0$ there exists $s \in A_2$ such that $y \leq s < y + \varepsilon$.

$\therefore (y - \ , y + \varepsilon) \cap A_2 \neq \emptyset \ \forall \varepsilon > 0$.

$\therefore y \in \overline{A_2}$.

Since A_2 is closed in A, $y \in A_2$ ………….. (2)

$\therefore y \in A_1 \cap A_2$ [By (1) & (2)].

This is a contradiction to $A_1 \cap A_2 = \emptyset$.

Hence A is connected.

2.7 Connectedness and continuity.

Theorem 2.7.1 Let M_1 be a connected metric space. Let M_2 be any metric space. Let $f : M_1 \to M_2$ be a continuous function. Then $f(M_1)$ is a connected subset of M_2.

i.e. continuous image of a connected set is connected.

Proof.

Let $f (M_1) = A$ so that f is a continuous function from M_1 on to A.

We claim that A is connected.

Suppose A is not connected.

Then, there exists a proper subset B of A which is both open and closed in A.

Hence $f^{-1}(B)$ is a proper subset of M_1 which is both open and in M_1.

$\therefore M_1$ is not connected which is a contradiction.

Hence A is connected.

Theorem 2.7.2 [intermediate value Theorem]

Let f be a real valued continuous function defined on an interval **I**. Then f takes every value between any two value it assumes.

Proof.

Let $a , b \in$ **I** and let $f(a) \neq f(b)$.

Without loss of generality we assume that f(a) < f(b).

Let c be a real number such that f(a) < c < f(b).

The interval **I** is a connected subset of **R**.

Since f is continuous, f(**I**) is a connected subset of **R** .

Hence f(**I**) is an interval.

Also f(a) , f(b) ∈ f(**I)**.

∴ [f(a) , f(b)] ⊆ f(**I) .**

∴ c ∈ f(**I) .** [Since f(a) < c < f(b)]

∴ c = f(x) for some x ∈ **I .**

Unit III

Compactness

3.1 Compact Metric Spaces.

Definition 3.1.1 Let M be a metric space. A collection of open sets $\{G_\alpha\}$ is said to be an **open cover** for M if $\cup\, G_\alpha = M$. A sub collection of $\{G_\alpha\}$ which itself is an open cover is called a **subcover.**

Definition 3.1.2 A metric space M is said to be **compact** if every open cover for M has a finite subcover.

i.e. for each collection of open sets $\{G_\alpha\}$ such that $\cup\, G_\alpha = M$, there exists a finite sub collection $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} = M$.

Theorem 3.1.3 Let M be a metric space. Let $A \subseteq M$. Then A is compact if and only if for every collection $\{G_\alpha\}$ of open sets in M such that $\cup\, G_\alpha \supseteq A$ there exists a finite sub collection $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

i.e. A is compact if and only if every open cover for A by sets open in M has a finite subcover.

Proof.

Let A be a compact subset of M.

Let $\{G_\alpha\}$ be a collection of open sets in M such that $\cup\, G_\alpha \supseteq A$.

Then $(\cup\, G_\alpha) \cap A = A$.

$\therefore \cup\, (G_\alpha \cap A) = A$.

Since G_α is open in M, $G_\alpha \cap A$ is open in A.

$\therefore \{G_\alpha \cap A\}$ is an open cover for A.

Since A is compact, this open cover has a finite subcover say

$\{G_{\alpha_1} \cap A, G_{\alpha_2} \cap A, \ldots, G_{\alpha_n} \cap A\}$.

$\therefore \bigcup_{i=1}^{n} (G_{\alpha_i} \cap A) = A$.

$\therefore (\bigcup_{i=1}^{n} G_{\alpha_i}) \cap A = A$.

$\therefore \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

Conversely, assume that for every collection $\{G_\alpha\}$ of open sets in M such that $\cup G_\alpha \supseteq A$ there exists a finite sub collection $\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

We have to prove A is compact.

Let $\{H_\alpha\}$ be an open cover for A.

Then H_α is open in A $\forall \alpha$.

$\therefore H_\alpha = G_\alpha \cap A$ where G_α is open in M $\forall \alpha$.

Now $\cup H_\alpha = A \Rightarrow \cup (G_\alpha \cap A) = A$.

$\Rightarrow (\cup G_\alpha) \cap A = A$.

$\Rightarrow \cup G_\alpha \supseteq A$.

Hence by our assumption, there exists a finite sub collection $\{G_{\alpha_1}, G_{\alpha_2}, \ldots\ldots, G_{\alpha_n}\}$ such that $\bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

$\therefore (\bigcup_{i=1}^{n} G_{\alpha_i}) \cap A = A$.

$\therefore \bigcup_{i=1}^{n} (G_{\alpha_i} \cap A) = A$.

$\bigcup_{I=1}^{n} H_{\alpha_I} = A$.

Thus $\{H_{\alpha_1}, H_{\alpha_2}, \ldots, H_{\alpha_n}\}$ is a finite subcover of the given open cover $\{H_\alpha\}$ of A.

$\therefore$ A is compact.

Theorem 3.1.4 Any compact subset A of a metric space (M , d) is closed.

Proof.

We shall prove that A^c is open.

Let $y \in A^c$.

Now, for each $x \in A$, $x \neq y$.

$\therefore d(x, y) = r_x > 0$ and $B(x, \frac{r_x}{2}) \cap B(y, \frac{r_x}{2}) = \emptyset$.

Clearly the collection $\{ B(x, \frac{r_x}{2}) / x \in A \}$ is an open cover for A by sets

open in M.

Since A is compact, there exists x_1 , x_2 , …. , x_n ∈ A such that

$\bigcup_{i=1}^{n} B(x, \frac{r_{x_i}}{2}) \supseteq A$ ……………………………….. (1)

Let $V_y = \bigcap_{i=1}^{n} B(y, \frac{r_{x_i}}{2})$.

Then V_y is an open set containing y.

Since $B(x, \frac{r_{x_i}}{2}) \cap B(y, \frac{r_{x_i}}{2}) = \emptyset$, $V_y \cap B(x, \frac{r_{x_i}}{2}) = \emptyset \ \forall$ i = 1, 2, …. , n .

∴ $V_y \cap [\bigcup_{i=1}^{n} B\left(x, \frac{r_{x_i}}{2}\right)] = \emptyset$.

∴ $V_y \cap A = \emptyset$. [By (1)]

∴ $V_y \subseteq A^c$.

Thus, for each y ∈ A^c there exists an open set V_y containing y such that $V_y \subseteq A^c$.

∴ $A^c = \bigcup_{y \in A^c} V_y$.

∴ A^c is open .

Hence A is closed.

Theorem 3.1.5 Any compact subset A of a metric space M is bounded.

Proof.

Let x ∈ A.

Now, { B(x , n) / n ∈**N** } is an open cover for A by sets open in M.

Since A is compact, there exists natural numbers n_1, n_2, … , n_k, such that $\bigcup_{i=1}^{k} B(x, n_k) \supseteq A$.

Let N = max { n_1, n_2, … , n_k}.

Then $\bigcup_{i=1}^{k} B(x, n_k) = B(x, N)$.

∴ B(x , N) ⊇ A.

Since B(x , N) is bounded and subset of a bounded set is bounded, A is bounded.

Theorem 3.1.6 A closed subset A of a compact metric space M is compact.

Proof.

Let $\{G_\alpha\}$ be a collection of open sets in M such that $\cup G_\alpha \supseteq A$.

$\therefore A^c \cup \cup G_\alpha = M$.

Since A is closed, A^c is open.

$\therefore \{G_\alpha\} \cup \{A^c\}$ is an open cover for M.

Since M is compact this open cover has a finite subcover say

$\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}, A^c\}$.

$\therefore (\bigcup_{i=1}^{n} G_{\alpha_i}) \cup A^c = M$.

$\therefore \bigcup_{i=1}^{n} G_{\alpha_i} \supseteq A$.

Hence A is compact.

Theorem 3.1.7 [Heine Borel Theorem]

Any closed interval [a , b] is a compact subset of **R.**

Proof.

Let $\{G_\alpha\}$ be a collection of open sets in **R** such that $\cup G_\alpha \supseteq$ **R**.

Let S = { x ∈ [a , b] / [a , x] can be covered by a finite number of G_α's. }

Clearly a ∈ S and hence S ≠ ∅.

Since S is bounded above by b , l.u.b of S exists.

Let c = l.u.b of S.

Clearly c ∈ [a , b].

$\therefore c \in G_{\alpha_1}$ for some index α_1.

Since G_{α_1} is open , there exists ε > 0 such that $B(x, \varepsilon) \subseteq G_{\alpha_1}$.

i.e. $(c - \varepsilon, c + \varepsilon) \subseteq G_{\alpha_1}$.

Choose $x_1 \in [a, b]$ such that $x_1 < c$ and $[x_1, c] \subseteq G_{\alpha_1}$.

Since $x_1 < c$, $[a, x_1]$ is covered by a finite number of G_α's.

These finite number of G_α's together with G_{α_1} covers [a , c].

∴ by the definition of S , $c \in S$.

Now, we claim that $c = b$.

Suppose $c \neq b$.

Then choose $x_2 \in [a , b]$ such that $x_2 > c$ and $[c , x_2] \subseteq G_{\alpha_1}$.

Since $[a , c]$ is covered by a finite number of G_α's , these finite number of G_α's together with G_{α_1} covers $[a , x_2]$.

∴ $x_2 \in S$, which is a contradiction to c is l.u.b of S [∵$x_2 > c$].

Hence $c = b$.

∴ $[a , x]$ can be covered by a finite number of G_α's.

∴ $[a , b]$ is a compact subset of **R** .

Theorem 3.1.8 A subset A or **R** is compact if and only if A is closed and bounded.

Proof.

If A is compact, then A is closed and bounded.

Conversely, assume that A is closed and bounded subset of **R** .

Since A is bounded, A has a lower bound and an upper bound say a and b respectively.

Then $A \subseteq [a , b]$.

Since A is closed in **R** , $A \cap [a , b]$ is closed in $[a , b]$.

I.e. A is closed in $[a , b]$.

Thus, A is a closed subset of the compact space $[a , b]$.

Hence A is compact.

3.2 Compactness and Continuity.

Theorem 3.2.1 Let M_1 be a compact metric space and M_2 be any metric space. Let $f : M_1 \rightarrow M_2$ be a continuous function. Then $f(M_1)$ is compact.

i.e. Continuous image of a compact metric space is compact.

Proof.

Without loss of generality we assume that f(M_1) = M_2.

Let $\{G_\alpha\}$ be a collection of open sets in M_2 such that $\cup G_\alpha = M_2$.

$\therefore \cup G_\alpha = f(M_2)$.

$\therefore f^{-1}(\cup G_\alpha) = M_1$.

$\therefore \cup f^{-1}(G_\alpha) = M_1$.

Since f is continuous, $f^{-1}(G_\alpha)$ is open in $M_1 \forall \alpha$.

$\therefore \{ f^{-1}(G_\alpha) \}$ is an open cover for M_1.

Since M1 is compact, this open cover has a finite subcover say $\left\{f^{-1}\left(G_{\alpha_1}\right), f^{-1}\left(G_{\alpha_2}\right), \ldots\ldots, f^{-1}\left(G_{\alpha_n}\right)\right\}$.

$\therefore f^{-1}(\bigcup_{i=1}^{n} G_{\alpha_i}) = M_1$.

$\bigcup_{i=1}^{n} G_{\alpha_i} = f(M_1) = M_2$.

Thus $\left\{G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n}\right\}$ is a finite subcover for the given open cover $\{G_\alpha\}$ of M_2.

Hence M_2 is compact.

Corollary 3.2.2 Let f be a continuous map from a compact metric space M_1 into any metric space M_2. Then f(M_1) is closed and bounded.

Proof.

Since f is continuous, f(M_1) is compact and hence closed and bounded.

Theorem 3.2.3 Any continuous mapping f defined on a compact metric space

(M_1, d_1) into any other metric space (M_2, d_2) is uniformly continuous on M_1.

Proof.

Let $\varepsilon > 0$ be given.

Let $x \in M_1$.

Since f is continuous at x, for $\varepsilon/2 > 0$, there exists $\delta_x > 0$ such that

$d_1(x , y) < \delta_x \Rightarrow d_2(f(x) , f(y)) < \varepsilon/2$ ……………….. (1)

Clearly, $\{ B(x , \frac{\delta_x}{2}) / x \in M_1\}$ is an open cover for M_1.

Since M_1 is compact, there exists $x_1 , x_2 , \ldots , x_n \in M_1$ such that

$\bigcup_{i=1}^{n} B(x_i , \frac{\delta_{x_i}}{2}) = M_1$.

Let $\delta = \min \{ \frac{\delta_{x_1}}{2} , \frac{\delta_{x_2}}{2} , \ldots , \frac{\delta_{x_n}}{2} \}$.

Now, we shall prove that $d_1(p , q) < \delta \Rightarrow d_2(f(p) , f(q)) < \varepsilon \ \forall \ p , q \in M_1$.

Let $p , q \in M_1$ such that $d_1(p , q) < \delta$

$P \in M_1 \Rightarrow P \in \bigcup_{i=1}^{n} B(x_i , \frac{\delta_{x_i}}{2})$

$\Rightarrow P \in B(x_i , \frac{\delta_{x_i}}{2})$ for some i such that $1 \leq i \leq n$

$\Rightarrow d_1(p , x_i) < \frac{\delta_{x_i}}{2} < \delta_{x_i}$

$\therefore$ by (1), $d_2(f(p) , f(x_i)) < \varepsilon/2$ ………………… (2)

Similarly, $d_2(f(q) , f(x_i)) < \varepsilon/2$ ………………… (3)

Now, $d_2(f(p) , f(q)) \leq d_2(f(p) , f(x_i)) + d_2(f(x_i) , f(q))$

$< \varepsilon/2 + \varepsilon/2$ [By (2) and (3)]

$\therefore d_2(f(p) , f(q)) < \varepsilon$.

Thus, $d_1(p , q) < \delta \Rightarrow d_2(f(p) , f(q)) < \varepsilon \ \forall \ p , q \in M_1$.

Hence f is uniformly continuous.

3.3 Equivalent forms of Compactness.

Definition 3.3.1 A collection **F** of subsets of a set M is said to have finite intersection property if the intersection of any finite number of elements of **F** is nonempty.

Theorem3.3.2 A metric space M is compact if and only if every collection of closed sets in M with finite intersection property has nonempty intersection.

Proof.

Suppose that M is compact.

Let $\{F_\alpha\}$ be a collection of closed subsets of M with finite intersection property.

We have to prove $\bigcap F_\alpha \neq \emptyset$.

Suppose $\bigcap F_\alpha = \emptyset$.

Then $(\bigcap F_\alpha)^c = M$.

$\therefore \bigcup F_\alpha{}^c = M$. [By De Morgan's laws]

Since each F_α is closed, each $F_\alpha{}^c$ is open.

Thus, $\{ F_\alpha{}^c \}$ is an open cover for M.

Since M is compact, this open cover has a finite subcover say

$\{F_{\alpha_1}{}^c, F_{\alpha_2}{}^c , \ldots\ldots , F_{\alpha_n}{}^c\}$.

$\therefore \bigcup_{i=1}^{n} F_{\alpha_i}{}^c = M$.

$\therefore(\bigcap_{i=1}^{n} F_{\alpha_i})^c = M$.

$\therefore \bigcap_{i=1}^{n} F_{\alpha_i} = \emptyset$.

This is a contradiction to the collection $\{F_\alpha\}$ has finite intersection property.

$\therefore \bigcap F_\alpha \neq \emptyset$.

Conversely, assume that every collection of closed sets in M with finite intersection property has nonempty intersection.

We have to prove M is compact.

Let $\{G_\alpha\}$ be an open cover for M.

$\therefore \bigcup G_\alpha = M$.

$\therefore (\bigcup G_\alpha)^c = \emptyset$.

$\therefore \bigcap G_\alpha{}^c = \emptyset$.

Since each G_α is open , each $G_\alpha{}^c$ is closed.

Hence **F= {** $G_\alpha{}^c$ } is a collection of closed sets whose intersection is empty.

$\therefore$ by hypothesis, this collection does not have finite intersection property.

Hence there exists a finite sub collection $\{G_{\alpha_1}{}^c, G_{\alpha_2}{}^c, \ldots, G_{\alpha_n}{}^c\}$ such that $\bigcap_{i=1}^{n} G_{\alpha_i}{}^c = \emptyset$.

$\therefore (\bigcup_{i=1}^{n} G_{\alpha_i})^c = \emptyset$.

$\therefore \bigcup_{i=1}^{n} G_{\alpha_i} = M$.

Thus the given open cover $\{G_\alpha\}$ of M has a finite subcover $\{ G_{\alpha_1}, G_{\alpha_2}, \ldots, G_{\alpha_n} \}$.

Hence M is compact.

Definition 3.3.3 A metric space M is said to be totally bounded if for every $\varepsilon > 0$, there exists a finite number of elements $x_1 , x_2 , \ldots , x_n \in M$ such that $B(x_1 , \varepsilon) \cup B(x_2 , \varepsilon) \cup \ldots B(x_n , \varepsilon) = M$.

A nonempty subset A of a metric space M is said to be totally bounded if the subspace A is totally bounded metric space.

Theorem 3.3.4 Any compact metric space is totally bounded.

Proof.

Let M be a compact metric space.

We have to prove M is totally bounded.

Let $\varepsilon > 0$ be given.

Now, $\{ B(x , \varepsilon) / x \in M \}$ is an open cover for M.

Since M is compact, there exists points $x_1 , x_2 , \ldots , x_n \in M$ such that

$M = B(x_1 , \varepsilon) \cup B(x_2 , \varepsilon) \cup \ldots \cup B(x_n , \varepsilon)$.

Hence M is totally bounded.

Theorem 3.3.5 Any totally bounded subset A of a metric space M is bounded.

Proof.

Let A be a totally bounded subset of a metric space M.

Then for given $\varepsilon > 0$, there exists points $x_1 , x_2 , \ldots , x_n \in A$ such that

$A = B_1(x_1, \varepsilon) \cup B_1(x_2, \varepsilon) \cup \ldots\ldots \cup B_1(x_n, \varepsilon)$ where $B_1(x_i, \varepsilon)$ are open balls in A.

Since open balls are bounded sets and finite union of bounded sets is bounded, A is bounded.

Note3.3.6 The converse of the above theorem is not true.

For,

Let M be an infinite set with discrete metric.

Then M is bounded.

Also, $B(x, 1) = \{ x \}$ for all $x \in M$.

Since M is infinite, M cannot be expressed as finite union of open balls of radius 1.

Hence M is not totally bounded.

Definition 3.3.7 Let (x_n) be a sequence in a metric space M. If $n_1 < n_2 < \ldots. < n_k < \ldots\ldots$ is a sequence of positive integers, then (x_{n_k}) is a subsequence of (x_n).

Theorem 3.3.8 A metric space M is totally bounded if and only if every sequence in M contains a Cauchy subsequence.

Proof.

Suppose that every sequence in M contains a Cauchy subsequence.

We have to prove M is totally bounded.

Let $\varepsilon > 0$ be given.

Choose $x_1 \in M$.

If $B(x_1, \varepsilon) = M$, then M is totally bounded.

If $B(x_1, \varepsilon) \neq M$, Then choose $x_2 \in B(x_1, \varepsilon) - M$ so that $d(x_1, x_2) \geq \varepsilon$.

If $B(x_1, \varepsilon) \cup B(x_2, \varepsilon) = M$, then M is totally bounded.

Otherwise, choose $x_3 \in [B(x_1, \varepsilon) \cup B(x_2, \varepsilon)] - M$ so that $d(x_3, x_1) \geq \varepsilon$ and $d(x_3, x_2) \geq \varepsilon$.

We proceed this process and if the process is terminated at a finite stage means M is totally bounded.

Suppose not, then we get a sequence (x_n) in M such that $d(x_n , x_m) \geq \varepsilon$ if $n \neq m$.

$\therefore$ (x_n) cannot be a Cauchy sequence, which is a contradiction.

Conversely, suppose that M is totally bounded.

Let $S_1 = \{ x_{11} , x_{12} , \ldots, x_{1n} , \ldots \}$ be a sequence in M.

If one of the terms in the sequence is repeated infinitely, then S_1 contains a constant subsequence which is in fact a Cauchy sequence.

So, we assume that no terms of S_1 is repeated infinitely so that the range of S_1 is infinite.

Since M is totally bounded, M can be covered by a finite number of open balls of radius $\frac{1}{2}$.

Hence one of these balls contains infinite number of terms of the sequence S_1.

$\therefore$ S_1 contains a subsequence $S_2 = \{ x_{21} , x_{22} , \ldots, x_{2n} , \ldots \}$ which lies within an open ball of radius $\frac{1}{2}$.

Similarly, S_2 contains a subsequence $S_3 = \{ x_{31} , x_{32} , \ldots, x_{3n} , \ldots \}$ which lies within an open ball of radius $\frac{1}{3}$.

We repeat the process of forming successive subsequences and finally we take the diagonal sequence $S = \{ x_{11} , x_{22} , \ldots, x_{nn} , \ldots \}$.

We claim that S is a Cauchy subsequence of S_1.

If $m > n$ then both x_{mm} and x_{nn} lie within an open ball of radius $\frac{1}{n}$.

$\therefore d(x_{mm} , x_{nn}) < \frac{2}{n}$.

$\therefore d(x_{mm} , x_{nn}) < \varepsilon \ \forall \ m , n \geq \frac{2}{\varepsilon}$.

Hence S is a Cauchy subsequence of S_1.

Thus every sequence in M has a convergent subsequence.

Corollary3.3.9 A nonempty subset of a totally bounded set is totally bounded.

Proof.

Let A be a totally bounded subset of a metric space M.

Let B be a nonempty subset of A.

Let (x_n) be a sequence in B.

Since $B \subseteq A$, (x_n) is a sequence in A.

Since A is totally bounded, (x_n) has a Cauchy subsequence.

Thus every sequence in B has a Cauchy subsequence.

$\therefore$ B is totally bounded.

3.4 Sequentially Compact.

Definition 3.4.1 A metric space M is said to be sequentially compact if every sequence in M has a convergent subsequence.

Theorem 3.4.2 Let (x_n) be a Cauchy sequence in a metric space M. If (x_n) has a subsequence (x_{n_k}) converges to x , then (x_n) converges to x.

Proof.

Suppose that (x_n) has a subsequence (x_{n_k}) which converges to x.

We have to prove $x_n \to x$.

Let $\varepsilon > 0$ be given.

Since (x_n) is a Cauchy sequence, there exists a positive integer N such that

$d(x_n , x_m) < \frac{\varepsilon}{2} \ \forall \ n , m \geq N_1$ ……………………… (1)

Since $x_{n_k} \to x$, there exists a positive integer N_2 such that

$d(x_{n_k} , x) < \frac{\varepsilon}{2} \ \forall \ n_k \geq N_2$ ……………………... (2)

Let $N = \max \{ N_1 , N_2 \}$. Fix $n_k \geq N$.

Now. $d(x_n , x) \leq d(x_n , x_{n_k}) + d(x_{n_k} , x)$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \forall \ n \geq N$

$\therefore d(x_n , x) < \varepsilon \ \forall \ n \geq N$.

$\therefore x_n \to x$.

Definition 3.4.3 A metric space M has Bolzano – Weierstrass property if every infinite subset of M has a limit point.

Theorem 3.4.4 In a metric space M the following are equivalent.

- (i) M is compact.
- (ii) M has Bolzano – Weierstrass property
- (iii) M is sequentially compact
- (iv) M is totally bounded and complete.

Proof.

(i) ⇒ (ii)

Let M be compact metric space.

Let A be an infinite subset of M.

Suppose that A has no limit point.

Let $x \in M$.

Since x is not a limit point if A, there exists an open ball $B(x , r_x)$ such that

$B(x , r_x) \cap (A – \{ x \}) = \emptyset$.

$B(x , r_x)$ contains at most one point of A (contains x if $x \in A$).

Now, $\{ B(x , r_x) / x \in M \}$ is an open cover for M.

Since M is compact, there exists points $x_1 , x_2 , \ldots , x_n \in M$ such that

$M = B(x_1 , r_{x_1}) \cup B(x_2 ,r_{x_2}) \cup \ldots \cup B(x_n , r_{x_n})$.

$\therefore A \subseteq B(x_1 , r_{x_1}) \cup B(x_2 ,r_{x_2}) \cup \ldots \cup B(x_n , r_{x_n})$.

Since each $B(x_1 , r_{x_i})$ has at most one point of A, A must be finite.

This is a contradiction to A is infinite.

Hence A has a limit point.

(ii) ⇒ (iii)

Suppose that M has Bolzano – Weierstrass property.

We have to prove M is sequentially compact.

Let (x_n) be a sequence in M.

If the range of (x_n) is finite , then a term of the sequence is repeated infinitely and hence (x_n) has a constant subsequence which is convergent.

Otherwise (x_n) has infinite number of distinct terms.

By hypothesis, this infinite set has a limit point say x.

$\therefore$ for any $r > 0$, the open ball $B(x , r)$ contains infinite number of terms of the sequence (x_n).

Choose a positive integer n_1 such that $x_{n_1} \in B(x , 1)$.

Now, choose $n_2 > n_1$ such that $x_{n_2} \in B(x , \frac{1}{2})$.

In general, for each positive integer k we choose $n_k > n_{k-1}$ such that $x_{n_k} \in B(x , \frac{1}{k})$.

Then (x_{n_k}) is a subsequence of (x_n) and $d(x_{n_k} , x) < \frac{1}{k} \forall k$.

$\therefore x_{n_k} \to x$.

Thus (x_{n_k}) is a convergent subsequence of (x_n).

Hence M is sequentially compact.

(iii) ⇒ (iv)

Suppose that M is sequentially compact.

Then every sequence in M has a convergent subsequence.

We have every Cauchy sequence is convergent.

Thus, every sequence in M has a Cauchy subsequence.

Hence M is totally bounded.

Now, we prove that M is complete.

Let (x_n) be a Cauchy sequence in M.

By hypothesis, (x_n) contains a convergent subsequence (x_{n_k}).

Let $x_{n_k} \to x$.

Then $x_n \to x$.

$\therefore$ M is complete.

(iv) ⇒ (i)

Suppose that M is totally bounded and complete.

We have to prove M is compact.

Suppose not.

Then there exists an open cover $\{G_\alpha\}$ for M which has no finite subcover.

Take $r_n = \frac{1}{2^n}$.

Since M is totally bounded, M can be covered by a finite number of open balls of radius r_1 .

Since M is not covered by a finite number of G_α's , at least one of these open balls say $B(x_1 , r_1)$ cannot be covered by finite number of G_α's .

Now, $B(x_1 , r_1)$ is totally bounded.

Hence as before we can find $x_2 \in B(x_1 , r_1)$ such that $B(x_2 , r_2)$ cannot be covered by finite number of G_α's .

Proceeding like this we get a sequence (x_n) in M such that $B(x_n , r_n)$ cannot be covered by finite number of G_α's and $x_{n+1} \in B(x_n , r_n)$.

Let m and n be positive integers with $n < m$.

Now, $d(x_n , x_m) \leq d(x_n , x_{n+1}) + d(x_{n+1} , x_{n+2}) + \ldots\ldots\ldots + d(x_{m-1} , x_m)$

$$< r_n + r_{n+1} + \ldots\ldots + r_{m-1}$$

$$< \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots\ldots\ldots\ldots + \frac{1}{2^{m-1}}$$

$$< \frac{1}{2^{n-1}} \left(\frac{1}{2^n} + \frac{1}{2^n} + \ldots\ldots \right)$$

$$< \frac{1}{2^{n-1}}$$

$\therefore$ (x_n) is a Cauchy sequence in M.

Since M is complete, there exists $x \in M$ such that $x_n \to x$.

Now, $x \in G_\alpha$ for some α.

Since G_α is open, there exists $\varepsilon > 0$ such that $B(x , \varepsilon) \subseteq G_\alpha$.

We have $x_n \to x$ and $r_n = \frac{1}{2^n} \to 0$.

$\therefore$ there exists a positive integer N such that

$d(x_n , x) < \frac{\varepsilon}{2}$ and $r_n < \frac{\varepsilon}{2}$ $\forall\, n \geq N$.

Fix $n \geq N$.

We claim that $B(x_n , r_n) \subseteq B(x , \varepsilon)$.

$y \in B(x_n , r_n) \Rightarrow d(x_n , y) < r_n < \frac{\varepsilon}{2}$

$\Rightarrow d(x_n , x) + d(x_n , y) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\Rightarrow d(x , y) < \varepsilon$

$\Rightarrow y \in B(x , \varepsilon)$.

$\therefore B(x_n , r_n) \subseteq B(x , \varepsilon) \subseteq G_\alpha$.

Thus, $B(x_n , r_n)$ is covered by a single G_α , which is a contradiction.

Hence M is compact.

UNIT IV

4.1 Complex number

Definition

A complex number z is of the form x+iy where x and y are real numbers and i is an imaginary unit with the property that i^2=1, x and y are called the real and imaginary part of z and we write x=Re z and y=Im z.

If x=0, the complex number z is called purely imaginary. If y=0 then z is real.

Two complex numbers are said to be equal iff they have the same real parts and the same imaginary parts.

Let C denote the set of all complex numbers.

Thus C is $\{x+iy/x, y\in R\}$

Definition

We define addition and multiplication in C as follows

Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$

$$z_1+z_2 =(x_1 + x_2) +i(y_1 + y_2)$$

$$z_1z_2 = (x_1x_2 - y_1 y_2) +i (x_1y_2 + x_2y_1)$$

Remark 1

If $z_1=x_1+iy_1$, and $z_2=x_2+iy_2 \neq 0$ then $\frac{z_1}{z_2} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + \frac{i\ y_1x_2-x_1y_2}{x_2^2+y_2^2}$

Remark 2

It is important to note that there is no order structure in the complex number system so that we cannot compare two complex numbers.

Remark 3

The complex number a+ib can also be represented by the ordered pair of real numbers (a, b).

4.1.2 Conjugation and modulus

Let z = x + iy be a complex number. Then the complex number x-iy is called the conjugate of z and it is denoted by $\overline{z}$.

The mapping f : $C\to C$ defined by f(z) = $\overline{z}$ is called the complex conjugation.

Note 1. z is real iff z = $\overline{z}$

2. $\overline{\overline{z}} = z$

3. $z + \overline{z} = 2 \text{ Re } z$ so that $x = \frac{z+\overline{z}}{2}$

4. $z - \overline{z} = 2i \text{ Im } z$ so that $y = \frac{z-\overline{z}}{2i}$

5. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

6. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

Theorem 4.1.2

If α is a root of the polynomial equation $f(z) = a_0z^n+a_1z^{n-1}+\ldots+a_{n-1}z+a_n=0$ where $a_0, a_1, \ldots, a_n \in R$ and $a_0 \neq 0$ then $\overline{\alpha}$ is also a root of $f(z)=0$

(ie.) The non-real roots of a polynomial equation with real co-efficients occur in conjugate pairs.

Proof

Since α is a root of $f(z)=0$, we have $f(\alpha)=0$

Hence $a_0\alpha^n+a_1\alpha^{n-1}+\ldots+a_{n-1}\alpha+a_n=0$

$\Rightarrow \overline{a_0\alpha^n + a_1\alpha^{n-1} + \cdots + a_{n-1}\alpha + a_n} = \overline{0}$

$\Rightarrow \overline{a}_0\, \overline{\alpha^n}+\overline{a}_1\, \overline{\alpha^{n-1}}+\ldots+\overline{a}_{n-1}\, \overline{\alpha}+\overline{a}_n=0$

$\Rightarrow a_0\overline{\alpha}^n+a_1\overline{\alpha}^{n-1}+\ldots+a_{n-1}\, \overline{\alpha}+\overline{a}_n=0$

$\Rightarrow a_0(\overline{\alpha})^n+a_1(\overline{\alpha})^{n-1}+\ldots+a_{n-1}(\overline{\alpha})+a_n=0$

$\Rightarrow f(\overline{\alpha})=0$ so that $\overline{\alpha}$ is also a root of $f(z)=0$.

Definition

Let $z = x+iy$ be a complex number. The modulus or absolute value of z denoted by $|z|$ is defined by $|z|=\sqrt{x^2 + y^2}$.

Remark

$|z|$ represents the distance between $z=(x, y)$ and the origin $O=(0, 0)$.

Theorem 4.1.3

i. $|z| \geq 0$ and $|z|=0$ iff $z=0$

ii. $z\overline{z} = |z|^2$

iii. $|z_1z_2| = |z_1|\, |z_2|$

iv. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ provided $z_2 \neq 0$

v. $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re} (z_1\, \overline{z}_2|$

vi. $|z_1-z_2|^2 = |z_1|^2 + |z_2|^2 - 2\text{Re}\,(z_1\,\bar{z}_2|$

vii. $|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

Solved Problems

Problem 1

Find the absolute value of $\frac{(1+3i)\,(1-2i)}{3+4i}$

Solution

$$\left|\frac{(1+3i)\,(1-2i)}{3+4i}\right| = \frac{|1+3i|\,|(1-2i)|}{3+4i}$$

$$= \frac{\sqrt{10}\sqrt{5}}{5}$$

$$= \frac{\sqrt{2x5}\sqrt{5}}{5}$$

$$= \frac{\sqrt{2}x5}{5} = \sqrt{2}$$

Problem 2

Find the condition under which the equation $az+b\bar{z}+c=0$ in one complex unknown has exactly one solution and compute that solution.

Solution

$$az+b\bar{z}+c=0 \qquad (1)$$

Taking conjugate we have,

$$\overline{az + b\bar{z} + c} = \bar{0}$$

$$\Rightarrow \bar{a}\bar{z}+\bar{b}z+\bar{c} = 0 \qquad (2)$$

$$(1)\ \ x\ \bar{a} \Rightarrow \bar{a}a\ z + \bar{a}b\ \bar{z} + \bar{a}c = 0 \qquad (3)$$

$$(2)\ \ x\ b \Rightarrow b\bar{b}\ z + b\bar{a}\ \bar{z} + b\bar{c} = 0 \qquad (4)$$

$$(3) - (4) \Rightarrow z(a\bar{a} - b\bar{b}) + \bar{a}c - b\bar{c} = 0$$

$$\Rightarrow z(|a|^2 - |b|^2) = b\bar{c} - \bar{a}c$$

Hence if $|a| \neq |b|$, the given equation has unique solution and the solution is given by $z = \frac{b\bar{c}-\bar{a}c}{|a|^2-|b|^2}$

Problem 3

If z_1 and z_2 are two complex numbers prove that $\left|\frac{z_1-z_2}{1-z_1\bar{z}_1}\right|=1$ if either $|z_1|=1$ or $|z_2|=1$. What exception must be made if $|z_1|=1$ and $|z_2|=1$.

Solution

Suppose $|z_1|=1$. Hence $|\bar{z}_1|=1$ and $z_1\,\bar{z}_1 = |z_1|^2 = 1$.

Now $\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right| = \left|\frac{z_1-z_2}{z_1\bar{z}_1-\bar{z}_{1-}z_2}\right|$

$= \left|\frac{z_1-z_2}{\bar{z}_1(z_1 z_2)}\right|$

$= \left|\frac{1}{\bar{z}_1}\right| = 1$

Similarly if $|\bar{z}_2|=1$, we have $\left|\frac{z_1-z_2}{1-\bar{z}_1 z_2}\right|=1$. If $|z_1|=1$ and $|z_2|=1$, then the result is true provided $1-\bar{z}_1 z_2 \neq 0$

ie. if $z_1-z_1\,\bar{z}_1 z_2 \neq 0$

ie. if $z_1 \neq |z_1|^2 z_2$

ie. if $z_1 \neq z_2$

Inequalities

Theorem 4.1.4

For any three complex numbers z_1, z_2 and z_3.

i. $-|z| \leq \text{Re } z \leq |z|$

ii. $-|z| \leq \text{Im } z \leq |z|$

iii. $|z_1+z_2| \leq |z_1|+|z_2|$. (Triangle inequality)

iv. $|z_1-z_2| \geq ||z_1|-|z_2||$

Proof

Let z = +iy

Hence $|z| = \sqrt{x^2+y^2}$

Now $-\sqrt{x^2+y^2} \leq x \leq \sqrt{x^2+y^2}$

and $-\sqrt{x^2+y^2} \leq y \leq \sqrt{x^2+y^2}$

$\therefore -|z| \leq \text{Re } z \leq |z|$ and $-|z| \leq \text{Im } z \leq |z|$

Hence (i) and (ii) are proved.

iii) Triangle inequality

$|z_1 + z_2| \leq |z_1| + |z_2|$

$|z_1 + z_2|^2 = (z_1+z_2)\,(\overline{z_1+z_2})$ $\qquad \because |z|^2 = z\bar{z}$

$= (z_1+z_2)\,(\overline{z_1+z_2})$

$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

$= |z_1|^2 + (z_1\bar{z}_2+\bar{z}_1 z_2)+|z_2|^2$

$= |z_1|^2 + (z_1\bar{z}_2+ \overline{z_1\bar{z}_2})+|z_2|^2$

$= |z_1|^2 + 2\ \mathrm{Re}(z_1\bar{z}_2) +|z_2|^2$

$\leq |z_1|^2 + 2\ |(z_1\bar{z}_2)|+|z_2|^2$

$= |z_1|^2 + 2\ |z_1|\ |\bar{z}_2|+|z_2|^2$

$= |z_1|^2 + 2\ |z_1|\ |z_2|+|z_2|^2$ $[\because |z_2| = |\bar{z}_2|]$

$= (|z_1| + |z_2|)^2$

Thus $|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2$

$\therefore |z_1 + z_2| \leq |z_1| +|z_2|$

iv) $|z_1 - z_2| \geq ||z_1| -|z_2||$

$z_1 = (z_1-z_2) + z_2$

$|z_1| = |(z_1-z_2) + z_2| \leq |z_1 - z_2| + |z_2|$

$\Longrightarrow |z_1| - |z_2| \leq |z_1 - z_2|$ (1)

$z_2 = (z_2 - z_1) + z_1$

$|z_2| = |(z_2-z_1) + z_1 | \leq |z_2 - z_1| + |z_1|$

$\Longrightarrow |z_2|-|z_1| \leq |z_2-z_1|$

$\Longrightarrow -(|z_1|-|z_2|) \leq |z_2-z_1|$

$\Longrightarrow |z_1|-|z_2| - |z_2-z_1|$ (2)

From (1) and (2)

$- |z_2-z_1| \leq |z_1|-|z_2| \leq |z_1-z_2|$

ie. $- |z_1-z_2| \leq |z_1|-|z_2| \leq |z_1-z_2|$

$\Longrightarrow -||z_1|-|z_2|| \leq |z_1-z_2|$

ie. $- |z_1-z_2| \geq ||z_1|-|z_2||$

Note

For any complex numbers $z_1, z_2, \ldots, z_n$ we have $|z_1+z_2+\ldots+z_n| \leq |z_1|+|z_2|$ $+\ldots+|z_n|$

Polar form of a complex number

Consider any non zero complex number $z=x+iy$.

Let (r, θ) denote the polar co-ordinates of the point (x, y)

Hence $x = r \cos \theta$ and $y = r \sin \theta$

$\therefore z = r (\cos \theta + \sin \theta)$

We notice that r = |z|=$\sqrt{x^2 + y^2}$ which is the magnitude of the complex number and θ is called the amplitude or argument of z and is denoted by arg z or amp z.

We note that the value of arg z not unique. If θ = arg z then θ + 2n π where n is any integer is also a value of arg z. The value of arg z lying in the range (-π, π) is called the principal value of arg z.

Theorem 4.1.5

If z_1 and z_2 are any two non zero complex numbers then

i. $-\arg z_1 = \arg \bar{z}_1$

ii. $\arg z_1 z_2 = \arg z_1 + \arg z_2$

ii. $\arg [\frac{z_1}{z_2}] = \arg z_1 - \arg z_2$

Proof

Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$

$\therefore \bar{z}_1 = r_1(\cos \theta_1 - i \sin \theta_1)$

$= r_1(\cos (-\theta_1) + i \sin (-\theta_1))$

Hence $\arg \bar{z}_1 = -\theta_1$

$= -\arg z_1.$

$\therefore \arg \bar{z}_1 = -\arg z_1$

ii) Let $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and

$z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$

$\Longrightarrow \arg z_1 = \theta_1$ and $\arg z_2 = \theta_2$

Now $z_1 z_2 = r_1 r_2(\cos \theta, + i \sin \theta) \quad (\cos \theta_2 + i \sin \theta_2) (\cos \theta_1 + i \sin \theta_1)$

$= r_1 r_2[(\cos (\theta_1+\theta_2)+ \sin\theta_1 \sin\theta_2) +i (\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)]$

$= r_1 r_2[(\cos\theta_1 \theta_2) + i \sin(\theta_1 + \theta_2)]$

$\therefore \arg z_1 z_2 = \theta_1 + \theta_2$

$= \arg z_1 + \arg z_2$

$\therefore \arg z_1 z_2 = \arg z_1 + \arg z_2$

iii) $\arg (\frac{z_1}{z_2}) = \arg z_1 - \arg z_2$

$\frac{z_1}{z_2} = \frac{r_1(\cos \theta_1 + i \sin \theta_1)}{r_2(\cos \theta_2 + i \sin \theta_2)}$

$$= \left[\frac{r_1}{r_2}\right]\left[\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + \sin\theta_2)} \text{ x } \frac{\cos\theta_2 + i\sin\theta_2}{\cos\theta_2 + i\sin\theta_2}\right]$$

$$= \left(\frac{r_1}{r_2}\right)\left[\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{(\cos^2\theta_2 + \sin^2\theta_2)}\right]$$

$$= \left(\frac{r_1}{r_2}\right)\left(\frac{(\cos\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)}{1}\right)$$

$$= \left(\frac{r_1}{r_2}\right)[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)$$

$$\arg\left[\frac{z_1}{z_2}\right] = \theta_1 - \theta_2$$

$$= \arg z_1 - \arg z_2$$

$$\therefore \arg\left[\frac{z_1}{z_2}\right] = \arg z_1 - \arg z_2$$

Theorem 4.1.6

Let $z=r(\cos\theta + i\sin\theta)$ be any non zero complex number and n be any integer. Then $z^n = r^n(\cos n\theta + i\sin n\theta)$.

Proof

We first prove this result for positive integers by induction on n.

When n=1

$z^1 = r^1(\cos\theta + i\sin\theta)$

ie. $z = r(\cos\theta + i\sin\theta)$ which is true.

Hence the theorem is true when n=1.

Suppose the result is true for n=m.

Hence $z^m = r^m(\cos m\theta + i\sin m\theta)$

To prove the result is true when n=m+1

Now $z^{m+1} = z^m z$

$= r^m(\cos m\theta + i\sin m\theta)\, r(\cos\theta + i\sin\theta)$

$= r^{m+1}[(\cos m\theta\cos\theta - \sin\theta\sin m\theta) + i(\cos m\theta\sin\theta + \sin m\theta\cos\theta)]$

$= r^{m+1}[\cos(m+1)\theta + i\sin(m+1)\theta]$

Hence the result is true for n=m+1

Hence $z^n = r^n[\cos n\theta + i\sin n\theta]$ for all positive integers n.

The result is obviously true if n=0

Now $z^{-1} = \frac{1}{z}$

$$= \frac{1}{r(\cos\theta + i\sin\theta)}$$

$$= \frac{1}{r} \times \frac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$$

$$= r^{-1}\left[\frac{\cos(-\theta) + i\sin(-\theta)}{\cos^2\theta + \sin^2\theta}\right]$$

$$= r^{-1}[\cos(-\theta) + i\sin(-\theta)]$$

∴ The result is true for n=-1. Hence it follows that the result is true for all negative integers.

Hence $z^n = r^n(\cos n\theta + i\sin n\theta)$ for all $n \in Z$.

Corollary: (De-Movire's theorem)

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

Solved Problem

Problem 1

For any three distinct complex numbers z, a, b the principal value of arg $\left[\frac{z-a}{z-b}\right]$ represents the angle between the line segment joining z and a and the line segment joining z and b taken in the appropriate sense.

Solution

Let A, B, P be the points in the complex plane representing the complex numbers a, b, z respectively.

Then $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$

$= z - a$

$\overrightarrow{BP} = \overrightarrow{OP} - \overrightarrow{OB}$

$= z - b$

∴ The complex numbers z-a, z-b are represented by the vectors $\overrightarrow{AP}$ and $\overrightarrow{BP}$ respectively.

Hence the principal value of arg $\left[\frac{z-a}{z-b}\right]$ gives the angle between the line segment AP and BP taken in the appropriate sense.

4.2 Circles and Straight lines

Equation of circles and straight lines in the complex plane can be expressed interms of z and $\bar{z}$.

General equation of circles

Equation of the circle with centre a and radius r is given by $|z-a|=r$

ie. $|z-a|^2 = r^2$

$\Rightarrow (z-a)(\overline{z-a}) = r^2$ $[\because |z|^2 = z\bar{z}]$

$\Rightarrow (z-a)(\bar{z} - \bar{a}) = r^2$

$\Rightarrow z\bar{z} - a\bar{z} - \bar{a}z + a\bar{a} - r^2 = 0$

This equation is of the form

$z\bar{z} + \bar{\alpha} z + \alpha \bar{z} + \beta = 0$ where β is a real number. Further any equation of the above form can be written as $|z+\alpha|^2 = \alpha\bar{\alpha} - \beta$ and hence represents a circle provided $\alpha\bar{\alpha} - \beta > 0$. It represents a circle with centre $-\alpha$ and radius $\sqrt{\alpha\bar{\alpha} - \beta}$.

Thus the general equation of a circle is given by $z\bar{z} + \bar{\alpha} z + \alpha \bar{z} + \beta = 0$ where β is real and $\alpha\bar{\alpha} - \beta > 0$.

General equation of straight lines

To find the general equation of the straight line passing through a and b, we note that $\arg\left[\frac{z-a}{z-b}\right]$ represents the angle between the lines joining a to z and b to z where z is any point on the line joining a and b.

$\therefore$ If z, a, b are collinear then $\arg\left[\frac{z-a}{z-b}\right] = 0$ or π

$\therefore \frac{z-a}{z-b}$ is real. Hence $\frac{z-a}{z-b} = \left[\overline{\frac{z-a}{z-b}}\right]$

$\therefore \frac{z-a}{z-b} = \left[\frac{\bar{z} - \bar{a}}{\bar{z} - \bar{b}}\right]$

$\Rightarrow (z-a)(\bar{z} - \bar{b}) = (z-b)(\bar{z} - \bar{a})$

$\Rightarrow z\bar{z} - a\bar{z} - \bar{b}z + a\bar{b} = z\bar{z} - \bar{a}z - b\bar{z} + \bar{a}b$

$\Rightarrow \bar{a}z - \bar{b}z - a\bar{z} + b\bar{z} + a\bar{b} - \bar{a}b = 0$

$\Rightarrow (\bar{a} - \bar{b})z - (a - b)\bar{z} + (a\bar{b} - \overline{a\bar{b}}) = 0$

$\Rightarrow (\bar{a} - \bar{b})z - (a - b)\bar{z} + 2i\,\mathrm{Im}(a\bar{b}) = 0$

$\left[\because \mathrm{Im}\, z = \frac{z-\bar{z}}{2i}\right] \Rightarrow z - \bar{z} = 2i\,\mathrm{Im}\, z]$

$\therefore i(\bar{a} - \bar{b})z - i(a-b)\bar{z} - 2\mathrm{Im}(a\bar{b}) = 0$. This equation is of the form $\bar{\alpha} z + \alpha \bar{z} + \beta = 0$ where $\alpha \neq 0$ and β is real.

Further any equation of the above form represents a straight line. This can be easily seen by changing the above equation into Cartesian form.

∴ The general equation of a straight line is given by $\overline{\alpha} z + \alpha \overline{z} + \beta = 0$. Where $\alpha \neq 0$ and β is real.

Theorem 4.2.1

Equation of the line joining a and b is $(\overline{a} - \overline{b}) z + (b-a) \overline{z} + (a \overline{b} - \overline{a} b) = 0$

Theorem 4.2.2

If a and b are two distinct complex numbers where $b \neq 0$, then the equation $z = a + t b$ where t is a real parameter represents a straight line passing through the point a and parallel to b.

Proof

Let z be any point on the line passing through a and parallel to b. The vectors represented by $z - a$ and b are parallel.

Hence $z - a = tb$ for some real number t. Hence $z = a + t b$, which is the equation of the required straight line.

Definition

Two points P and Q are called reflection points for a given straight line ℓ iff ℓ is the perpendicular bisector of the segment PQ.

Theorem 4.2.3

Two points z_1 and z_2 are reflection points for the line $\overline{\alpha} z + \alpha \overline{z} + \beta = 0$ iff $\overline{\alpha} z_1 + \alpha \overline{z}_2 + \beta = 0$.

Proof

Let z_1 and z_2 be reflection points for the straight line $\overline{\alpha} z + \alpha \overline{z} + \beta = 0$ (1)

To prove that $\overline{\alpha} z_1 + \alpha \overline{z}_2 + \beta = 0$

For any point z on the line we have

$|z - z_1| = |z - z_2|$ [∵ z_1, z_2 are reflection points]

$\Rightarrow |z - z_1|^2 = |z - z_2|^2$

$\Rightarrow (z - z_1) . (\overline{z - z_1}) = (z - z_2) (\overline{z - z_2})$ [∵ $|z_1|^2 = z \overline{z}$]

$\Rightarrow (z - z_1) (\overline{z} - \overline{z}_1) = (z - z_2) (\overline{z} - \overline{z}_2)$

$\Rightarrow z \overline{z} - z \overline{z}_1 - z_1 \overline{z} + z_1 \overline{z}_1 = z \overline{z} - z \overline{z}_2 - z_2 \overline{z} + z_2 \overline{z}_2$

$\Rightarrow z \overline{z}_2 - z \overline{z}_1 + z_2 \overline{z} - z_1 \overline{z} + z_1 \overline{z}_1 - z_2 \overline{z}_2 = 0$

$\Rightarrow z (\overline{z}_2 - \overline{z}_1) + \overline{z} (z_2 - z_1) + z_1 \overline{z}_1 - z_2 \overline{z}_2 = 0$ (2)

Since the equation is true for any point z on the given line it may be regarded as the equation of the given line.

∴ From (1) and (2) we get

$$\frac{\overline{\alpha}}{\overline{z}_2-\overline{z}_1} = \frac{\alpha}{z_2-z_1} = \frac{\beta}{z_1\overline{z}_1 - z_2\overline{z}_2} = k \text{ (say)}$$

∴ $\alpha = k(z_2 - z_1)$; $\overline{\alpha} = k(\overline{z}_2 - \overline{z}_1)$ and $\beta = k(z_1\overline{z}_1 - z_2\overline{z}_2)$

∴ $\overline{\alpha}z_1 + \alpha\overline{z}_2 + \beta = k[z_1(\overline{z}_2 - \overline{z}_1) + \overline{z}_2(z_2 - z_1 + z_1\overline{z}_1 - z_2\overline{z}_2]$

$= 0$

∴ $\overline{\alpha}z_1 + \overline{\alpha}z_2 + \beta = 0$

Conversely, suppose $\overline{\alpha}z_1 + \alpha\overline{z}_2 + \beta = 0$ (3)

Subtracting (3) from (1) we get

$\overline{\alpha}(z - z_1) + -\alpha(\overline{z} - \overline{z}_2) = 0$

$\Longrightarrow \overline{\alpha}(z - z_1) = -\alpha(\overline{z} - \overline{z}_2)$

Taking modulus on both sides

$\Longrightarrow |\overline{\alpha}||z - z_1| = |\alpha||\overline{z} - \overline{z}_2|$

$\Longrightarrow |z - z_1| = |\overline{z} - \overline{z}_2| = |\overline{z - z_2}|$ $\lceil \because |\alpha| = |\overline{\alpha}| \rceil$

$\Longrightarrow |z - z_1| = |z - z_2|$

∴ $z_1$and z_2 are reflection points for the line $\overline{\alpha}z + \alpha\overline{z} + \beta = 0$

Definition

Two points P and Q are said to be inverse points with respect to a circle with centre 0 and radius r if Q lies on the ray OP and OP. OQ = r^2.

Theorem 4.2.4

z_1 and z_2 are inverse points with respect to a circle $z\overline{z} + \overline{\alpha}z + \alpha\overline{z} + \beta = 0$

iff $z_1.\overline{z}_2 + \overline{\alpha}z_1 + \overline{\alpha}z_2 + \beta = 0$

Proof

Suppose z_1 and z_2 are inverse points with respect to the

circle $z\overline{z} + \overline{\alpha}z + \alpha\overline{z} + \beta = 0$ (1)

(1) can be rewritten as

$|z + \alpha|^2 = \alpha\overline{\alpha} - \beta$

∴ The centre of the circle is $-\alpha$ and radius is $\sqrt{\alpha\overline{\alpha} - \beta}$

Since

z_1 and z_2 are inverse points w.r. to (1)

we have, $\arg(z_1+\alpha) = \arg(z_2+\alpha)$ (2)

and $|z_1+\alpha|\,|\overline{z_2+\alpha}| = \alpha\bar{\alpha} - \beta$ (3)

$\therefore \arg(z_1+\alpha)\,\overline{z_2+\alpha} = \arg(z_1+\alpha) + \arg\overline{(z_2+\alpha)}$

$= \arg(z_1+\alpha) - \arg\overline{(z_2+\alpha)}$

$= 0$ [$\because$ by (2)]

$\therefore (z_1+\alpha)\,\overline{(z_2+\alpha)}$ is a +ve real number.

Hence using (3) we get $(z_1+\alpha)\,\overline{z_2+\alpha} = \alpha\bar{\alpha} - \beta$

$\Longrightarrow (z_1+\alpha)(\bar{z}_2 + \bar{\alpha}) = \alpha\bar{\alpha} - \beta$

$\therefore z_1\bar{z}_2 + \bar{\alpha} z_1 + \alpha\bar{z}_2 + \beta = 0$

Converse can be similarly proved.

Note 1:

Let z_1, z_2, z_3 and z_4 be four distinct points which are either con-cyclic or collinear. Then $\arg\left[\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}\right]$ is either 0 or π depending on the relative positions of the points.

Hence $\frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$ is purely real.

Note 2 :

The equation $pz\bar{z} + \bar{\alpha}z + \alpha\bar{z} + \beta = 0$ (1)

Where p and β are real and $\alpha\bar{\alpha} - p\beta \geq 0$ can be taken as the joint equation of the family of circles and straight lines. When $p \neq 0$, it represents a circle. When $p=0$, it represents a straight line. Further z_1 and z_2 are inverse points or reflection points w.r.to (1) iff $pz_1\bar{z}_2 + \bar{\alpha} z_1 + \alpha\bar{z}_2 + \beta = 0$

Solved problems

Problem 1 : Prove that the equation $\left|\frac{z-z_1}{z-z_2}\right| = \lambda$ where λ is a non negative parameter represents a family of circles such that z_1 and z_2 are inverse points for every member of the family.

Solution:

Given, $\left|\frac{z-z_1}{z-z_2}\right| = \lambda \Longrightarrow \left|\frac{z-z_1}{z-z_2}\right|^2 = \lambda^2$

$\Longrightarrow \left[\frac{z-z_1}{z-z_2}\right]\left[\frac{\overline{z-z_1}}{z-z_2}\right] = \lambda^2$

$\Rightarrow [\frac{z-z_1}{z-z_2}] \frac{\bar{z}-\bar{z}_1}{\bar{z}-\bar{z}_2} = \lambda^2$

$\Rightarrow (z - z_1)(\bar{z} - \bar{z}_1) = \lambda^2 (z - z_2)(\bar{z} - \bar{z}_2)$

$\Rightarrow z\bar{z} - \bar{z}_1 z - z\bar{z} + z_1\bar{z}_1 = \lambda^2(z\bar{z} - \bar{z}_2 z - z_2\bar{z} - + z_2\bar{z}_2)$

$\Rightarrow (1-\lambda^2) z\bar{z} + (\lambda^2\bar{z}_2 - \bar{z}_1) z + (z_2\lambda^2 - z_1)\ (z_1\bar{z}_1 - \lambda^2 z_2\bar{z}_2) = 0$ (1)

(1) represents a circle when $\lambda \neq 1$

Using Note 2, it can be verified that z_1 and z_2 are inverse points w.r.to (1). When $\lambda=1$, the given equation represents a straight line which is the perpendicular bisector of the line segment joining z_1 and z_2. Clearly z_1 and z_2 are reflection points for this line.

Problem 2

Prove that arg $[\frac{z-a}{z-b}] = \mu$ where μ is a real parameter, represents a family of circles every member of which passes through a and b.

Solution

For any fixed value μ, arg $[\frac{z-a}{z-b}] = \mu$ is the locus of a point z such that the angle between the lines joining a to z and b to z is μ.

Clearly this locus is the arc of a circle passing through a and b the remaining part of the circle is represented by the equation arg $[\frac{z-a}{z-b}] = \mu+\pi$. Hence the result follows.

Exercise

1. Show that the inverse point of any point α with respect to the unit circle $|z|=1$ is $\frac{1}{\bar{\alpha}}$.
2. Find the inverse point of $-i$ with respect to the circle $2z\bar{z}+(i-1)z-(i+1)\bar{z} = 0$.

4.3 Regions in the complex plane.

Definition

Let z_0 be any complex number. Let ε be a +ve real number. Then the set of all points z satisfying $|z-z_0| < \varepsilon$ is called a neighbourhood of z_0 and is represented by $N_\varepsilon(z_0)$ or $S(z_0, \varepsilon)$. Thus $N_\varepsilon(z_0) = \{z/1z-z_0| < \varepsilon\}$.

Note 1: $|z - z_0| < \varepsilon$ represents the interior of the circle with centre z_0 and radius ε.

Note 2: $|z - z_0| \leq \varepsilon$ represents the set of points on and inside the circle with centre z_0 and radius ε and is called the closed circular disc with centre z_0 and radius ε.

Definition

Let $S \subseteq C$. Let $z_0 \in S$. Then z_0 is said to be an interior point of S if there exists a neighbourhood $N_\varepsilon(z_0)$ such that $N_\varepsilon(z_0) \subseteq S$.

S is called an open set if every point of S is an interior point of S.

Definition

Let $S \subseteq C$. Let $z_0 \in S$. Then z_0 is called a limit point of S if every neighbourhood of z_o contains infinitely many points of S.

S is called a closed set if it contains all its limit points.

Remark

A set S is closed iff its complement C-S is open.

Definition

Let $S \subseteq C$. Let $z_0 \in C$. Then z_0 is called a boundary point of S if z_o is a limit point of both S and C-S. Thus z_o is a boundary point of S iff every neighbourhood of z_o contains infinitely many points of S and infinitely many points of C-S.

Definition

Let $S \subseteq C$. Then S is called a bounded set if there exist a real number k such that $|z| \leq k$ for all $z \in S$.

Definition

Let $S \subseteq C$ then S is called a connected set if every pair of points in S can be joined by a polygon which lies in S.

Definition

A non empty open connected subset of C is called a region in C.

Example

a) Let $D = \{z / \text{Re}z > 1\}$

Let $z = x+iy$. Then $D = \{z / x > 1\}$

$\therefore$ D is nonempty, open and connected.

$\therefore$ D is a region in C.

Here D is the half plane as shown in the figure.

Example

Let $D = \{z / |z-2+i| \leq 1\}$

i.e. D is the set of all complex number satisfying $|z-(2-i)| \leq 1$. Clearly D represents the closed disc with centre 2-i and radius 1. Also D is a connected and bounded set. But the points which lie on the circle $|z-(2-i)| = 1$ are not interior points of D. Hence D is not open. Hence D is not a region.

Example

Let $D = \{z / \text{Im } z/ > 1\}$

Let $D = x + iy$

$D = \{z/|y| > 1\}$

$= \{z/y > 1 \text{ or } y < -1\}$

$= \{z/y > 1\} \cup \{z/y < -1\}$

Clearly D is the union of two half planes and it is unbounded as shown in the figure.

Obviously if z_1 is any complex number with Im $z_1 > 1$ and z_2 is any complex number with $\text{Im}z_2 < -1$ then z_1 and z_2 cannot be joined by a polygon entirely lying in D. Hence D is not connected. Hence D is not a region.

Example

$D = \{z/0 < \arg z < \pi/4\}$ is a region in C.

Example

Let $D = \{z/0 < \arg z < \frac{\pi}{4} \text{ and } |z| > 1\}$ D is as shown in the figure. Clearly D is an unbounded region in C.

Example

Let $D = \{z/1 < |z| < 2\}$ D is the region bounded by the circles $|z|=1$ and $|z|=2$. Such a region is called an annulus or annular region.

Exercise

1. For each of the following subsets of C sketch the set and determine whether it is a region.

   a) $\text{Im}z > 1$ b) $|z| > 0, 0 \leq \arg z \leq \frac{\pi}{4}$

   c) $|2z+3| > 4$ d) $|z-4| \geq |z|$

   e) $0 < |z-z_0| < \delta$ where z_0 is a fixed point and δ is a +ve number.

2. If the points z_1, z_2, z_3 are the vertices of an equilateral triangle prove that $z_1^2+z_2^2+z_3^2 = z_1z_2+ z_2z_3+ z_3z_1$
3. If z is a variable point and Re $\left[\frac{z-4}{z-2_i} \right] = 0$ prove that the locus of z is a circle.

4.4 Analytic functions

Definition

A function f defined in a region D of the complex plane is said to be analytic at a point $a \in D$ if f is differentiable at every point of some neighbourhood of a.

Thus f is analytic at a if there exist $\varepsilon>0$ such that f is differentiable at every point of the disc $s(a, \varepsilon)=\{z/|z-a|< \varepsilon\}$.

If f is analytic at every point of a region D then f is said to be analytic in D.

Definition

A function which is analytic at every point of the complex plane is called an entire function or integral function. For example any polynomial is an entire function.

Remark

If f is analytic at a point a then f is differentiable at a. However the converse is not true.

For example, $f(z)=|z|^2$ is differentiable only at z=0.

Hence f is differentiable at z=0 but not analytic at z=0.

Remark

f(z) is analytic in a region D if and only if the real and imaginary parts of f(z) have continuous first order partial derivatives that satisfy the Cauchy-Riemann equation $u_x=v_y$ and $u_y= -v_x$ for all points in D.

Further it follows that if f(z) is analytic in D then u and v have continuous partial derivatives of all orders.

Theorem 4.4.1

An analytic function in a region D with its derivative zero at every point of the domain is constant.

Proof

Let f(z) = u(x, y) + iv(x, y) be analytic is D in $f^1(z) = 0$ for all $z \in D$.

Since $f^1(z) = u_x+iv_x = v_y-iu_y$

We have $u_x = u_y = v_x = v_y = 0$

∴ u(x, y) and v (x, y) are constant functions and hence f(z) is constant.

Remark

The above theorem is not true if the domain of f(z) is not a region.

For example let D = {z/|z| < 1} ∪ {z/|z| > 2}.

D is not a connected subset of C so that D is not region.

Let f: D-C be define by $f(z) = \{\begin{smallmatrix} 1 \text{ if } |z|<1 \\ 2 \text{ if } |z|>2 \end{smallmatrix}$ clearly $f^1(z) = 0$ for all points z∈D and f is not a constant function in D.

Solved Problems

Problem 1

An analytic function in a region with constant modulus is constant.

Solution

Let f(z) = u(x, y) +iv (x,y) be analytic in a domain D.

Given |f(z)| is constant.

∴ $u^2 + v^2 = c$ where c is a constant (1)

[∵ z = u + iv

$\Rightarrow |z| = \sqrt{u^2 + v^2}$]

Differentiating equation (1)

Partially w.r. to x.

$2u\, u_x + 2v\, v_x = 0$

$\Rightarrow uu_x + vv_x = 0$ (2)

Differentiating equation (1) partially w.r. to y.

$2u\, u_y + 2vv_y = 0$

$\Rightarrow uu_y + vv_y = 0$ (3)

Using C.R. equation $u_x = v_y$ and $u_y = -v_x$ in (2) and (3) we get,

$uu_x – vu_y = 0$ (4)

$uu_y + vu_x = 0$ (5)

(4) x u $\Rightarrow u^2 u_x - uv\, u_y = 0$

(5) x v $\Rightarrow uv\, u_y + v^2 u_x = 0$

Adding $(u^2 + v^2)\, u_x = 0$

$\Rightarrow u_x = 0 \quad [\therefore u^2 + v^2 = \text{constant}]$

Similarly we can prove that $v_x = 0$

$\therefore f^1(z) = u_x + iv_x$

$= 0$

i.e. $f^1(z) = 0$

Hence f is constant.

Problem 2

Any analytic function f(z) = u+iv with arg f(z) = constant is itself a constant.

Solution

Given arg f(z) = constant

$\Rightarrow \tan^{-1}\left(\frac{v}{u}\right) = c$; where c is a constant.

$\Rightarrow \frac{v}{u} = k$ where k is a constant.

$\therefore v = ku$

Differentiating partially w.r.to x and w.r. to y

$v_x = k\, u_x \qquad (1)$

$v_y = k\, u_y \qquad (2)$

(1) $\Rightarrow \quad k = \frac{v_x}{u_x}$

(2) $\Rightarrow \quad v_y = k\, u_y$

i.e. $v_y = \frac{v_x}{u_x} \cdot u_y$

$\Rightarrow u_x v_y = v_x u_y$

$\Rightarrow u_x u_x - v_x u_y = 0$

$\Rightarrow u_x u_x - u_y(-u_y) =$ (using C.R equations $u_x = v_y$ and $v_x = -u_y$)

$\Rightarrow u_x^2 + u_y^2 = 0$

$\Rightarrow u_x = 0$ and $u_y = 0$

Hence u is constant.

Similarly we can prove that v is constant.

∴ f = u + iv is constant.

Problem 3

If f(z) and $\overline{f(z)}$ are analytic in a region D, show that f(z) is constant in that region.

Solution

Let f(z) = u(x, y) + iv (x, y)

∴ $\overline{f(z)}$ = u(x, y) – iv(x, y)

= u(x, y) + i(-v(x, y))

Since f(z) is analytic in D, C.R. equation are satisfied.

∴ We have $u_x = v_y$ and $u_y = -v_x$.

Since $\overline{f(z)}$ is analytic in D, C.R. equations are satisfied.

∴ We have $u_x = -v_y$ and $u_y = v_x$

Adding we get, $2u_x = 0$ and $2u_y = 0$

$\Longrightarrow u_x = 0$ and $u_y = 0$

Hence $u_x = 0 = v_x$

∴ $f^1(z) = u_x + iv_x = 0$

∴ f(z) is constant in D.

Problem 4

Prove that the functions f(z) and $f(\bar{z})$ are simultaneously analytic.

Solution

Suppose f(z) = u(x, y) + iv(x, y) is analytic in a region D.

Then the first order partial derivatives of u and v are continuous and satisfy the C.R. equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad (1)$$

$$\frac{\partial v}{\partial x} = \frac{-\partial u}{\partial y} \qquad (2)$$

Now $f(\bar{z})$ = u(x, -y) + iv (x, -y)

∵ z = x+iy $\Longrightarrow \bar{z}$ = x – iy

$\overline{f(\bar{z})}$ = u(x, -y) - iv (x, -y)

= $u_1(x,y) + iv_1(x, y)$ where $u_1(x,y) = u(x, -y)$

and $v_1(x, y) = -v(x, -y)$

Hence $\frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v_1}{\partial x}$ (using(1))

and $\frac{\partial u_1}{\partial y} = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v_1}{\partial x}$

∴ The first order partial derivatives of u_1 and v_1 are continuous and satisfy the Cauchy-Riemann equations in D.

∴ $\overline{f(\bar{z})}$ is analytic in D.

Similarly if $\overline{f(\bar{z})}$ is analytic in D then f(z) in also analytic in D.

Problem 5

If $\frac{\partial^2}{\partial x\,\partial y} = \frac{\partial^2}{\partial y\,\partial x}$, prove that

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\,\frac{\partial^2}{\partial x\,\partial \bar{z}}$$

Solutions

Let z = x+ iy

∴ $x = \frac{1}{2}(z+\bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$

Hence $\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$

$$= \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)$$

∴ $\frac{\partial^2}{\partial z\,\partial \bar{z}} = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \bar{z}}\right)\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\cdot\left(\frac{\partial}{\partial \bar{z}}\right)\frac{\partial y}{\partial z}$$

$$= \frac{\partial}{\partial x}\left[\frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)\right]\frac{1}{2} + \frac{\partial}{\partial y}\left[\frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right)\right] x\ \frac{1}{2i}$$

$$= \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + i\,\frac{\partial^2}{\partial x\,\partial y}\right] + \frac{1}{4}\left[\frac{\partial^2}{\partial y\,\partial x}\ x\ \frac{1}{i} + \frac{\partial^2}{\partial y^2}\right]$$

$$= \frac{1}{4}\left[\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] + i\,\frac{\partial^2}{\partial x\,\partial y} + \frac{1}{i}\,\frac{\partial^2}{\partial x\,\partial y}\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x\,\partial y}\left(i + \frac{1}{i}\right)\right]$$

$$= \frac{1}{4}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]$$

$$\Longrightarrow \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\,\frac{\partial^2}{\partial z\,\partial \bar{z}}$$

Exercise

1. Prove that an analytic function whose real part is constant is itself is constant.
2. Prove that an analytic function whose imaginary part is constant is itself a constant.
3. It f = u+iv is analytic in a region D and uv is constant in D then prove that f reduces to a constant.
4. If f = u+iv is analytic in a region D and $v = u^2$ in D then prove that f reduces to a constant.
5. Determine the constants a and b in order that the function $f(z) = (x^2+ay^2-2xy)+i\,(bx^2-y^2+2xy)$ should be analytic. Find $f^1(z)$.
6. Test whether the following functions are analytic.

   (i) $z^3 + z$. (ii) e^x (cosy +I sin y)

   (ii) e^x (cos y – i sin y) (iv) e^{-x} (cos y – i sin y)

Answers

4. a=-1 b=1 $f^1(z) = (1+i)z^2$ 6) (i) yes.

   (ii) Yes (iii) No. (iv) Yes.

4.5 The Cauchy-Riemann Equations

Theorem 4.5.1

Let f(z) = u(x, y) +iv (x, y) be differentiate at a point $z_0 = x_0 + iy_0$ then u(x, y) and v(x,y) have first order partial derivatives $u_x(x_0, y_0)$, $u_y(x_0, y_0)$, $v_x(x_0, y_0)$ and $v_y(x_0, y_0)$ at (x_0, y_0) and these partial derivatives satisfy the Cauchy-Riemann equations (C.R. equations) given by $u_x(x_0, y_0) = v_y(x_0, y_0)$ and $u_y(x_0, y_0) = -v_x(x_0, y_0)$.

Also $f^1(z_0) = u_x(x_0, y_0) + i\, v_x(x_0, y_0)$.

$= v_y(x_0, y_0) - i\, u_y(x_0, y_0)$.

Proof

Since f(z) = u(x, y) +i v(x, y) is differentiable at $z_0 = x_0 + i\, y_0$, $\lim_{h\to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exist and hence the limit is independent of the path in which h approaches to zero.

Let $h = h_1 + i\, h_2$.

$z_0+h = x_0 + i\, y_0 + h_1 + i\, h_2$

$= x_0 + h_1 + i\,(y_0 + h_2)$

Now $\frac{f(z_0+h)-f(z_0)}{h}$

$= \frac{u(x_0+h_1, y_0+h_2)+ i\, v(x_0+h_1, y_0+h_2)-u\,(x_0,y_0)- i\, v(x_0,y_0)}{(h_1+ i\, h_2)}$

$= \left(\frac{u(\, x_0+h_1, y_0+h_2)- u(x_0, y_0)}{(h_1+ i\, h_2)}\right)+i\left(\frac{v(\, x_0+h_1, y_0+h_2)- v(x_0, y_0)}{(h_1+ i\, h_2)}\right)$

Suppose $h\to 0$ along the real axis so that $h=h_1$.

Then $f^1(z_0) = \lim_{h\to 0} \frac{f(z_0+h_1)-f(z_0)}{h_1}$

$= \lim_{h_1\to 0} \frac{u(z_0+h_1\,,y_0)-u(z_0y_0)}{h_1}$

$+i \lim_{h_1\to 0} \frac{v(x_0+h_1,y_0)-v(z_0y_0)}{h_1}$

$= u_x(x_0, y_0) + i\, v_x(x_0, y_0)$ (1)

Now suppose $h\to 0$ along the imaginary axis so that $h = i\, h_2$

$\therefore f^1(z_0) = \lim_{i\,h_2\to 0} \frac{f(z_0+h_2)-f(z_0)}{i\,h_2}$

$= \lim_{h_2\to 0} \frac{u(x_0, y_0+h_2)-u(z_0,y_0)}{i\,h_2} + i \lim_{h_2\to 0}\left(\frac{v(x_0, y_0+h_2)-v(x_0,y_0)}{i\,h_2}\right)$

$= \left(\frac{u_y(x_0,y_0)}{i}\right)+i\left(\frac{v_y(x_0,y_0)}{i}\right)$

$= \frac{1}{i}\, u_y(x_0, y_0) + v_y(x_0, y_0)$

$= -i\, u_y(x_0, y_0) + v_y(x_0, y_0)$ (2)

From (1) and (2) we get

$f^1(z_0) = u_x\,(x_0, y_0) + i\, v_x(x_0, y_0) = v_y(x_0, y_0) - i\, u_y(x_0, y_0)$

Equating real and imaginary parts we get

$u_x(x_0, y_0) = v_y(x_0, y_0)$

$u_y(x_0, y_0) = -v_x(x_0, y_0)$

Remark 1

Since $f^1(z) = u_x+ i\, v_x = v_y - i\, u_y$.

We have $|f^1(z)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2$

Also $|f^1(z)|^2 = u_x^2 + u_y^2 = v_x^2 + v_y^2$

Further $|f^1(z)|^2 = u_x\, v_y - u_y\, v_x$

$= \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$= \frac{\partial(u,v)}{\partial(x,y)}$

Remark 2

The Cauchy-Riemann equations provide a necessary condition for differentiability at a point. Hence if the C.R.equations are not satisfied for a complex function at any point then we can conclude that the function is not differentiate.

For example, consider the function

$f(z) = \bar{z} = x - iy$

Hence $u(x, y) = x$ and $v(x, y) = -y$

$\therefore u_x(x, y) = 1$ and $v_y(x, y) = -1$

$\therefore u_x \neq v_y$ so that C.R. equations are not satisfied at any point z.

Hence the function $f(z) = \bar{z}$ is nowhere differentiable.

Remark 3

The C.R. equations are not sufficient for differentiability at a point.

Theorem 4.5.2

Let $f(z) = u(x, y) + iv(x, y)$ be a function defined in a region D such that u, v and their first order partial derivatives are continuous in D. If the first order partial derivatives of u, v satisfy the Cauchy-Riemann equations at a point $(x, y) \in D$ then f is differentiable at $z = x+iy$.

Proof

Since u(x, y) and its first order partial derivatives are continuous at (x, y), we have by the mean value theorem for functions of two variables.

$$u(x + h_1, y + h_2) - u(x, y) = h_1 u_x(x, y) + h_2 u_y(x, y) + h_1 \varepsilon_1 + h_2 \varepsilon_2 \qquad (1)$$

where ε_1 and $\varepsilon_2 \to 0$ as h_1 and $h_2 \to 0$

Similarly

$$v(x + h_1, y + h_2) - v(x, y) = h_1 v_x(x, y) + h_2 v_y(x, y) + h_1 \varepsilon_3 + h_2 \varepsilon_4 \qquad (2)$$

where

$\varepsilon_3, \varepsilon_4 \to 0$ as h_1 and $h_2 \to 0$

let $h=h_1 + ih_2$

$z = x + iy$

$\therefore z+h = x+h_1 + i(y+h_2)$

Then $\frac{f(z+h) - f(z)}{h}$

$= \frac{1}{h}$ [u(x+h$_1$, y+h$_2$) +i v(x+h$_1$, y+h$_2$)] - [u(x, y) +i v(x,y)]

$= \frac{1}{h}$ {u(x+h$_1$, y+h$_2$) -u(x, y)] +i [v (x+h$_1$, y+h$_2$) –v (x,y)}

$= \frac{1}{h}$ [{h$_1$ u$_x$(x,y) +h$_2$ u$_y$(x,y) + h$_1$ ε_1 + h$_2$ ε_2]

+i{h$_1$ v$_x$(x, y) +h$_2$ v$_y$(x, y) +h$_1$ ε_3 + h$_2$ ε_4} [using (1) and (2)]

$= \frac{1}{h}$ [h$_1${ u$_x$(x,y) +iv$_x$(x,y)}+h$_2${(u$_y$ (x,y) + iv$_y$(x,y)}

+h$_1$(ε_1 + i ε_3) +h$_2$(ε_2 + i ε_4)]

$= \frac{1}{h}$ [h$_1${ u$_x$(x,y) -iu$_y$(x,y)}+h$_2$ {u$_y$ (x,y) + iu$_x$(x,y)}

+h$_1$(ε_1 + ε_3) +h$_2$(ε_2 + ε_4)] using C.R. equation

$= \frac{1}{h}$ [({h$_1$+i h$_2$) u$_x$(x,y) -i(h$_1$+i h$_2$) (u$_y$ (x,y) + h$_1$(ε_1+iε_3) +h$_2$(ε_2 + ε_4)]

$= \frac{1}{h}$ [(h u$_x$(x,y) – i h u$_y$ (x,y) + h$_1$(ε_1+iε_3) +h$_2$(ε_2 + i ε_4)]

$= u_x(x,y) - i\, u_y(x,y) + \frac{h_1}{h}(\varepsilon_1+i\varepsilon_3) + \frac{h_2}{h}(\varepsilon_2 + i\,\varepsilon_4)$

Now, Since $\left|\frac{h_1}{h}\right| \le 1$, $\frac{h_1}{h}(\varepsilon_1+ i\varepsilon_3) \to 0$ as $h \to 0$

Similarly $\frac{h_2}{h}(\varepsilon_2+ i\varepsilon_4) \to 0$ as $h \to 0$

$\therefore \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = u_x(x, y) - iu_y(x, y)$

Hence f is differentiable.

Example 1

Let $f(z) = e^x (\cos y + i \sin y)$

$\therefore u(x, y) = e^x \cos y$ and $v(x, y) = e^x \cos y$

Then $u_x(x, y) = e^x \cos y$ and $v_y(x, y) = e^x \cos y$.

$\therefore u_y(x, y) = -e^x \sin y$ and $v_x(x, y) = e^x \sin y$

$\therefore u_y(x, y) = -v_y(x, y)$

Thus the first order partial derivatives of u and v satisfy the Cauchy-Riemann equations at every point.

Further u(x, y) and v(x, y) and their first order partial derivatives are continuous at every point. Hence f is differentiable at every point of the complex plane.

Example 2

Let $f(z) = |z|^2$

$\therefore f(z) = u(x, y) + iv(x, y) = x^2 + y^2$

$\therefore u(x, y) = x^2 + y^2$ and $v(x, y) = 0$

Hence $u_x(x, y) = 2x$, $u_y(x, y) = 2y$

$v_x(x, y) = 0 = v_y(x, y)$

Clearly the Cauchy-Riemann equations are satisfied at z=0.

Further u and v and their first order partial derivatives are continuous and hence f is differentiable at z=0.

Also we notice that the C.R. equations are not satisfied at any point $z \neq 0$ and hence f is not differentiable at $z \neq 0$.

Thus f is differentiable only at z = 0.

Theorem 4.5.3

(Complex form of C R equations)

Let $f(z) = u(x, y) + iv(x, y)$ be differentiable then the C R equations can be put in the complex form as $f_x = -ify$.

Proof

Let $f(z) = u(x, y) + iv(x, y)$

Then $f_x = u_x + iv_x$

and $f_y = u_y + iv_y$

Hence $f_x = -if_y \Leftrightarrow u_x + iv_x = -i(u_y + iv_y)$

$\Leftrightarrow u_x + iv_x = v_y - iu_y$

$\Leftrightarrow u_x = v_y$ and $v_x = -u_y$

Thus the two C.R. equations are equivalent to the equation $f_x = -if_y$.

Theorem 4.4

(C.R equations in polar co-ordinates)

Let $f(z) = u(r, \theta) + iv(r, \theta)$ be differentiable at $z = re^{i\theta} \neq 0$. Then $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

$\frac{\partial v}{\partial r} = -\frac{1}{r} \cdot \frac{\partial u}{\partial \theta}$. Further $f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$

Proof

$Z = re^{i\theta} \neq 0$

$= r(\cos\theta + \sin\theta)$

$\therefore x = r\cos\theta$ and $y = r\sin\theta$

Hence $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} . \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} . \frac{\partial y}{\partial r}$

i.e., $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x} . \cos\theta + \frac{\partial u}{\partial y} . \sin\theta$ (1)

Also $\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial x} . \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} . \frac{\partial y}{\partial \theta}$

$= \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}\cos\theta$

$\therefore \frac{1}{r}\frac{\partial v}{\partial \theta} = -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta$

$= \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta$ (using C R equations)

$= \frac{\partial u}{\partial r}$ (using (1))

Thus $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

Similarly we can prove that $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

Now $r\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial r}\right) = r\left[\left(\frac{\partial u}{\partial x} . \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y} . \frac{\partial y}{\partial r}\right) + i\left(\frac{\partial v}{\partial x} . \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} . \frac{\partial y}{\partial r}\right)\right]$

$= r\left[\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) + i\left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\right]$

$= r\cos\theta\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + r\sin\theta\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)$

$= x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + i\,y\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right)$

$= x f^1(z) + iy f^1(z)$

$= (x + iy) f^1(z)$

$= z f^1(z)$

$\therefore f^1(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$

Theorem 4.5.5

If f(z) is a differentiable function, the C.R. equation can be put in the form $\frac{\partial f}{\partial \bar{z}} = 0$.

Proof

$\frac{\partial f}{\partial \bar{z}} = \frac{\partial f}{\partial x} . \frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y} . \frac{\partial y}{\partial \bar{z}}$

$= \frac{\partial f}{\partial x}(1/2) + \frac{\partial f}{\partial y}(-1/2i)$

Thus $\frac{\partial f}{\partial \bar{z}} = 0 \Leftrightarrow \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$ which is the complex form of the C.R. equations.

Thus the C.R. equations can be put in the form $\frac{\partial f}{\partial \bar{z}} = 0$

Solved Problems

Problem 1

Verify Cauchy-Riemann equations for the function $f(z) = z^3$.

Solution

$f(z) = z^3 = (x + iy)^3$

$= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

$\therefore u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2y - y^3$

$\therefore u_x = 3x^2 - 3y^2$ and $v_x = 6xy$

$u_y = -6xy$ and $v_y = 3x^2 - 3y^2$

Here $u_x = v_y$ and $u_y = -v_x$.

Hence the Cauchy-Riemann equations are satisfied.

Problem 2

Prove that the following functions are nowhere differentiable.

(i) $f(z) = \text{Re } z$ (ii) $f(z) = e^x(\cos y - i \sin y)$

Solution

(i) $f(z) = \text{Re } z$.

i.e. $f(z) = x$

$\therefore u(x, y) = x$ and $v(x, y) = 0$

$\therefore u_x = 1$ and $v_x = 0$

$u_y = 0$ and $v_y = 0$

Since $u_x \neq v_y$ the C R equations are not satisfied at any point.

Hence f(z) is nowhere differentiable.

(ii) $f(z) = e^x(\cos y - i \sin y)$

$= e^x \cos y - ie^x \sin y$

$\therefore u(x, y) = e^x \cos y$ and $v(x, y) = -e^x \sin y$

$\therefore u_x = e^x \cos y$ and $v_x = -e^x \sin y$

$u_y = -e^x \sin y$ and $v_y = -e^x \cos y$

$u_x \neq v_y$ and $u_y \neq -v_x$

$\therefore$ C.R equations are not satisfied at any point and hence f(z) is no where differentiable.

Problem 3

Prove that $f(z)=\begin{cases} \frac{z\text{Re } z}{|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z=0 \end{cases}$ is continuous at z = 0 but not differentiable at z=0.

Solution

First we shall prove that $\lim_{z\to 0} f(z)=0$.

Now $|f(z) - 0| = |f(z)|$

$= \left|\frac{z\text{Re } z}{|z|}\right|$

$= \frac{|z|\,|\text{Re } z|}{|z|}$

$= |\text{Re } z|$

Further $|\text{Re } z| \leq |z|$.

$\therefore$ For any given $\varepsilon > 0$, if we choose $\delta = \varepsilon$, we get,

$|z| = |z-0| < \delta \Longrightarrow |f(z)-0| = |\text{Re } z| \leq |z| < \varepsilon$

i.e. $|z-0| < \delta \Longrightarrow |f(z)-0| < \varepsilon$

Hence f is continuous at z=0

Now we prove that f(z) is not differentiable at z = 0

$$\frac{f(z)-f(0)}{z-0} = \frac{z\text{ Re } z}{z\,|z|}$$

$$= \frac{\text{Re } z}{|z|} = \frac{x}{\sqrt{x^2+y^2}} \quad \text{where } z = z + i\,y$$

Along the path y = mx,

$$\frac{f(z)-f(0)}{z-0} = \frac{x}{\sqrt{x^2+m^2x^2}} = \frac{1}{\sqrt{1+m^2}}$$

$\therefore$ The value of the limit depends on m and hence on the path along which $z \to 0$

$\therefore \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ does not exist.

$\therefore$ f(z) is not differentiable at z = 0.

Problem 4

Prove that f(z) = z Im z is differentiable only at z=0 and find $f'(0)$.

Solution

$f(z) = z \text{ Im } z$

$= (x+iy)\, y = xy + i\, y^2$

$\therefore u(x, y) = xy$ and $v(x, y) = y^2$

$\therefore u_x = y, v_x = 0, u_y = x$ and $v_y = 2y$

Clearly the C.R. equations are satisfied only at z = 0.

Further all the first order partial derivatives are continuous.

Hence f(z) is differentiable only at z=0.

$f^1(z) = u_x + iv_x$

$\therefore f^1(0) = u_x(0,0) + iv_x(0, 0) = 0 + 0 = 0$

Problem 5

Show that $f(z) = \begin{cases} \frac{xy^2(x+i\,y)}{x^2+y^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

is not differentiable at z = 0

Solution

$$\frac{f(z)-f(0)}{z-0} = \frac{\frac{xy^2(x+i\,y)}{x^2+y^4} - 0}{x+i\,y-0}$$

$$= \frac{xy^2(x+i\,y)}{x^2+y^4} \times \frac{1}{x+i\,y} = \frac{xy^2}{x^2+y^4}$$

Along the path $x = my^2$

$$\frac{f(z)-f(0)}{z-0} = \frac{my^4}{m^2y^4+y^4} = \frac{m}{m^2+1}$$

The value of the limit depends on m and hence depends on the path along which $z \to 0$

$\therefore \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$ does not exist.

$\therefore$ f(z) is not differentiable at z = 0

Problem 6

Prove that the function $f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

Satisfies C.R. equations at the origin but $f^1(0)$ does not exist.

Solution

$$f(z) = \begin{cases} \dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

$$f(z) = \begin{cases} \dfrac{x^3-y^3+i\,(x^3+y^3)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$

Here $u(x, y) = \frac{x^3-y^3}{x^2+y^2}$ and $v(x, y) = \frac{x^3+y^3}{x^2+y^2}$ if $(x, y) \neq (0, 0)$ and $u(0, 0) = v(0, 0) = 0$

Now, $u_x(x, y) = \lim_{h\to 0} \frac{u(x+h,y)-u(x,y)}{h}$

$\therefore u_x(0, 0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h}$

$$= \lim_{h\to 0} \frac{\frac{h^3}{h^2}-0}{h} = \lim_{h\to 0} \frac{h^3}{h^3} = 1$$

$u_y(x, y) = \lim_{h\to 0} \frac{u(x,y+h)-u(x,y)}{h}$

$u_y(0, 0) = \lim_{h\to 0} \frac{u(0,\ h)-u(0,0)}{h}$

$$= \lim_{h\to 0} \frac{-\frac{h^3}{h^2}}{h} = \lim_{h\to 0} -\frac{h^3}{h^3} = 1$$

Similarly we can prove that $v_x(0, 0) = 1$ and $v_y(0, 0)=1$

Thus $u_x(0, 0) = v_y(0, 0) = 1$ and

$u_y(0, 0) = -v_x(0, 0) = -1$. So that

C.R. equations are satisfied at z = 0.

Now $\frac{f(z)-f(0)}{z-0} = \dfrac{\frac{x^3-y^3+i\,(x^3+y^3)}{x^2+y^2}-0}{x+i\,y-0}$

$$= \frac{x^3-y^3}{(x^2+y^2)(x+i\,y)} + \frac{i\,(x^3+y^3)}{(x^2+y^2)(x+i\,y)}$$

Along the path y = mx we have y=mx

$$\frac{f(z)-f(0)}{z-0} = \frac{x^3-m^3x^3}{(x^2+m^2x^2)(x+i\,mx)} + i\,\frac{x^3+m^3x^3}{(x^2+m^2x^2)(x+i\,mx)}$$

$$= \frac{x^3(1-m^3)}{x^3(1+m^2)(1+i\,m)} + \frac{i\,x^3(1+m^3)}{x^3(1+m^2)(1+im)}$$

$$= \frac{1-m^3}{(1+m^2)(1+i\,m)} + \frac{i\,(1+m^3)}{(1+m^2)(1+im)}$$

Hence the value of the limit depends on the path along which $z \to 0$

Thus $\lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$ does not exist.

Hence f is not differentiable at 0.

Problem 7

Prove that f(z) = sin x cos hy + i cos x sin hy is differentiable at every point.

Solution

f(z) =sin x cos hy + i cos x sin hy

∴ u(x, y) = sin x cos hy and v(x, y) = cos x sin hy.

u_x = cos x cos hy and v_x = -sin x sin hy.

u_y = sin x sin hy and v_y = cos x cos hy.

∴ $u_x = v_y$ and $u_y = -v_x$ for all x, y

Hence C.R equation are satisfied at every point.

Further all the first order partial derivatives are continuous.

Hence f(z) is differentiable at every point.

Problem 8

Find constants a and b so that the function $f(z) = a(x^2-y^2) + i\, b\, xy + c$ is differentiable at every point.

Solution

Here $u(x, y) = a(x^2-y^2)+c$ and v(x, y) = b xy

$u_x = 2ax$; $v_x\ b_y$

$u_y = -2ay$ and $v_y = bx$

Clearly $u_x = v_y$ and $u_y = -v_x$ iff 2a=b.

∴ C.R equations are satisfied at all points iff 2a=b.

∴ The function f(z) is differentiable for all values of a, b with 2a=b

Problem 9

Show that $f(z) = \sqrt{r}\,(\cos \theta/2 + i \sin \theta/2)$ where r>0 and $0<\theta< 2\pi$ is

differentiable and find $f^1(z)$.

Solution

$f(z) = \sqrt{r}\,(\cos \theta/2 + i \sin \theta/2)$

$u = \sqrt{r}\,(\cos {}^{\theta}/_{2})$ and $v = \sqrt{r}\,\sin {}^{\theta}/_{2}$

$\therefore \frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos({}^{\theta}/_{2})$ and $\frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}}\sin({}^{\theta}/_{2})$

$\frac{\partial u}{\partial \theta} = \frac{-\sqrt{r}}{2}\sin({}^{\theta}/_{2})$ and $\frac{\partial v}{\partial \theta} = [\frac{\sqrt{r}}{2}\cos({}^{\theta}/_{2})]$

Now $\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\,[\frac{\sqrt{r}}{2}\cos {}^{\theta}/_{2})]$

$= \frac{1}{2\sqrt{r}}\cos({}^{\theta}/_{2})$

$= \frac{\partial u}{\partial r}$

Thus $\frac{\partial u}{\partial r} = \frac{1}{r}\,\frac{\partial v}{\partial \theta}$

Similarly $\frac{\partial v}{\partial r} = -\frac{1}{r}\cdot\frac{\partial u}{\partial \theta}$

$= \frac{1}{2\sqrt{r}}\sin({}^{\theta}/_{2})$

Hence the C.R. equations in polar form) are satisfied.

Further all the first order partial derivatives are continuous.

Hence $f^1(z)$ exist

Also $f^1(z) \quad = \frac{r}{z}(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r})$

$= \frac{r}{z}\,(\frac{1}{2\sqrt{r}}\cos({}^{\theta}/_{2}) + \frac{i}{2\sqrt{r}}\sin({}^{\theta}/_{2})$

$= \frac{r}{2\sqrt{r}\,2}\,[\cos({}^{\theta}/_{2}) + i\sin({}^{\theta}/_{2})\,]$

$= \frac{1}{2z}\,[\sqrt{r}\,(\cos({}^{\theta}/_{2}) + i\sin({}^{\theta}/_{2})\,]$

$= \frac{1}{2z}\,x\,\sqrt{z} = \frac{1}{2\sqrt{z}}$

Hence $f^1(z) = \frac{1}{2\sqrt{z}}$

[$\because z = re^{i\theta}$

i.e. $z = r\,(\cos\theta + \sin\theta)$

$\sqrt{z} = \sqrt{r}\,(\cos\theta + \sin\theta)^{1/2}$

$= \sqrt{r}\cos(\theta/2) + i\sin(\theta/2)\,]$

4.6 Harmonic functions

Definition

Let u(x, y) be a function of two real variables x and y defined in a region D. u(x, y) is said to be a harmonic function if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and this equation is called Laplace's equation.

Theorem 4.6.1

The real and imaginary parts of an analytic function are harmonic functions.

Proof

Let f(z) = u(x, y) +iv (x, y) be an analytic function.

Then u and v have continuous partial derivatives of first order which satisfy the C.R. equation given by $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Further $\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}$ and $\frac{\partial^2 v}{\partial x\,\partial y} = \frac{\partial^2 v}{\partial y\,\partial x}$

Now $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 v}{\partial x\,\partial y} - \frac{\partial^2 v}{\partial y\,\partial x} = 0$$

Thus u is a harmonic function. Similarly we can prove that v is a harmonic function.

Remark 1

Laplace's equation provides a necessary condition for a function to be the real or imaginary part of an analytic function.

For example if u(x, y) = $x^2 + y$,

$$\frac{\partial^2 u}{\partial x^2} = 2;\ \frac{\partial^2 u}{\partial y^2} = 0$$

and $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2$

Thus u(x, y) is not harmonic function and hence it cannot be the real part of any analytic function.

Definition

Let f = u+iv be an analytic function in a region D. Then v is said to be a conjugate harmonic function of u.

Theorem 4.6.2

Let f = u+iv be an analytic function in a region D. Then v is a harmonic conjugate of u if and only if u is a harmonic conjugate of –v.

Proof

Let v be a harmonic conjugate of u.

Then f = u+iv is analytic

∴ if = iu-v is also analytic.

Hence u is a harmonic conjugate of –v. Similarly we can prove the converse part.

Theorem 4.6.3

Any two harmonic conjugates of a given harmonic function u in a region D differ by a real constant.

Proof

Let u be a harmonic function. Let v and v* be two harmonic conjugates of u Then u+iv and u+iv* are analytic in D.

Since u+iv is analytic in D, by C.R.

equation $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (1)

since u+iv* is analytic in D, by C.R

equation $\frac{\partial u}{\partial x} = \frac{\partial v*}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v*}{\partial x}$ (2)

From (1) & (2)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v*}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\frac{\partial v*}{\partial x}$$

$$\therefore \frac{\partial v}{\partial y} = \frac{\partial v*}{\partial y} \text{ and } \frac{\partial v}{\partial x} = \frac{\partial v*}{\partial x}$$

Hence $\frac{\partial}{\partial y}$ (v-v*) = 0 and $\frac{\partial}{\partial x}$ (v-v*) = 0

∴ v-v* = c (a constant)

∴ v=v*+c where c is a real constant.

Remark

The Cauchy-Riemann equation can be used to obtain a harmonic conjugate of a given harmonic function.

Milne-Thompson Method

Let u(x, y) be a given harmonic function. Let f(z) = u(x, y) +iv(x,y) be an analytic function.

Then $f^1(z) = u_x(x,y) + i\, v_x(x, y)$

$= u_x(x,y) - i\, u_y(x, y)$

Let $\theta_1 (x, y) = u_x(x, y)$ and $\theta_2 (x, y) = u_y(x, y)$

We have $x = \frac{z+\bar{z}}{2}$ and $y = \frac{z-\bar{z}}{2i}$

Hence $f^1(z) = \varphi_1 (\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}) - i\varphi_2(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i})$

Putting $z = \bar{z}$ we obtain $f^1(z) = \varphi_1(z,0) - i\, \varphi_2 (z,0)$

Hence $f(z) = \int [\varphi_1(z, 0) - i\, \varphi_2(z, 0)]\, dz + c$

Note

It can be proved in a similar way that the analytic function f(z) with a given harmonic function v(x, y) as imaginary part is given by $f(z) = \int [\aleph_1(z, 0) + i\aleph_2(z, 0)]\, dz + c$ where $\aleph_1(x, y) = v_y$ and $\aleph_2(x, y) = v_x$.

Solved Problems

Problem 1

Prove that $u=2x-x^3+3xy^2$ is harmonic and find its harmonic conjugate. Also find the corresponding analytic function.

Solution

$u = 2x-x^3+3xy^2$

$\therefore u_x = 2-3x^2+3y^2;\ u_{xx} = -6x$

$u_y = 6xy;\ u_{yy} = 6x$

$\therefore u_{xx}+ u_{yy} = -6x+6x=0$

Hence u is harmonic.

Let v be the harmonic conjugate of u.

$\therefore$ f(z) = u+i v is analytic.

By Cauchy-Riemann equations we have

$v_y = u_x = 2\text{-}3x^2+3y^2$

ie $v_y = 2\text{-}3x^2+3y^2$

Integrating w.r. to y we get

$v=2y-3x^2y+y^3+ \lambda(x)$ (1)

where $\lambda(x)$ is an arbitrary function of x

$\therefore v_x = -6xy + \lambda'(x)$

Now $v_x = -u_y$

$\Longrightarrow -6xy + \lambda'(x) = -6xy$

$\Longrightarrow \lambda'(x) = 0 \Longrightarrow \lambda(x) = c$ where c is a constant.

Thus $v = 2y-3x^2y+y^3+c$ [From (1)]

Now $f(z) = (2x-x^3+3xy^2)+i(2y-3x^2y+y^3)+ic$

$= 2(x+iy) – [(x^3-3xy^2)+i(3x^2y-y^3)]+ic$

$= 2z – z^3 + ic$

$\therefore f(z)=2z-z^3+ic$ is the required analytic function

Problem 2

Show that $u = \log\sqrt{x^2+y^2}$ is harmonic and determine its conjugate and hence find the corresponding analytic function f(z).

Solution

$u=\log\sqrt{x^2+y^2} = \frac{1}{2}\log(x^2+y^2)$

$\therefore u_x = \frac{1}{2} . \frac{1}{x^2+y^2} . 2x = \frac{x}{x^2+y^2}$

$u_{xx} = \frac{(x^2+y^2)-x.2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

similarly $u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$

obviously $u_{xx} + u_{yy} = 0$ and hence u is harmonic

Let v be a harmonic conjugate of u.

$\therefore f(z)=u+iv$ is an analytic function.

By C.R. equation we have,

$v_y = u_x$

$= \frac{x}{x^2+y^2}$

ie. $v_y = \frac{x}{x^2+y^2}$

Integrating w.r.t. to y we get

$v = \tan^{-1}(y/x) + \theta(x)$ where $\varphi(x)$ is an arbitrary function of x.

Now $v_x = \frac{1}{1+\frac{y^2}{2^2}}\left(\frac{-y}{x^2}\right) + \varphi'(x)$

Also $v_x = -u_y$

$\Longrightarrow \frac{-y}{x^2+y^2} + \varphi'(x) = \frac{-y}{x^2+y^2}$

$\Longrightarrow \varphi'(x) = 0$

Hence $\varphi(x) = c$

$\therefore v = \tan^{-1}(y/x)+c$

$\therefore f(x) = u+iv = \log\sqrt{x^2+y^2} + i\,[\tan^{-1}(y/x)+c]$

Problem 3

Show that $u(x, y) = \sin x \text{ co shy} + 2 \cos x \sin hy + x^2-y^2+4xy$ is harmonic. Find an analytic function f(z) interms of z with the given u for its real part.

Solution

$u_x = \cos x \cos hy - 2 \sin x \sin hy + 2x + 4y$

$u_{xx} = -\sin x \cos hy - 2 \cos x \sin hy + 2$

$u_y = \sin x \sin hy + 2 \cos x \cos hy -2y + 4x$

$u_{yy} = \sin x \cos hy + 2 \cos x \sin hy -2$

$\therefore u_{xx} + u_{yy} = 0$

Hence u is harmonic

Now let $\varphi_1(x, y) = u_x$ and $\varphi_2(x, y) = u_y$.

$\therefore \varphi_1(z, 0) = \cos z \cos h\, 0-2 \sin z \sin h\, 0 + 2z$

$= \cos z + 2z$

Similarly $\varphi_2(z, 0) = 2 \cos z +4z$

$\therefore f(z) = \int [\varphi_1(z, 0) - i\, \varphi_2(z, 0)]dz$

$= \int [\cos z + 2z - i\,(2 \cos z + 4z)\, dz$

$= \sin z + z^2 - 2\, i \sin z - 2\, iz^2 + c$

Problem 4

Find the analytic function

$f(z) = u + iv$ if $u+v = \frac{\sin 2x}{\cos h2y - \cos 2x}$

Solution

$$u + v = \frac{\sin 2x}{\cos h2y - \cos 2x} \quad (1)$$

$$\therefore u_x+v_x = \frac{(\cos h\, 2y - \cos 2x)\,(\cos 2x)\, 2 - \sin 2x(0+\sin 2x\, 2)}{(\cos h\, 2y - \cos 2x)^2}$$

$$\therefore u_x+v_x = \frac{2(\cos h\, 2y - \cos 2x)\,\cos 2x - 2\sin^2 2x}{(\cos h\, 2y - \cos 2x)^2} \quad (2)$$

and

$$u_y+v_y = \frac{(\cos h\, 2y - \cos 2x) \text{ x } 0 - \sin 2x \text{ x } (2 \sin hy)}{(\cos h\, 2y - \cos 2x)^2}$$

$$\Longrightarrow u_y+v_y = \frac{-2\sin 2x \sin h\, 2y}{(\cos h\, 2y - \cos 2x)^2} \quad (3)$$

Since the required function $f(z) = u+iv$ is to be analytic, u and v satisfy the C.R. equation $u_x = v_y$ and $u_y = -v_x$.

Using these equations in (2), we get,

$$u_x - u_y = \frac{2(\cos h\, 2y - \cos 2x)\cos 2x - 2\sin^2 2x}{(\cos h\, 2y - \cos 2x)^2}$$

$$\therefore u_x(z, 0) - u_y(z, 0) = \frac{2(1-\cos 2z)\cos 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{(2-2\cos 2z)\cos 2z - 2\sin^2 2z}{(1-\cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2} = \frac{-2}{1-\cos 2z}$$

$$= -\frac{2}{2\sin^2 z} = -\operatorname{cosec}^2 z \quad (4)$$

Using C.R. equations in (3) we get

$$u_y + u_x = \frac{-2\sin 2x \sin h\, 2y}{(\cos h\, 2y - \cos 2x)^2}$$

$$\therefore u_y(z, 0) + u_x(z, 0) = 0 \quad (5)$$

Now adding (4) and (5) we get

$$2\, u_x(z, 0) = -\operatorname{cosec}^2 z$$

$$\therefore u_x(z, 0) = \frac{-1}{2}\operatorname{cosec}^2 z \quad (6)$$

Subtracting (4) from (5) we get

$2u_y(z, 0) = \text{cosec}^2 z$

$\Rightarrow u_y(z, 0) = \frac{1}{2}\text{cosec}^2 z$ (7)

Now f(z) = u(z, 0) + iv (z, 0).

$\Rightarrow f^1(z) = u_x(z, 0) + i\, v_x\, (z, 0)$

$= u_x(z, 0) - i\, u_y\, (z, 0)$

$= \frac{-1}{2}\text{cosec}^2 z - i\,\frac{1}{2}\text{cosec}^2 z$

ie) $f^1(z) = \frac{-1}{2}(1+i)\,\text{cosec}^2 z$

Integrating w.r. to z, we have

$f(z) = (\frac{1+i}{2})\cot z + c$

Problem 5

Given $v(x, y) = x^4-6x^2y^2+y^4$ find f(z) = u(x, y)+iv(x, y) such that f(z) is analytic

Solution

$v(x, y) = x^4-6x^2y^2+y^4$

$v_x = 4x^3-12xy^2$

$v_{xx} = 12x^2-12y^2$

$v_y = -12x^2y+4y^3$

$v_{yy} = -12x^2+12y^2$

$v_{xx}+v_{yy} = 12x^2-12y^2-12x^2+12y^2$

$= 0$

$\therefore$ v(x, y) is harmonic.

Let f(z) = u+iv be the required analytic function.

By Cauchy – Riemann equations $u_x = v_y$

$\therefore u_x = -12x^2y+4y^3$

$\therefore$ Integrating with respect to x we get $u=-4x^3y+4xy^3+\lambda(y)$ where $\lambda(y)$ is an arbitrary function of y.

$\therefore u_y = -4x^3+12xy^2+\lambda'(y) = -v_x$.

$\therefore -(4x^3-12xy^2)=-4x^3+12xy^2+\lambda'(y)$

$\Rightarrow \lambda'(y) = 0$ so that $\lambda(y) =$ c where c is a constant.

$\therefore u = -4x^3y+4xy^3+ c$

$\therefore f(z) = (-4x^3y+4xy^3+ c) +i(x^4-6x^2y^2+y^4)$

$= i[(x^4 - 6x^2y^2+y^4) +i(4x^3y-4xy^3)]+c$

$= i[(x+iy)^4]+c$

$= i z^4+c$

Problem 6

Find the analytic function f(z) = u+iv given $u-v = e^x (\cos y - \sin y)$

Solution

$u-v = e^x (\cos y - \sin y)$ (1)

Diff w.r.to x, $u_x - v_x = e^x (\cos y - \sin y)$ (2)

Differentiate (1) w.r. to y,

$u_y-v_y = e^x (-\sin y - \cos y)$

ie $u_y - v_y = -e^x (\sin y + \cos y)$ (3)

Since the required function is to be analytic, it has to satisfy the C.R equations.

∴ using C.R. equations in (3) we get,

$-v_x-u_x=-e^x (\sin y + \cos y)$ (4)

Solving (2) and (4) we get

$u_x = e^x \cos y$ (5)

and $v_x = e^x \cos y$ (6)

Integrating (6) w.r. to x, we get,

$v = e^x \sin y + f(y)$

$\therefore v_y = e^x \sin y + f^1(y)$

$\Rightarrow u_x = e^x \cos y + f^1(y)$ $[\therefore v_y = u_x]$

$\Rightarrow e^x \cos y = e^x \cos y + f^1(y)$

$\Rightarrow f^1(y) = 0$

Hence $f(y) = c_1$ where c_1 is a constant.

$\therefore v = e^x \sin y + c_1$

From (1) $u = e^x \cos y + c_2$

Now, f(z) = u+ iv

$= e^x \cos y + c_2 + i (e^x \sin y + c_1)$

$= e^x (\cos y + i \sin y) + (c_2 + i c_1)$

$= e^x \cdot e^{iy} + \alpha$ where α is a complex constant.

$= e^{x+iy} + \alpha$

$= e^z + \alpha$

Problem 7

Find the constant a so that $u(x, y) = ax^2-y^2+xy$ is harmonic. Find the analytic function f(z) for which u is the real part. Also find its harmonic conjugate.

Solution

$u = ax^2-y^2+xy$

Given that u is harmonic

Hence it satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Now $\frac{\partial u}{\partial x} = 2a\,x+y$

$$\frac{\partial^2 u}{\partial x^2} = 2a$$

$$\frac{\partial u}{\partial y} = -2y+x$$

$$\frac{\partial^2 u}{\partial y^2} = -2$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \implies 2a-2 = 0$$

$$\implies a = 1$$

$\therefore u = x^2-y^2+xy$

Hence $u_x = 2x+y$ and $u_y = -2y+x$

Let $\varphi_1(x, y) = u_x = 2x+y$

and $\varphi_2(x, y) = u_y = -2y+x$

$\therefore \varphi_1(z, 0) = 2z$ and $\varphi_2(z, 0)=z$

$\therefore f(z) = \int [\varphi_1(z, 0)-i\varphi_2(z, 0)]=dz$

$= \int (2z-iz)\, dz$

$= z^2 - \frac{iz^2}{2} + c$

$= (x+iy)^2 - \frac{i(x+iy)^2}{2} + c$

$= (x^2-y^2+2i\,xy) - \frac{i}{2}(x^2-y^2+2i\,xy)$

$= (x^2-y^2+xy)+i\ (2xy + \frac{y^2-x^2}{2}) + c$

$\therefore v(x, y) = 2xy + (\frac{y^2-x^2}{2})$ is the harmonic conjugate of u(x, y)

Problem 8

If f(z) is analytic prove that $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})|f(z)|^2 = 4|f^1(z)|^2$

Solution

Let f(z) = u+iv

$|f(z)| = \sqrt{u^2 + v^2}$

$|f(z)|^2 = u^2+v^2 = \varphi$ (say)

and $f^1(z) = u_x + i\ v_x$

Also $\varphi = u^2 + v^2$

$\therefore \frac{\partial\varphi}{\partial x} = 2u.u_x + 2vv_x$

$\frac{\partial^2\varphi}{\partial x^2} = 2\ [u.u_{xx}+u_x.u_x+v.v_{xx}+v_x.v_x]$

$= 2\ [u_x^2+uu_{xx}+ v_x^2 + vv_{xx}]$

Similarly $\frac{\partial^2\varphi}{\partial y^2} = 2\ [u_y^2+uu_{yy}+ v_y^2 + vv_{yy}]$

$= 2\ [v_x^2+uu_{yy}+ u_x^2 + vv_{yy}]$

[Using C.R equation]

Since u and v are harmonic,

$u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$

$\therefore \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 2[u_x^2 + uu_{xx}+ v_x^2 + vv_{xx}+v_x^2+ uu_{yy}+u_x^2+v.v_{yy}]$

$= 2\ u_x^2 + 2v_x^2 + u(u_{xx}+u_{yy}) + 2v_x^2+2u_x^2 + v(v_{xx} + vv_{yy})$

$= 4[u_x^2 + v_x^2]$

$= 4[u_x+i\ v_x|^2$

$= 4[f^1(z)\ |^2$

Exercise

1. If $u+v = (x-y)\ (x^2+4xy+y^2)$ and f(z) = u+iv, find the analytic function f(z) in terms of z.
2. Find the real part of the analytic function whose imaginary part is $e^{-x}\ [2\ xy \cos y + (y^2-x^2) \sin y]$. Construct the analytic function.

3. Prove that the function u=sin hx sin y is harmonic. Also find the harmonic conjugate.

Answers :

1. z^3+c
2. $u = e^{-x} [(x^2-y^2)\cos y+2xy \sin y]$; $f(z) = e^{-x} [(x^2-y^2) + 2i\, xy] (\cos y - i \sin y)$
3. $v = -\cos h\, x \cos y$

4.7 Bilinear Transformations

Translation : w = z+b

Consider the transformation w = z+b. If z = x+iy, w = u+iv and $b = b_1+ib_2$ then the image of the point (x, y) in the z-plane is the point $(x+b_1, y+b_2)$ in the w-plane.

Under this transformation the image of any region is simply a translation of that region. Hence the two regions have the same shape, size and orientation. In particular the image of a straight line is a straight line and the image of a circle with centre a and radius r is a circle with centre a+b and radius r.

We note that ∞ is the only fixed point of this translation when b≠0.

Rotation w = az where |a|=1.

Consider the transformation w = az where |a|=1.

Let $z = re^{i\theta}$ and $a=e^{i\alpha}$ so that |a|=1.

$\therefore w = az=e^{i\alpha} (re^{i\theta}) = re^{i(\theta+\alpha)}$

∴ A point with polar co-ordinates (r, θ) is the z-plane is mapped to the point (r, θ+ α) in the w-plane. Hence this transformation represents a rotation through an angle α=arg α about the origin. Under this transformation also straight lines are mapped into straight lines and circles are mapped into circles.

We note that 0 and ∞ are the two fixed points of this transformation.

Inversion : $w = \frac{1}{z}$

Consider the transformation $w = \frac{1}{z}$

Put $z = re^{i\theta}$

$$w = \frac{1}{x} = \frac{1}{re^{i\theta}}$$

i.e. $w = \left(\frac{1}{r}\right) e^{-i\theta}$

This transformation can be expressed as a product of two transformations $T_1(z) = (\frac{1}{r})\, e^{i\theta}$

and $T_2(z) = re^{-i\theta} = \bar{z}$

For, $(T_1 \circ T_2)(z) = T_1(T_2(z))$

$= T_1(re^{-i\theta})$

$= (\frac{1}{r})\, e^{-i\theta} = \frac{1}{z}$

The transformation $T_1(z) = (\frac{1}{r})\, e^{i\theta}$ represents the inversion with respect to the unit circle $|z|=1$ and $T_2(z) = \bar{z}$ represents reflection about the real axis.

Hence the transformation $w = \frac{1}{z}$ is the inversion w.r.to the unit circle followed by the reflection about the real axis.

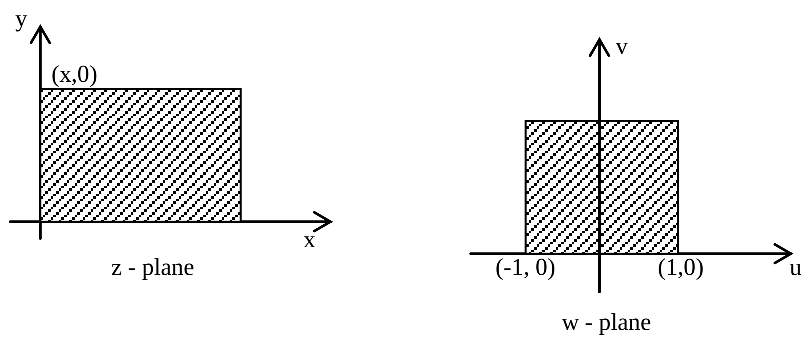
Here points outside the unit circle are mapped into points inside the unit circle and vice versa. Points on the circle are reflected about the real axis.

However the family of circles and lines are again mapped into the family of circles and lines.

We note that the fixed points of the transformation $w = \frac{1}{z}$ are 1 and -1.

Problem 1

Show that the region in the z-plane given by $x>0$ and $0<y<2$ is mapped into the region in the w-plane given by $-1<u<1$ and $v>0$ under the transformation $w = iz+1$.



Solution

Let $z=x+iy$ and $w=u+iv$

$w = iz+1$

$\Rightarrow w = i(x+iy) +1$

$\Rightarrow$ u+iv = (-y+ix) +1

$\Rightarrow$ u+iv = 1-y+ix

$\therefore$ u = 1-y and v=x

x > 0 $\Leftrightarrow$ v > 0

y > 0 $\Rightarrow$ 1-u > 0

$\Rightarrow$ 1> u

ie u < 1

y < 2 $\Rightarrow$ 1-u < 2

$\Rightarrow$ -1< u

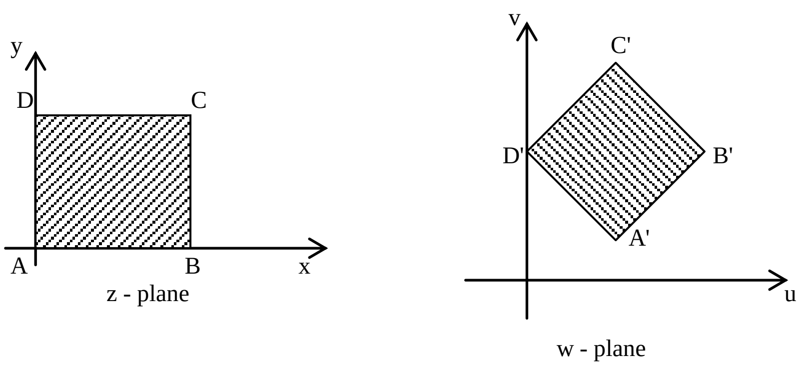
$\therefore$ -1 < u < 1

$\therefore$ x > 0 ad 0 < y < 2 $\Leftrightarrow$ v > 0 and -1 < u < 1.

Hence the given region is mapped into the region v>0 and -1<u<1 as shown in the figure.

Problem 2

Find the image of the square region with vertices (0, 0), (2, 0), (2, 2), (0, 2) under the transformation w = (1+i) z+ (2+i).



Solution

w = (1+i)z + (2+i)

Under this transformation,

A(0, 0) is mapped into A^1=(1+i) (0+0i)+2+i = 2+i = (2,1)

B(2, 0) is mapped into B^1=(1+i) (2+0i)+2+i = 2+2i+2+i = 4+3i = (4, 3)

C(2, 2) is mapped into C^1=(1+i) (2+2i)+2+i = 2+2i+2i = 2+5i = (2,5)

D(0, 2) is mapped into D^1 = (1+i) (0+2i)+2+i = 2i-2+2+i = 3i = (0,3)

∴ The required image region is another square $A^1 B^1 C^1 D^1$ as given in the figure.

Problem 3

Show that by means of the inversion $w = \frac{1}{z}$, the circle given by $|z-3|=5$ is mapped into the circle $|w+\frac{3}{16}| = \frac{5}{16}$

Solution

The circle $|z-3|=5$ is mapped into $|\frac{1}{w} - 3| = 5$ [Since $z = \frac{1}{w}$]

Now $|\frac{1}{w} -3| = 5 \Longrightarrow |\frac{1}{u+iv} -3| = 5$

$\Longrightarrow |\frac{1-3u-3iv}{u+iv}| = 5$

$\Longrightarrow |(1-3u)-3iv| = 5|u+iv|$

$\Longrightarrow (1-3u)^2 +9v^2 = 25(u^2 + v^2)$

$\Longrightarrow 1-6u + 9u^2 +9v^2 = 25u^2 + 25v^2$

$\Longrightarrow 16u^2 + 16v^2 + 6u-1= 0$

$\Longrightarrow u^2 + v^2 + \frac{6}{16}u - \frac{1}{16} = 0$

This is a circle with centre $(-\frac{3}{16}, 0)$ and radius $\sqrt{(\frac{3}{16})^2 + \frac{1}{16}}$

[Since centre = (-g, -f), radius = $\sqrt{g^2 + f^2 - c}$]

$= \sqrt{\frac{9}{256} + \frac{1}{16}}$

$= \sqrt{\frac{9+16}{256}} = \sqrt{\frac{25}{256}} = \frac{5}{16}$

Hence the image circle in the w-plane is given by the equation $|w+\frac{3}{16}|=\frac{5}{16}$

Problem 4

Find the image of the circle $|z-3i|=5$ under the map $w = \frac{1}{z}$.

Solution

The image of the circle $|z-3i| =3$ under the transformation $w = \frac{1}{z}$ is given by the equation $|\frac{1}{w} - 3i| =3$

Now $|\frac{1}{w} - 3i| =3$

$\Longrightarrow \left| \frac{1}{u+iv} - 3i \right| = 3$

$\Longrightarrow \left| \frac{1-3i(u+iv)}{u+iv} \right| = 3$

$\Longrightarrow \frac{|1-3iu+v|}{u+iv} = 3$

$\Longrightarrow |(1+3v)-3iu| = 3|u+iv|$

$\Longrightarrow \sqrt{(1+3v)^2 + (3u)^2} = 3\sqrt{u^2+v^2}$

Squaring on both sides,

$(1+3v)^2 + (3u)^2 = 9(u^2 + v^2)$

$\Longrightarrow 1+6v+9v^2+9u^2 = 9u^2+9v^2$

$\Longrightarrow 6v+1 = 0$ which represents a straight line.

Hence the image of the circle $|z-3i|=3$ under $w = \frac{1}{z}$ in the z-plane is the straight line $6v+1 = 0$ in the w-plane.

Problem 5

Find the image of the strip $2<x<3$ under $w = \frac{1}{z}$.

Solution

The transformation $w = \frac{1}{z}$ can be written in Cartesian coordinates as $z = \frac{1}{w}$

$x+iy = \frac{1}{u+iv}$

$= \frac{1}{u+iv} \text{ x } \frac{u-iv}{u-iv}$

i.e.) $x+iy = \frac{u-iv}{u^2+v^2}$

$\Longrightarrow x = \frac{u}{u^2+v^2}, \; y = \frac{-v}{u^2+v^2}$

Now $x > 2 \Longrightarrow \frac{u}{u^2+v^2} > 2$

$\Longrightarrow u > 2(u^2 + v^2)$

$\Longrightarrow 2u^2 + 2v^2 - u < 0$

$\Longrightarrow u^2 + v^2 - \frac{u}{2} < 0$

Now $u^2 + v^2 - \frac{u}{2} = 0$ is the equation of a circle with centre $(\frac{1}{4}, 0)$ and radius $\frac{1}{4}$.

Now $x < 3$

$\Longrightarrow \frac{u}{u^2+v^2} < 3$

$\Rightarrow\ 3(u^2 + v^2) > u$

$\Rightarrow\ 3(u^2 + v^2) - u > 0$

$\Rightarrow\ u^2 + v^2 - \frac{u}{3} > 0$

$2g = -\frac{1}{3} \Rightarrow g = -\frac{1}{6}$, f=0; c=0

$\therefore$ centre $(-g, -f) = (\frac{1}{6}, 0)$

radius $= \sqrt{g^2 + f^2 - c} = \sqrt{(\frac{1}{6})^2 + 0 - 0} = \frac{1}{6}$

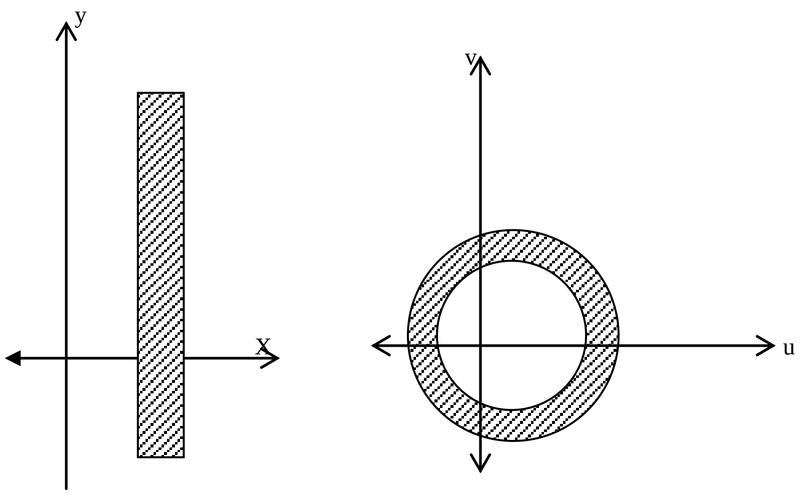
$u^2 + v^2 - \frac{u}{3} = 0$ is the equation of the circle with centre $(\frac{1}{6}, 0)$ and radius $\frac{1}{6}$.

$\therefore$ The region x > 2 mapped into a region represented by $u^2 + v^2 - \frac{u^2}{2} < 0$, which is the interior of the circle with centre $(\frac{1}{4}, 0)$ and radius $\frac{1}{4}$.

Also the region x < 3 is mapped into the exterior of the circle with centre $(\frac{1}{6}, 0)$ and radius $\frac{1}{6}$.

$\therefore$ The strip 2 < x < 3 is mapped. Onto the region bounded by the circles $u^2 + v^2 = \frac{u}{2}$ and

$u^2 + v^2 = \frac{u}{3}$ in the w. plane



z – plane w - plane

Bilinear Transformation

A transformation of the form $w = T(z) = \frac{az+b}{cz+d}$ …. (1) where a, b, c, d are complex constants and $ad-bc \neq 0$ is called bilinear transformation or Mobius transformation.

We define $T(\infty) = \frac{a}{c}$ and $T(-\frac{d}{c}) = \infty$. Hence T become a 1-1 onto map of the extended complex plane onto itself.

The inverse of (1) is given by

$w = \frac{az+b}{cz+d}$

$\Longrightarrow w(cz+d) = az + b$

$\Longrightarrow w\,cz + wd = az + b$

$\Longrightarrow w\,cz - az = -dw + b$

$\Longrightarrow z(cw - a) = -dw + b$

$\Longrightarrow z = \frac{-dw+b}{cw-a}$

$\therefore z = T^{-1}(w) = \frac{-dw+b}{cw-a}$ which is also a bilinear transformation.

Note :

All the elementary transformation (translation, rotation, magnification or contraction, Inversion) are bilinear transformations.

Theorem:

Any bilinear transformation can be expressed as a product of translation, rotation magnification or contraction and inversion.

Proof

Let $w = T(z) = \frac{az+b}{cz+d}$ where $ad-bc \neq 0$ be the given bilinear transformation

Case (i) c=0

Hence $d \neq 0$ [$\because ad - bc \neq 0$]

$\therefore (1) \Longrightarrow w = \frac{az+b}{d}$

$= (\frac{a}{d})z + (\frac{b}{d})$

Now, let $T_1(z) = (\frac{a}{d})z$ and $T_2(z) = z + (\frac{b}{d})$

T_1 and T_2 are elementary transformations and $(T_2 \circ T_1)(z) = T_2(T_1(z))$

$= T_2 \, [(\frac{a}{d}) \, z] = \frac{a}{d} z + \frac{b}{d}$

$= T(z)$

ie. $(T_2 \circ T_1)(z) = T(z)$

case (ii) $c \neq 0$

$$w = \frac{az+b}{cz+d}$$

$$= \frac{az+\frac{ad}{c}+b-\frac{ad}{c}}{c[z+\frac{d}{c}]}$$

$$= \frac{a\left[z+\frac{d}{c}\right]+b-(\frac{ad}{c})}{c[z+\frac{d}{c}]}$$

$$= \frac{a}{c} + \frac{b-(\frac{ad}{c})}{cz+d}$$

Now let $T_1(z) = cz+d$

$T_2(z) = \frac{1}{z}$

$T_3(z) = (b - \frac{ad}{c}) \, z$

$T_4(z) = z + (\frac{a}{c})$

$$(T_4 \circ T_3 \circ T_2 \circ T_1)(z) = T_4 \circ T_3 \circ T_2(T_1(z))$$

$$= T_4 \circ T_3 \circ T_2(cz+d)$$

$$= T_4 \circ T_3 \, (\frac{1}{cz+d})$$

$$= T_4 \, [(\frac{bc-ad}{c}) \, (\frac{1}{cz+d})]$$

$$= (\frac{bc-ad}{c}) \, (\frac{1}{cz+d}) + \frac{a}{c}$$

$$= (\frac{bc-ad+acz+ad}{c(cz+d)})$$

$$= \frac{(ac+b)}{(cz+d)} = \frac{az+b}{cz+d} = T(z)$$

Hence the theorem.

Solved Problems

Problem 1

Show that the transformation $w = \frac{5-4z}{4z-2}$ maps the unit circle $|z| = 1$ into the circle of radius unity and centre $-\frac{1}{2}$

Solution

$w = \frac{5-4z}{4z-2}$

$4wz - 2w = 5-4z$

$(4w+4)z = 5+2w$

$Z = \frac{5+2w}{4w+4}$

Now, $|z| = 1$

$\Longrightarrow z\,\bar{z} = 1$

$\Longrightarrow \left(\frac{5+2w}{4w+4}\right)\left(\frac{5+2\bar{w}}{4\bar{w}+4}\right) = 1$

$\Longrightarrow (5+2w)(5+2\bar{w}) = (4w+4)(4\bar{w}+4)$

$\Longrightarrow 25+4w\bar{w} + 10w + 10\bar{w} = 16w\bar{w} +16+16w + 16\bar{w}$

$\Longrightarrow 12w\bar{w} + 6\bar{w} + 6w - 9 = 0$

$\Longrightarrow w\bar{w} + \frac{1}{2}\bar{w} + \frac{1}{2}w - \frac{3}{4} = 0$

This represents the equation of the circle with center $-\frac{1}{2}$ and radius $\sqrt{\frac{1}{4}+\frac{3}{4}} = 1$. Hence the result.

[∵ Equation of the circle is $z\bar{z} + \bar{\alpha}z + \alpha\,\bar{z} + \beta = 0$

∴ centre = $-\alpha$ and radius $r = \sqrt{\alpha\,\bar{\alpha} - \beta}$]

Problem 2

Show that the transformation $w = \frac{2z+3}{z-4}$ maps the circle $z\bar{z} - 2(z + \bar{z}) = 0$ into a straight line given by $2(w+\bar{w})+3=0$

Solution

$w = \frac{2z+3}{z-4}$

∴ $w(z-4) = 2z+3$

$z(w-2) = 3 + 4w$

∴ $z = \frac{3+4w}{w-2}$

The image of the circle $z\bar{z} - 2(z + \bar{z}) = 0$ is

$\left(\frac{3+4w}{w-2}\right)\left(\overline{\frac{3+4w}{w-2}}\right) - 2\left[\frac{3+4w}{w-2} + \left(\overline{\frac{3+4w}{w-2}}\right)\right] = 0$

$\Longrightarrow (\frac{3+4w}{w-2})\ (\frac{3+4\overline{w}}{\overline{w}-2})$-2 $[\frac{3+4w}{w-2} + \frac{3+4\overline{w}}{\overline{w}-2}] = 0$

$\Longrightarrow \frac{9+12\overline{w}+12w+16w\overline{w}}{(w-2)(\overline{w}-2)}) - \frac{2[(3+4w)(\overline{w}-2)+(w-2)(3+4\overline{w})]}{(w-2)(\overline{w}-2)} = 0$

$\Longrightarrow \frac{16w\overline{w}+12\overline{w}+12w+9}{(w-2)(\overline{w}-2)}) - \frac{2[(3\overline{w}-6+4\overline{w}-8w)+(3w+4w\overline{w}-6-8\overline{w})]}{(w-2)(\overline{w}-2)} = 0$

$\Longrightarrow \frac{16w\overline{w}+12\overline{w}+12w+9-2(8w\overline{w}-5w-5\overline{w}-12)}{(w-2)(\overline{w}-2)} = 0$

$\Longrightarrow$ 12$\overline{w}$+12w+9+10w+10$\overline{w}$+24=0

$\Longrightarrow$ 22$\overline{w}$+22w+33 = 0

$\Longrightarrow$ 2(w+$\overline{w}$) +3 = 0 which is obviously a straight line.

Problem 3

Show that w = $\frac{z-1}{z+1}$ maps the imaginary axis in the z-plane onto the circle |w|=1. What portion of the z-plane corresponds to the interior of the circle |w|=1.

Solution

|w|=1

$\Leftrightarrow |\frac{z-1}{z+1}| = 1$

$\Leftrightarrow$ |z-1| = |z+1|

$\Leftrightarrow$ |x+iy-1| = |x+iy+1|

$\Leftrightarrow \sqrt{(x-1)^2+y^2} = \sqrt{(x+1)^2+y^2}$

$\Leftrightarrow (x-1)^2+y^2 = (x+1)^2+y^2$

$\Leftrightarrow x^2-2x+1 = x^2+2x+1$

$\Leftrightarrow$ 4x = 0

$\Leftrightarrow$ x = 0

Hence the transformation w = $\frac{z-1}{z+1}$ maps the imaginary axis x=0 onto the unit circle |w|=1.

Also since the point z=1 is mapped to w=0, it follows that the half plane x>0 is mapped onto the interior of the circle |w|=1.

Exercise

1. Show that the transformation $w = \frac{i-iz}{1+z}$ maps the unit circle $|z|=1$ into the real axis of the w-plane.
2. Show that the transformation $w = \frac{iz+2}{4z+i}$ maps the real axis in the z-plane to a circle in the w-plane. Find the centre and radius of the circle.

4.8 Cross Ratio

Definition

Let z_1, z_2, z_3, z_4 be four distinct points in the extended complex plane. The cross ratio of these four points denoted by (z_1, z_2, z_3, z_4) is defined by

$$(z_1, z_2, z_3, z_4) = \begin{cases} \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} \text{ if none of } z_1, z_2, z_3, z_4 \text{ is } \infty \\ \frac{(z_1-z_3)}{(z_1-z_4)} \text{ if } z_2 = \infty \\ \frac{(z_2-z_4)}{(z_1-z_4)} \text{ if } z_3 = \infty \\ \frac{(z_1-z_3)}{(z_2-z_3)} \text{ if } z_4 = \infty \\ \frac{(z_2-z_4)}{(z_2-z_3)} \text{ if } z_1 = \infty \end{cases}$$

Theorem 4.8.1

Any bilinear transformation preserver cross ratio.

Proof

Let $w = \frac{az+b}{cz+d}$, $ab-bc \neq 0$ be the given bilinear transformation

Let z_1, z_2, z_3, z_4 be four distinct points.

Let their images under this transformation bet w_1, w_2, w_3, w_4 respectively.

We assume that all the z_i and w_i are different from ∞.

Claim

$(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4)$

We have $w_i = \frac{az_i+b}{cz_i+d}$ (i=1, 2, 3, 4)

Now $w_1 - w_3 = \frac{az_1+b}{cz_1+d} - \frac{az_3+b}{cz_3+d}$

$$= \frac{(az_1+b)\,(cz_3+d)-(cz_1+d)((az_3+b)}{(cz_1+d)(cz_3+d)}$$

$(acz_1z_3+adz_1+bcz_3+bd)$

$$= \frac{-(acz_1z_3+bcz_1+adz_3+bd)}{(cz_1+d)(cz_3+d)}$$

$$= \frac{(ad-bc)z_1+(bc-ad)z_3}{(cz_1+d)(cz_3+d)}$$

$$= \frac{(ad-bc)x(z_1-z_3)}{(cz_1+d)(cz_3+d)}$$

$= k_1\ (z_1\text{-}z_3)$ (say)

Similarly $w_2 - w_4 = k_2(z_2 - z_4)$

$\therefore (w_1\text{-}w_3)\ (w_3\text{-}w_4) = k_1\ k_2(z_1\text{-}z_3)\ (z_2\text{-}z_4)$

$= k(z_1\text{-}z_3)\ (z_2\text{-}z_4)$

Similarly we can prove that

$(w_1 - w_4)\ (w_2\text{-}w_3) = k(z_1 - z_4)\ (z_2 - z_3)$

$$\therefore \frac{(w_1-w_3)\ (w_2-w_4)}{(w_1-w_4)\ (w_2-w_3)} = \frac{(z_1-z_3)\ (z_2-z_4)}{(z_1-z_4)\ (z_2-z_3)}$$

$\therefore (w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$

Hence the claim.

The proof is similar if one of the z_i or w_i is ∞.

Note 1

Four distinct point z_1, z_2, z_3, z_4 are collinear or concyclic iff (z_1, z_2, z_3, z_4) is real.

Note 2

The bilinear transformation which map the three points z_1, z_2, z_3 to three points w_1, w_2, w_3 is given by $(z, z_1, z_2, z_3) = (w, w_1, w_2, w_3)$

Solved Problems

Problem 1

Find the bilinear transformation which maps the points $z_1=2$, $z_2=i$, $z_3=-2$ onto $w_1=1$, $w_2=i$, $w_3=-1$ respectively.

Solution

Let the image of any point z under the required bilinear transformation be w. Since bilinear transformation preserves cross ratio we have,

$(w, 1, i, -1) = (z, 2, i, -2)$

$$\frac{(w-i)\ (1+1)}{(w+1)\ (1-i)} = \frac{(z-i)\ (2+2)}{(z+2)\ (2-i)}$$

$$\frac{2(w-i)}{w-i\,w+1-i} = \frac{4(z-i)}{2z-iz+4-2i}$$

(w-i) (2z-iz+4-2i) = (2z-2i) (w-iw+1-i)

$\Longrightarrow$ 2zw-iwz+4w-2iw-2iz-z-4i-2

=2zw-2iwz+2z-2iz-2iw-2w-2i-2

$\Longrightarrow$ iwz-3z+6w-2i = 0

$\Longrightarrow$ w(iz+6) = 3z+2i

$\Longrightarrow$ w = $\frac{3z+2i}{iz+6}$

This is the required bilinear transformation.

Problem 2

Find the bilinear transformation which maps z_1, z_2, z_3 to w_1, w_2, w_3 respectively where $z_1=\infty$, $z_2=i$, $z_3=0$ and $w_1=0$, $w_2=i$, $w_3=\infty$

Solution

Let the image of any point z under the required bilinear transformation be w. Since bilinear transformation preserves cross ratio we have

$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$

$\Longrightarrow$ (w, 0, i, ∞) = (z, ∞, i, 0)

$\Longrightarrow$ $\frac{w-i}{0-i} = \frac{z-i}{z-0}$

$\Longrightarrow$ zw-iz = -iz-1

$\Longrightarrow$ w = $\frac{-1}{z}$ which is the required bilinear transformation.

Problem 3

Find the bilinear transformation which maps the points $z_1=0$, $z_2=-i$ and $z_3=-1$ into $w_1=i$, $w_2=1$ and $w_3=0$ respectively.

Solution

Let the image of any point z under the required bilinear transformation be w. Since bilinear transformation preserves cross ratio we have

(z, 0, -i, -1) = (w, i, 1, 0)

$\therefore \frac{(z+i)(0+1)}{(z+1)(0+i)} = \frac{(w-1)(i-0)}{(w-0)(i-1)}$

$\Longrightarrow$ w(i-1)(z+i) = i^2(w-1)(z+1)

$\Longrightarrow$ w(zi-1–z-i) = - (wz+w-z-1)

$\Longrightarrow$ wzi-1w–zw-wi = -wz-w+z+1

$\Longrightarrow$ wi(z-1) = z+1

$\Longrightarrow w = \frac{z+1}{i(z-1)}$

$\therefore$ w = -i ($\frac{z+1}{z-1}$) which is the required bilinear transformation.

Problem 4

Determine the bilinear transformation which maps 0, 1, ∞ into i, -1, -i respectively. Under this transformation show that the interior of the unit circle of the z-plane maps onto the half plane left to the v-axis (left half of the w-plane).

Solution

The required bilinear transformation is given by the equation,

(w, i, -1, -i) = (z, 0, 1 ∞)

$\therefore \frac{(w+1)(i+1)}{(w+i)(i+1)} = \frac{z-1}{0-1}$

$\Longrightarrow \frac{2i(w+1)}{wi+w-1+i} = 1-z$

$\Longrightarrow$ 2iw+2i = wi+w-1+i-ziw-zw+z-iz

$\Longrightarrow$ wi-w+zwi+zw=-i-1+z-iz

$\Longrightarrow$ w[(i-1)+z(i+1)] = z(1-i)-(1+i)

$$\therefore w = \frac{z(1-i)-(1+i)}{z(1+i)-(1-i)}$$

$$= \frac{z-(\frac{1+i}{1-i})}{z-(\frac{1-i}{1+i})}$$

$$= \frac{z-i}{z-{}^{1}/_{i}} = \frac{z-i}{(z+i)}$$

$\therefore$ The required bilinear transformation is $w = \frac{z-i}{(z+i)}$

The equation of the left half of the w-plane and the interior of the unit circle in z-plane are Re w < 0 and |z|<1 respectively.

Now Re w < 0 $\Leftrightarrow$ Re $\left(\frac{z-i}{z+i}\right) < 0$

$\Leftrightarrow$ Re [$\frac{(z-i)(\bar{z}-i)}{|z+i|^2}$) < 0

$\Leftrightarrow$ Re [(z-i) ($\bar{z}$-i) < 0

$\Leftrightarrow$ Re $[(z\bar{z}\text{-}i\ (z+\bar{z})\text{-}1)] < 0$

$\Leftrightarrow$ Re $(z\bar{z})$ $-1 < 0$ $\quad [\because i(z + \bar{z})$ is imaginary]

$\Leftrightarrow [z]^2 < 1$

$\Leftrightarrow [z] < 1$

$\therefore$ The left half plane is mapped into the interior of the unit circle.

4.9 Fixed Points of Bilinear Transformations

If w = f(z) is any transformation from the z-plane to w-plane, the fixed points of the transformation are the solutions of the equation z=f(z).

Consider a bilinear transformation given by $w = \frac{az+b}{cz+d}$ where $ad\text{-}bc \neq 0$

The fixed points or invariant points of the bilinear transformation are given by the roots of the equation $z = \frac{az+b}{cz+d}$.

i.e. $cz^2 + (d\text{-}a)z\text{-}b = 0$

case (i) $c \neq 0;\ (d\text{-}a)^2+4bc \neq 0 \Longrightarrow$ 2 finite fixed points

case (ii) $c \neq 0;\ (d\text{-}a)^2+4bc = 0 \Longrightarrow$ one finite fixed point.

case (iii) $c = 0;\ a \neq d \Longrightarrow \infty$ and one finite fixed point.

case (iv) $c = 0,\ a = d \Longrightarrow \infty$ is the only fixed point.

Theorem 4.9.1

Any bilinear transformation having two finite fixed points α and β can be written in the form $\frac{w-\alpha}{w-\beta} = k\ (\frac{z-\alpha}{z-\beta})$.

Proof

Let T be the given bilinear transformation having α and β as fixed points. Let the image of any point γ under T be δ.

Then the bilinear transformation T is given by (w, δ, α, β) = (z, γ, α, β).

$$\therefore \frac{(w-\alpha)(\delta-\beta)}{(w-\beta)(\delta-\alpha)} = \frac{(z-\alpha)(\gamma-\beta)}{(z-\beta)(\gamma-\alpha)}$$

$$\Longrightarrow \frac{w-\alpha}{w-\beta} = k\,\frac{(z-\alpha)}{(w-\beta)} \text{ where } k = \frac{(\gamma-\beta)(\delta-\alpha)}{(\gamma-\alpha)(\delta-\beta)}$$

Definition

Let T be a linear transformation with two finite fixed points α, β. If $k = \frac{(\gamma-\beta)(\delta-\alpha)}{(\gamma-\alpha)(\delta-\beta)}$ is real, T is called hyperbolic and if $|k| = 1$, T is called elliptic.

Theorem 4.9.2

Any bilinear transformation having ∞ and $\alpha \neq \infty$ as fixed points can be written in the form $w - \alpha = k(z- \alpha)$.

Proof

Let T be the given bilinear transformation having ∞ and α as fixed points. Let the image of any point γ under T be δ.

Then the bilinear transformation is given by $(w, \delta, \alpha, \infty) = (z, \gamma, \alpha, \infty)$

$\therefore \frac{w-\alpha}{\delta-\alpha} = \frac{z-\alpha}{\gamma-\alpha}$

$\Rightarrow w-\alpha = \left(\frac{\delta-\alpha}{\gamma-\alpha}\right)(z-\alpha)$

$\Rightarrow w-\alpha = k(z-\alpha)$ where $k = \frac{\delta-\alpha}{\gamma-\alpha}$

Definition

A bilinear transformation with only one finite fixed point is called parabolic.

Theorem 4.9.3

Any bilinear transformation having ∞ as the only fixed point is a translation.

Proof

Let $w = \frac{az+b}{cz+d}$ be the bilinear transformation having ∞ as the only fixed point.

Then $c=0$ and $a=d$

$\therefore$ The bilinear transformation reduces to the form $w = \frac{az+b}{a}$

$\therefore$ $w = z + \left(\frac{b}{a}\right)$ which is a translation.

Theorem 4.9.4

Let C be a circle or a straight line and z_1, z_2 be inverse points or reflection points with respect to C. Let w_1, w_2 and C_1 be the images of z_1, z_2 and C under a bilinear transformation. Then w_1 and w_2 are inverse points or reflection points with respect to C_1. (i.e.) a bilinear transformation preserves inverse points.

Proof

Let the equation of C be

$\rho z\bar{z} + \alpha\bar{z} + \bar{\alpha}z + \beta = 0$ (1)

since z_1 and z_2 are inverse points w.r. to C by a theorem, we have

$\rho z_1\bar{z}_2 + \alpha\bar{z}_2 + \bar{\alpha}z_1 + \beta = 0$ (2)

Let the given bilinear transformation be $w = \frac{az+b}{cz+d}$ where ad-bc ≠ 0

∴ $z = \frac{dw-b}{-cw+a}$

Under the given bilinear transformation (1) is transformed into

$$\rho \left[\frac{dw-b}{-cw+a}\right]\left[\frac{\bar{d}\,\bar{w}-\bar{b}}{-\bar{c}\,\bar{w}+\bar{d}}\right] + \alpha\left[\frac{\bar{d}\bar{w}-\bar{\beta}}{-\bar{c}\,\bar{w}+\bar{a}}\right] + \bar{\alpha}\left[\frac{dw-\beta}{-cw+a}\right] + \beta = 0 \qquad (3)$$

Also (2) is transformed into,

$$\rho \left[\frac{dw_1-b}{-cw_1+a}\right]\left[\frac{\bar{d}\bar{w}_2-\bar{b}}{-\bar{c}\bar{w}_2+\bar{a}}\right] + \alpha\left[\frac{\bar{d}\bar{w}_2-\bar{b}}{-\bar{c}\bar{w}_2+\bar{a}}\right] + \bar{\alpha}\left[\frac{dw_1-b}{-cw_1+a}\right] + \beta = 0 \qquad (4)$$

clearly (4) is the condition for w_1 and w_2 to be the inverse points with respect to (3). Hence the theorem.

Note:

We shall regard the centre of the circle and ∞ as inverse points with respect to the circle.

Solved Problems

Problem 1

Find the invariant points of the transformation $w = \frac{z}{2-z}$.

Solution

The invariant points of w = f(z) are got from f(z) = z.

∴ $f(z) = z \Rightarrow z = \frac{z}{2-z}$

$2z - z^2 - z = 0$

$\Rightarrow z - z^2 = 0$

$\Rightarrow z(1 - z) = 0$

$\Rightarrow z = 0$ or $z = 1$

∴ The invariant points are 0, 1.

Problem 2

Find the invariant points of the transformation $w = \frac{1}{z-2i}$

Solution

f(z) = z

$\Rightarrow z = \frac{1}{z-2i}$

$\Rightarrow z^2 - 2iz - 1 = 0$

$\Longrightarrow (z-i)^2 = 0$

$\Longrightarrow (z-i) = 0$

$\Longrightarrow z = i$

Hence i is the only fixed point.

Problem 3

Prove that the transformation $w = \bar{z}$ is not a bilinear transformation.

Solution

Any bilinear transformation, other than the identity transformation has two fixed points. However the transformation $w = \bar{z}$ has infinitely many fixed points, namely all real numbers. Hence it is not a bilinear transformation.

4.10 Special Bilinear Transformation

Theorem 4.10.1

A bilinear transformation $w = \frac{az+b}{cz+d}$ where $ad-bc \neq 0$ maps the real axis into itself if and only if a, b, c, d are real. Further this transformation maps the upper half plane. Im $z \geq 0$ into the upper half plane Im $w \geq 0$ if and only if $ad-bc > 0$.

Proof

Suppose a, b, c, d are real.

Then obviously z is real $\Longrightarrow$ w is also real.

$\therefore$ The real axis is mapped into itself.

Conversely consider any bilinear transformation T that maps the real axis into itself.

$\therefore$ There exist real number x_1, x_2, x_3 such that $T(x_1)=1$, $T(x_2)=0$ and $T(x_3)= \infty$

$\therefore$ The bilinear transformation T is given by $(z, x_1, x_2, x_3) = (w, 1, 0, \infty)$

$$\Longrightarrow \frac{(z-x_2)(x_1-x_3)}{(z-x_3)(x_1-x_2)} = \frac{w-0}{1-0} = w$$

$\therefore w = \frac{az+b}{cz+d}$ where $a = x_1-x_3$; $b= -x_2(x_1-x_3)$, $c=(x_1-x_2)$ and $d=-x_3(x_1-x_2)$

Since x_1, x_2, x_3 are real, a, b, c, d are also real.

Now $2i \,\text{Im} w = w-\bar{w}$ $\quad [\because \text{Im } w = \frac{w-\bar{w}}{2i}]$

$$\Longrightarrow 2i \text{ Im } w = \frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}$$

$$= \frac{(acz\bar{z}+ adz +bc\bar{z}+ bd)- (acz\bar{z}+ ad\bar{z} + bcz +bd)}{(cz+d)(c\bar{z}+d)}$$

$= \frac{ad(z-\bar{z}) + bc(\bar{z}-z)}{|cz+d|^2}$

$= \frac{(ad-bc)(z-\bar{z})}{|cz+d|^2}$

$= 2i\ [\ \frac{(ad-bc)}{|cz+d|^2}\ ]$ Im z [$\because$ Im z = $\frac{z-\bar{z}}{2i}$]

$\therefore$ Im w = $\frac{(ad-bc)}{|cz+d|^2}$ Im z.

$\therefore$ The upper half plane Im z ≥ 0 is mapped onto the upper half plane.

Im w $\geq 0 \Leftrightarrow$ ad-bc>0.

Theorem 4.10.2

Any bilinear transformation which maps the unit circle |z|=1 onto the unit circle |w|=1 can be written in the form $w=e^{i\lambda}\ [\ \frac{z-\alpha}{\bar{\alpha}z-1}\ ]$ where k is real.

Further this transformation maps the circular disc $|z| \leq 1$ onto the circular disc $|w| \leq 1$ iff $|\alpha|<1$.

Proof

Let w = $\frac{az+b}{cz+d}$ where ad-bc $\neq 0$ be any bilinear transformation which maps |z|=1 onto |w|=1

0 to ∞ are inverse points with respect to the circle |w|=1.

Hence their pre-images (- $\frac{b}{a}$) and (- $\frac{d}{c}$) are inverse points with respect to |z|=1.

$\therefore (-\frac{b}{a})\ (-\frac{\bar{d}}{c}) = 1$ [using theorem in 4.2]

$\therefore$ If $\alpha = -(\frac{b}{a})$ then $(\frac{1}{\bar{\alpha}}) = -\frac{d}{c}$

$\therefore\ w = \frac{az+b}{cz+d}$

$= \frac{a[(z-(-\frac{b}{a})]}{c[z-(-\frac{d}{c})}$

$= (\frac{a}{c})\ [\frac{z-\alpha}{z-\frac{1}{\bar{\alpha}}}]$

$= (\frac{a\bar{\alpha}}{c})\ (\frac{z-\alpha}{\bar{\alpha}z-1})$

Now let |z| = 1 Hence |w|=1

$\therefore 1=|w| = |\frac{a\bar{\alpha}}{c}|\ |\frac{z-\alpha}{\bar{\alpha}z-1}|$

$= |\frac{a\bar{\alpha}}{c}|\ |\frac{\bar{z}-\bar{\alpha}}{\bar{\alpha}z-z\bar{z}}|$ [since $z\bar{z}=1$]

$= \left| \frac{a\bar{\alpha}}{c} \right| \left| \frac{\bar{z} - \bar{\alpha}}{\bar{\alpha} - \bar{z}} \right|$

$= \left| \frac{a\bar{\alpha}}{c} \right|$

Thus $\left| \frac{a\bar{\alpha}}{c} \right| = 1$

$\therefore \frac{a\bar{\alpha}}{c} = e^{i\lambda}$ for some real number λ

$\therefore w = e^{i\lambda} \left(\frac{z-\alpha}{\bar{\alpha}z-1} \right)$ where λ is real.

Now $w\bar{w} - 1 = e^{i\lambda} \left(\frac{z-\alpha}{\bar{\alpha}z-1} \right) e^{-i\lambda} \left[\frac{\bar{z} - \bar{\alpha}}{\alpha\bar{z}-1} \right] - 1$

$= \frac{(z-\alpha)(\bar{z} - \bar{\alpha})}{(\bar{\alpha}z-1)(\alpha\bar{z}-1)}) - 1$

$= \frac{(z\bar{z} - \bar{\alpha}z - \alpha\bar{z} - \alpha\bar{\alpha}) - (\alpha\bar{\alpha}z\bar{z} - \bar{\alpha}z - \alpha\bar{z} + 1)}{(\bar{\alpha}z-1)(\alpha\bar{z}-1)}$

$= \frac{z\bar{z} - (1-\alpha\bar{\alpha}) + \alpha\bar{\alpha} - 1)}{|\alpha\bar{z}-1|^2}$

$= \frac{(1-\alpha\bar{\alpha})(z\bar{z}-1)}{|\alpha\bar{z}-1|^2}$

The transformation maps $|z| \leq 1$onto $|w| \leq 1$

$\Leftrightarrow 1 - \alpha\bar{\alpha} > 0$

$\Leftrightarrow \alpha\bar{\alpha} < 1$

$\Leftrightarrow |\alpha| < 1$

Theorem 4.10.3

Any bilinear transformation which maps the real axis onto unit circle $|w|=1$ can be written in the form $w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}} \right)$ where λ is real.

Further this transformation maps the upper half plane Im $z \geq 0$ onto the unit circular disc $|w| \leq 1$ iff Im $\alpha > 0$.

Proof

Let $w = \frac{az+b}{cz+d}$ where $ad-bc \neq 0$ be any bilinear transformation which maps the real axis onto the unit circle $|w|=1$. 0 and ∞ are inverse points with respect to the unit circle $|w|=1$.

Hence their pre-images $-\left(\frac{b}{a} \right)$ and $-\left(\frac{d}{c} \right)$ are reflection points with respect to the real axis.

$\therefore$ If $\alpha = -\left(\frac{b}{a} \right)$ then $\bar{\alpha} = -\left(\frac{d}{c} \right)$

Now w = $\frac{az+b}{cz+d}$

$= \left(\frac{a}{c}\right)\left[\frac{z+\frac{b}{a}}{z+\frac{d}{c}}\right]$

$= \left(\frac{a}{c}\right)\left[\frac{z-\alpha}{z-\bar{\alpha}}\right]$

Now suppose z is real.

Hence |w|=1

$\therefore \left|\frac{a}{c}\right| \left|\frac{z-\alpha}{z-\bar{\alpha}}\right| = 1$

Now since z is real, $z = \bar{z}$ and hence

$|z - \alpha| = |\overline{z - \alpha}|$

$= |\bar{z} - \bar{\alpha}|$

$= |z - \bar{\alpha}|$

$\therefore \left|\frac{a}{c}\right| = 1$. Hence $\frac{a}{c} = e^{i\lambda}$, λ is real.

$\therefore w = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right)$ where λ is real the required transformation

Now $w\bar{w} - 1 = e^{i\lambda}\left(\frac{z-\alpha}{z-\bar{\alpha}}\right) e^{-i\lambda}\left(\frac{\bar{z}-\bar{\alpha}}{\bar{z}-\alpha}\right) - 1$

$= \left[\frac{z-\alpha}{z-\bar{\alpha}}\right]\left[\frac{\bar{z}-\bar{\alpha}}{\bar{z}-\alpha}\right] - 1$

$= \frac{(z\bar{z}-\bar{\alpha}z-\alpha\bar{z}+\alpha\bar{\alpha})-(z\bar{z}-\alpha z-\bar{\alpha}\bar{z}+\alpha\bar{\alpha})}{|z-\bar{\alpha}|^2}$

$= \frac{(\alpha z+\bar{\alpha}\bar{z}-\alpha\bar{z}+\bar{\alpha}z)}{|z-\alpha|^2}$

$= \frac{(\alpha-\bar{\alpha})(z-\bar{z})}{|z-\alpha|^2}$

$= \frac{(2i\text{Im}\,\alpha)(2i\text{Im}z)}{|z-\alpha|^2}$

$= \frac{-4\text{Im}z\ \text{Im}\,\alpha}{|z-\alpha|^2}$

∴ The bilinear transformation maps the upper half plane Im z ≥ 0 onto the disc |w| ≤ 1 iff Im α > 0.

Solved Problems

Find the general bilinear transformation which maps the unit circle |z|=1 onto |w|=1 and the points z=1 to w=1 and z=-1 to w=-1.

Solution

We know any bilinear transformation which maps $|z|=1$ onto $|w|=1$ is of the form $w = e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$ where λ is real. Since 1 and -1 are again mapped to 1, -1 respectively we have

$$1 = e^{i\lambda}\left(\frac{1-\alpha}{\bar{\alpha}-1}\right) \qquad (1)$$

$$-1 = e^{i\lambda}\left(\frac{-1-\alpha}{-\bar{\alpha}-1}\right) = e^{i\lambda}\left(\frac{1+\alpha}{(1+\bar{\alpha})}\right) \qquad (2)$$

Dividing (1) by (2)

$$-1 = \left(\frac{1-\alpha}{\bar{\alpha}-1}\right)\left(\frac{1+\bar{\alpha}}{(1+\alpha)}\right)$$

$$-(\bar{\alpha}-1)(1+\alpha) = (1-\alpha)(1+\bar{\alpha})$$

$\Longrightarrow -[\bar{\alpha} + \alpha\bar{\alpha}-1-\alpha] = 1+ \bar{\alpha} - \alpha - \alpha\bar{\alpha}$

$\Longrightarrow -\bar{\alpha} - \alpha\bar{\alpha}+1+\alpha = 1+ \bar{\alpha} - \alpha - \alpha\bar{\alpha}$

$\Longrightarrow 2\alpha-2\bar{\alpha} = 0$

$\Longrightarrow \alpha = \bar{\alpha} \qquad (3)$

Using (3) in (1) we get

$$1= e^{i\lambda}\left(\frac{1-\alpha}{\alpha-1}\right)$$

$$\Longrightarrow e^{i\lambda} = -1$$

$\therefore$ The required transformation is $w = \frac{\alpha-z}{\alpha z-1}$.

Problem 2

Prove that the transformation given by $\bar{\alpha}wz-bw + \bar{b}z+a=0$ maps the unit circle $|z|=1$ onto the unit circle $|w|=1$ if $|b| \neq |a|$.

Solution

$$\bar{a}wz - bw - \bar{b}z+a = 0$$

$$\therefore w = \frac{\bar{b}z-a}{\bar{a}z-b}$$

Now $w\bar{w} -1 = \left(\frac{\bar{b}z-a}{\bar{a}z-b}\right)\left(\frac{b\bar{z}-\bar{a}}{a\bar{z}-\bar{b}}\right) -1$

$$= \frac{(b\bar{b}z\bar{z}-\bar{a}\,\bar{b}z-ab\,\bar{z}+a\bar{a})-(\bar{a}az\bar{z}-\overline{ab}\,z-ab\,\bar{z}+b\bar{b})}{(\bar{a}z-b)(a\bar{z}-\bar{b})}$$

$$= \frac{|b|^2z\bar{z}+|a|^2-|a|^2z\overline{\bar{z}-|b|^2}}{|(\bar{a}z-b)|^2}$$

$$= \frac{z\bar{z}(|b|^2-|a|^2)-(|b|^2-|a|^2)}{|\bar{a}z-b|^2}$$

$$= \frac{(|b|^2-|a|^2)(z\bar{z}-1)}{|\bar{a}z-b|^2}$$

If $|b| \neq |a|$ then $w\bar{w}-1=0 \Leftrightarrow z\bar{z}-1=0$

∴ The unit circle $|z|=1$ is mapped onto the unit circle $|w|=1$ if $|b| \neq |a|$.

UNIT V

5.1 Complex Integration

Definition

Let $f(t) = u(t)+iv(t)$ be a continuous complex valued function defined on $[a, b]$.

We define $\int_a^b f(t)dt = \int_a^b u(t)dt+i\int_a^b v(t)dt$

Remark

1. $Re\int_a^b f(t)dt =\int_a^b Re[f(t)]dt.$
2. $Im\int_a^b f(t)dt = \int_a^b Im[f(t)]dt.$
3. $\int_a^b [f(t)dt + g(t)]dt = \int_a^b f(t)dt + \int_a^b g(t)dt$.
4. $\int_a^b cf(t)dt = c\int_a^b f(t)dt$ where c is any complex constant.

Lemma :

$|\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$

Proof

Let $\int_a^b f(t)dt = re^{i\theta}$

$|\int_a^b f(t)dt|=|re^{i\theta}| = |r(\cos\theta + i \sin \theta)|$

$= |r (\cos\theta + ir \sin \theta|$

$= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r$

$= e^{-i\theta} \int_a^b f(t)dt$

$= Re(e^{-i\theta} \int_a^b f(t)dt)$ (since r is real)

$= Re(\int_a^b e^{-i\theta} f(t)dt))$

$= \int_a^b Re(e^{-i\theta} f(t)dt))$

$\leq \int_a^b |e^{-i\theta} f(t)|dt$

$= \int_a^b |e^{-i\theta} ||f(t)|dt$

$= \int_a^b |f(t)|dt$

$[\therefore |e^{-i\theta} | = 1]$

Thus $|\int_a^b f(t)\,dt| \leq \int_a^b |f(t)|dt$

Definition

Let C be a piecewise differentiable curve given by the equation z=z(t) where a≤t≤b. Let f(z) be a continuous complex valued function defined in a region containing the curve C. We define $\int_C f(z)dz = \int_a^b f(z(t))\, z'(t)dt$.

Example 1

$\int_C \frac{dz}{z-a} = 2\pi i$ where C is the circle with centre a, radius r given by the equation $z = a+re^{it}$, $0 < t < 2\pi$.

$\int \frac{dz}{z-a} = \int_0^{2\pi} \frac{rie^{it}}{re^{it}}\, dt$

$= i\int_0^{2\pi} dt = 2\pi i$

Remark

1. $\int_{-C} f(z)dz = -\int_C f(z)dz$

2. $\int_c f(z)dz = -\int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \ldots + \int_{C_n} f(z)dz$ where

$C = C_1 + C_2 + \ldots. + C_n$.

Definition

Let C be a pricewise differentiable curve given by the equation z=z(t) where $a \leq t \leq b$. Then the length ℓ of C is defined by $\ell = \int_a^b |z'(t)|dt$.

Example 2

Consider the circle C with centre a and radius r. The parameter equation of C is given by $z=a+re^{it}$ where $0< t < 2\pi$.

$z'(t) = ire^{it}$

$\ell = \int_0^{2\pi} |z'(t)|dt$

$\therefore\ \ell = \int_0^{2\pi} |ire^{it}|dt$

$= \int_0^{2\pi} r\,dt = r(t)_0^{2\pi}$

$= 2\pi r$.

Theorem 5.1.1

$|\int_C f(z)dz| \leq M\ell$ where

$M = \max\{|f(z)|/z\in C\}$ and ℓ is the length of C.

Proof

Suppose C is given by the equation z = z(t) where $a \leq t \leq b$. By definition of M, we have $|f(z(t))| \leq M$ for all t, $a\leq t\leq b$. … (1)

$$\text{Now } \left|\int_c f(z)dz\right| = \left|\int_a^b f(z(t))z'(t)dt\right|$$

$$\leq \int_a^b |f(z(t))z'(t)|dt$$

$$= \int_a^b |f(z(t))||z'(t)|dt$$

$$\leq \int_a^b M|z'(t)|dt \text{ using (1)}$$

$$= M \int_a^b |z'(t)|dt = M\ell$$

$\therefore \left|\int_c f(z)dz\right| \leq M\ell$.

Solved Problem

1. Prove that $\int_C \frac{dz}{(z-a)^n} = \begin{cases} 0 & \text{if } n \neq 1 \\ 2\pi i & \text{if } n=1 \end{cases}$ where C is the circle with centre a and radius r and $n\in Z$.

Solution

The parametric equation of the circle C is given by $z-a=re^{i\theta}$, $0 \leq t \leq 2\pi$

$\frac{dz}{dt} = z'(t) = ire^{it}$.

$\Rightarrow dz = ire^{it}\, dt$

$$\text{Now } \int_c \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{ire^{it}}{(re^{it})^n} dt$$

$$= \frac{i}{r^{n-1}} \int_0^{2\pi} e^{i(1-n)t} dt$$

$$= \frac{i}{r^{n-1}} \left[\frac{e^{i(1-n)t}}{i(1-n)}\right]_0^{2\pi} \quad \text{provided } n \neq 1$$

$$= \frac{i}{(1-n)r^{n-1}} \left[e^{i(1-n)2\pi} - e^0\right]$$

$$= \frac{1}{(1-n)r^{n-1}} [1-1]$$

$$= 0$$

If n=1, $\int_C \frac{dz}{z-a} = 2\pi i$ (Refer example (1)]

Problem 2

Let C be the arc of the circle |z|=2 from z=2 to z=2i that lies in the first quadrant. Without actually evaluating the integral show that $\left|\int_C \frac{dz}{z^2+1}\right| = \frac{\pi}{3}$

Solution

Let $f(z) = \frac{1}{z^2+1}$

Since C is the circular arc of radius 2 lying in the first quadrant, the length ℓ of C is given by

$\ell = \frac{1}{4}(2\pi \text{ x } 2) = \pi$

Also on C, $|z^2+1| = |z^2-(-1)| \geq |z^2| - |-1|$

$= |z|^2-1$

$= 4-1=3$

Thus $|z^2+1| \geq 3$

$\Longrightarrow \left|\frac{1}{z^2+1}\right| \leq \frac{1}{3}$

∴ By theorem 5.1.1 $\left|\int_c \frac{dz}{z^2+1}\right| \leq \frac{\pi}{3}$

Problem 3

Show that $\int_C |z|^2dz = -1+i$ where C is the square with vertices O (0,0), A(1,0), B(1,1) and C(0, 1)

Solution

$C=C_1+C_2+C_3+C_4$ where C_1,C_2,C_3 and C_4 are the line segments OA, AB, BC and CO as shown in the figure. The parametric equation of C_1 is given by x=t and y=0 where $0\leq t\leq 1$.

Hence z(t)=t and z′(t)=1

$\therefore \int_{C_1} |z|^2dz = \int_0^1 t^2dt = \left[\frac{t^3}{3}\right]_0^1 = \frac{1}{3}$

The parametric equation of C_2 is given by y=t and x=1 where $0\leq t\leq 1$. Hence

Z(t) = 1+it

$\Longrightarrow z'(t)=i$

$\therefore \int_{C_2} |z|^2dz = \int_0^1 |1+it|^2 i\,dt$

$= i\int_0^1 (1+t^2)dt$

$= i\,[t+\frac{t^3}{3}]_0^1$

$= \frac{4i}{3}$

The parametric equation of C_3 is given by y=1 and x=1-t; $0 \leq t \leq 1$

Hence z(t) = (1-t)+i

$\Longrightarrow z'(t) = -1$

$\therefore \int_{C_3} |z|^2dz = \int_0^1 |(1-t)^2+1](-1)dt$

$= -\int_0^1 (t^2-2t+2)dt$

$= -[\frac{t^3}{3} - 2\,\frac{t^2}{2} + 2t]_0^1$

$= -\frac{4}{3}$

The parametric equation of C_4 is given by x=0, y=1-t, $0 \leq t \leq 1$

Hence z(t)=i(1-t) and z′(t)=-i

$\therefore \int_{C_4} |z|^2dz = \int_0^1 (1-t)^2\,(-i)dt$

$= -i\,[\frac{(1-t)^3}{3}]_0^1 = -\frac{i}{3}$

Hence $\int_C f(z)dz = \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3}$

$= -1 + i$

5.2 Cauchy’s Integral theorem

Definition

Let p(x, y) and q(x,y) be two real valued functions. Then the differential equation p(x, y)dx+q(x, y)dy = 0 is said to be exact if there exist a function u(x, y) such that $\frac{\partial u}{\partial x}$ = p and $\frac{\partial u}{\partial y}$ = q.

Note

$\int_C$ pdx+qdy depends only on the end points of C if and only if the integrand is exact.

Theorem 5.2.1

Let f(z) be a continuous complex valued function defined on a region D. Then $\int_C$ f(z)dz depends only on the end points of C if and only if there exists an analytic function F(z) such that F′(z)=f(z) in D.

Proof

$\int_C f(z)dz = \int_C f(z)(dx+idy)$ [since $z = x+iy$]

$= \int_C f(z)\, dx+if$

$\int_C f(z)dz$ depends only on the end points of C if and only if there exist a function F(z) defined on D such that $\frac{\partial F}{\partial x} = f(z)$ and $\frac{\partial F}{\partial y} = if(z)$.

$\therefore \frac{\partial F}{\partial x} = \frac{1}{i}\frac{\partial F}{\partial y}$ so that $\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$ which is the complex form of the Cauchy Rieman equation for F(z).

Since f(z) is continuous, the partial derivatives of F(z) are also continuous and hence F(z) is analytic in D and $F'(z)=f(z)$. Hence the theorem.

Corollary 1

Let f(z) be a continuous complex valued function defined on a region D then $\int_C f(z)\, dz = 0$ for every closed curve C lying in D iff there exist an analytic function F(z) such that $F'(z)=f(z)$ in D.

Corollary 2

$\int_C (z-a)^n dz = 0$ for every closed curve C provided $n \geq 0$.

Theorem 5.2.2 (Cauchy's theorem)

Let f be a function which is analytic at all points inside and on a simple closed curve C. Then $\int_C f(z)\, dz = 0$

Proof

Let D be the closed region consisting of all points interior to C together with the points on C.

Let $\varepsilon > 0$ be given.

Let C_j(j=1, 2, …, n) denote the boundaries of the squares and partial squares covering D such that there exist a point z_j lying inside or on C_j satisfying

$$\left|\frac{f(z)-f(z_j)}{z-z_j} f'(z_j)\right| < \varepsilon \qquad (1)$$

for all z distinct from z_j and lying within or on c_j.

$$\text{Let } \delta_j(z) = \begin{cases} \frac{f(z)-f(z_j)}{z-z_j} - f'(z_j) & \text{if } z \neq z_j \\ 0 & \text{if } z = z_j \end{cases}$$

Clearly $\delta j(z)$ is a continuous function

$$f(z) = f(z_j)-z_j f'(z_j)+ z f'(z_j)+(z-z_j)\ \delta j(z)$$

$$\therefore \int_{C_j} f(z)dz = \int_{C_j} f(z_j)dz-\int_{C_j} z_j f'(z_j)dz +\int_{C_j} z f'(z_j)dz +\int_{C_j} (z-z_j)\ \delta_j(z)dz$$

$$= f(z_j)\int_{C_j} dz-z_j f'(z_j)\int_{C_j} dz + f'(z_j) \int_{C_j} zdz + \int_{C_j} (z-z_j)\ \delta_j(z)dz$$

$$= \int_{C_j} (z-z_j)\ \delta_j(z)dz. \quad [\text{since } \int_{C_j} dz=0 \text{ and } \int_{C_j} zdz=0$$

$$\therefore \sum_{j=1}^{n} \int_{C_j} f(z)dz = \sum_{j=1}^{n} \int_{C_j} (z-z_j)\ \delta_j(z)dz \qquad (2)$$

Now in the sum $\sum_{j=1}^{n} \int_{C_j} f(z)dz$ the integrals along the common boundary of every pair of adjacent sub regions cancel each other (since the integral is taken in one direction along that line segment in one subregion and in the opposite direction in the other) (refer figure)

Hence only the integrals along the arcs which are the parts of C remain.

$$\therefore \sum_{j=1}^{n} \int_{C_j} f(z)dz = \int_{C} f(z)dz$$

$$\therefore \text{From (2)} \int_{C} f(z)dz = \sum_{j=1}^{n} \int_{C_j} (z-z_j)\ \delta_j(z)dz$$

$$\therefore \left|\int_{C} f(z)dz\right| = \left|\sum_{j=1}^{n} \int_{C_j} (z-z_j)\ \delta_j(z)dz\right| \leq \sum_{j=1}^{n} \int_{C} |(z-z_j)\delta_j(z)|dz$$

$$= \sum_{j=1}^{n} \int_{C_j} |z-z_j|\ |\delta_j(z)|dz$$

$$\therefore \left|\int_{C} f(z)dz\right| \leq \sum_{j=1}^{n} \int_{C_j} |z-z_j|\ |\delta_j(z)|dz \qquad (3)$$

Now if C_j is a square and s_j is the length of its side then $|z-z_j| < \sqrt{2}\ s_j$ for all z on C_j. Also from (1) we have $|\delta_j(z)|<\varepsilon$ and hence $\int_{C_j} |z-zj||\delta_j(z)|dz < (\sqrt{2}s_j\varepsilon)\ (4s_j)$ [by theorem 5.1.1]

$$= 4\ (\sqrt{2}A_j\varepsilon) \qquad (4)$$

Where A_j is the area of the squre C_j.

Similarly for a partial square with boundary C_j if l_j is the length of the arc of C which forms a part of C_j.

We have $\int_{C_j} |z-zj||\delta_j(z)|dz < (\sqrt{2}s_j)\varepsilon(4\ s_j+l_j)<(4\sqrt{2}A_j\ \varepsilon+\sqrt{2}\ sl_j$ (5)

Where S is the length of a side of some square containing the entire region D as well as all the squares originally used in covering D.

We observe that the sum of all A_j's that occur in the right hand side of (4) and (5) do not exceed S^2 and the sum of all the ℓ_j's is equal to L (the length of C) using (4) and (5) in (3) we obtain

$|\int_C f(z)dz| < (4\sqrt{2}S^2 + \sqrt{2}\ SL)\varepsilon$

$= k\varepsilon$ where $k = 4\sqrt{2}S^2 + \sqrt{2}\ SL$ is a constant.

Thus $|\int_C f(z)dz| < k\ \varepsilon$

Since ε is arbitrary we have $\int_C f(z)dz = 0$

Definition

A region D is said to be simply connected if every simple closed curve lying in D encloses only points of D.

Definition

A region which is not a simply connected is said to be multiply connected region.

Theorem 5.2.3 (Cauchy's theorem for simply connected regions)

Let f be a function which is analytic in a simply connected region D. Let C be any simple closed curve lying within D. Then $\int_C f(z)dz = 0$.

Theorem 5.2.4 (Cauchy's theorem for multiply connected regions)

Let C be a simple closed curve. Let C_j (j=1, 2, …, n) be a finite number of simple closed curves lying in the interior of C such that the interiors of C_j's are disjoint. Let D be the closed region consisting of all parts within and on C except the points interior to each C_j. Let B denote the entire oriented boundary of D consisting of C and all the Cj described in a direction such that the points of D are to the left of B. Let f be a function which is analytic in D. Then $\int_B f(z)dz = 0$.

5.3 Cauchy's Integral Formula

Theorem 5.3.1

Let f(z) be a function which is analytic inside and on a simple closed curve C. Let z_0 be any point in the interior of C. Then $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$.

Proof

Choose a circle C_o with centre z_0 and radius r_0 such that C_0 lies in the interior of C.

Now z_0 is the only point inside C at which the function $\frac{f(z)}{z-z_0}$ is not analytic and hence is analytic in the region D consisting of all points inside and on C except the points interior to C_0.

Hence $\int_C \frac{f(z)dz}{z-z_0} = \int_{C_0} \frac{f(z)dz}{z-z_0}$

$= \int_{C_0} \left(\frac{f(z)-f(z_0)+f(z_0)}{z-z_0}\right) d_z$

$= \int_{C_0} \frac{f(z)-f(z_0)}{z-z_0} d_z + \int_{C_0} \frac{f(z_0)}{z-z_0} dz$

$= \int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right) d_z + f(z_0) \int_{C_0} \frac{(dz)}{z-z_0}$

$= \int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right) d_z + f(z_0)\,(2\pi i)$

Thus $\int_C \frac{f(z)dz}{z-z_0} = \int_{C_0} \frac{f(z)-f(z_0)}{z-z_0} dz + (2\pi i)\, f(z_0)$ (1)

Claim

$\int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right) dz = 0$

Since f(z) is analytic inside and on C, it is continuous at z_0.

∴ Given $\varepsilon > 0$ there exist. $\delta>0$ such that $|z-z_0) < \delta \implies |f(z) - f(z_0)|<\varepsilon$

If we choose $r_0 < \delta$, then $|z-z_0| < r_0 \implies |f(z)-f(z_0)|<\varepsilon$

Hence $\left|\int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right) dz\right| < \left(\frac{\varepsilon}{r_0}\right)(2\pi\, r_0)$ [By theorem 5.1.1]

Thus $\left|\int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right) dz\right| < 2\pi\, \varepsilon$

Since ε is arbitrary we have $\int_{C_0} \left(\frac{f(z)-f(z_0)}{z-z_0}\right) dz=0$

Hence the claim.

From (1), we get $\int_C \frac{f(z)}{z-z_0} d_z = 2\pi i\, f(z_0)$

∴ $f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_0} dz.$

Theorem 5.3.2

Let f(z) be analytic in a region D bounded by two concentric circles C_1 and C_2 and on the boundary. Let z_0 be any point in D. Then

$f(z_0) = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{z-z_0} dz - \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{z-z_0}$

Example

Consider $\int_C \frac{dz}{z-3}$ where C is the circle |z-2|=5

Let f(z)=1

The point z=3 lies inside C.

Hence by Cauchy's integral formula,

$\int_C \frac{dz}{z-3} = 2\pi i\ f(3) = 2\pi i$

Example 2

Let C denote the unit circle |z|=1

Then $\int_C \frac{e^z}{z} dz = \int_C \frac{e^z}{z-0} dz$

$\int_C \frac{e^z}{z-0} dz = 2\pi i\ e^0 = 2\pi i$

Solved Problems

Problem 1

Evaluate using Cauchy's integral formula $\frac{1}{2\pi i} \int_C \frac{z^2+5}{z-3} dz$. Where C is |z|=4

Solution

$f(z) = z^2+5$ is analytic inside and on |z| = 4 and z = 3 lies inside it.

∴ By Cauchy's integral formula,

$\frac{1}{2\pi i} \int_C \frac{z^2+5}{z-3} dz = f(3)$

$= 3^2+5=14$

Problem 2

Evaluate $\int_C \frac{e^z}{z^2+4} dz$ where C is positively oriented circle |z-i|=2

Solution

$\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$

$= \frac{1}{4i} (\frac{1}{z-2i} - \frac{1}{z+2i})$ by partial fraction.

Now, 2i lies inside C and by Cauchy's integral formula we have $\int_C \frac{e^z}{z-2i} dz = 2\pi i\ e^{2i}$

Also -2i lies outside C and hence $\int_C \frac{e^z}{z^2+4} dz$ is analytic inside and on C.

Hence by Cauchy's theorem $\frac{e^z}{z+2i}$ dz = 0

$\therefore \int_C \frac{e^z}{z^2+4}\,dz = \frac{1}{4i}\,(2\pi i\, e^{2i} - 0) = \frac{\pi}{2}\, e^{2i}$

Problem 3

Evaluate $\int_C \left(\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\right)dz$

Where C is the circle |z|=3

Solution

By partial fraction

$$\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

Let $f(z) = \sin \pi z^2 + \cos \pi z^2$

Then f(z) is analytic inside and on C and the points 1 and 2 lie inside C. Hence by Cauchy's integral formula

$$\int_C \frac{f(z)}{z-1}\,dz = 2\pi i f(1)$$

$$= 2\pi i(\sin \pi + \cos \pi)$$

$$= 2\pi i(0+(-1)) = -2\pi i$$

Similarly $\int_C \frac{f(z)}{z-2}\,dz = 2\pi i f(2)$

$$= 2\pi i(\cos 4\pi + \sin 4\pi)$$

$$= 2\pi i(1+0) = 2\pi i$$

Hence $\int_C \frac{f(z)}{(z-1)(z-2)}\,dz = 2\pi i - (-2\pi i) = 4\pi i$

Problem 4

Evaluate $\frac{z\,dz}{(9-z^2)(z+i)}$ where C is the circle |z|=2 taken in the positive sense.

Solution

Let $f(z) = \frac{z}{9-z^2}$

Clearly f(z) is analytic with in and on C. By Cauchy's integral formula,

$$\int_C \frac{z\,dz}{(9-z^2)(z+i)} = \int_C \frac{f(z)}{z+i}\,dz$$

$$= 2\pi i\, f(-i) = 2\pi i\left(\frac{-i}{10}\right) = \frac{\pi}{5}$$

Exercise

1. Evaluate $\int_C \frac{zdz}{z^2-1}$ where C is the positively oriented circle |z|=2. Ans : 2πi
2. Show that $\frac{1}{2\pi i}\int_C \frac{e^{zt}\,dt}{z^2+1}$ = sin t if t > 0 and C is the circle |z|=3.

Theorem 5.3.3 (Morera's theorem)

If f(z) is continuous in a simply connected domain D and if $\int_C$ f(z)dz=0 for every simple closed curve C lying in D then f(z) is analytic in D. (This theorem is the converse of Cauchy's theorem)

Proof

By corollary 1 of theorem 5.2.1 there exists an analytic function F(z) such that $F^1(z)=f(z)$ in D.

Also we know the derivative of an analytic function is an analytic function.

Hence $F^1(z)$ is analytic in D

∴ f(z) is analytic in D.

Theorem 5.3.4

Let f be analytic inside and on a simple closed curve C. Let z be any point inside C. Then $f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\varepsilon)}{(\zeta-z)^2}\,d\zeta$

Proof

By Cauchy's integral formula we have $f(z) = \frac{1}{2\pi i}\int_C \frac{f(\varepsilon)}{\zeta-z}\,d\zeta$

$$\therefore \frac{f(z+h)-f(z)}{h} = \frac{1}{h}\left[\frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-(z+h)}\,d\zeta - \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta-z}\,d\zeta\right.$$

$$= \frac{1}{h(2\pi i)}\int_C \frac{f(\zeta)}{\zeta-z-h} - \frac{f(\zeta)}{\zeta-z})\,d\zeta$$

$$= \frac{1}{h2\pi i}\int_C\left[\frac{hf(\zeta)}{(\zeta-z-h)(\zeta-z)}\right]d\zeta$$

$$= \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)} \qquad (1)$$

Now $\int_C \frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)} - \int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2}$

$$= \int_C\left[\frac{f(\zeta)}{(\zeta-z-h)(\zeta-z)} - \frac{f(\zeta)}{(\zeta-z)^2}\right]d\zeta$$

$$= \int_C \frac{f(\zeta)}{(\zeta-z)}\left[\frac{1}{\zeta-z-h} - \frac{1}{\zeta-z}\right]d\zeta$$

$$= \int_C \frac{f(\zeta)}{(\zeta-z)}\left[\frac{\zeta-z-(\zeta-z-h)}{(\zeta-z-h)(\zeta-z)}\right]d\zeta$$

$$= h \int_C \frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)^2}$$

$$\therefore \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2} = \frac{h}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)^2}$$

$$\therefore \frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2} = \frac{h}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)^2} \qquad (2) \qquad [\text{using } (1)]$$

Now let M denote the maximum value of $|f(\zeta)|$on C. Let L be the length of C and d be the shortest distance from z to any point on the curve C.

$\therefore$ For any point ζ on C we have $|\zeta-z| \geq d$ and $|\zeta-z-h| \geq |\zeta-z| - |h| \geq d-|h|$

Hence $\left|\frac{f(\zeta)}{(\zeta-z)^2(\zeta-z-h)}\right| \leq \frac{M}{d^2(d-|h|)}$

From (2) we get

$$\left|\frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2}\right| \leq \frac{|h|}{2\pi}\left(\frac{ML}{d^2(d-|h|)}\right)$$

$$\therefore \lim_{h\to 0}\left(\frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2}\right) = 0$$

$$\therefore \lim_{h\to 0}\left(\frac{f(z+h)-f(z)}{h}\right) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2}$$

$$\therefore f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2})$$

Remark

By using induction on n we can prove that for any positive integer n we have

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta-z)^{n+1}}) \, d\zeta$$

Note

Thus an analytic function has derivatives of all orders and the derivate of an analytic function is again analytic.

Thorem 5.3.5

(Cauchy's Inequality)

Let f(z) be analytic inside and on the circle C with centre z_0 and radius r. Let M denote the maximum of $|f(z)|$ on C. Then $|f^{(n)}(z_0) \leq \frac{n!\,M}{r^n}$

Proof

M denote the maximum of $|f(z)|$ on C.

$\therefore |f(z)| \leq M$ on C

We have $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{n+1}}$

$|f^{(n)}(z_0)| = \left| \frac{n!}{2\pi i} \int_C \frac{f(z)dz}{(z-z_0)^{n+1}} \right|$

$\leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} (2\,\pi r)$ (by theorem 5.1.1)

$= \frac{n!\,M}{r^n}$

Hence $|f^{(n)}(z_0)| \leq \frac{n!\,M}{r^n}$

Theorem 5.3.6 (Liouville's theorem)

A bounded entire function in the complex plane is constant.

Proof

Let f(z) be a bounded entire function. Since f(z) is bounded there exist a real number M such that $|f(z)| \leq M$ for all z. Let z_0 be any complex number and r>0 be any real number.

By Cauchy's inequality we have $|f'(z_0)| \leq \frac{M}{r}$. Taking the limit as $r \to \infty$ we get $f'(z_0)=0$. Since z_0 is arbitrary, $f'(z_0)|=0$ for all z in the complex plane.

∴ f(z) is a constant function.

Theorem 5.3.7

(Fundamental theorem of algebra)

Every polynomial of degree ≥ 1 has atleast one zero (root) in C.

Proof

Let f(z) be a polynomial of degree ≥1. Suppose f(z) has no zero in C. Then $f(z) \neq 0$ for all z.

Further f(z) is an entire function in the complex plane.

∴ $\frac{1}{f(z)}$ is also an entire function.

Also as $z \to \infty$, $f(z) \to \infty$

∴ $\frac{1}{f(z)} \to 0$ as $z \to \infty$

Solved Problems

Problem 1

Evaluate $\int_C \frac{z^3 dz}{(2z+i)^3}$ where C is the unit circle.

Solution

$$\int_C \frac{z^3 dz}{(2z+i)^3} = \frac{1}{8}\int_C \frac{z^3 dz}{(z+\frac{i}{2})^3}$$

Let f(z) = z^3. Then f ′(z)=$3z^2$ and f ′′(z)=6z.

Also $\frac{-i}{2}$ lies inside C

Hence $\int_C \frac{z^3 dz}{(2z+i)^3} = \frac{1}{8}\left(\frac{2\pi i}{2!}\right) f''\left(\frac{-i}{2}\right)$

$$= \frac{2\pi i}{16} \times 6 (- {}^{i}/_{2})$$

$$= \frac{3\pi}{8}$$

Problem 2

Evaluate : $\int_C \frac{\sin\ 2zdz}{(z-\frac{\pi i}{4})^4}$ where C is |z|=1,

Solution

Let f(z) = sin 2z

Since f(z) is analytic and $\frac{\pi i}{4}$ lies inside C

$\therefore \int_C \frac{\sin 2z}{(z-\pi i)^4} dz = \frac{2\pi i}{3!} f'''(\frac{\pi i}{4})$

Now f′(z) = 2 cos 2z, f′′(z)=-4sin 2z

f′′′ (z) = -8 cos 2z

Hence f′′′ $(\frac{\pi i}{4})$ = -8 cos $(\frac{\pi i}{2})$

= -8 cos h $({}^{\pi}/_{2})$

$\therefore \int_C \frac{\sin z}{(z-\pi i)^4} dz = \frac{-8\pi i}{3}$ cos h $({}^{\pi}/_{2})$

Problem 3

Evaluate $\int_C \frac{e^z}{(z+2)(z+2)^2}$ dz where C is |z|=3

Solution

$$\frac{1}{(z+2)(z+1)^2} = \frac{(z+2)-(z+1)}{(z+2)(z+1)^2}$$

$$= \frac{1}{(z+1)^2} - \frac{1}{(z+2)(z+1)}$$

$$= \frac{1}{(z+1)^2} - \frac{1}{z+1} + \frac{1}{z+2}$$

$\int_C \frac{e^z}{(z+2)(z+1)^2} dz = \int_C \frac{e^z}{(z+2)} dz - \int_C \frac{e^z}{(z+1)} dz + \int_C \frac{e^z}{(z+1)^2} dz$

We note that z=-2, -1 lie in the interior of C

Let $f(z) = e^z$. It is analytic in C.

Also $f'(z) = e^z$

By Cauchy's integral formula,

$\int_C \frac{e^z}{z+2} dz = 2\pi i\ f(-2)$

$= 2\pi i\ e^{-2}$

$\int_C \frac{e^z}{z+1} dz = 2\pi i\ f(-1) = 2\pi i\ e^{-1}$

$\int_C \frac{e^z}{(z+1)^2} dz = \left(\frac{2\pi i}{1!}\right) f'(-1) = 2\pi i\ e^{-1}$

$\therefore \int_C \frac{e^z}{(z+2)(z+1)^2} dz = 2\pi i\ [e^{-2}-e^{-1}+e^{-1}]$

$= 2\pi i\ e^{-2}$

Exercise

1. Evaluate $\int_C \frac{(e^z+z \sin h\ z)dz}{(z-\pi i)^2} dz$ where C in the circle |z|=4. [Ans : $-2\pi i(1+\pi i)$]
2. Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle |z|=2. [Ans : $\frac{8\pi i e^{-2}}{3}$]

5.4 Taylor's Series

Theorem 5.4.1 (Taylor's theorem)

Let f(z) be analytic in a region D containing z_0. Then f(z) can be represented as a power series in $z-z_0$ given by $f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots$

$\dots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^2 + \dots$

The expansion is valid in the largest open disc with centre z_0 contained in D.

Proof

Let r>0 be such that the disc $|z-z_0|<r$ is contained in D.

Let $0 < r_1 < r$. Let C_1 be the circle $|z-z_0| = r_1$. By Cauchy's integral formula, we have

$$f(z) = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)}{(\zeta-z)} d\zeta \qquad (1)$$

Also by theorem on higher derivatives we have,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{(\zeta-z)^{n+1}} \qquad (2)$$

Now $\frac{1}{\zeta-z} = \frac{1}{(\zeta-z_0)-(z-z_0)}$

$$= \frac{1}{(\zeta-z_0)\left(1-\frac{z-z_0}{\zeta-z_0}\right)}$$

$$= \frac{1}{(\zeta-z_0)}\left[1+\left(\frac{z-z_0}{\zeta-z_0}\right)+\left(\frac{z-z_0}{\zeta-z_0}\right)^2+\ldots+\left(\frac{z-z_0}{\zeta-z_0}\right)^{n-1}+\frac{\left(\frac{z-z_0}{\zeta-z_0}\right)^n}{1-\left(\frac{z-z_0}{\zeta-z_0}\right)}\right]$$

(using the identity $\frac{1}{1-\alpha} = 1+\alpha+\alpha^2+\ldots.+\alpha^{n-1}+\frac{\alpha^n}{1-\alpha}$

$$= \frac{1}{\zeta-z_0}+\frac{z-z_0}{(\zeta-z_0)^2}+\frac{(z-z_0)^2}{(\zeta-z_0)^3}+\ldots.+\frac{(z-z_0)^{n-1}}{(\zeta-z_0)^n}+\frac{(z-z_0)^n}{(\zeta-z_0)^n(\zeta-z)}$$

Now multiplying throughout by $\frac{f(\zeta)}{2\pi i}$, integrating over C_1 and using (1) and (2) we get

$$\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{\zeta-z} = \frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{\zeta-z_0}+\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^2}\,.\,(z-z_0)+\ldots+\frac{(z-z_0)^{n-1}}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^n}+R_n \quad (3)$$

where $R_n = \frac{(z-z_0)^n}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{(\zeta-z)(\zeta-z_0)^n}$

$$\Longrightarrow f(z) = f(z_0)+f'(z_0)(z-z_0)+\frac{f''(z_0)}{2!}(z-z_0)^2+\ldots+\frac{f^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1}+R_n$$

Here ζ lies on C_1 and z lies in the interior of C_1 so that $|\zeta-z_0|=r_1$ and $|z-z_0|<r_1$

$\therefore |\zeta-z|=|(\zeta-z_0)-(z-z_0)| \geq |\zeta-z_0|-|z-z_0| = r_1-|z-z_0|$

$\therefore \frac{1}{|\zeta-z|} \leq \frac{1}{r_1-|z-z_0|}$

Let M denote the maximum value of $|f(z)|$ on C_1.

Then $|R_n| \leq \frac{|z-z_0|^n}{2\pi}\;\frac{M(2\pi r_1)}{(r_1-|z-z_0|)\,r_1^n}$

[Since By theorem, $|\int_{C_1} f(z)dz| \leq Ml$]

$$= \frac{m|z-z_0|}{(r_1-|z-z_0|)}\left(\frac{|z-z_0|}{r_1}\right)^{n-1}$$

Also $\frac{|z-z_0|}{r_1} < 1$

Hence $\lim_{n\to\infty} R_n=0$

$\therefore$ Taking limit as $n\to\infty$ in (3) we get, $f(z)=f(z_0)+\frac{f'(z_0)}{2!}(z-z_0)+f''(z_0)x(z-z_0)^2+\ldots$

$+\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n+\ldots$

Note 1

The above series is called Taylor's series of f(z) about the point z_0. The expansion is valid in some neighbourhood of z_0.

Note 2

The Taylor series expansion of f(z) about the point zero is called the Maclaurin's series. Thus the Maclaurin's series of f(z) is given by

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) : \ldots + \frac{z^n}{n!} f^{(n)}(0) + \ldots.$$

Example 1

The Taylor's series for $f(z) = \frac{1}{z}$ about z=1 is given by,

$$\frac{1}{z} = f(1) + \frac{f'(1)}{1!}(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \frac{f'''(1)}{3!}(z-1)+\ldots.$$

Now $f(z) = \frac{1}{z} \Longrightarrow f(1)=1$

$f'(z) = -\frac{1}{z^2} \Longrightarrow f'(1)-1$

$f''(z) = \frac{2}{z^3} \Longrightarrow f''(1)=2$

$f'''(z) = -\frac{6}{z^4} \Longrightarrow f'''(1)=-6$

Hence the Taylor's series expansion for $\frac{1}{z}$ about 1 is $\frac{1}{z} = 1-(z-1) + (z-1)^2-(z-1)^3+\ldots.$

This expansion is valid in the disc $|z-1|<1$.

Example 2

Let $f(z) = e^z$

Then $f^{(n)}(z)=e^z$ for all n and hence $f^{(n)}(0)=1$.

Hence the Maclaurin's series for e^z is given by $e^z = 1+ \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots + \frac{z^n}{n!} + \ldots$

Maclaurin's series expansion of some of the standard functions are given below

1. $e^{-z} = 1 - \frac{z}{1!} + \frac{z^2}{2!} \ldots +(-1)^n \frac{z^n}{n!} + \ldots \ (|z| < \infty)$
2. $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} \ldots \ (|z| < 0)$
3. $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots + (-1)^{n-1} \frac{z^{2n-2}}{(2n-2)!} \ldots \ (|z| < \infty)$
4. $\sin hz = \frac{z}{1!} + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots + \frac{z^{2n-1}}{(2n-1)!} \ldots \ (|z| < \infty)$
5. $\cos hz = 1+ \frac{z^2}{2!} + \frac{z^4}{4!} + \ldots + \frac{z^{2n}}{(2n)!} + \ldots \ (|z| < \infty)$

6. $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots + (-1)^n z^n + \dots \; (|z|<1)$

7. $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots + z^n + \dots \; (|z|<1)$

8. $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} \dots (-1)^{n-1} \frac{z^n}{n} + \dots \; (|z|<1)$

9. $\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} \dots - \frac{z^n}{n} + \dots \; (|z|<1)$

Solved Problems

Problem 1

Expand cos z into a Taylor's series about the point $z = \frac{\pi}{2}$ and determine the region of convergence.

Solution

Let f(z) = cos z.

∴ The Taylor's series for f(z) about $z = \frac{\pi}{2}$ is $f(z) = f(\frac{\pi}{2}) + \frac{(z-\frac{\pi}{2})}{1!} f'(\frac{\pi}{2}) + \frac{(z-\pi/2)^2}{2!}$

$$f''(\pi/2) + \frac{(z-\pi/2)^3}{3!} f'''(\pi/2) + \dots$$

Now f(z) = cos z. Hence $f(\pi/2) = 0$.

$f'(z) = -\sin z$. Hence $f'(\pi/2) = -\sin \pi/2 = -1$

$f''(z) = -\cos z$. Hence $f''(\pi/2) = -\cos \pi/2 = 0$

$f'''(z) = \sin z$. Hence $f'''(\pi/2) = \sin \pi/2 = 1$

∴ The Taylor's series for cos z about $z = \pi/2$ is $\cos z = -\frac{(z-\pi/2)}{1i}$

$$+ \frac{(z-\pi/2)^3}{3!} - \frac{(z-\pi/2)^5}{5!} + \dots$$

The expansion is valid throughout the complex plane.

Problem 2

Expand f(z) = sin z a Taylor's series about $z = \frac{\pi}{4}$ and determine the region of convergence of this series.

Solution

The Taylor's series for f(z) about $z = \frac{\pi}{4}$ is $f(z) = f(\pi/4) + \frac{(z-\pi/4)}{1!} f'(\pi/4)$

$$+ \frac{(z-\pi/4)^2}{2!} f''(\pi/4) + \dots.$$

Here f(z) = sin z. Hence $f(\pi/4) = \frac{1}{\sqrt{2}}$

$f'(z) = \cos z$. Hence $f'(\pi/4) = \frac{1}{\sqrt{2}}$

$f''(z) = -\sin z$. Hence $f''(\pi/4) = -\sin \pi/4 = \frac{-1}{\sqrt{2}}$

$f'''(z) = -\cos z$. Hence $f'''(\pi/4) = -\frac{1}{\sqrt{2}}$

∴ The Taylor's series for sin z

about $z = \pi/4$ is $\sin z = \frac{1}{\sqrt{2}} + \frac{(z-\pi/4)}{1!}(\frac{1}{\sqrt{2}}) - \frac{(z-\pi/4)^2}{2!}(\frac{1}{\sqrt{2}}) + \ldots$

$= \frac{1}{\sqrt{2}}[1 + \frac{(z-\pi/4)}{1!} - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \ldots]$

This expansion is valid in the entire complex plane.

Problem 3

Expand $f(z) = \frac{z-1}{z+1}$ as a Taylor's series

(i) about the point z = 0

(ii) about the point z = 1. Determine the region of convergence in each case.

i. $f(z) = \frac{z-1}{z+1}$

$= (z-1)(1+z)^{-1}$

$= (z-1)(1-z+z^2-z^3+\ldots)$ if $|z|<1$

$= (z-z^2+z^3-\ldots) - (1-z+z^2-z^3\ldots)$

$= -1+2z-2z^2+2z^3+\ldots$

The region of convergence is $|z| < 1$

ii) $f(z) = \frac{z-1}{z+1}$

$= \frac{z-1}{(2+z-1)}$

$= \frac{z-1}{2(1+\frac{z-1}{2})} = \frac{z-1}{2}(1+\frac{z-1}{2})^{-1}$

$= \frac{z-1}{2}[1 - \frac{z-1}{2} + (\frac{z-1}{2})^2 - (\frac{z-1}{2})^3 + \ldots]$ if $|\frac{z-1}{2}|<1$

$= \frac{z-1}{2} - \frac{(z-1)^2}{2^2} \frac{(z-1)^3}{2^3} \ldots$

The region of convergence is given by $|\frac{z-1}{2}| < 1$ which is same as the circular disc $|z-1|<2$

Problem 4

i. $\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n$ when $|z+1|<1$

ii. $\frac{1}{z^2} = \frac{1}{4} + \frac{1}{4}\sum_{n=1}^{\infty} (-1)^n (n+1)\left(\frac{z-1}{2}\right)^n$ when $|z-2|<2$

Solution

i. $\frac{1}{z^2} = \frac{1}{[1-(z+1)]^2}$

$= [1-(z+1)]^{-2}$

$= 1+2(z+1)+3(z+1)^2+4(z+1)^3 + \ldots$ if $|z+1|<1$

$= 1+\sum_{n=1}^{\infty} (n+1)(z+1)^n$ when $|z+1|<1$

ii. $\frac{1}{z^2} = \frac{1}{[z-2+2]^2}$

$= \frac{1}{[2(1+\frac{z-2}{2})]^2}$

$= \frac{1}{4}\left(1 + \frac{z-2}{2}\right)^{-2}$

$= \frac{1}{4}\left[1-2\left(\frac{z-2}{2}\right) + 3\left(\frac{z-2}{2}\right)^2 - \ldots\right]$ if $\left|\frac{z-2}{2}\right| < 1$

$= \frac{1}{4} - \frac{1}{4} \times 2\left(\frac{z-2}{2}\right) + \frac{1}{4} \times 3\left(\frac{z-2}{2}\right)^2 - \ldots$

$= \frac{1}{4} + \frac{1}{4}\sum_{n=1}^{\infty} (-1)^n (n+1)\left(\frac{z-2}{2}\right)^n$

Here the region of convergence is $\left|\frac{z-2}{2}\right| < 1$ which is the same as the circular disc $|z-2|<2$.

Problem 5

Expand ze^{2z} in a Taylor's series about $z=-1$ and determine the region of convergence.

Solution

Let $f(z) = ze^{2z}$

$= z\, e^{2(z+1-1)}$

$= z\, e^{2(z+1)} \cdot e^{-2}$

$= \frac{1}{e^2}\left[(z+1-1)e^{2(z+1)}\right]$

$= \frac{1}{e^2}\left[(z+1)e^{2(z+1)} - e^{2(z+1)}\right]$

$= \frac{1}{e^2}\left[(z+1)\left\{1+ \frac{2(z+1)}{1!} + \frac{[2(z+1)]^2}{2!} + \ldots\right\} - \left\{1+ \frac{2(z+1)}{1!} + \frac{[2(z+1)]^2}{2!} + \ldots\right\}\right]$

$= \frac{1}{e^2}$ [{(z+1) + $\frac{2(z+1)^2}{1!}$ + $\frac{2^2(z+1)^3}{2!}$ + …}- {1+ $\frac{2(z+1)}{1!}$ + $\frac{2^2(z+1)]^2}{2!}$ +…}]

$= \frac{1}{e^2}$ [-1+ (1- $\frac{2}{1}$) (z+1) + ($\frac{2}{1!}$ - $\frac{2^2}{2!}$) $(z+1)^2$ + ($\frac{2^2}{2!}$ - $\frac{2^3}{3!}$) $(z+1)^3$+…]

This expansion is valid throughout the complex plane.

Problem 6

Find the Taylor's series to represent $\frac{z^2-1}{(z+2)(z+3)}$ in |z| < 2

Solution

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{z^2-1}{z^2+5z+6}$$

$$= 1- \frac{5z+7}{z^2+5z+6}$$

$$= 1- \left[\frac{-3}{z+2} + \frac{8}{z+3} \right]$$

$$= 1+ \frac{3}{z+2} - \frac{8}{z+3}$$

$$= 1+ \frac{3}{2(1+\frac{z}{2})} - \frac{8}{3(1+\frac{z}{3})}$$

$$= 1+ \frac{3}{2}(1+ \frac{z}{2})^{-1} - \frac{8}{3}(1+ \frac{z}{3})^{-1}$$

$$= 1+ \frac{3}{2}(1- \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \dots) - \frac{8}{3}(1- \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^2} + \dots)$$

$$= (1+ \frac{3}{2} - \frac{8}{3}) + (- \frac{3}{2^2} + \frac{8}{3^2})z + (\frac{3}{2.2^2} - \frac{8}{3.3^2}) z^2 + \dots$$

$$= - \frac{1}{6} + \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{8}{3^{n+1}} - \frac{3}{2^{n+1}} \right) z^n$$

And the expansion is valid in |z|<2.

5.5 Laurent's series

Any function which is analytic in a region containing the annulus $r_1<|z-z_0|<r_2$ can be represented in a series of the form $\sum_{-\infty}^{\infty} a_n(z-z_0)^n$

Theorem (Laurent's theorem)

Let C_1 and C_2 denote respectively the concentric circles $|z-z_0|=r_1$ and $|z-z_0|=r_2$ with $r_1 < r_2$. Let f(z) be analytic in a region containing the circular annulus $r_1<|z-z_0|<r_2$. Then f(z) can be represented as a convergent series of positive and negative powers of $z-z_0$ given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n(z-z_0)^n \text{ . where}$$

$$b_n = \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)d\zeta}{(\zeta-z_0)^{-n+1}} \text{ and } a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}}$$

Proof

Let z be any point in the circular annulus $r_1<|z-z_0|<r_2$. Then by theorem 5.3.2

we have $f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)d\zeta}{\zeta-z} - \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)d\zeta}{\zeta-z}$

$$\therefore f(z) = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)d\zeta}{\zeta-z} + \frac{1}{2\pi i}\int_{C_1} \frac{f(\zeta)d\zeta}{\zeta-z} \qquad (1)$$

As in the proof of Taylor's theorem, we have

$$\frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)d\zeta}{\zeta-z} d\zeta = a_0+a_1(z-z_0)+a_2(z-z_0)^2 + \dots + a_{n-1}(z-z_0)^{n-1}+R_n(z) \qquad (2)$$

where $a_n = \frac{1}{2\pi i}\int_{C_2} \frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}}$ and

$$R_n(z) = \frac{(z-z_0)^n}{2\pi i}\int_{C_2} \frac{f(\zeta)d\zeta}{(\zeta-z_0)^n(\zeta-z)}$$

Now $\frac{1}{z-\zeta} = \frac{1}{z-z_0+z_0-\zeta}$

$$= \frac{1}{z-z_0-(\zeta-z_0)}$$

$$= \frac{1}{(z-z_0)[1-\frac{\zeta-z_0}{z-z_0}}$$

$$= \frac{1}{(z-z_0)}\left(1+\left(\frac{\zeta-z_0}{(z-z_0)}\right)+\left(\frac{\zeta-z_0}{(z-z_0)}\right)^2+\dots+\left(\frac{\zeta-z_0}{(z-z_0)}\right)^{n-1}+\frac{\left(\frac{\zeta-z_0}{(z-z_0)}\right)^n}{1-\left(\frac{\zeta-z_0}{(z-z_0)}\right)}\right)$$

Multiplying by $\frac{f(\zeta)}{2\pi i}$ and integrating over c_1,

$$\text{we get, } \int_{c_1} \frac{f(\zeta)d\zeta}{\zeta-z} = \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_{n-1}}{(z-z_0)^{n-1}} + s_n(z) \qquad (3)$$

where $b_n = \frac{1}{2\pi i}\int_{c_1} \frac{f(\zeta)d\zeta}{(\zeta-z_0)^{-n+1}}$; $S_n = \frac{1}{2\pi i(z-z_0)^n}\int_{c_1} \frac{f(\zeta)(\zeta-z_0)^n d\zeta}{z-\zeta}$

From (1), (2) and (3) we get $f(z) = a_0+a_1(z-z_0)+ \dots +a_{n-1}(z-z_0)^{n-1}+ \frac{b}{z-z_0}$

$$+ \frac{b_2}{(z-z_0)^2} + \dots + \frac{b_{n-1}}{(z-z_0)^{n-1}} + R_n(z) + s_n(z) \qquad (4)$$

The required result follows if we can prove that $R_n \to 0$ and $S_n \to 0$ as $n \to \infty$.

Now, if $\zeta \in C$, then $|\zeta-z_0|=r_1$ and

$|z-\zeta|=|(z-z_0)-(\zeta-z_0)| \geq |z-z_0|-r_1$

If $\zeta \in C_2$, then $|\zeta-z_0| = r_2$ and

$|\zeta-z| = |(\zeta-z_0) - (z-z_0)| \geq r_2-|z-z_0|$

Now let M denote the maximum value of |f(z)| in $C_1 \cup C_2$.

Then $|R_n| \leq \frac{|z-z_0|^{n-1}}{2\pi} \frac{M(2\pi\, r_2)}{r_2^n(\, r_2-|z-z_0|)}$ [$|\int_c f(z)dz| \leq m\ell$]

$\leq \frac{M|z-z_0|}{(r_2-|z-z_0|)} (\frac{|z-z_0|}{r_2})^{n-1}$

Since $\frac{|z-z_0|}{r_2}$ <1, $R_n \to 0$ as $n \to \infty$

Also $|S_n| \leq \frac{M\, r_1^n\, (2\pi\, r_1)}{|z-z_0|^n\, 2\pi(|z-z_0|-r_1}$

$\leq \frac{M\, r_1}{(|z-z_0|-r_1)} (\frac{r_1}{|z-z_0|})^n$

Since $\frac{r_1}{|z-z_0|} < 1$, $S_n \to 0$ as $n \to \infty$

Hence by taking limit $n \to \infty$ in (4) we get

$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n(z\text{-}z_0)^n$

Hence the theorem.

Solved Problems

Problem 1

Find the Laurent's series expansion of $f(z) = z^2e^{1/z}$ about z=0.

Solution

$f(z) = z^2e^{1/z}$ about z=0

Clearly f(z) is analytic at all point $z \neq 0$.

Now, $f(z) = z^2[1+ \frac{1}{z} + \frac{(\frac{1}{z})^2}{2!} + \frac{(\frac{1}{z})^3}{3!} + \ldots]$

$= z^2[1+ \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{2!z^3} + \ldots]\ 3!\ z^3$

$= z^2+z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \ldots]$

This is the required Laurent's series expansion for f(z) at z=0.

Problem 2

Expand $\frac{-1}{(z-1)(z-2)}$ as a power series in z in the regions (i) |z|<1 (ii) 1<|z|<2 (iii) |z|>2.

Solution

Let $f(z) = \frac{-1}{(z-1)(z-2)}$

By splitting into partial fractions, we have f(z) = $\frac{1}{(z-1)} - \frac{1}{(z-2)}$

(i) The only points where f(z) is not analytic are 1 and 2. Hence f(z) is analytic in |z|<1 and hence can be represented as a Taylor's series in |z|<1.

$\therefore f(z) = \frac{1}{(z-1)} - \frac{1}{(z-2)}$

$= - \frac{1}{1-z} + \frac{1}{2-z}$

$= -(1-z)^{-1} + \frac{1}{2\,(1-\frac{z}{2}}$

$= -(1-z)^{-1} + \frac{1}{2}\ (1- \frac{z}{2})^{-1}$

$= -(1+z + z^2 + \ldots + z^n + \ldots) + \frac{1}{2}(1+ \frac{z}{2} + \frac{z^2}{4} + \ldots + \frac{z^n}{2^n} + \ldots)$

$= \sum_{n=0}^{\infty} [-z^n + \frac{1}{2}(\frac{z}{2})^n]$

$= \sum_{n=0}^{\infty} (\frac{1}{2^{n+1}} -1)\, z^n$

(ii) f(z) is analytic in the annular region 1<|z|<2 and hence can be expanded as a Laurent's series in this region.

$f(z) = \frac{1}{(z-1)} - \frac{1}{(z-2)}$

$= \frac{1}{z(1-\frac{1}{z})} + \frac{1}{2(1-\frac{z}{2})}$

$= \frac{1}{z}(1- \frac{1}{z})^{-1} + \frac{1}{2}(1- \frac{z}{2})^{-1}$

$= \frac{1}{z}[1+(\frac{1}{z}) + (\frac{1}{z})^2 + \ldots] + \frac{1}{2}[1+(\frac{z}{2}) + (\frac{z}{2})^2 + \ldots]$

$[\because |\frac{1}{z}| < 1 \text{ and } |\frac{z}{2}| < 1]$

$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$

This gives the Laurent's series expansion in 1<|z|<2.

(iii) f(z) is analytic in the domain |z|>2 and in this domain we have $|2/z| < 1$.

Hence $f(z) = \frac{1}{z}[\frac{1}{1-\frac{1}{z}}] - \frac{1}{z}[\frac{1}{1-(\frac{2}{z})}]$

$= \frac{1}{z}[1- (\frac{1}{z})]^{-1} - \frac{1}{z}[1- (\frac{2}{z})]^{-1}$

$= \frac{1}{z}[(1+ (\frac{1}{z}) + (\frac{1}{z})^2 + \ldots) - (1+(\frac{2}{z}) + (\frac{2}{z})^2 + \ldots]$

$= \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$

Problem 3

Find the Laurent's series for $\frac{z}{(z+1)(z+2)}$ about z = -2.

Let f(z) = $\frac{z}{(z+1)(z+2)}$

$= \frac{-1}{z+1} + \frac{2}{z+2}$

$= \frac{-1}{(z+2)-1} + \frac{2}{z+2}$

$= \frac{1}{1-(z+2)} + \frac{2}{z+2}$

$= [1-(z+2)]^{-1} + \frac{2}{z+2}$

$= [1+(z+2) + (z+2)^2 \ldots] + \frac{2}{z+2}$

$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \ldots.$

Problem 4

If f(z) = $\frac{z+4}{(z+3)(z-1)^2}$ find Laurent's series expansion

in (1) 0 < |z-1|<4 and (ii) |z-1|>4.

Solution

Let f(z) = $\frac{z+4}{(z+3)(z-1)^2}$ By expressing f(z) into partial fractions we get

$f(z) = \frac{1}{16(z+3)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$

(i) 0 < |z-1|<4

Hence $0 < \left|\frac{z-1}{4}\right| < 1$

$f(z) = \frac{1}{16(z-1+4)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$

$= \frac{1}{64\left(1+\frac{z-1}{4}\right)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$

$= \frac{1}{64}\left(1+\frac{z-1}{4}\right)^{-1} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$

Since $\left|\frac{z-1}{4}\right| < 1$, we have

$f(z) = \frac{1}{64}\left[1-\left(\frac{z-1}{4}\right) + \left(\frac{z-1}{4}\right)^2 - \left(\frac{z-1}{4}\right)^3 + \ldots\right] - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$

$$= \frac{5}{4(z-1)^2} - \frac{1}{16(z-1)} + \frac{1}{64} - \frac{1}{64}\left[\frac{z-1}{4} - \left(\frac{z-1}{4}\right)^2 + \left(\frac{z-1}{4}\right)^3 + \ldots\right]$$

Problem 5

Find the Laurent's series expansion of the function $\frac{z^2-1}{(z+2)(z+3)}$ valid in the annular region $2 < |z| < 3$.

Solution

Let $f(z) = \frac{z^2-1}{(z+2)(z+3)}$

By splitting f(z) into partial fractions, we get $f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$

f(z) is analytic in the annular region $2<|z|<3$.

Hence f(z) can be expanded as a Laurent's series in that region.

$$f(z) = 1 + \frac{3}{z(1+\frac{2}{z})} - \frac{8}{3(1+\frac{z}{3})}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left(1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \ldots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 + \ldots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n$$

$$= 1 + 3\sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n}{z^{n+1}} - 8\sum_{n=0}^{\infty} \frac{(-1)^n\, z^n}{3^{n+1}}$$

Problem 6

For the function $f(z) = \frac{2z^3+1}{z(z+1)}$ find (i) a Taylor's series valid in a neighbourhood of z=i and (ii) a Laurent's series valid with in an annulus of which centre is the origin.

Solution

$$f(z) = \frac{2z^3+1}{z(z+1)}$$

$$= 2z - 2 + \frac{1}{z} + \frac{1}{z+1} \text{ (by partial fraction)}$$

$$= 2(z-1) + \frac{1}{z} + \frac{1}{z+1} \qquad (1)$$

$$= g(z) + h(z) + j(z)$$

Where $g(z) = 2(z-1)$, $h(z)= \frac{1}{z}$ and $j(z) = \frac{1}{z+1}$

Taylor's expansion for g(z) about z=i is obviously $2(i-1) + 2(z-i)$.

Taylor's expansion for h(z) about z=i is given by $h(z)=h(i) + \sum_{n=1}^{\infty} \frac{h^{(n)}(i)}{n!}(z-i)^n$

Here $h(i) = \frac{1}{i}$, $h^{(n)}(z) = \frac{(-1)^n n!}{z^{n+1}}$ so that $h^{(n)}(i) = \frac{(-1)^n n!}{i^{n+1}}$

$\therefore h(z) = \frac{1}{i} + \sum_{n=1}^{\infty} \frac{(-1)^n n!}{i^{n+1} n!}(z-i)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (z-i)^n}{i^{n+1}}$.

Similarly we can prove that $j(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z-i)^n}{(1+i)^{n+1}}$

Hence the Taylor's expansion for f(z) is

$$f(z) = 2(i-1)+2(z-i) + \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{i^{n+1}} + \frac{(-1)^n}{(1+i)^{n+1}}\right](z-i)^n$$

(ii) $f(z) = 2z-2 + \frac{1}{z} + (1+z)^{-1}$ (from (1))

$= 2z-2 + \frac{1}{z} + (1-z+z^2-z^3+\ldots)$ if $|z|<1$

$\therefore$ In the annulus $0 < |z|<1$ the Laurent's expansion is given by

$f(z) = \frac{1}{z} -1+z+z^2-z^3+z^4 \ldots$

5.6 Singularities

Definition

A point a is called a singular point or a singularity of a function f(z) if f(z) is not analytic at a and f is analytic at some point of every disc $|z-a|<r$.

Example 1

Consider the function $f(z) = \frac{1}{z}$

Then $f'(z) = -\frac{1}{z^2}$ for all $z \neq 0$

Thus f(z) is analytic except at z=0.

$\therefore$ z=0 is a singular point of f(z).

Example 2

Consider the function $f(z) = \frac{1}{z(z-i)}$. 0 and i are singular points for f(z).

Definition A

A point a is called an isolated singularity for f(z) if

i. f(z) is not analytic at z=a and
ii. there exist r>0 such that f(z) is analytic in $0<|z-a|<r$.

(ie) the neighbourhood $|z-a|<r$ contains no singularity of f(z) except a.

Example 1

$f(z) = \frac{z+1}{z^2(z^2+1)}$ has three isolated singularities z = 0, i, -i.

Example 2

Consider the function $f(z) = \frac{1}{\sin z}$. The singular points are 0, ±π, ±2π, … and these are isolated singular points.

Types of singularities

Let a be an isolated singularity for a function f(z). Let r>0 be such that f(z) is analytic in $0<|z-a|<r$. In this domain the function f(z) can be represented as a Laurent series given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty} a_n(z-a)^n \text{ where}$$

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-a)^{n+1}} \text{ and } b_n = \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-a)^{-n+1}}$$

The series consisting of the negative powers of z-a in the above Laurent series expansion of f(z) is given by $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ and is called the principal part or singular part of f(z) at z=a.

The singular part of f(z) at z=a determines the character of the singularity. There are three types of singularities. They are (i) Removable singularities (ii) Poles (iii) Essential singularities.

Definition

Let a be an isolated singularity for f(z). Then a is called a removable singularity if the principal part of f(z) at z=a has no terms.

Note

If a is a removable singularity for f(z) then the Laurent's series expansion of f(z) about z=a is given by

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

$$= a_0+a_1(z-a)+ \ldots.+a_n(z-a)^n+ \ldots$$

Hence $\lim_{z\to a} f(z) = a_0$

Hence by defining $f(a) = a_0$ the function f(z) becomes analytic at a.

Example 1

Let $f(z) = \frac{\sin z}{z}$. Clearly 0 is an isolated singular point for f(z).

Now $\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots.\right)$

$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots)$

Here the principal part of f(z) at z = 0 has no terms.

Hence z = 0 is a removable singularity

Also $\lim_{z \to 0} \frac{\sin z}{z} = 1$. Hence the singularity can be removed by defining f(0)=1 so that the extended function becomes analytic at z = 0.

Example 2

Let $f(z) = \frac{z - \sin z}{z^3}$

z = 0 is an isolated singularity

Further $\frac{z - \sin z}{z^3} = \frac{1}{z^3} [z-(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots]$

$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \ldots$

∴ z = 0 is a removable singularity. By defining $f(0) = \frac{1}{6}$ the function becomes analytic at z = 0.

Definition

Let a be an isolated singularity of f(z). The point a is called a pole if the principal part of f(z) at z=a has a finite number of terms. If the principal part of f(z) at z=a is given by $\frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \ldots + \frac{b_r}{(z-a)^r}$. where $b_r \neq 0$. We say that a is a pole of order r for f(z).

Note :

A pole of order 1 is called a simple pole and a pole of order 2 is called double pole.

Example 1

Consider $f(z) = \frac{e^z}{z}$

$\frac{e^z}{z} = \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \ldots$

Here the principal part of f(z) at z=0 has a single term $\frac{1}{z}$. Hence z=0 is a simple pole of f(z).

Example 2

$f(z) = \frac{\cos z}{z^2}$ has a double pole at z = 0

For,

$\frac{\cos z}{z^2} = \frac{1}{z^2}(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots)$

$= \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} - \ldots$

The principal part of f(z) at z=0 contains the term $\frac{1}{z^2}$. Hence z=0 is a double pole of f(z).

Definition

Let a be an isolated singularity of f(z). The point a is called an essential singularity of f(z) at z=a if the principal part of f(z) at z=a has an infinite number of terms.

Example

Let f(z) = $e^{½}$. Obviously z=0 is an isolated singularity for f(z).

Further $e^{½} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \ldots$

The principal part of f(z) has infinite number of terms. Hence $e^{\frac{1}{z}}$ has an essential singularity at z=0.

Theorem 5.6.1 (Riemann's theorem)

Let f be a function which is bounded and analytic through out a domain $0<1z-z_0|< \delta$. Then either f is analytic at z_0 or else z_0 is a removable singular point of f.

Proof

Consider the Laurent's series for the function in the given domain about z_0. The co-efficient b_n of $\frac{1}{(z-z_0)^n}$ is given by $b_n = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}$ where C is the circle $|z-z_0|=r$ where $r < \delta$.

Now since f is bounded there exist a positive real number M such that $|f(z)| \leq M$ in $0<|z-z_0|< \delta$.

$\therefore |b_n| \;= |\frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-z_0)^{-n+1}}|$

$\leq \frac{1}{2\pi}\frac{M(2\pi r)}{r^{-n+1}}$ [by theorem 5.1.1]

$= Mr^n$

Since it is true for every r such that $0 < r < \delta$, taking limit $r \to 0$ we get $b_n=0$. Hence the Laurent's series for f(z) has no principal part.

Hence z_0 is a removable singular point for f(z).

Problem 1

Determine and classify the singular points of $f(z) = \frac{z}{e^z - 1}$

Solution

The singularities of f(z) are given by the values of z for which e^z-1=0.

Hence $z = 2n\pi i$, $n \in Z$ are the singularities of f(z).

Now $e^z - 1 = (1 + z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots) - 1$

$$= z + \frac{z^2}{2!} + \ldots + \frac{z^n}{n!} + \ldots$$

$\lim_{z \to 0} \frac{z}{e^z - 1} = 1$

Hence 0 is a removable singularity for f(z). Also $\lim_{z \to 2n\pi i} \left(\frac{z}{e^z - 1}\right) = \infty$ if $n \neq 0$ and hence $2n\pi i$, $n \neq 0$ are simple poles of f(z).

Problem 2

Determine and classify the singularities of $f(z) = \sin\left(\frac{1}{z}\right)$.

Solution

Clearly 0 is the only singularity of f(z).

Also $f(z) = \frac{1}{z} - \frac{1}{3i\,z^3} + \frac{1}{5i\,z^5} \ldots$

Thus the principal part of f(z) at z=0 has infinitely many terms and hence 0 is an essential singularity for f(z).

5.7 Residues

Definition

Let a be an isolated singularity for f(z). Then the residue of f(z) at a is defined to be the co-efficient of $\frac{1}{z-a}$ in the Laurent's series expansion of f(z) about a and is denoted by Res [f(z); a]. Thus Res [f(z); a] = $\frac{1}{2\pi i} \int_C f(z)dz = b_1$ where C is a circle $|z-a|=r$ such that f is analytic in $0<|z-a|<r$.

Example

Consider $f(z) = \frac{e^z}{z^2}$

$\frac{e^z}{z^2} = \frac{1}{z^2}\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \ldots\right)$

$$= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} +$$

∴ f(z) has a double pole at z=0

∴ Res{f(z); 0} = co-efficient of $\frac{1}{z} = 1$

The following lemmas provide methods for calculation of residues.

Lemma 1

If z=a is a simple pole for f(z) then Res{f(z); a} = $\lim_{z\to a}$ (z-a) f(z).

Lemma 2

If a is a simple pole for f(z) and f(z) = $\frac{g(z)}{z-a}$ where g(z) is analytic at a and g(a) ≠ 0 then Res{f(z); a} = g(a).

Lemma 3

If a is a simple pole for f(z) and if f(z) is of the form $\frac{h(z)}{k(z)}$ where h(z) and k(z) are analytic at a and h(a) ≠ 0 and k(a) = 0 then Res{f(z); a} = $\frac{h(a)}{k'(a)}$

Lemma 4

Let a be a pole of order m>1 for f(z) and let f(z) = $\frac{g(z)}{(z-a)^m}$ where g(z) is analytic at a and g(a) ≠ 0. Then Res {f(z); a} = $\frac{g^{(m-1)}(a)}{(m-1)!}$

Solved Problems

Problem 1

Calculate the residue of $\frac{z+1}{z^2-2z}$ at it poles.

Solution

Let f(z) = $\frac{z+1}{z^2-2z}$

ie. f(z) = $\frac{z+1}{z(z-2)}$

∴ z=0 and z=2 are simple poles for f(z).

Res {f(z);0} $= \lim_{z\to 0} (z-0) \left[\frac{z+1}{z(z-2)} \right]$

$= \lim_{z\to 0} z \left[\frac{z+1}{z(z-2)} \right]$

$= \lim_{z\to 0} \frac{z+1}{z-2} = -\frac{1}{2}$

Res {f(z); 2} = $\lim_{z \to 2}$ (z-2) [$\frac{z+1}{z(z-2)}$]

$= \lim_{z \to 2} \frac{z+1}{z} = \frac{3}{2}$

Problem 2

Find the residue at z = 0 of $\frac{1+e^z}{z\cos z + \sin z}$

Solution

Let f(z) = $\frac{1+e^z}{z\cos z + \sin z}$

Clearly 0 is a pole of order 1 for f(z).

∴ Res {f(z); 0} = $\lim_{z \to 0} \frac{h(z)}{k'(z)}$ where

h(z) = 1+ e^z and k(z)=z cos z + sin z

k′(z) = -z sin z + cos z + cos z

= -z sin z + 2 cos z

∴ Res {f(z); 0} = $\frac{2}{2}$ = 1

Problem 3

Find the residue of $\frac{1}{(z^2+G^2)^2}$ at z=ai

Solution

Let f(z) = $\frac{1}{(z^2+a^2)^2}$

$= \frac{1}{(z+ai)^2(z-ai)^2}$

z = ai and z = -ai are poles of order 2 for f(z)

Let g(z) = $\frac{1}{(z+ai)^2}$

∴ g′(z) = $\frac{-2}{(z+ai)^3}$

∴ Res {f(z); ai} = g′(ai)

$= \frac{-2}{(ai+ai)^3} = \frac{-2}{(2ai)^3}$

$= \frac{-2}{8a^3i^3} = \frac{-2}{-8a^3i}$

$= \frac{2i}{8a^3i^2} = \frac{-i}{4a^3}$

Problem 4

Find the poles of $f(z) = \frac{z^2+4}{z^3+2z^2+2z}$ and determine the residues at the poles.

Solution

$$f(z) = \frac{z^2+4}{z^3+2z^2+2z}$$

$$= \frac{z^2+4}{z(z^2+2z+2)}$$

$$\text{ie } f(z) = \frac{z^2+4}{z(z+1-i)\,(z+1+i)}$$

∴ 0, i-1,-1-i are simple poles for f(z).

Hence $f(z) = \frac{h(z)}{k(z)}$ where

$h(z) = z^2 + 4$ and $k(z) = z^3 + 2z^2 + 2z$

Hence $k'(z) = 3z^2+4z+2$

$\text{Res }\{f(z); 0\} = \frac{h(0)}{k'(0)} = \frac{4}{2} = 2$

$\text{Res }\{f(z); i\text{-}1\} = \frac{h(i-1)}{k'(i-1)}$

$$= \frac{(i-1)^2+4}{3(i-1)^2+4(i-1)+2}$$

$$= \frac{-1-2i+1+4}{3(-1-2i+1)+4i-4+2}$$

$$= \frac{4-2i}{-2i-2} = \frac{(2-i)}{(-i-1)}$$

$$= \frac{(2-i)(-1+i)}{(-1-i)(-1+i)}$$

$$= \frac{-2+2i+i+1}{1-i+i+1}$$

$$= \frac{3i-1}{2}$$

Similarly $\text{Res }\{f(z);-1-i\} = \frac{-(1+3i)}{2}$

Problem 5

Find the residue of $\frac{e^z}{z^2(z^2+9)}$ at its poles.

Solution

Let $f(z) = \frac{e^z}{z^2(z^2+9)}$

Here z=0 is a double pole and z=3i and z=-3i are simple poles for f(z). To find the Res {f(z); 0} let $g(z)=\frac{e^z}{z^2+9}$. Clearly g(z) is analytic at z=0 and g(0) ≠ 0.

Also $g'(z) = e^z\,[\frac{(e^z+9)-2z}{(z^2+9)^2}]$

∴ Res {f(z); 0} $= \frac{g'(0)}{1!}$ (by lemma 4)

$= \frac{1}{9}$

Now, to find Res{f(z); 3i}; let $f(z)=\frac{h(z)}{k(z)}$

So that $h(z) = e^z$ and $k(z) = z^2\,(z^2+9)$

Then $k'(z) = z^2 x 2z+(z^2+9)2z$

$= 2z^3+2z^3+18z$

$= 4z^3+18z$

∴ Res{f(z), 3i} $= \frac{h(3i)}{k'(3i)}$

$= \frac{e^{3i}}{4(3i)^3+18(3i)}$

$= \frac{e^{3i}}{-108i+54i}$

$= \frac{i(\cos 3+i\sin\ 3)}{54}$

Similarly Res {f(z);-3i} $= \frac{-(\sin\ 3+i\cos\ 3)}{54}$

5.8 Cauchy's Residue Theorem

Statement

Let f(z) be a function which is analytic inside and on a simple closed curve C except for a finite number of singular points $z_1, z_2, \ldots, z_n$ inside C. Then

$\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res}\{f(z); z_j\}$

Proof

Let $C_1, C_2, \ldots, C_n$ be circles with centres $z_1, z_2, \ldots, z_n$ respectively such that all circles are interior to C and are disjoint with each other. By Cauchy's theorem for multiply connected regions we have,

$\int_C f(z)dz = \int_{C_1} f(z)dz+ \int_{C_2} f(z)dz+ \ldots+\int_{C_n} f(z)dz +$

$= 2\pi i\,\text{Res}\{f(z);z_1\} + 2\pi i\,\text{Res}\{f(z);z_2\}+\ldots+2\pi i\,\text{Res}\{f(z);z_n\}$

(By definition of residue)

$= 2\pi i \sum_{j=1}^{n} \text{Res}\{f(z);z_j\}$

Solved Problems

Problem 1

Evaluate $\int_C \frac{dz}{2z+3}$ where C is |z|=2

Solution

$z=-\frac{3}{2}$ is the simple pole of f(z) which lies inside the circle |z|=2.

Res f(z); $\frac{-3}{2} = \lim_{z\to-\frac{3}{2}} \frac{h(z)}{k'(z)}$ where h(z)=1 and k(z)=2z+3

∴ k′(z)=2

∴ Res {f(z); $\frac{-3}{2}$} = $\frac{1}{2}$

∴ By Residue theorem $\int_C f(z)dz = 2\pi i(½)$

$= \pi i$

Problem 2

Evaluate $\int_C \frac{dz}{z^2 e^z}$ where C ={z: |z|=1}

Solution

Given integral can be written as $\int_C f(z)dz$ where $f(z) = \frac{e^{-z}}{z^2}$

∴ f(z) has pole of order 2 at z=0 which lies inside the circle |z|=1.

Let $g(z) = e^{-z}$

Hence $g'(z) = -e^{-z}$

By Lemma 4, Res [f(z); 0] = $\frac{g'(0)}{1!} = -1$

∴ By residue theorem,

$\int_C f(z)dz = \int_C \frac{dz}{z^2 e^2} = 2\pi i(-1)$

$= -2\pi i$

Problem 3

Prove that $\int_C \frac{e^{2z}}{(z+1)^3} dz = \frac{4\pi i}{e^2}$ where C is $|z| = \frac{3}{2}$

Solution

Let $f(z) = \frac{e^{2z}}{(z+1)^3}$

f(z) has a pole of order 3 at z=-1

Res {f(z);-1} = $\frac{g''(-1)}{2i}$ where g(z)=e^{2z}

Now $g'(z) = 2e^{2z}$

Now $g''(z) = 4e^{2z}$

Res {f(z);-1} = $\frac{4e^{-2}}{2i} = \frac{2}{e^2}$

∴ By residue theorem, $\int_C f(z)dz = 2\pi i(\frac{2}{e^2})$

$= \frac{4\pi i}{e^2}$

Problem 4

Evaluate using (i) Cauchy's integral formula. (ii) residue theorem

$\int_C \frac{z+1}{z^2+2z+4} dz$ where C is the circle |z+1+i|=2.

Solution

Clearly C is a circle with centre a=-(1+i) and radius 2.

Now $\frac{z+1}{z^2+2z+4} = \frac{z+1}{z^2+2z+1+3}$

$= \frac{z+1}{(z+1)^2+(\sqrt{3})^2}$

$= \frac{z+1}{(z+1+i\sqrt{3})(z+1-i\sqrt{3})}$

$= \frac{z+1}{[z-(-1-i\sqrt{3})][z-(-1+i\sqrt{3})]}$

$Z_0 = -1+i\sqrt{3}$ and $z_1 = -1-i\sqrt{3}$ are the singular points of the given integrated $\frac{z+1}{z^2+2z+4}$

Now $|z_0-a| = |(-1+ i\sqrt{3})-[-(1+i)]|$

$= |-1+ i\sqrt{3}+1+i|$

$= |i(\sqrt{3}+1)| = \sqrt{3} +1>2$

and $|z_1-a| = |-1-i\sqrt{3}-[-(1+i)]|$

$= |-1-i\sqrt{3}+1+i|$

$= |i(1-\sqrt{3})|$

$= \sqrt{3}-1<2$

∴ $z_1 =-1-i\sqrt{3}$ lies inside C

1. By using Cauchy integral formula.

Consider $f(z) = \frac{z+1}{z-(-1-i\sqrt{3})}$

We note that f(z) is analytic at all points inside C.

∴ By Cauchy's integral formula,

$\frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_1}dz = f(z_1)$

i.e $\frac{1}{2\pi i} \int_C \frac{(z+1)dz}{[z-(1-i\sqrt{3}][z-(-1+i\sqrt{3})]} = f(-1-i\sqrt{3})$

i.e $\frac{1}{2\pi i} \int_C \frac{(z+1)dz}{z^2+2z+4} = \frac{(-1-i\sqrt{3})+1}{(-1-i\sqrt{3})-(-1+i\sqrt{3})}$

$= \frac{-i\sqrt{3}}{-2i\sqrt{3}}$

$= \frac{1}{2}$

∴ $\int_C \frac{(z+1)dz}{z^2+2z+4} = \frac{1}{2}(2\pi i) = \pi i$

ii. By using residue theorem

$f(z) = \frac{z+1}{z^2+2z+4}$

since $z = -1-i\sqrt{3}$ lies inside C

Res $\{f(z); -1-i\sqrt{3}\} = \frac{h(-1-i\sqrt{3})}{k'(-1-i\sqrt{3})}$ where h(z)=z+1

and $k(z) = z^2+2z+4$

$\Rightarrow k'(z) = 2z+2$

∴ Res$\{f(z); -1-i\sqrt{3}\} = \frac{-1-i\sqrt{3}+1}{2(-1-i\sqrt{3})+2}$

$= \frac{-i\sqrt{3}}{-i2\sqrt{3}} = \frac{1}{2}$

∴ By residue theorem $\int_C f(z)dz = \frac{2\pi i}{2} = \pi i$.

5.9 Evaluation of Definite Integrals

Type 1

$\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$ where $f(\cos\theta, \sin\theta)$ is rational function of cos θ and sin θ.

To evaluate this type of integral we substitute $z=e^{i\theta}$. As θ varies from 0 to 2π, z describes the unit circle |z|=1.

Also $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2}$ and

$$\sin \theta = \frac{e^{1\theta}-e^{-i\theta}}{2} = \frac{z-z^{-1}}{2i}$$

substituting these values in the given integrand, the integral is transformed into $\int_C \theta(z)dz$ where $\theta(z) = f\left[\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2}\right]$ and C is the positively oriented unit circle $|z|=1$. The integral $\int_C \theta(z)dz$ can be evaluated using the residue theorem.

Solved Problem

Problem 1

Evaluate $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$

Solution Let $I = \int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$

put $z = e^{i\theta}$

Then $dz = e^{i\theta} . id\theta$

$= izd\theta.$

and $\sin\theta = \frac{z-z^{-1}}{2i}$

The given integral is transformed to

$I = \int_C \frac{dz}{iz[5+4\left(\frac{z-z^{-1}}{2i}\right)}$ where C is the unit circle $|z|=1$

$$= \int_C \frac{dz}{iz\,\frac{[10i+4z-\frac{4}{z}]}{2i}}$$

$$= \int_C \frac{2dz}{z\left[\frac{4z^2+10i-4}{z}\right]}$$

$$= \int_C \frac{dz}{2z^2+5iz-2}$$

Let $f(z) = \frac{1}{2z^2+5iz-2}$

$$= \frac{1}{2z^2+4iz+iz-2}$$

$$= \frac{1}{2z(z+2i)+i(z+2i)}$$

$$= \frac{1}{(z+2i)(2z+i)} = \frac{1}{(z+2i)2(z+\frac{i}{2})}$$

Clearly -2i and $\frac{-i}{2}$ lies inside C.

Also Res $\{f(z); \frac{-i}{2}\} = \lim_{z\to ½} \frac{1}{2(z+2i)(z+\frac{i}{2})}$ x $(z+\frac{i}{2})$

$$= \lim_{z\to ½} \frac{1}{2(z+2i)} = \frac{1}{3i}$$

Hence by Cauchy's residue theorem.

$I = 2\pi i(\frac{1}{3i}) = \frac{2\pi}{3}$

Problem 2

Prove that $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}$ $(-1< a < 1)$

Solution

Put $z= e^{i\theta}$

Then $\sin\theta = \frac{z-z^{-1}}{2i}$ where $dz = izd\theta$

$\int^{2\pi} \frac{d\theta}{1+a\sin\theta} = \int_C \frac{dz}{iz[1+(\frac{z-z^{-1}}{2i})}$ where C is the unit circle.

$$= \int_C \frac{dz}{iz[\frac{2i+az-\frac{a}{z}}{2i}]}$$

$$= \int_C \frac{2dz}{az^2+2iz-a}$$

Let $f(z) = \frac{2}{az^2+2iz-a}$

The poles of f(z) are given by

$$z = \frac{-2i \pm \sqrt{-4+4a^2}}{2a}$$

$$= \frac{-i \pm i\sqrt{1-a^2}}{a} \quad \text{[since -1<a<1]}$$

Let $z_1 = \frac{-i+ i\sqrt{1-a^2}}{a}$ and $z_2 = \frac{-i- i\sqrt{1-a^2}}{a}$

We note that $|z_2| = \frac{1+\sqrt{1-a^2}}{|a|} > 1$ [$\therefore -1 < a < 1$]. Also since $|z_1 z_2| = 1$, it follows that $|z_1|<1$. Hence there are no singular points on C and $z = z_1$ is the only simple pole inside C.

Res $\{f(z): z_1\} = \lim_{z\to z_1} (z-z_1)\left[\frac{2/a}{(z-z_1)(z-z_2)}\right]$

$$= \left[\frac{2/a}{z-z_2}\right]$$

$$= \frac{2}{a}\left[\frac{1}{\left(\frac{-i+i\sqrt{1-a^2}}{a}\right)-\left(\frac{-i-i\sqrt{1-a^2}}{a}\right)}\right]$$

$$= \frac{2}{a}\left[\frac{a}{(2i\sqrt{1-a^2})}\right]$$

$$= \frac{1}{i\sqrt{1-a^2}}$$

By residue theorem $\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = 2\pi i\left[\frac{1}{(i\sqrt{1-a^2})}\right]$

$$= \frac{2\pi i}{(\sqrt{1-a^2})}$$

Problem 3

Prove that $I = \int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} = \frac{\pi}{\sqrt{a^2+1}}$ [a>0]

Solution

$$I = \int_0^{\pi} \frac{a\,d\theta}{a^2+\left(\frac{1-\cos 2\theta}{2}\right)}$$

$$= \int_0^{\pi} \frac{2a\,d\theta}{2a^2+1-\cos 2\theta)}$$

$$= \int_0^{2\pi} \frac{a\,d\varphi}{2a^2+1-\cos\varphi} \quad \text{(putting } 2\theta = \theta)$$

Put $z = e^{i\varphi}$ then $\cos\varphi = \frac{z+z^{-1}}{2}$

$dz = i\,e^{i\varphi}\,d\theta$

$dz = iz\,d\varphi$

$$I = \frac{1}{i}\int_C \frac{a\,dz}{z[2a^2+1-\left(\frac{z+z^{-1}}{2}\right)}$$

$$= \frac{1}{i}\int_C \frac{a\,dz}{z[\frac{2(2a^2+1)-z-z^{-1}}{2})}$$

$$= \frac{2a}{i}\int_C \frac{dz}{z[2(2a^2+1)-z-\frac{1}{z}]}$$

$$= \frac{2a}{i}\int_C \frac{dz}{[2(2a^2+1)z-z^2-1}$$

$$= 2ai\int_C \frac{dz}{z^2-2(2a^2+1)z+1}$$

$$= 2ai\int_C f(z)dz \qquad (1)$$

Where $f(z) = \frac{1}{z^2-2(2a^2+1)z+1}$ and C is the unit circle $|z|=1$.

Poles of f(z) are the roots of $z^2-2(2a^2+1)z+1=0$

$$z = \frac{2(2a^2+1)\pm\sqrt{4(2a^2+1)^2-4}}{2}$$

$$= \frac{2[(2a^2+1)\pm\sqrt{4a^4+4a^2+1-1}}{2}$$

ie.z = $(2a^2+1) \pm 2a\sqrt{a^2+1}$

Let $z_1 = 2(a^2+1) + 2a\sqrt{a^2+1}$;$z_2=(2a^2+1)-2a\sqrt{a^2+1}$ clearly $|z_1|>1$ and $|z_1 Z_2|=1$ so that $|z_2|<1$. Hence the only pole inside C is $z=z_2$.

Res $\{f(z)\ z_2] = \lim_{z\to z_2} (z-z_2)\frac{1}{(z-z_1)(z-z_2)}$

$$= \frac{1}{z_2-z_1}$$

$$= \frac{1}{(-4a)\sqrt{a^2+1}}$$

From (1), by residue theorem,

$$I = 2\pi i\ [\frac{2ai}{-4a\sqrt{a^2+1}}]$$

$$= [\frac{\pi}{\sqrt{a^2+1}}]$$

Exercise

1. Prove that $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta} = \frac{\pi}{6}$
2. Prove that $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$

Type 2

$\int_{-\infty}^{\infty} f(x)\,dx$ where $f(x) = \frac{g(x)}{h(x)}$ and g(x), h(x) are polynomials in x and the degree of h(x) exceeds that of g(x) by atleast two.

To evaluate this type of integral we take $f(z) = \frac{g(z)}{h(z)}$. The poles of f(z) are determined by the zeros of the equation h(z)=0.

Case (i) No pole of f(z) lies on the real axis.

We choose the curve C consisting of the interval [-r, r] on the real axis and the semi circle |z|=r lying in the upper half of the plane.

Here r is chosen sufficiently large so that all the poles lying in the upper half of the plane are in the interior of C. Then we have

$\int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz$. Where C_1 is the semi circle.

Since deg h(x) – deg f(x) ≥ 2 it follows that $\int_{C_1} f(z)dz \to 0$ as $r \to \infty$ and hence $\int_{C_1} f(z)dz = \int_{-\infty}^{\infty} f(x)dx$.

∴ $\int_{-\infty}^{\infty}$ f(x)dx can be evaluated by evaluating $\int_{C-}$ f(z)dz which in turn can be evaluated by using Cauchy's residue theorem.

Case (ii) f(z) has poles lying on the real axis

Suppose a is a pole lying on the real axis. In this case we indent the real axis by a semi circle C_2 of radius ε with centre a lying in the upper half plane where ε is chosen to be sufficiently small. (refer figure)

Such an indenting must be done for every pole of f(z) lying on the real axis. It can be proved that $\int_{C_2}$ f(z)dz = -πi Res {f(z); a}. By taking limit as r→∞ and ε→0 we obtain the value of $\int_{-\infty}^{\infty}$ f(x)dx.

Solved Problems

Problem 1

Use contour integration method to evaluate $\int_0^{\infty} \frac{dx}{1+x^4}$

Solution

Let $f(z) = \frac{1}{1+z^4}$

The poles of f(z) are given by the roots of the equation $z^4+1=0$ which are the four fourth roots of -1.

$z^4=-1$

$z = (-1)^4 = (\cos \pi + i \sin \pi)^4$

$= \cos (2n+1)\frac{\pi}{4} + \sin (2n+1)\frac{\pi}{4}$, n=0,1,2,3

$= e^{i\pi/4}, e^{i3\pi/4}, e^{i5\pi/4}, e^{i7\pi/4}$ which are all simple poles.

We choose the contour C consisting of the interval [-r, r] on the real axis and the upper semi-circle |z|=r which we denote by C_r.

∴ $\int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz$ (1)

The poles of f(z) lying inside the contour C are obviously $e^{i\pi/4}$ and $e^{i3\pi/4}$ only. We find the residues of f(z) at these points.

Res $\{f(z); e^{i\pi/4}\} = \frac{h(e^{i\pi/4})}{k'(e^{i\pi/4})}$ where h(z)=1

and $k(z) = z^4+1$

$k'(z) = 4z^3$

$\Longrightarrow k'(e^{i\pi/4}) = 4(e^{i\pi/4})^3$

$= 4e^{i3\pi/4}$

$\therefore$ Res $\{f(z); e^{i\pi/4}\} = \frac{1}{4e^{i3\pi/4}} = \frac{e^{-i3\pi/4}}{4}$

Similarly Res $\{f(z); e^{i3\pi/4}\} = \frac{e^{-i9\pi/4}}{4}$

By residue theorem,

$\int_C f(z)dz = 2\pi i$ (sum of the residues at the poles)

$= 2\pi i\, [\frac{e^{-i3\pi/4}}{4} + \frac{e^{-i9\pi/4}}{4}]$

$= \frac{2\pi i}{4}\, [\cos(3\pi/4) - i\sin(3\pi/4) + \cos(9\pi/4) - i\sin(9\pi/4)]$

$= \frac{\pi i}{2}\, [(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}) + (\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}})]$

$= \frac{\pi i}{2}\, [\frac{-2i}{\sqrt{2}}]$

$= \frac{\pi}{\sqrt{2}}$

From (1), $\int_{-r}^{r} \frac{dx}{1+x^4} + \int_{c_1} f(z)dz = \frac{\pi}{\sqrt{2}}$

As $r \to \infty$, $\int_{c_1} f(z)dz \to 0$

$\therefore \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$

$\therefore 2\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$ [$\because \frac{1}{1+x^4}$ is an even function]

$\therefore \int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

Problem 2

Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\, dx$

Solution

Let $f(z) = \frac{z^2-z+2}{z^4+10z^2+9}$

Poles of f(z) are the zeros of $z^4+10z^2+9=0$, $z^4+10z^2+9 = (z^2+9)(z^2+1)$

$\therefore z = \pm 3i, \pm i$

Hence z = 3i, -3i, i, -i are the simple poles of f(z)

Choose the contour C as shown in the figure

$\int_c f(z)dz = \int_{-r}^{r} f(x)dx + \int_{c_1} f(z)dz$ (1)

The poles of f(z) lying within C are i and 3i and both of them are simple poles.

Res{f(z)i} = $\frac{h(i)}{k'(i)}$ where h(z) = z^2-z+2 and k(z) =z^4+10z^2+9.

$\Longrightarrow$ k′(z) = 4z3+20z

$\therefore$ Res{f(z)i} = $\frac{(i)^2-i+2}{4(i)^3+20i}$

$= \frac{1-i}{-4i+20i} = \frac{1-i}{16i}$

Similarly Res {f(z); 3i} = $\frac{7+3i}{48i}$

$\therefore \int_c f(z)dz =$ 2πi (sum of the residues at the poles)

$= 2\pi i \left(\frac{1-i}{16i} + \frac{7+3i}{48i} \right)$

$= 2\pi i \left(\frac{3-3i+7+3i}{48i} \right)$

$= 2\pi i \left(\frac{10i}{48i} \right) = \frac{5\pi}{12}$

From (1) $\int_{-r}^{r} \frac{x^2-x+2}{x^4-10x^2+9} dx = \frac{5\pi}{12}$

Problem 3

Prove that $\int_0^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}$

Solution

Since $\frac{1}{x^6+1}$ is an even function we have $\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2 \int_0^{\infty} \frac{dx}{x^6+1}$

Now let f(z) = $\frac{1}{x^6+1}$

The poles of f(z) are given by the roots of the equation $x^6+1 = 0$ which are the sixth roots of -1.

Z = $(-1)^{1/6}$

By De Movre's theorem, they are given by $e^{i\pi/6}$, $e^{i3\pi/6}$, $e^{i5\pi/6}$, $e^{i7\pi/6}$ and $e^{i11\pi/6}$ and they are simple poles.

Now choose the contour C consisting of the interval [-r, r] on the real axis and the upper semi circle |z|=r which we denote by C_1.

The poles of f(z) lying inside C are $e^{i\pi/6}$, $e^{i3\pi/6}$ and $e^{i5\pi/6}$

Res {f(z); $e^{i5\pi/6}$} = $\frac{h(e^{i\pi/6})}{k'(e^{i\pi/6})}$

and k(z) = z^6+1

$\Rightarrow k'(z) = 6z^5$

$\Rightarrow k'(e^{i\pi/6}) = 6(e^{i\pi/6})^5$

$= 6\, e^{i5\pi/6}$

Res {f(z); ($e^{i\pi/6}$} = $\frac{1}{6\, e^{i5\pi/6}}$

$= \frac{1}{6} e^{-i5\pi/6}$

Similarly Res{f(z), $e^{i3\pi/6}$} $\frac{1}{6} e^{-i5\pi/6}$ and Res {f(z), $e^{i5\pi/6}$}= $\frac{1}{6} e^{-\frac{25\pi}{6}}$

∴ By residue theorem,

$\int_c f(z)dz\ \ 2\pi i$ (sum of the residues at the 3 poles)

$= 2\pi i\, [\, \frac{1}{6} e^{-5i\pi/6} + = \frac{1}{6} e^{-5i\pi/6} + \frac{1}{6} e^{-25i\pi/6}]$

$= \frac{2\pi i}{6}$ [(cos $\frac{5\pi}{6}$ – sin $\frac{5\pi}{6}$)+(cos $\frac{5\pi}{2}$ – i sin $\frac{5\pi}{2}$) + (cos $\frac{25\pi}{6}$ – i sin $\frac{25\pi}{6}$)]

$= \frac{\pi i}{3}$ [(- $\frac{\sqrt{3}}{2}$ – $\frac{i}{2}$)+(0 - i) + ($\frac{\sqrt{3}}{2}$ – $\frac{i}{2}$)]

$= \frac{\pi i}{3}$ (-i -i) $\frac{2\pi}{3}$

From (1)

$\int_{-r}^{r} \frac{dx}{x^6+1} + \int_{C_1}^{r} f(z)dz\ \frac{2\pi}{3}$

As r→∞, the integral over C_1→ 0

$\therefore \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}$

$\therefore \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}$

Exercise

1. Using the method of contour integration evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^4+4)} dx$
2. Prove that $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^4+b^2)}\ \frac{\pi}{a+b}$
3. Evaluate i = $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$

   Ans : (1) $\frac{\pi}{3}$ (3) $\frac{\pi}{4a^3}$