B.Sc. MATHEMATICS – I YEAR

DJM1A : CALCULUS AND DIFFERENTIAL EQUATIONS

SYLLABUS

UNIT -1

Curvature – radius of curvature - Cartesian and polar - centre of curvature - Involute and evolute - Asymptotes in Cartesian and polar co-ordinates.

UNIT – 2

Evaluation of double and triple integrals-Jacobeans, change of variables.

UNIT -3

First order differential: equations of higher degree- solvable for p, x and y- Clairaut's form/ linear differential equations of second order- Particular integrals for functions of the form, X^n, e^{ax}, $e_{ax}(f(x))$. Second order differential equations with variable coefficients.

UNIT – 4

Laplace transform – Inverse transform – Properties-Solving differential equations. Simultaneous equations of first order using Laplace transform.

UNIT – 5

Partial differential equations of first order – formation – different kinds of solution – four standard forms- Lagranges method.

Books :

1) Calculus Vol.1, Vol.2 & Vol.3, By T. K. Manickavachagompillai & others.

2) Calculus Vol.1, & Vol.2, By S. Arumugam and Isaac.

DJM1A : CALCULUS AND DIFFERENTIAL EQUATIONS

CONTENTS

UNIT -1 : CURVATURE

Curvature – radius of curvature - Cartesian and polar - centre of curvature - Involute and evolute - Asymptotes in Cartesian and polar co-ordinates.

CURVATURE

1. 1 Curvature and radius of curvature

The curvedness of a curve at a point ρ on it is measured by the rate of change of Ψ with respect to s, where Ψ is the angle made by the tangent at ρ with the x-axis and s is the arcual distance of ρ from a fixed point Q on the curve, that is by dΨ/ds.

This rate is called the curvature of the curve at ρ.

Curvature of a circle

Consider a circle as in the figure whose centre is C and radius a. Let Ψ be the angle made by the tangent at any point ρ with the x-axis. If the arcual distance of ρ from O is s, then s = aΨ. This is the intrinsic eqn of the circle.

Differentiating this w.r.t 's', we get

$$1 = a\frac{d\Psi}{ds}.$$
$$\therefore \frac{d\Psi}{ds} = \frac{1}{a}$$

So, in the case of circle, the curvature is a constant which is the reciprocal of the radius.

1.2 Radius of curvature

The reciprocal of Curvature of a curve at a point is called the radius of curvature of the curve at the point. So it is $\frac{ds}{d\Psi}$.

The radius of Curvature of a circle is its radius.

Notation

Radius of Curvature is denoted by ρ.

Remark :1

In the case of a straight line the change of Ψ is zero and hence $\frac{d\Psi}{ds} = 0, \rho = \frac{ds}{d\Psi} = \infty$

Remark : 2

If the curve is such that, as 's' increases, Ψ increases, then $\frac{d\Psi}{ds}$ is +ve and, so ρ is +ve.

ie) if the curve is concave, ρ is +ve otherwise is –ve In general, ρ is given as its absolute value, namely $|\rho|$.

1.3. Cartesian formula for the radius of curvature

We know that $\frac{dy}{dx} = \tan\Psi$

$$\therefore \frac{d^2y}{dx^2} = \sec^2\Psi.\frac{d\Psi}{dx} = \sec^2\Psi\frac{d\Psi}{ds}\frac{ds}{dx}.$$

$$\therefore \frac{ds}{d\Psi} = \frac{\sec^3\Psi}{\frac{d^{2y}}{dx^2}} as \frac{dx}{ds} = \cos\Psi$$

$$= \frac{(1+\tan^2\Psi)^{3/2}}{\frac{d^2y}{dx^2}}$$

$$= \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

$$\therefore \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}}$$

Examples:

1. What is the radius of curvature of the curve $x^4 + y^4 = 2$ at the point (1,1)?

Soln:

Given the curve $x^4 + y^4 = 2$

Differentiating the above equation, we get

$$4x^3 + 4y^3\frac{dy}{dx} = 0.$$

$$4x^3 = -4y^3\frac{dy}{dx}.$$

$$\therefore \frac{dy}{dx} = -\frac{x^3}{y^3}.$$

Differentiating this once again, we get

$$\frac{d^2y}{dx^2} = \frac{3\left(x^3\frac{dy}{dx} - x^2y\right)}{y^4}.$$

At the point (1,1), $\frac{dy}{dx} = -1, and\ \frac{d^2y}{dx^2} = -6.$

$$\therefore \rho = \frac{(1+1)^{3/2}}{6} = -\frac{\sqrt{2}}{3}.$$

2. Show that the radius of curvature at any point of the catenary $y = c\cosh\frac{x}{c}$ is equal to the length of the portion of the normal intercepted between the curve and the axis of x.

Soln:

Given $y = c\cosh\frac{x}{c}$

Differentiating the above equation, we get

$$\frac{dy}{dx} = \sinh\frac{x}{c}$$

Now,` $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \left(1+\sinh^2\frac{x}{c}\right)^{3/2} = \cosh^3\frac{x}{c}$

Also $\frac{d^2y}{dx^2} = \frac{1}{c}\cosh\frac{x}{c}.$

$$\text{Here } \rho = \frac{\cosh^3 \frac{x}{c}}{\frac{1}{c}\cosh\frac{x}{c}} = c\cosh^2\frac{x}{c} = \frac{y^2}{c}$$

Again at any point (x,y)

$$\text{the normal } = y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2} = y\cosh\frac{x}{c} = \frac{y^2}{c}$$

$\therefore$ Radius of curvature = length of the normal.

3. If a curve is defined by the parametric equation x=f (θ) and y=φ(θ), prove that the curvature is $\frac{1}{\rho} = \frac{x'y''-y'x''}{(x'2+y'2)^{3/2}}$

Soln:

where dashes denote differentiation with respect to θ.

$$\frac{dy}{dx} = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{y'}{x'}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \frac{d}{d\theta}\left(\frac{y'}{x'}\right)\frac{d\theta}{dx}$$

$$= \frac{y''x'-y'x''}{x'^2}\,\frac{1}{x'}$$

$$= \frac{y''x'-y'x''}{x'^3}$$

$$\therefore \frac{1}{\rho} = \frac{\frac{d^{2y}}{dx^2}}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{y''x'-y'x''}{x'^3\left[1+\frac{y'^2}{x'^2}\right]^{3/2}}$$

$$= \frac{x'y''-y'x''}{(x'^2+y'^2)^{3/2}}.$$

4. Prove that the radius of curvature at any point of the cycloid x =a (θ + sinθ) and y = a (1 - cosθ) is 4 a cos $\frac{\theta}{2}$.

Soln:

From the given equations ,

x =a (θ + sinθ)

differentiation with respect to θ.

$$\frac{dx}{d\theta} = a(1+\cos\theta)$$

$$\frac{d^2x}{d\theta^2} = -a\sin\theta$$

y = a (1 - cosθ)

differentiation with respect to θ.

$$\frac{dy}{d\theta} = a\sin\theta$$

$$\frac{d^2y}{d\theta^2} = a\cos\theta.$$

Substituting the values in the formula obtained in the previous example, we get

$$\frac{1}{\rho} = \frac{a(1+\cos\theta)a\cos\theta - a\sin\theta(-a\sin\theta)}{\left[a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right]^{3/2}}$$

$$= \frac{a^2(1+\cos\theta)}{a^3\left[2(1+\cos\theta)\right]^{3/2}}$$

$$= \frac{2\cos^2\theta/2}{a\left[4\cos^2\theta/2\right]^{3/2}} = \frac{1}{4a\cos^{\theta/2}}$$

$$\therefore \rho = 4a\cos\frac{\theta}{2}.$$

5. Find ρ at the point ‘t’ of the curve x = a (cos t + t sin t); y = a (sin t – t cos t)

Soln:

Given the curve

x = a (cos t + t sin t); y = a (sin t – t cos t)

$$\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = at\cos t.$$

$$\frac{dy}{dt} = a(\cos t - \cos t + t\sin t) = at\sin t.$$

$$\therefore \frac{dy}{dx} = \tan t.$$

Differentiating with respect to x,

$$\frac{d^2y}{dx^2} = \frac{d}{dt}(\tan t)\frac{dt}{dx} = \sec^2 t\frac{1}{at\cos t} = \frac{1}{at\cos^3 t}$$

$$\therefore \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{1+\tan^2 t)^{3/2}}{\frac{1}{at\cos^3 t}} = at.$$

(The formula of Ex.3 can also be employed)

Exercise 1:

1. Find the radius of curvature for the curves

(a) $y = e^x$ at the point where it crosses the y – axis

(b) $\sqrt{x}+\sqrt{y} = 1 at (1/4, 1/4)$

(c)$y^2 = x^3 + 8$ at the point (-2, 0).

(d) xy = 30 at the point (3,10)

(e) $(x^2 + y^2)^2 = a^2 (y^2 – x^2)$ at the point (0, a)

Polar form.

Let r = f(θ) be the given curve in polar coordinates.

∴ x = r cos θ and y = r sin θ, may be regarded as the parametric equations of the given curve the parameter being θ.

$$\therefore \frac{dx}{d\theta} = \cos\theta\frac{dr}{d\theta} - r\sin\theta$$

and $\frac{dy}{d\theta} = \sin\theta\frac{dr}{d\theta} + r\cos\theta$

$$\therefore \frac{d^2x}{d\theta^2} = \cos\theta\frac{d^2r}{d\theta^2} - 2\sin\theta\frac{dr}{d\theta} - r\cos\theta \, and$$

$$\frac{d^2y}{d\theta^2} = \sin\theta\frac{d^2r}{d\theta^2} - 2\cos\theta\frac{dr}{d\theta} - r\sin\theta$$

Substituting these values in the formula for ρ in parametric from and simplifying we get

$$\rho = \frac{(r^2 + r_1^2)^{\frac{3}{2}}}{r^2 + 2r_1^2 - rr_2} \, where \; r_1 = \frac{dr}{d\theta} \, and \; r_2 = \frac{d^2r}{d\theta^2}.$$

1.4 The coordinates of the centre of curvature

Let the centre of curvature of the curve y = f (x) corresponding to the point P (x,y) be X and Y.

X = ON

= OQ – NQ = OQ – MP

= X - PC sin Ψ = x - ρ sin Ψ.

Y = NC

= NM + MC

= QP + PC cos Ψ = y + ρ cos Ψ.

If y_1 and y_2 denote $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ we know that

$$\rho = \frac{(1+y_1^2)^{3/2}}{} and \tan\psi = y_1$$

$$\therefore \cos\psi = \frac{1}{\sqrt{1+y_1^2}} and \sin\psi = \frac{y_1}{\sqrt{1+y_1^2}}$$

$$\therefore X = X - \frac{(1+y_1^2)^{3/2}}{y_2}\frac{y_1}{(1+y_1^2)^{1/2}} = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$Y = y + \frac{(1+y_1^2)^{3/2}}{y_2}\frac{1}{(1+y_1^2)^{1/2}} = y + \frac{(1+y_1^2)}{y_2}$$

The locus of the centre of curvature for a curve is called the evolute of the curve.

Examples.

1. Find the co-ordinates of the centre of curvature of the curve xy = 2 at the point (2,1).

   Soln:

   Given the curve xy = 2

Here y = $\frac{2}{x}$

Differentiating with respect to ‘x’ we get

$$\frac{dy}{dx} = -\frac{2}{x^2} and \frac{d^2y}{dx^2} = \frac{4}{x^3}.$$

$$\therefore At\ (2,1)\ the\ values\ of\ \frac{dy}{dx}\ and\ \frac{d^2y}{dx^2}\ are respectively\ -1/2\ and\ 1/2.$$

$$\therefore X = 2 + \frac{(1+\frac{1}{4})\times\frac{1}{2}}{\frac{1}{2}} = 3\frac{1}{4}.$$

$$Y = 1 + \frac{1+\frac{1}{4}}{\frac{1}{2}} = 3\frac{1}{2}.$$

$\therefore$ The centre of curvature is $(3\frac{1}{4}, 3\frac{1}{2})$.

2. Show that in the parabola $y^2 = 4ax$ at the point t, $\rho = -2a\ (1+t^2)^{3/2}$, $X=2a + 3a\,t^3$, $Y = -2a\,t^3$. Deduce the equation of the evolutes.

Soln:

$x = a\,t^2$, $y = 2\,at$.

$$\therefore \frac{dx}{dt} = 2at, \frac{dy}{dt} = 2a$$

$$\therefore \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{1}{t}\right) \div \frac{dx}{dt}$$

$$= -\frac{1}{t^2} \div 2at = -\frac{1}{2at^3}$$

$$\therefore \rho = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} \div \frac{d^2y}{dx^2} = -2a(1+t^2)^{3/2}$$

$$X = x - \frac{1+\left(\frac{dy}{dx}\right)^2 \frac{dy}{dx}}{\frac{d^2y}{dx^2}} = at^2 - \frac{\left(1+\frac{1}{t^2}\right)\frac{1}{t}}{-\frac{1}{2at^3}}$$

$$= 2a + 3at^2$$

$$Y = y + \frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}} = 2at + \frac{1+\frac{1}{t^2}}{-\frac{1}{2at^2}} = -2at$$

Eliminating t from X and Y

$$y = -2a\left(\frac{X-2a}{3a}\right)^{3/2}$$

Squaring both sides and simplifying, we get

$$27aY^2 = 4(X-2a)^3$$

The locus of (X,Y) is $27ay^2 = 4\,(x-2a)^3$

The curve is called a semi – cubical parabola.

3. Find the evolute of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.Any point on the ellipse is (a cos θ, b sin θ)

Soln:

$$x = a\cos\theta; \frac{dx}{d\theta} = -a\sin\theta$$

$$y = b\sin\theta; \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}(-\frac{b}{a}\cot\theta)=\frac{b}{a}\cos ec^2\theta\frac{d\theta}{dx}$$

$$\frac{d\theta}{dx}=\frac{1}{-a\sin\theta}$$

$$=-\frac{b}{a^2}\cos ec^3\theta$$

$$X=x-\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}\frac{dy}{dx}}{\frac{d^2y}{dx^2}}$$

$$=a\cos\theta-\frac{\left(1+\frac{b^2}{a^2}\cot^2\theta\right)\left(\frac{b}{a}\cot\theta\right)}{\frac{b}{a^2}\cos ec^3\theta}$$

$$=\frac{(a^2-b^2)\cos^3\theta}{a}$$

$$Y=y+\frac{1+\left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

$$=b\sin\theta-\frac{1+\frac{b^2}{a^2}\cot^2\theta}{\frac{b}{a^2}\cos ec^3\theta}$$

$$=-\frac{a^2-b^2}{b}\sin^3\theta.$$

$$\cos\theta=\left(\frac{aX}{a^2-b^2}\right)^{1/3} \textit{ and}$$

$$\sin\theta=\left(\frac{-bY}{a^2-b^2}\right)^{1/3}$$

To eliminate θ, squaring and adding, we get

$$\left(\frac{ax}{a^2-b^2}\right)^{2/3}+\left(\frac{-by}{a^2-b^2}\right)^{2/3}=1$$

$$i.e.,\left(\frac{ax}{a^2-b^2}\right)^{2/3}+\left(\frac{by}{a^2-b^2}\right)^{2/3}=1$$

$\therefore$ The locus of (X,Y) is the four cusped hypocycloid.

$$(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$$

4. Show that the evolute of the cycloid

$$x=a(\theta-\sin\theta);\ y=a(1-\cos\theta) \textit{ is another cycloid}.$$

Soln:

Given

$$x=a(\theta-\sin\theta)$$

Differentiating with respect to θ

$$\frac{dx}{d\theta}=a(1-\cos\theta)$$

$$y = a(1-\cos\theta)$$

Differentiating with respect to θ

$$\frac{dy}{d\theta} = a\sin\theta.$$

$$\therefore \frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \cot\frac{\theta}{2}.$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\cot\frac{\theta}{2}\right) = -\frac{1}{2}\cos ec^2\frac{\theta}{2}.\frac{d\theta}{dx}$$

$$= -\frac{1}{4a\sin^4\frac{\theta}{2}}$$

$$X = x + \frac{(1+\cot^2\frac{\theta}{2})\cot\frac{\theta}{2}}{\frac{1}{4a\sin^4\frac{\theta}{2}}}$$

$$= a(\theta - \sin\theta) + 2a\sin\theta$$

$$= a(\theta - \sin\theta)$$

$$Y = y + \frac{1+\cot^2\frac{\theta}{2}}{-\frac{1}{4a\sin^4\frac{\theta}{2}}}$$

$$= a(1-\cos\theta) - 2a(1-\cos\theta)$$

$= -a(1-\cos\theta)$

$\therefore$ The locus of (X,Y) is

$$x = a(\theta + \sin\theta); y = -a(1-\cos\theta)$$

This is also a cycloid.

5. Find the centre of curvature of y = x^2 at the origin.

Solution. We have y = x^2.

$\therefore y_1 = 2x \, and \, y_2 = 2$

$\therefore At\,(0,0), y_1 = 0 \, and \, y_2 = 2$

Let (x, y) be the centre of curvature at (0,0).

$$\therefore X = x - \frac{y_1}{y_2}(1+y_1^2) = 0.$$

$$Y = y + \frac{1+y_1^2}{y_2} = \frac{1}{2}$$

$\therefore$ Centre of curvature is (0, $\frac{1}{2}$).

6. Find the evolute of the curve given by $x = a\cos^3\theta \, and \; y = \sin^3\theta.$

Solution. We have $x = a\cos^3\theta \, and \; y = \sin^3\theta.$

$\therefore y_1 = -\tan\theta \, and \, y_2 = (1/3a)\sec^4\theta \cos ec\,\theta$

Let (X, Y) be the centre of curvature.

$$\therefore X = x - \frac{y_1}{y_2}(1+y_1^2) = a\cos^3\theta + \frac{3a\tan\theta(1+\tan^2\theta)}{\sec^4\theta\,\mathrm{cos}\,ec\,\theta}$$

$$= a\cos^3\theta + \_3a\sin^2\theta\cos\theta ......[1]$$

$$Y = y + \frac{1+y_1^2}{y_3} = a\sin^3\theta + \frac{3a\tan\theta(1+\tan^2\theta)}{\sec^4\theta\,\mathrm{cos}\,ec\,\theta}$$

$$= a\sin^3\theta + 3a\cos^2\theta\sin\theta ......[2]$$

Now, to find the equation of the evolute, we have to eliminate θ from [1] and [2] we have

$$X + Y = a(\cos\theta + \sin\theta)^3.$$

$$X - Y = a(\cos\theta + \sin\theta)^3.$$

$$(X+Y)^{\frac{2}{3}} + (X-Y)^{\frac{2}{3}} = a^{\frac{2}{3}}(2) = 2a^{\frac{2}{3}}$$

$$\therefore \textit{The locus of } (X,Y) \textit{ is } (x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}.$$

7.Find the evolute of the parabola $y^2 = 4ax$

Solution. We have $y^2 = 4ax$

$\therefore y_1 = 2a/y \textit{ and } y_2 = -4a^2/y3$

Let (X, Y) be the centre of curvature.

$$\therefore X = x - \frac{y_1}{y_2}(1+y_1^2) = x + \frac{y^2+4a^2}{2a}$$

$$= 3x + 2a \ (by\ 1) .....[2]$$

$$Y = y + \frac{1+y_1^2}{y_2} = -\frac{y^3}{4a^2}$$

$$= -\frac{2x^{\frac{3}{2}}}{\sqrt{a}} \ (by\ 1) \ .....[3]$$

From[2] and [3] eliminating x , we have

$$Y^2 = \frac{4x^3}{a} = \frac{4(X-2a)^3}{27a}$$

$$\therefore 27aY^2 = 4(X-2a)^3$$

$$\therefore \ The\ locus\ of\ (X, Y)\ is\ 27ay^2 = 4(x-2a)^3$$

Theorem 8 The normal to a given curve is tangent to its evolute.

Proof. We know that the coordinates of the centre of curvature of the given curve are given by

$$X = x - \frac{y_1}{y_2}(1+y_1^2$$

$$Y = y + \frac{1+y_1^2}{y_2}$$

These two equations can be taken as the parametric equation of the evolute with x as parameter.

$$\therefore \frac{dX}{dx} = 1 - \left(\frac{y_1}{y_2}\right) 2y_1y_2 - (1+y_1^2)\left\{\frac{y_{2^2} - y_1y_3}{y_{2^2}}\right\}$$

$$=1-2y_1^2-(1+y_1^2)(1-\frac{y_1y_3}{y_2^{\,2}})$$

$$=-3y_1^{\,2}+\frac{y_1y_3}{y_2^{\,2}}+\frac{y_1^{\,3}y_3}{y_2^{\,2}})$$

$$=-\frac{y_1}{y_2^{\,2}}(3y_3^{\,2}-y_3-y_1^{\,2}y_3).$$

$$Now\frac{dY}{dx}=y_1+\left\{\frac{2y_1y_{2^2}-(1+y_1^{\,2})\,y_3}{y_2^{\,4}}\right\}3y_3^{\,2}-y_3-y_1^{\,2}y_3$$

$$=\frac{1}{y_2^{\,2}}(3y_1y_2^{\,2}-y_3-y_1^{\,2}y_3)$$

$$\therefore\frac{dY}{dX}=-\frac{1}{y_1} \qquad \text{....(1)}$$

But $\frac{dY}{dX}$ is the slope of the tangent to the evolute and y_1 is the slope of the tangent to the given curve at the corresponding point and their product is -1 (by 1).

∴ Tangent to the evolute is normal to the given curve.

Exercises

1. Find the coordinates of the center of curvature at the indicated points.

[a] $y = x^2$ at $(\frac{1}{3},\frac{1}{4})$

[b] $xy = c^2$ at (c,c).

[c] x = a (cos t + t sin t), y=a (sin t – t cos t) at 't'.

[d] y = x log x at the point where y' = 0.

Exercise 2:

1. Find the coordinates of the centres of curvature at given points on the curves :

(1) $y = x^2;\ (1/2, 1/4)$

(2) $xy = c^2;\ (c, c).$

(3) $y = \log\sec x;\ \left(\frac{\pi}{3}, \log 2\right)$

2. Prove that the circle of curvature at the point $(t^2, 2t)$ of the curve $y^2=4x$ cuts the curve again at a point whose ordinate is – 6t. Calculate the coordinates of the centre of curvature.

1.5 Evolute and involute.

We have already defined evolute of a curve as the locus of the centre of curvature and deduced the equations of the evolute of the parabola and ellipse.

If the evolute itself be regarded as the original curve, a curve of which it is the evolute is called an involute.

It may be noted that there is but one evolute but an infinite number of involutes.

Radius of curvature when the curve is given in polar co-ordinates

Let us assume that the equation of the curve in polar coordinates be r = f (θ).

In the figure,

$$\psi = \theta + \phi.$$

$$\therefore \frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta}.$$

We have proved that

$$\tan\phi = r\frac{d\theta}{dt} = \frac{r}{\left(\frac{dr}{d\theta}\right)}.$$

Differentiating w.r.t θ, we get

$$Sec^2\varphi \frac{d\varphi}{d\theta} = \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$$

$$\therefore \frac{d\varphi}{d\theta} = \frac{1}{\sec^2\varphi}\frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$$

$$= \frac{1}{1 + \frac{r^2}{\left(\frac{dr}{d\theta}\right)^2}}\frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}$$

$$= \frac{\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2 + r^2}$$

$$\frac{d\varphi}{d\theta} = 1 + \frac{d\varphi}{d\theta}$$

$$=1+\frac{\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2+r^2}$$

$$=\frac{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2+r^2}$$

We have proved in the previous chapter that

$$=\frac{ds}{d\theta}=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{1/2}$$

$$\rho=\frac{ds}{d\psi}=\frac{ds}{d\theta}.\frac{d\theta}{d\psi}$$

$$=\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{1/2}\frac{\left(\frac{dr}{d\theta}\right)^2+r^2}{r^2+2\left(\frac{dr}{d\theta}\right)^2-\left(r\frac{d^2r}{d\theta^2}\right)}$$

$$=\frac{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}$$

Examples .

1. Find the radius of curvature of the cardioid r = a (1-cosθ).

Soln:

Given r = a (1-cosθ).

Differentiating w.r.t θ, we get

$$\frac{dr}{d\theta} = a\sin\theta, and \frac{d^2r}{d\theta^2} = a\cos\theta.$$

$$\therefore \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{3/2} = \left[a^2(1-\cos\theta)^2 + (a^2\sin^2\theta) \right]^{3/2}$$

$$= 8a^3\sin^3\frac{\theta}{2}.$$

$$r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2} = a^2(1-\cos\theta)^2 + 2a^2\sin^2\theta - a^2\cos\theta(1-\cos\theta)$$

$$= 6a^2\sin^2\frac{\theta}{2}$$

$$\therefore \rho = \frac{8a^3 Sin^3\frac{\theta}{2}}{6a^2\sin^2\frac{\theta}{2}} = \frac{4}{3}a\sin\frac{\theta}{2}$$

$$= \frac{2}{3}\sqrt{2ar}$$

.

2. Show that the radius of curvature of the curve

$$r^n = a^n Cos\, n\theta\, is \frac{a^n r^{-n+1}}{n+1}$$

Soln:

Taking logarithms on both sides and differentiating, we get

$$\frac{n}{r}\frac{dr}{d\theta}=-\frac{n\sin n\theta}{\cos n\theta}$$

$$\therefore \frac{dr}{d\theta}=-r\tan n\,\theta$$

Differentiating once again w.r.t. θ, we get

$$\frac{d^2r}{d\theta^3}=-\frac{dr}{d\theta}\tan n\theta-nr\sec^2 n\theta$$

$$=r\tan^2 n\theta-nr\sec^2 n\theta$$

$$\rho=\frac{(r^2+r^2\tan^2 n\theta}{r^2+2r^2\tan^2 n\theta-r^2\tan^2 n\theta+nr^2\sec^2 n\theta}$$

$$=\frac{r^3\sec^3 n\theta}{(n+1)r^2\sec^2 n\theta}=\frac{r}{(n+1)\cos n\theta}$$

$$=\frac{r.a^n}{(n+1)r^n}=\frac{a^n r^{-n+1}}{n+1}$$

Particular cases.

i) Putting n = 2, we get Bernouilli's lemniscate;

$$\rho=\frac{a^2}{3\ r}$$

ii) when n = -2 we have a rectangular hyperbola.

$$\rho=\frac{r^3}{a^2}$$

iii) when n = ½, we get cardioids; $\rho = \frac{2}{3}\sqrt{ar}$

iv) when n = -1/2, we get a parabola, $\rho = \frac{2r^{3/2}}{\sqrt{a}}$

v) when n =1. we get a circles ; $\rho = \frac{a}{2}$.

1.6 Asymptotes in Cartesian and polar co-ordinates.

Asymptotes of polar curves.

Theorem: For the polar curves of the form $\frac{1}{r} = f(\theta)$ the asymptotes are $r\sin(\theta - \theta_i) = \frac{1}{f'(\theta_i)}$

where θ_i 's are the roots of the equation f(θ) = 0.

Proof: Let P (r,θ) be any point on the curve $\frac{1}{r} = f(\theta)$ ………………… (1)

As P tends to infinity along the curve r → ∞. Hence from (1) we note that

when r → ∞, f(θ) → 0.

Let the roots of the equation f(θ) = 0

be θ_i where i = 1,2,3 ……….

Hence θ_i 's are the only directions along which the branches of the curve tend to infinity. Let us consider the branch corresponding to the value $\theta = \theta_1$.

For this branch $\theta \to \theta_1$, r → ∞.

We know that the polar equation of any line is p = r cos(θ - α)

where p is the length of the perpendicular from the pole on the line and α is the angle which this perpendicular makes with the initial line and (r,θ) is any point on the line.

Consider an asymptote to the branch corresponding to $\theta = \theta_1$.

Let it be $p = r\cos(\theta - \alpha)$ ………………. (2)

For this branch corresponding to $\theta = \theta_1$. the equation of the asymptote will be determined if we find p and α.

Draw ON perpendicular to the line (2).

Then ON = p and $\angle XON = \alpha$.

Draw PM perpendicular to the line (2) and PL perpendicular to ON.

$\therefore$ PM = LN = ON - OL = p-OP $\cos(\theta - \alpha)$

$= p - r\cos(\theta - \alpha)$.

$\therefore \frac{PM}{r} = \frac{p}{r} - \cos(\theta - \alpha)$ ……………………… (3).

Since (2) is an asymptote of (1), by definition, PM $\to 0$ as P $\to \infty$.

(i.e) As $r \to \infty$, $f(\theta) \to 0$ as $\theta \to \theta_1$ so that $\frac{p}{r} \to 0$

$\therefore$ From (3) $\lim_{\theta \to \theta_1} \frac{PM}{r} = \lim_{\theta \to \theta_1} \left[\frac{p}{r} - \cos(\theta - \alpha)\right]$

(i.e) $0 = 0 - \lim_{\theta \to \theta_1} [\cos(\theta - \alpha)]$

$\therefore$ $0 = \cos(\theta_1 - \alpha)$. Hence $\theta_1 - \alpha = \frac{\pi}{2}$.

$\therefore \alpha = \theta_1 - \frac{\pi}{2}$. ………………………… (4)

Also using $\frac{1}{r} = f(\theta)$ in (3) we get $PM = p - \frac{\cos(\theta - \alpha)}{f(\theta)}$

Taking limit as $\theta \to \theta_1$ we have $0 = \lim_{\theta\to\theta_1}\left[p - \frac{\cos(\theta-\alpha)}{f(\theta)}\right]$

$$= p - \lim_{\theta\to\theta_1}\frac{\cos(\theta-\alpha)}{f(\theta)}$$

$$\therefore\ p = \lim_{\theta\to\theta_1}\frac{\cos(\theta-\alpha)}{f(\theta)} = \lim_{\theta\to\theta_1}\left[\frac{\cos\left(\theta-\theta_1+\frac{\pi}{2}\right)}{f(\theta)}\right] \text{(Using (4))}$$

$$= \lim_{\theta\to\theta_1}\left[\frac{-\sin(\theta-\theta_1)}{f(\theta)}\right]$$

$$= -\lim_{\theta\to\theta_1}\left[\frac{\cos(\theta-\theta_1)}{f'(\theta)}\right]$$

(Using L-Hospital's rule)

$$= -\left[\frac{\cos 0}{f'(\theta_1)}\right]$$

$$\therefore\ p = -\frac{1}{f'(\theta_1)}.$$

Substituting the values of p and α in (2) we get

$$-\frac{1}{f'(\theta_1)} = r\cos\left(\theta-\theta_1+\frac{\pi}{2}\right) = -r\sin(\theta-\theta_1)$$

$$\therefore\ r\sin(\theta-\theta_1) = \frac{1}{f'(\theta_1)}.$$

This is the required asymptote corresponding to $\theta = \theta_1$.

Similarly the other asymptotes corresponding to the other roots of $f(\theta) = 0$ can be got.

Working rule to find the asymptotes of the polar curves.

1. Write the polar equation in the form $\frac{1}{r} = f(\theta)$.

2. Find the roots of $f(\theta) = 0$. Let the roots be $\theta_1, \theta_2, \theta_3 \ldots\ldots$

3. Find $f'(\theta)$ and calculate $f'(\theta)$ at $\theta = \theta_1, \theta_2, \theta_3 \ldots\ldots$

4. Then write the equations of the asymptotes as $r\sin(\theta-\theta_1) = \frac{1}{f'(\theta_1)}, r\sin(\theta-\theta_2) = \frac{1}{f'(\theta_2)},$

$r\sin(\theta-\theta_3) = \frac{1}{f'(\theta_3)} \ldots\ldots\ldots$

Solved problems.

Problem 1. Find the equations of the asymptotes of the following curves

(i) $r\,\theta = a$ (ii) $r \log \theta = a$.

Solution. (i): The equation of the given curve in the form $\frac{1}{r} = f(\theta)$

we have $\frac{1}{r} = \frac{\theta}{a}$ so that $f(\theta) = \frac{\theta}{a}$.

Now, $f(\theta) = 0 \Rightarrow \frac{\theta}{a} = 0$. Hence $\theta = 0$.

Also, $f'(\theta) = \frac{1}{a}$. Hence $f'(0) = \frac{1}{a}$.

$\therefore$ The equation of the asymptote is $r\sin(\theta - 0) = \frac{1}{f'(0)} = \frac{1}{a}$.

(i.e) r sin θ = a.

(ii) The equation of the given curve in the form $\frac{1}{r} = f(\theta)$

we have $\frac{1}{r} = \frac{\log\theta}{a}$ so that $f(\theta) = \frac{\log\theta}{a}$.

Now, f(θ) = 0 ⇒ $\frac{\log\theta}{a} = 0$. Hence $\theta = 1$.

Also, $f'(\theta) = \frac{1}{a\theta}$. Hence $f'(1) = \frac{1}{a}$.

$\therefore$ The equation of the asymptote is $r\sin(\theta - 1) = \frac{1}{f'(1)} = a$.

(i.e) r sin(θ-1) = a .

Problem 2. Find the equation of the asymptotes of the curve
$r(\theta^2 - \pi^2) = 2a\theta$

Soln: The equation of the given curve in the form
$\frac{1}{r} = f(\theta)$

we have $\frac{1}{r} = \frac{\theta^2 - \pi^2}{2a\theta}$ so that $f(\theta) = \frac{\theta^2 - \pi^2}{2a\theta}$.

Now, $f(\theta) = 0 \Rightarrow \frac{\theta^2 - \pi^2}{2a\pi} = 0$

$\Rightarrow \theta^2 = \pi^2$.

$\Rightarrow \theta = \pm\pi.$

Also, $f'(\theta) = \frac{1}{2a}\left[\frac{2\theta^2 - (\theta^2 - \pi^2)}{\theta^2}\right] = \frac{\theta^2 + \pi^2}{2a\theta^2}$

$\therefore\ f'(\pi) = \frac{\pi^2 + \pi^2}{2a\pi^2} = \frac{1}{a}.\ Also\ f'(-\pi) = \frac{1}{a}.$

$\therefore$ The equation of the asymptote corresponding to θ = π is

$$r\sin(\theta - \pi) = \frac{1}{f'(\pi)} = a.$$

(i.e) -r sin θ = a. Hence r sin θ + a = 0 ……….(1) Similarly the asymptote corresponding to θ = -π is

$$r\sin(\theta + \pi) = \frac{1}{f'(-\pi)} = \frac{1}{a}.$$

(i.e) r sin θ + a = 0 which is same as (1).

Hence there is only one asymptote for the given curve.

Problem 3. Find the equation of the asymptotes of the curve $r = \frac{a}{1-\cos\theta}$

Solution. The equation of the given curve in the form $\frac{1}{r} = f(\theta)$

we have $\frac{1}{r} = \frac{1-\cos\theta}{a}\ so that\ f(\theta) = \frac{1-\cos\theta}{a}.$

Now, $f(\theta) = 0 \Rightarrow \frac{1-\cos\theta}{a} = 0$

$\Rightarrow 1 - \cos\theta = 0$

$\Rightarrow \cos\theta = 1$

$\Rightarrow \theta = 2n\pi \pm 0$ where $n \in Z$

$\Rightarrow \theta = 2n\pi$ where $n \in Z$.

Now, $f'(\theta) = \dfrac{\sin\theta}{a}$. Hence $f'(2n\pi) = \dfrac{\sin 2n\pi}{a} = 0$ for all $n \in Z$.

$\therefore \dfrac{1}{f'(2n\pi)} = \infty$ for all $n \in Z$

Hence the curve has no asymptotes.

Problem 4. Find the equation of the asymptotes of the curve r cos θ = a sin θ.

Solution. Writing the equation of the given curve in the form $\dfrac{1}{r} = f(\theta)$.

we have $\dfrac{1}{r} = \dfrac{\cos\theta}{a\sin\theta}$ so that $f(\theta) = \dfrac{\cos\theta}{a\sin\theta}$.

Now, f(θ) = 0 ⇒ cos θ = 0

$$\Rightarrow \theta = (2n+1)\frac{\pi}{2} \text{ where } n \in Z.$$

Since $f(\theta) = \dfrac{\cot\theta}{a}$ we have $f'(\theta) = -\dfrac{\operatorname{cosec}^2\theta}{a} = -\dfrac{1}{a\sin^2\theta}$

Now, $f'\left[(2n+1)\dfrac{\pi}{2}\right] = \dfrac{1}{-a\sin^2\left[(2n+1)\dfrac{\pi}{2}\right]}$ for all $n \in Z$.

$$= -\frac{1}{a\sin^2(n\pi + \frac{\pi}{2})}$$

$$= -\frac{1}{a\left[(-1)^n \sin \pi/2\right]^2} = -\frac{1}{a(-1)^{2n}} = -\frac{1}{a}.$$

$\therefore$ The asymptotes are given by $r\sin\left[\theta - (2n+1)\frac{\pi}{2}\right] = \frac{1}{f'\left[(2n+1)\frac{\pi}{2}\right]}, n \in Z$

(i.e) $-r\sin\left[(2n+1)\frac{\pi}{2} - \theta\right] = -a$

(i.e) $r\sin\left[(2n+1)\frac{\pi}{2} - \theta\right] = a \text{ } for\, all\, n \in Z.$

(i.e) $r\sin\left(\frac{\pi}{2} - \theta\right) = \pm a$

(i.e) r cos θ = ±a

(i.e) $r\cos\theta = \frac{a}{(-1)^n}.$

For different values of n ϵ Z we get only two asymptotes r cos θ = -a and r cos θ = a.

Hence the asymptotes are r cos θ = ±a.

Exercises.

Find the asymptotes of the following curves.

(i) $r = \frac{a\theta}{\theta - 1}$ (ii) $r(1 - e^{\theta}) = a$

Tracing of curves f(x,y) = 0 (Cartesian Coordinates)

Suppose a curve is represented in terms of Cartesian coordinates by the equation $f(x,y) = 0$. The following points provide useful informations regarding the shape and nature of the curve.

I. Symmetry of the curve.

(a) Symmetry about the x - axis:

A curve $f(x,y) = 0$ is symmetric about the x-axis if $f(x,-y) = f(x,y)$.

Example. $y^2 = 4ax$; $x^2 + y^2 = a^2$; $y^4 + y^2 + x^3 = 0$ are curves which are symmetric about the x - axis.

But $x^2 + y^2 = ay$ is not symmetric about the x-axis.

(b) Symmetry about the y - axis.

A curve $f(x,y) = 0$ is symmetric about the y - axis if $f(-x,y) = f(x,y)$

Example. $x^2 = 4ay$; $x^2 + y^2 = a^2$; $y = x^4 + x^2 + a$ are symmetric about y - axis. But $x^2 + y^2 = ax$ is not symmetric about y - axis.

Note. $x^2 + y^2 = a^2$ is symmetric about x - axis and y - axis. In this case the equation involves even and only even powers of x as well as y.

(c) Symmetry about the line y = x

If $f(x,y) = f(y,x)$ then the curve is symmetric about the line $y = x$.

Example. $x^2 + y^2 = a^2$; $x^3 + y^3 = 3axy$; $xy = c^2$ are symmetric about the line $y = x$.

(d) Symmetric about the origin. (Symmetric in opposite quadrants)

If f(-x,-y) = f(x,y) then the curve is symmetric about the origin (symmetric in opposite quadrants)

Examples. $x^2 + y^2 = a^2$; $xy = c^2$ are symmetric about the origin.

$x^3 + y^3 = 3axy$; $y^2 = x^3$ are not symmetric about the origin.

Note. From the above examples the equation of the circle has all symmetric properties we have discussed so far.

Points of intersection with the coordinate axes.

To obtain the points where the curve f(x,y) = 0 intersects the x - axis put y = 0 in the equation and solve for x. Similarly, to find the points where the curve intersects the y - axis put x = 0 in the equation and solve for y.

Examples. The curve $x^2 + y^2 = a^2$ crosses the x - axis at (a,0) and (-a,0) and crosses the y - axis at (0,a) and (0,-a).

The curve $y^2 = 4ax$ passes through the origin.

IV. Tangents to the curve.

(a) Tangents at the origin.

If the origin is found to be a point on the curve then the tangents at the origin are obtained by equating to zero the lowest degree terms occuring in the equation.

Example. $y^2 = 4ax$ passes through the origin and the lowest degree term occuring in it is 4ax which when equated to zero becomes 4ax = 0 (i.e) $x = 0$. Hence y - axis is the tangent to the parabola at the origin.

Also $x^3 + y^3 = 3axy$ passes through the origin at which $x = 0$ and $y = 0$ are the tangents.

For the curve $a^2y^2 = a^2x^2 - x^4$, $y = \pm x$ are the tangents at the origin.

(b) Tangents at any other point (h,k) other than the origin.

Find $\frac{dy}{dx}$ at (h,k) and it gives the slope of the tangent to the curve at this point. This will be useful to deside the nature of the tangent - whether parallel to the x - axis or y - axis or inclined tangent.

V. Asymptotes.

The concept of asymptotes described in the previous chapter will be helpful to know about the asymptotes in tracing any curve.

(a) Asymptotes parallel to the x - axis.

These are obtained by equating to zero the coefficient of the highest power of x.

Example. $(y+a)x^2 + x - 1 = 0$ has an asymptote $y = -a$ parallel to the x - axis.

(b) Asymptotes parallel to the y - axis.

These are obtained by equating to zero the coefficient of the highest power of y.

Example. $y^2(4 - x^2) = x^3 - 1$ has asymptotes $4 - x^2 = 0$ (i.e) $x = 2$ and $\lambda = -2$ are two asymptotes parallel to the y - axis.

(c) Inclined asymptotes.

Taking y = mx + c as an asymptote we can find m and c by substituting y = mx + c in the equation and equating to zero the various powers of x starting from the highest power.

Polar Coordinates : (Tracing a curve f(r,θ) = 0):

To trace curve given in terms of polar coordinates by the equation f(r,θ) = 0.

I. Symmetry of the curve.

(a) Symmetry about the initial line:

The curve f(r,θ) = 0 is symmetric about the initial line θ = 0 if f(r, -θ) = f(r, θ).

Example. r = a(1+cosθ); r = a(1-cosθ); r = a cos2θ Symmetric about the initial line.

However r = a(1-sinθ) is not Symmetric about the initial line.

Symmetry about the pole

.

The curve is symmetric about the pole if f(-r,θ) = f(r, θ).

Example: $r^2 = a^2 \cos 2\theta$; $r^2 = a^2 \sin 2\theta$ are Symmetric about the pole.

.

(c) Symmetry about θ = $\frac{\pi}{2}$

The curve f(r,θ) = 0 is symmetric about the line θ = $\frac{\pi}{2}$ (y-axis) if f(r, π, -θ) = f(r, θ). :

Example : r = a(1+sinθ) ; r = a sin3θ are symmetric about θ = $\frac{\pi}{2}$

Tangents at the pole.

We put r = θ in the equation of the curve and solve the resulting equation for θ.If there exists a real solution α for θ,then the curve passes through the pole and the line θ=α is a tangent to the curve at the pole.

Region in which the curve lies.

1) If the maximum value of r is a, then the curve lies within the circle r = a.

2) If there exist values of θ for which r2<0 so that r becomes imaginary then the curve does not exist for those values of θ.

Example: $r^2=a^2 \sin 2\theta$ does not exist if $\frac{\pi}{2} < \theta < \pi$

Value of ϕ.

The angle ϕ which a tangent at(r, θ) makes with the initial line is found from the formula $\tan\phi = r\frac{d\theta}{dr}$

Asymptotes:

If there is no finite value α for θ such that $r \to \infty$, then the curve f(r, θ) = 0 has no asymptotes

Points on the curve:

Giving different values for θ we can get different points on the curve which will be of use in tracing the curve and ascertain whether r increases or decreases in the region.

Tracing a curve x = f(t), y = g(t) **(parametric equations)**

(i) Suppose x = f(t), y = g(t) are parametric eqations of a curve where t is the parameter.

If it is possible to eliminate the parameter between the two equations and get the

Cartesian form of the curve we proceed as in the case of Cartesian coordinates.

(ii) If the parameter t can not be easily eliminated

(a) Find $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

(b) Give different values to the parameter t and find x, y, $\frac{dy}{dx}$. This gives different points on the curve end slopes of the tangents at these points.

(c) We plot the points and trace the curve.

Solved Problems:

Problem 1 : Trace the curve $(x)^{2/3} + (y)^{2/3} = (a\ )^{2/3}$ **(four cusped cycloid or asteroid)**

Solution:

Given the curve $(x)^{2/3} + (y)^{2/3} = (a\ )^{2/3}$ ……………………….(1)

Clearly the curve is symmetrical about both the axes. Hence it is enough to discuss

The nature of the curve in the first guadrant only.

To find the points of intersection of the curve with x-axis, we put y=0 in eqn (1)

We get $(x)^{2/3} = (a^2)^{2/3}$

Therefore $x^2 = a^2$ and hence x=± a.

Hence the curve meets the x-axis at (0,a) and (-a,0).

Similarly, the curve meets the y-axis at (a,0) and (0,-a).

From eqn (1) $\left(\frac{y}{a}\right)^{2/3} = 1 - \left(\frac{x}{a}\right)^{2/3}$,we see that if $|x| > a$,then

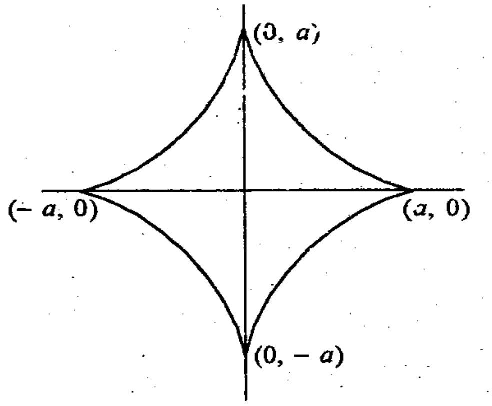
$\left(\frac{y}{a}\right)^{2/3}$ < 0 and hence y is imaginary.

Hence the curve does not lie beyond x = ± a

Similarly, the curve does not lie beyond y = ± a

$$Also, \left(\frac{dy}{dx}\right) = -\sqrt[3]{\frac{y}{x}}$$

$\sqrt[3]{\ }\ \sqrt[3]{\ }$



$$\left(\frac{dy}{dx}\right) = 0$$

At (a, 0) and hence x-axis is a tangent to the two branches of the curve at (a, 0) lying in the first and fourth quadrants.

Hence the curve has a cusp of first kind at (a, 0).

Similarly, the curve has cusps of first kind at (0, a), (-a, 0), (0,-a).

Hence the curve is known as four cusp hypocycloid.

Also, the curve is concave in [0,a].

Hence the form of the curve is as shown in the fig.

Note: The parametric equation of this curve can be taken as $x = a\cos^3\theta$; $y = a\sin^3\theta$.

Problem 2 : Trace the curve $y^2 (2a - x) = x^3$ **(cissoid)**

Solution. $y^2 (2a - x) = x^3$ ……………………………… [1]

Since [1] contains even power of y the curve is symmetrical about the x-axis.

Obviously it passes through the origin.

The tangents at the origin are given by $y^2 = 0$ and they are real and coincident. Hence the origin is a cusp.

The curve meets the x-axis and y-axis only at the origin.

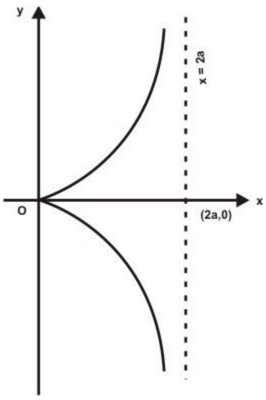
Equating the coefficient of the highest degree term in y to zero we get x - 2a = 0. The asymptotes parallel to the y - axis is x - 2a = 0 and this is the only asymptote to the curve.

Writing the given equation as $y = x\sqrt{\frac{x}{2a-x}}$. (Considering the positive root) we see that y is imaginary when x < 0 or when x > 2a.

Hence the curve does not lie to the left of the y - axis and to the right of the line x = 2a.

As x increases from 0 to 2a y increases from 0 to ∞.

Hence the form of the curve is as shown in the figure and the curve is called cissoid.



Problem 3: Trace the curve r = a(1+cos θ) where a > 0 **(cardioid).**

Solution. We note the following from the equation of the given curve. The curve is symmetric about the initial line.

When θ = π we have r = 0. Hence the curve passes through the pole and further θ = π is the tangent at the pole.

Let φ be the angle made by the tangent at (r, θ) with the initial line.

Now, $\tan\phi = r\frac{d\theta}{dr} = \frac{a(1+\cos\theta)}{-a\sin\theta} = -\cot\left(\frac{\theta}{2}\right) = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right).$

$\therefore \theta = \frac{\pi}{2} + \frac{0}{2}$ Hence when θ = 0, $\phi = \frac{\pi}{2}$ and r = 2a.

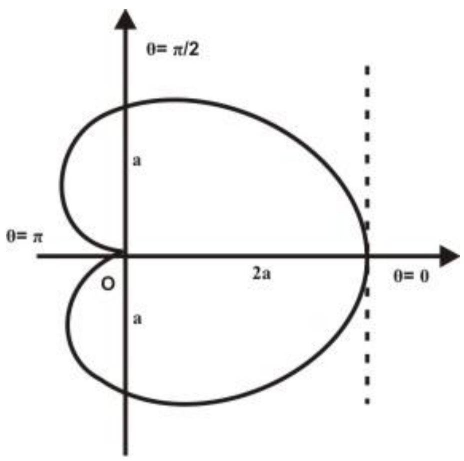
Thus the tangent at (2a, 0) is perpendicular to the initial line.

Since the maximum value of r is 2a, no portion of the curve lies to the right of the tangent at (2a, 0) and hence the curve lies within the circle r = 2a.

The following table gives a set of points lying on the curve.

θ	0	π/4	π/2	π	- π/2	- π/4
r	2a	$a\,(1+\frac{1}{\sqrt{2}})$	a	0	a	$a\,(1+\frac{1}{\sqrt{2}})$

When θ increases from 0 to 2π, r is positive and it decreases from 2a to 0. The form of the curve is as shown in the figure and it is a cardioid.



Problem 4 : Trace the curve $r^2 = a^2 \cos 2\theta$ **(Lemniscate of Bernoulli)**

Solution. The curve is symmetric about the pole and the initial line.

Negative values of cos 2 θ give imaginary values of r. Hence the curve lies in the two quadrants bounded by $\theta = \frac{7\pi}{4}, \theta = \frac{\pi}{4}; and\, \theta = \frac{3\pi}{4}, \theta = \frac{5\pi}{4}.$

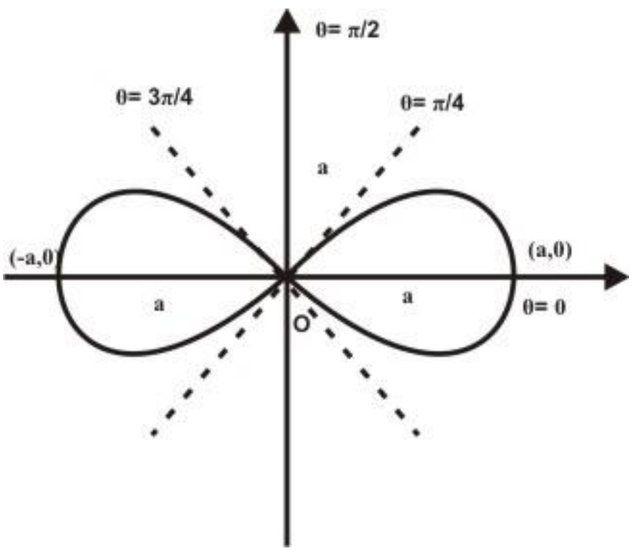
The lines $\theta = \frac{\pi}{4}, \theta = \frac{3\pi}{4}; \theta = \frac{5\pi}{4}, \theta = \frac{7\pi}{4}$ are tangents to the curve at the people.

Some points on the curve are given below.

θ	0	π/4	3π/4	5π/4	7π/4
r	A	0	0	0	0

When θ increases from $\theta = \frac{7\pi}{4}$ to 2π (=0), r incerases from 0 to a and when θ increases from 0 to $\theta = \frac{\pi}{4}$ r decreases from a to 0.

The form of the curve is as shown in the figure below.



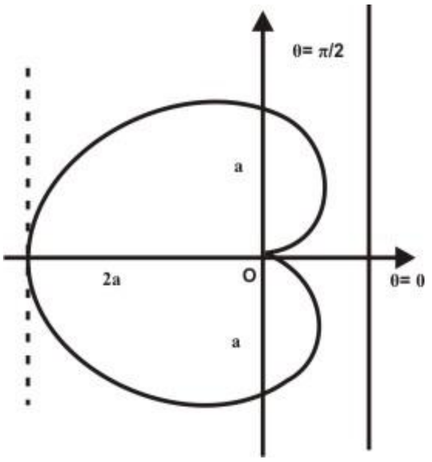
Problem 5 : Trace the curve r = a(1-cosθ)

Soln: The curve is symmetric about the initial line.

It passes through the pole.

$\tan\phi = \tan(\theta/2)$. Hence $\phi = \theta/2$.

At $(2a, \pi)$, the tangent is perpendicular to the initial line.



Problem 6 : Trace the curve r = sin 3 θ.

Solution. The curve is symmetric about the line $\theta = \frac{\pi}{2}$

$r = 0 \Rightarrow \sin 3\theta = 0 \Rightarrow 3\theta = 0$ or multiple of π

$$\Rightarrow \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}.$$

For θ = 0 the curve passes through the pole.

Further θ = 0, θ = π ; $\theta = \frac{\pi}{3}, \theta = \frac{4\pi}{3}; and\ \theta = \frac{2\pi}{3}, \theta = \frac{5\pi}{3}$ are the tangents to the curve at the pole

Since $|\sin 3\theta| \le 1$, $r \le a$. Hence the curve lies entirely within the circle r = a.

We get different points on the curve as shown in the table.

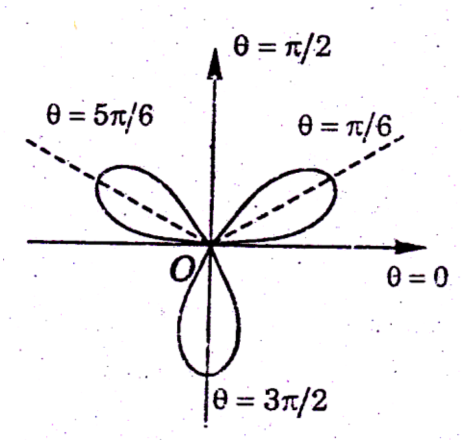
θ	0	π/6	π/3	π/2	2π/3	5π/6	π
r	0	a	0	-a	0	A	0

When $0 \leq \theta \leq \pi/3$ the curve has one loop around $\theta = \pi/6$.

When $4\pi/3 \leq \theta \leq 5\pi/3$ the curve has another loop around $\theta = 3\pi/2$.

When $2\pi/3 \leq \theta \leq \pi$ the curve has yet another loop around $\theta = 5\pi/6$.

The form of the courve as shown in the figure and it is known as three leaved rose.



UNIT – 2 : EVALUATION OF DOUBLE INTEGRALS

Evaluation of double and triple integrals-Jacobeans, change of variables.

EVALUATION OF DOUBLE INTEGRALS

1.1 Double Integrals:

Let f(x,y) be a continuous function defind on a closed rectangle R = {(x,y)/ a ≤ x ≤ b and c ≤ y ≤ d}.

For any fixed x∈[a,b] consider the integral $\int_c^d f(x,y)dy$.

The value of this integral depends on x and we get a new function of x. This can be integrated with respect to x and we get $\int_c^b \left[\int_c^d f(x,y)\,dy\right]dx$.

This is called an iterated integral.

Similarly we can define another integral $\int_c^d \left[\int_a^b f(x,y)dx\right]dy$.

For continuous functions f(x,y) we have

$$\iint_R f(x,y)\,dx\,dy = \int_c^b \left[\int_c^d f(x,y)\,dy\right]dx = \int_c^d \left[\int_a^b f(x,y)\,dx\right]dy$$

We omit the proof of this result.

If f(x,y) is continuous on a bounded region S and if S is given by S = { (x,y)/ a ≤ x ≤ b and φ_1(x) ≤ y ≤ φ_2(x)} where φ_1 and φ_2 are two continuous functions definded on [a, b] then

$$\iint_S f(x,y)dx\,dy = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy\right]dx$$

The inerated integral in the right hand side is also written in the form

$$\int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)dy.$$

Similarly if S = { (x,y) / c ≤ y ≤ d and $\varphi_1(y) \le x \le \varphi_2(y)$}

then

$$\iint_S f(x,y)dx\,dy = \int_c^d \left[\int_{\varphi_1(y)}^{\varphi_2(y)} f(x,y)dx\right] dy$$

If S cannot be written in neither of the above two forms we devide S into finite number of subregions such that each of the subregion can be represented in one of the above forms and we get the double integral over S by adding the integrals over these subregions.

Hence to evaluate $\iint_D f(x,y)\,dx\,dy$ we first convert it to an iterated integral of the two forms given above.

Solved Problems.

Problem 1. Evaluate $I = \int_0^1 \int_0^2 xy^2\,dy\,dx.$

Solution.

$$Now \int_0^1 \int_0^2 xy^2\,dy\,dx = \int_0^1 \left[\frac{1}{3}xy^3\right]_0^2 dx = \frac{1}{3}\int_0^1 8x\,dx = \frac{8}{3}\left[\frac{x^2}{2}\right]_0^1 = \frac{4}{3}$$

$$Therefore I = \int_0^1 \int_0^2 xy^2\,dy\,dx = \frac{4}{3}.$$

Problem :2 Evaluate $I = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx.$

Solution. $$I = \int_0^{4a} \left[\frac{xy^2}{2} \right]_{x^2/4a}^{2\sqrt{ax}} dx$$

$$= \frac{1}{2}\int_0^{4a} x\left[4ax - \frac{x^4}{16a^2}\right]dx = \frac{1}{2}\left[\frac{4ax^3}{3} - \frac{x^6}{96a^2}\right]_0^{4a} = \frac{64a^4}{3}$$

Problem : 3 Evaluate $I = \int_0^a \int_0^b \left(x^2 + y^2\right) dx\, dy$

Solution. $$I = \int_0^a \left[\frac{x^3}{3} + xy^2\right]_0^b dy = \int_0^a \left[\frac{b^3}{3} + by^2\right] dy$$

$$= \left[\frac{b^3 y}{3} + \frac{by^3}{3}\right]_0^a$$

= 1/3 ab (a^2 + b^2).

Problem 4. Evaluate $I = \int_0^a dx \int_0^{b\sqrt{\left[1-\left(x^2/a^2\right)\right]}} x^3 y\, dy$

Solution. $$I = \int_0^a \left[\frac{1}{2}x^3y^3\right]_0^{b\sqrt{\left[1-\left(x^2/a^2\right)\right]}} dx$$

$$= \frac{1}{2}\int_0^a b^2x^3\left(1 - \frac{x^2}{a^2}\right)dx = \frac{1}{2}b^2\left[\frac{1}{4}x^4 - \frac{x^6}{6a^2}\right]_0^a$$

$$= \frac{a^4b^2}{24}.$$

Problem 5 : Evaluate $I = \int_0^{1/2} \int_0^1 \frac{x}{\sqrt{\left(1 - x^2y^2\right)}} dy\, dx.$

Solution. $I=\int_0^{1/2} x\left[\frac{\sin^{-}1(xy)}{x}\right]_0^1 dx$

$=\int_0^{1/2}\sin^{-1}x\,dx=\left[x\sin^{-1}x+\sqrt{1-x^2}\,\right]_0^{1/2}$ (integration by parts)

$=\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1.$

Problem 6 : Evaluate $I=\int_0^{\pi}\int_0^{a\cos\theta} r\sin\theta\,dr\,d\theta$

Solution. $I=\int_0^{\pi}\sin\theta\left[\frac{1}{2}r^2\right]_0^{a\cos\theta} d\theta$

$=\frac{1}{2}\int_0^{\pi}a^2\cos^2\theta\sin\theta\,d\theta=-\frac{1}{2}a^2\int_0^{\pi}\cos^2\theta\,d(\cos\theta)$

$=-\frac{1}{6}a^2\left[\cos^3\theta\right]_0^{\pi}$

= 1/3 a^2

Problem 7. Evaluate $\int_0^{\pi/2}\int_0^{\infty}\frac{r}{\left(r^2+a^2\right)^2}dr\,d\theta$

Solution Let $I=\int_0^{\pi/2}\int_0^{\infty}\frac{r}{\left(r^2+a^2\right)^2}dr\,d\theta$

$=\int_0^{\pi/2}\left[\int_0^{\infty}\frac{\frac{1}{2}d(r^2)}{(r^2+a^2)^2}\right]d\theta=\int_0^{\pi/2}\frac{1}{2}\left[\frac{-1}{r^2+a^2}\right]_0^{\infty}d\theta$

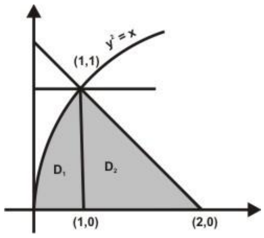
$=\frac{1}{2}\int_0^{\pi/2}\frac{d\theta}{a^2}=\left[\frac{\theta}{2a^2}\right]_0^{\pi/2}$

$$= \frac{\pi}{4a^2}.$$

Change of order of Integration:

Problem 1. Evaluate $I = \iint_D xy\,dy\,dx$ where D is the region bounded by the curve $x = y^2$, $x = 2-y$, $y=0$ and $y = 1$.

Solution. The given region bounded by the curves is given in the figure.



In this region x various from 0 to 2. When $0 \le x \le 1$, for fixed x y varies from 0 to x. when $1 \le x \le 2$,y various from 0 to 2-x.

The region D can be subdivided into two regions D_1 and D_2 as shown in the figure.

$$\iint_D xy\,dx\,dy = \iint_{D1} xy\,dy\,dx + \iint_{D2} xy\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} xy\,dy\,dx + \int_1^2 \int_0^{2-x} xy\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{2}xy^2\right]_0^{\sqrt{x}} dx + \int_1^2 \left[\frac{1}{2}xy^2\right]_0^{2-x} dx.$$

$$= \frac{1}{2}\int_0^1 x^2 dx + \frac{1}{2}\int_1^2 x(2-x)^2 dx$$

$$= \left[\frac{1}{6}x^3\right]_0^1 + \frac{1}{2}\left[2x^2 + \frac{1}{4}x^4 - \frac{4}{3}x^3\right]_1^2$$

$$= \; = \frac{1}{6} + \frac{1}{2}[(8 + \frac{1}{4}\times 16 - \frac{4}{3}\times 8) - (2 + \frac{1}{4} - \frac{4}{3})]$$

=9/24 (verify).

Problem 2. Evaluate $\iint_D x^2y^2dxdy$ where D is the circular disc $x^2+y^2 \le 1$.

Solution. In D, x varies from -1 to 1. For a fixed x. y varies from $-\sqrt{(1-x^2)}$ to $\sqrt{(1-x^2)}$

$$\therefore \iint_D x^2y^2dx\,dy = \int_{-1}^{1}\int_{-\sqrt{(1-x^2)}}^{\sqrt{(1-x^2)}} x^2y^2dy\,dx$$

$$= 4\int_0^1\int_0^{\sqrt{(1-x^2)}} x^2y^2dy\,dx$$

$$= 4\int_0^1\left[\frac{1}{3}x^2y^3\right]_0^{\sqrt{(1-x^2)}} dx$$

$$= 4\int_0^1 x^2(1-x^2)^{3/2}dx$$

$$= \frac{4}{3}\int_0^{\pi/2}\sin^2\theta\cos^4\theta\,d\theta \quad \text{(Putting x=Sin}\theta\text{)}$$

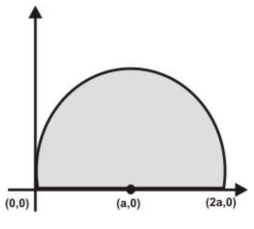
$$= \frac{4}{3}\left(\frac{1.3.1}{2.4.6}\right)\left(\frac{\pi}{2}\right) = \frac{\pi}{24}.$$

Problem 3. Change the order of integration in

$$I = \int_0^{\pi/2}\int_0^{2a\cos\theta} f(r,\theta)r\,dr\,d\theta$$

Solution We have r =2a cosθ represents a circle with centre (a, 0) and radius a.

Since $0 \le \theta \le \pi/2$ the region of integration is the semicircular disc lying in the first quadrant.



In this region r varies from 0 to 2a.

Futher r = 2a cosθ implies θ = $\cos^{-1}$ (r/2a).

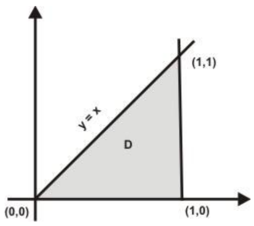
Hence for each fixed r,θ varies from 0 to $\cos^{-1}$(r/2a).

Hence $I = \int_0^{2a}\int_0^{\cos^{-1}(r/2a)} f(r,\theta)\,r\,d\theta\,dr$.

Problem 4. Evaluate $I = \iint_D e^{y/x} dxdy$ where D is the region bounded by the straight lines y=x; y=0 and x=1.

Solution. The rigion D is a triangle as shown in the figure.

In this region x varies from 0 to 1. For each fixed x,y varies from 0 to x.



$$\therefore I = \int_0^1 \int_0^x e^{y/x} dy\, dx$$

$$= \int_0^1 \left[x e^{y/x} \right]_0^x dx$$

$$= \int_0^1 x(e-1)dx = \frac{1}{2}(e-1).$$

Problem 5: Evaluate $\iint_D x^2 y^2\, dx\, dy$ where D is the circular disc $x^2+y^2 \le 1$.

Solution. In D, x varies from -1 to 1.

For a fixed x,y varies from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$

$$\therefore \iint_D x^2 y^2 dx\, dy = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 y^2 dy\, dx$$

$$= 4\int_0^1 \int_0^{\sqrt{1-x^2}} x^2 y^2 dy\, dx$$

$$= 4\int_0^1 \left[\frac{1}{3} x^2 y^3 \right]_0^{\sqrt{1-x^2}} dx$$

$$= 4\int_0^1 x^2 (1-x^2)^{3/2} dx$$

$$= \frac{4}{3} \int_0^{\pi/2} \sin^2\theta \cos^4\theta d\theta \text{ (Putting x= sin}\theta)$$

$$= 4\left(\frac{1.3.1}{2.4.6}\right)\left(\frac{\pi}{2}\right)$$

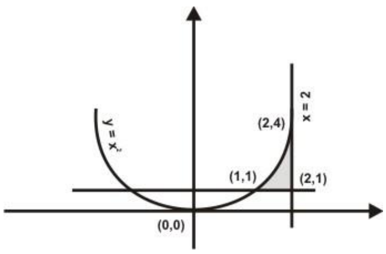
$= \pi/24.$

Problem 6. Evaluate $\iint_D \left(x^2 + y2\right)dx\,dy$ where D is the region bounded by y=x^2, x = 2 and y=1.

Solution. The region of integration is as shown in the figure.

In this region x varies from 1 to 2 and for each fixed x,y varies from 1 to x^2.

$$\therefore \iint_D \left(x^2 + y2\right)dx\,dy = \int_1^2 \int_1^{x^2} \left(x^2 + y^2\right)dy\,dx.$$



$$= \int_1^2 \left[x^2 y + \frac{1}{3}y^3\right]_1^{x^2} dx$$

$$= \int_1^2 \left(x^4 + \frac{1}{3}x^6\right)dx$$

$$= \left[\frac{1}{5}x^5 + \frac{1}{21}x^7\right]_1^2 = \frac{1286}{105}$$

Problem 7. Evaluate $I = \int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx.$

Solution. We notice that we must integrate first w.r.t.x. Hence we change the order of integration. The region of integration is as shown in the figure. (We note the D is an unbounded region and the given integral is an improper double integral).

In the region D, y varies from 0 to ∞. For each fixed y,x varies from 0 to y.

$$\therefore I = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\,dx\,dy = \int_0^\infty \left[\frac{e^{-y}}{y}x\right]_0^y dy$$

$$= \int_0^\infty e^{-y} dy = \left[-e^{-y}\right]_0^\infty$$

= 1.

Exercises.

1. Evaluate the following integrals.

(a) $\int_0^1\int_0^2 (x+2)\,dy\,dx$ $\qquad$ $b) I = \int_0^1\int_0^2 (x^2+y^2)\,dx\,dy$

1.2 TRIPLE INTEGRALS:

The definition triple integrals for a funtion f(x,y,z) defind over a region D is R^3 is analogus to the definition of double integral is defind is the definition . we replace rectangles by parallelopipeds and area by volume to obtain the corresponding definition of triple integrals.

A triple integral of a function defind over a region D is denoted by

$\iiint_D f(x,y,z)\,dx\,dy\,dz$ or $\iiint_D f(x,y,z)\,dV$ or $\iiint_D f(x,y,z)\,d(x,y,z)$.

The triple integral can be expressed as an integrated integrals in several ways.

For example is a region D in R^3 is given by

D = {(x,y,z)/ a ≤ x ≤ b; $\phi_1(x) \le y \le \phi_2(x)$ $\psi_1(x,y) \le z \le \psi_2(x,y)$ then

$$\iiint_D f(x,y,z)dx\;dy\;dz = \int_a^b \int_{\Phi_1(x)}^{\Phi_2(x)} \int_{\Psi_1(x,y}^{\Psi_2(x,y)} f(x,y,z)dz\;dy\;dx$$

This can also be written as

$$\int_a^b \int_{\Phi_1(x)}^{\Phi_2(x)} dy \int_{\Psi_1(x,y}^{\Psi_2(x,y)} f(x,y,z)dz$$

Similarly under suitable conditions a given triple integral can be expressed as an iterated integral in five other ways by permuting the variables.

Solved Problems.

Problem 1. Evaluate $I = \int_0^a \int_0^x \int_0^y xyz \, dz \, dy \, dx$

Solution:

$$I = \int_0^a \int_0^x \left[\frac{1}{2} xyz^2 \right]_0^y dy \, dx$$

$$= \frac{1}{2} \int_0^a \int_0^x xy^3 \, dy \, dx = \frac{1}{2} \int_0^a \left[\frac{1}{4} xy^4 \right]_0^x dx$$

$$= \frac{1}{8} \int_0^a x^5 dx = \frac{1}{8} \left[\frac{1}{6} x^6 \right]_0^a$$

$= a^6/48$

Problem 2. Evaluate $I = \iiint_D xyz \, dx \, dy \, dz$ where D is the region bounded by the positive octant of the sphere $x^2+y^2+z^2 = a^2$.

Solution:

The projection of the given region in the x-y plane (z = 0) is the region bounded by
the circle $x^2+y^2 = a^2$ and lying in the first quadrant as shown in the figure.

In the given region x varies from 0 to a. For a fixed x,y varies from 0 to $\sqrt{(a^2 - x^2)}$.

For a fixed (x,y), z varies from 0 to $\sqrt{(a^2 - x^2 - y^2)}$.

$$\therefore I = \int_0^a \int_0^{\sqrt{(a^2-x^2)}} \int_0^{\sqrt{(a^2-x^2-y^2)}} xyz\, dz\, dy\, dx$$

$$= \frac{1}{2}\int_0^a \int_0^{\sqrt{(a^2-x^2)}} xy(a^2 - x^2 - y^2)\, dy\, dx$$

$$= \frac{1}{8}\int_0^a x(a^2 - x^2)^2\, dx \text{ (Verify)}$$

$$= -\frac{1}{16}\left[\frac{1}{3}(a^2 - x^2)^3\right]_0^a$$

$= a^6/48.$

Problem 3. Evaluate $I = \int_0^{\log a} \int_0^{x} \int_0^{x+y} e^{x+y+z} dz\, dy\, dx$.

Solution. $I = \int_0^{\log a} \int_0^{x} \left[e^{x+y+z}\right]_0^{x+y} dy\, dx.$

$$= \int_0^{\log a} \int_0^{x} \left[e^{2(x+y)} - e^{x+y}\right] dy\, dx$$

$$= \int_0^{\log a} \left[\frac{1}{2}e^{2(x+y)} - e^{x+y}\right]_0^x dx$$

$$= \int_0^{\log a} \left[\frac{1}{2}e^{4x} - \frac{3}{2}e^{2x} + e^x\right] dx$$

$$= \left[\frac{1}{8}e^{4x} - \frac{3}{4}e^{2x} + e^x\right]_0^{\log a} dx$$

$$= \frac{1}{8}a^4 - \frac{3}{4}a^2 + a - \frac{3}{8}.$$

Problem 4. Evaluate $I = \iiint_D \frac{dx\, dy\, dz}{(x+y+z+1)^3}$ Where D is the region bounded by the planes x = 0, y = 0, z = 0 and x+y+z = 1.

Solution. The given region is a tetrahedron. The projection of the given region in the x-y plane is the triangle bounded by the lines x = 0, y = 0 and x+y=1 as the shown in the figure.

In the given region x varies from 0 to 1. For each fixed x,y varies from 0 to 1-x. For each fixed (x,y), z varies from 0 to 1-x-y.

$$\therefore I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dz\, dy\, dx}{(x+y+z+1)^3}$$

$$= -\frac{1}{2}\int_0^1 \int_0^{1-x} \left[(x+y+z+1)^{-2}\right]_0^{1-x-y} dy\, dx$$

$$= -\frac{1}{2}\int_0^1 \int_0^{1-x} \left[\frac{1}{4} - (x+y+z+1)^{-2}\right] dy\, dx$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{1}{4}y + (x+y+1)^{-1}\right]_0^{1-x} dx$$

$$= -\frac{1}{2}\int_0^1 \left\{\frac{1}{4}(1-x) + \frac{1}{2} - (x+1)^{-1}\right\} dx$$

$$= -\frac{1}{2}\left[\frac{1}{4}x - \frac{1}{8}x^2 + \frac{1}{2}x - \log(x+1)\right]_0^1$$

$$= \frac{1}{2}\log 2 - \frac{5}{16}.$$

Excersies.

1. Evaluate the following triple integrals.

a) $\int_0^1 \int_0^1 \int_{\sqrt{x^2+y^2}}^2 xyz\, dz\, dy\, dx$

b) $\int_0^a \int_0^b \int_0^c (x+y+z)\, dx\, dy\, dz$

c) $\int_0^1 dx \int_0^2 dy \int_1^2 x^2 yz\, dz$

1.3 JACOBIANS:

In this section we introduce the concept of Jacobian of a transformation which plays an important role in change of variables in double and triple integrals.

Definition. Consider the transformation given by the equations

x = x(u, v, w); y = y(u, v, w); z =z (u, v, w) where the functions x,y,z have continuous first order partial derivatives.

The Jacobian *J* of the transformation is defind by

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

The jacobian is also denoted by $J = \frac{\partial(x, y, z)}{\partial(u, v, w)}$

For a transformation in two variables x = x (u,v) and y = y(u,v) the Jacobian is given by a determinant of order two. Hence $J = \frac{\partial(x, y)}{\partial(u, v)}$.

Examples.

1. The transformation from cartesian coordinates (x,y) to polar coordinates (r,θ)

Soln:

Given that x = rcosθ and y = r sinθ.

$\frac{\partial x}{\partial r} = \cos\theta : \frac{\partial y}{\partial r} = \sin\theta$

$\frac{\partial x}{\partial \theta}$ = -r sinθ : $\frac{\partial y}{\partial \theta}$ = r cosθ

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$= r\cos^2\theta + r\sin^2\theta = r(\cos^2\theta + \sin^2\theta) = r.$

2. The transformation from cartesian coordinates (x,y,z) to spherical polar coordinates (r,θ,φ)

Soln: Given that

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

Here $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$.

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{vmatrix}$$

$$= \begin{vmatrix} sin\theta cos\varphi & rcos\theta cos\varphi & -rsin\theta sin\varphi \\ sin\theta sin\varphi & rcos\theta sin\varphi & rsin\theta cos\varphi \\ cos\theta & -rsin\theta & 0 \end{vmatrix}$$

$= \sin\theta\cos\varphi[0+r^2\sin^2\theta\cos\varphi]-r\cos\theta\cos\varphi[0-r\sin\theta\cos\theta\cos\varphi]$

$-r\sin\theta\sin\varphi[-r\sin^2\theta\sin\varphi-r\cos^2\theta\sin\varphi]$

$= r^2\sin\theta$. (on simplification).

3. The transformation from cartesian coordinates (x,y,z) to Cylindrical coordinates (r,θ,z)

Soln:

given that x = rcosθ, y = rsinθ, z = z.

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

= cosθ(rcosθ)+rsinθ(sinθ) = r

4. Consider the transformation x +y = u, 2x-3y = v.

Soln:

Given x +y = u, 2x-3y = v.

$$\therefore x = \frac{1}{5}(3u+v) \text{ and } y = \frac{1}{5}(2u-v)$$

$$\therefore J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{vmatrix} = -\frac{1}{5}.$$

Solved Problems.

Problem 1. If x+y+z=u; y+z=uv; z=uvw then find *J*.

Solution. From the given three transformations we get x = u-uv; y = uv-uww; z = uvw.

$$\text{Now } J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$=\begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$= (1-v)[u^2v(1-w)+ u^2vw]+u[v^2u(1-w)+uv^2w]$

$=(1-v)[u^2v- u^2v w+ u^2vw]+u[v^2u- v^2u w+uv^2w]$

$= u^2v- u^2v w+ u^2vw-u^2v^2+u^2v^2w-u^2v^2w+v^2u^2-v^2u^2w+u^2v^2w$

$= u^2v$

Problem 2. If $u=x^2-y^2$ and $v = 2xy$ prove that $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4\sqrt{u^2+v^2}}$

Solution. Consider $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix}$

$=4x^2+4y^2 = 4(x^2+y^2)$............. (1)

We have $(x^2 + y^2)^2 = (x^2-y^2)^2 + (2xy)^2 = u^2+v^2$.

$\therefore x^2 + y^2 = \sqrt{u^2+v^2}$ ………..(2)

From (1) and (2) we get $\frac{\partial(u,v)}{\partial(x,y)} = 4\sqrt{u^2+v^2}$

$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{4\sqrt{u^2+v^2}}$.

Excercises.

1. Prove that $\frac{\partial(u,v)}{\partial(x,y)} . \frac{\partial(x,y)}{\partial(u,v)} = 1$.

1.4 CHANGE OF VARIABLES IN DOUBLE AND TRIPLE INTEGRALS.

The evaluation of a double or a triple integral sometimes becomes easier when we transform the given variables into new variables.

We state without proof the following theorem regarding change of variables in double and triple intergrals.

Theorem 4.1.

Consider a transformation given by the equation x = x (u,v) and y = y(u,v) where x and y have continuous first order partial derivatives. Let the region D in the x-y plane be mapped into the region D* in the u-v plane. Further we assume that the Jacobian of the transformation J≠0 for all points in D. Then $\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f\left[x(u,v), y(u,v)\right]\left|J\right| du\,dv.$

Similarly for triple integrals we have

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{D^*} f\left[x(u,u,w), y(u,v,w), z(u,v,w)\right]\left|J\right| du\,dv\,dw$$

We now proceed to evaluate some double and triple integrals by making appropriate change of variables.

Solved Problems.

Problem 1. Evaluate $I = \iint_D \frac{xy\,dx\,dy}{\sqrt{x^2+y^2}}$ by transforming to plor coordinates Where D is the region enclosed by the circles $x^2+y^2 = a^2$ and $x^2+y^2 = 4a^2$ in the first quardrant.

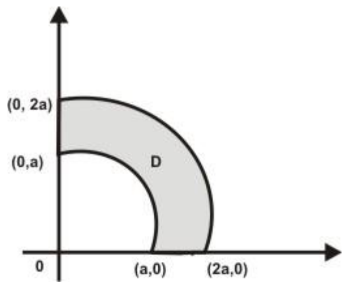
Solution:

Put x = rcosθ and y = rsinθ

We know that J= r.

Further in the given domain D,

0 ≤ θ ≤ π/2 and a ≤ r ≤ 2a.



$$\therefore I = \int_0^{\pi/2}\int_a^{2a}\left(\frac{r\cos\theta\, r\sin\theta}{r}\right) r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \cos\theta\sin\theta\left[\frac{1}{3}r^3\right]_0^{2a} d\theta$$

$$= \frac{7a^3}{3}\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta$$

$$= \frac{7a^3}{3}\int_0^{\pi/2} \sin\theta\, d(\sin\theta)$$

$$= \frac{7}{6}a^3\left[\sin^2\theta\right]_0^{\pi/2}$$

$$= \frac{7a^3}{6}.$$

Problem 2. Evaluate the improper integral $I = \int_0^\infty e^{-x^2}dx$.

Solution. $I^2 = II = \left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-y^2}dy\right).$

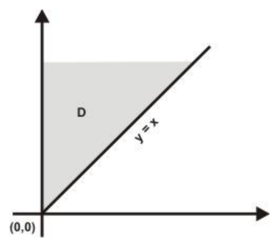
$$= \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dx\,dy.$$

Put x = rcosθ and y = rsinθ. Hence J = r.

The region of integration is the first quadrant.

Hence r varies from 0 to ∞ and θ varies from 0 to π/2.

$$\therefore I^2 = \int_0^\infty\int_0^{\pi/2} e^{-r^2} r\, d\theta\, dr = \frac{\pi}{2}\int_0^\infty e^{-r^2} r\, dr.$$



$$= \frac{\pi}{2}\int_0^\infty -\frac{1}{2}e^{-r^2}d(-r^2) = \frac{\pi}{2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty$$

= π/2 (1/2) = π/4

$$\therefore I = \frac{\sqrt{\pi}}{2}.$$

Problem 3. Prove that $I = \iint_D \left(\frac{1-x^2-y^2}{1+x^2+y^2}\right)^{1/2} dx\,dy = \frac{\pi}{4}\left(\frac{\pi}{2}-1\right).$

Where D is the positive quadrant of the circle $x^2+y^2=1$.

Solution. Put x = rcosθ and y = r sinθ.

J = r.

Futher in D, $0 \le r \le 1$ and $0 \le \theta \le \pi/2$.

$$\therefore I = \int_0^1 \int_0^{\pi/2} \left(\frac{1-r^2}{1+r^2} \right)^{1/2} r \, d\theta \, dr$$

$$= \frac{\pi}{2} \int_0^1 \left(\frac{1-r^2}{1+r^2} \right)^{1/2} r \, dr$$

$$= \frac{\pi}{2} \int_0^1 \left(\frac{1-r^2}{\sqrt{1-r^4}} \right) r \, dr$$

$$= \frac{\pi}{4} \int_0^1 \left(\frac{1-t}{\sqrt{1-t^2}} \right) dt \qquad \text{(by putting } r^2 = t\text{)}$$

$$= \frac{\pi}{4} \left[\sin^{-1} t + (1-t^2)^{1/2} \right]_0^1$$

$$= \frac{\pi}{4} \left(\frac{\pi}{2} - 1 \right).$$

Problem 4. Prove that

$$\iint_D \left(\frac{a^2b^2 - b^2x^2 - a^2y^2}{a^2b^2 + b^2x^2 + a^2y^2} \right)^{1/2} dx \, dy = \frac{\pi}{4} \left(\frac{\pi}{2} - 1 \right) ab$$

Where D is the Positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution. Put x = au and y = bv.

$$\therefore J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab.$$

Let D* be the image of D under the above transformation. Then D* is the region bounded by the unit circle $u^2+v^2 = 1$ in the first quadrant.

$$\text{Now, } I = \iint_D \left[\frac{1-(x^2/a^2)-(y^2/b^2)}{1+(x^2/a^2)+(y^2/b^2)} \right]^{1/2} dx \, dy$$

$$= \iint_D \left[\frac{1-u^2-v^2}{1+u^2+v^2}\right]^{1/2} |J| \, du\, dv$$

$$= ab \iint_D \left[\frac{1-u^2-v^2}{1+u^2+v^2}\right]^{1/2} du\, dv$$

$$= ab\frac{\pi}{4}(\tfrac{\pi}{2}-1). \qquad \text{(by problem 3).}$$

Problem 5. Evaluate $\iint_D \sqrt{x+y}\, dx\, dy$ where D is the parallelogram bounded by the lines x+y = 0; x+y = 1; 2x-3y = 0 and 2x-3y = 4.

Solution.

Put x+y = u and 2x-3y = v.

Then J = -1/5 (using by above example 1.3 of 4)

Also D is transformed into the rectangle bounded by the lines u = 0; u = 1; v = 0 and V = 4.

$$\therefore \int_0^1\int_0^4 \sqrt{u}\left(-\frac{1}{5}\right) dv\, du = -\frac{1}{5}\int_0^1 \sqrt{u}\,[v]_0^4\, du$$

$$= -\frac{4}{5}\left[\frac{2}{3}u^{3/2}\right]_0^1$$

= -8/15

Problem 6. Evaluate $I = \iiint_D xyz\, dx\, dy\, dz$ where D is the positive octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Solution.

Put x = au, y = bv and z = cw.

$$\therefore J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc.$$

Let D* be the image of D under the above Transformation. The D* is the positive octant of the sphere $u^2+v^2+w^2 = 1$.

$$\therefore I = \iiint_{D^*} abc\, uvw du\, du\, dw$$

$$= a^2b^2c^2 \iiint_{D^*} uvw\; du\; dv\; dw.$$

Now, put u = r sinθ cosφ

v = r sinθ sinφ

w = r cosθ

Then $J = r^2 \sin\theta$. (above example 1.3 of 2)

$$\therefore I = a^2b^2c^2 \int_0^1 \int_0^{\pi/2} \int_0^{\pi/2} r^5 \sin^3\theta \cos\theta \cos\varphi \sin\varphi \, d\varphi\, d\theta\, dr$$

$$= a^2b^2c^2 \int_0^1 r^5 dr \int_0^{\pi/2} \sin^3\theta \cos\theta \, d\theta \int_0^{\pi/2} \sin\varphi \cos\varphi \, d\varphi.$$

$$= a^2b^2c^2 \left[\frac{1}{6} r^6\right]_0^1 \left[\frac{1}{4}\sin^4\theta\right]_0^{\pi/2} \left[\frac{1}{2}\sin^2\varphi\right]_0^{\pi/2}$$

$$= \frac{a^2b^2c^2}{48}.$$

Excercise 1. Evaluate the following double integrals using change of variables or otherwise over the region indicated.

a). $\iint_D \sqrt{(x^2+y^2)}\, dx\, dy$;D is the a region bounded by the circle $x^2 + y^2 = a^2$.

b). $\iint_D e^{-(x^2+y^2)}\, dx\, dy$ D is the region bounded by the circle $x^2 + y^2 = a^2$.

c). $\iint_D \sqrt{(x^2+y^2)}\, dx\, dy$ D is the region in the x-y plane bounded by $x^2 + y^2 = 4$ and $x^2+y^2 = 9$.

UNIT – 3 : FIRST ORDER DIFFERENTIAL

First order differential: equations of higher degree- solvable for p, x and y- Clairaut's form/ linear differential equations of second order- Particular integrals for functions of the form, X^n, e^{ax}, $e_{ax}(f(x))$. Second order differential equations with variable coefficients.

1.1 Equations of the first order, but of higher degree.

TYPE A:- Equations solvable for p $(= \frac{dy}{dx})$.

We shall denote $\frac{dy}{dx}$ hereafter by p.

Let the equation of the first order and of the n^{th} degree in p be

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots + P_{n=0} \quad \ldots(1)$$

where $P_1, P_2 \ldots\ldots P_n$ denote functions of x and y.

Suppose the first number of (1) can be resolved into factors of the first degree of the form

$$(p - R_1)\,(p - R_2)\,(p - R_3)\,\ldots.\,(p - R_n)$$

Any relation between x and y which makes any of these factors vanish is a solution of (1).

Let the primitives of $p - R_1 = 0$, $p - R_2 = 0$, etc be

$$\varphi_1(x, y, c_1) = 0,\ \varphi_2(x, y, c_2) = 0 \ldots \varphi_n(x, y, c_n) = 0.$$

respectively, where $c_1, c_2, \ldots\ldots c_n$ are arbitrary constants. Without any loss of generality, we can replace $c_1, c_2, \ldots.c_n$ by c, where c is an arbitrary constant. Hence the solution of (1) is

$$\varphi_1(x, y, c).\ \varphi_2(x, y, c) \ldots \varphi_n(x, y, c) = 0.$$

Examples.

1) Solve $x^2p^2 + 3xyp + 2y^2 = 0$.

Soln:

Solving for p, p = $-\frac{y}{x} or = -\frac{2y}{x}$.(Quadratic eqn)

$\frac{dy}{dx} = -\frac{y}{x} \quad ; \frac{dy}{dx} + \frac{y}{x} = 0$

$\frac{dx}{x} + \frac{dy}{y} = 0$

Integrating,

log x+log y =log c

Therefore xy = c … (1)

$\frac{dy}{dx} = -\frac{2y}{x} \quad ; \frac{dy}{dx} + \frac{2y}{x} = 0$

$$\frac{dy}{y} + \frac{2dx}{x} = 0$$

Integrating,

log y+2logx = log c

ie) log y + log x^2 = log c

$yx^2 = c$ …(2)

The solution is $(xy - c)(yx^2 - c) = 0$.

2) Solve $p^2 + \left(x + y - \frac{2y}{x}\right)p + xy + \frac{y^2}{x^2} - y - \frac{y^2}{x} = 0.$

Soln:

Solving for p, $p = \frac{y}{x} - y (or) p = \frac{y}{x} - x$ (Quadratic eqn)

$$\frac{dy}{dx} = \left(\frac{y}{x} - y\right)(or)\frac{dy}{dx} = \frac{y}{x} - x$$

$$\frac{dy}{y} = \left(\frac{1}{x} - 1\right)dx\, or\, \frac{dy}{dx} - \frac{y}{x} = -x$$

Integrating on,

log y = log x –x+logc

ie)logy-log x = -x+log c

log $\frac{y}{x}$ = -x + log c

$\frac{y}{x} = ce^{-x}$

i.e., $y = cxe^{-x}$

The second equation is linear in y. Hence the solution is $ye^{-\int \frac{dx}{x}} = -\int xe^{-\int \frac{dx}{x}} dx + c$

i.e., $\frac{y}{x} = -x + c$

i.e., $y = - x^2 + cx$

The general solution is $(y-cxe^{-x})(y+x^2-cx) = 0$.

TYPE B :-Let the differential equation (1) of 2 be put in the form f (x, y, p) = 0. When it cannot be resolved into rational linear factors as in 5.1, it may be either solved for y or x.

Equations solvable for y.

f(x,y,p) = 0 can be put in the form

$$y = F(x,p) \qquad \text{....(1)}$$

Differentiating with respect to x, $p = \varphi\left(x,p,\frac{dp}{dx}\right)$

This, being an equation in the two variables p and x, can be integrated by any of the foregoing methods. Hence we obtain

$$\Psi(x,p,c) = 0 \qquad \text{....(2)}$$

Eliminating p between (1) and (2), the solution is got.

Equations solvable for x:

f (x, y,ρ) = 0 can be put in the form

$$x= F(y, \rho) \qquad \text{....(1)}$$

Differentiating with respect to $y, \frac{1}{\rho} = \varphi\left(y, p, \frac{dp}{dy}\right)$.

Integrating leads to $\Psi(y, p, c) = 0$ ....(2)

Eliminating ρ between (1) and (2), the solution of (1) is got.

Examples

1) Solve $xp^2-2yp+x=0$:

Soln:

Solving for y, $y = x\frac{(p^2+1)}{2p}$

Differentiating with respect to x, $[\frac{dy}{dx} = p]$

$$p = \frac{p^2+1}{2p} + x\frac{p^2-1}{2p^2}\frac{dp}{dx}$$

$$\frac{p^2-1}{2p} = x\frac{dp}{dx} \times \frac{(p^2-1)}{2p^2}$$

$$\therefore \frac{dx}{x} = \frac{dp}{p}$$

Integrating, p = cx.

Eliminating p between this and the given equation, the solution is

$$2\,cy = c^2x^2+1.$$

2) Solve $x = y^2 + \log p$ … (1)

Soln:

(This is easily solvable for x only)

Diffenentiating with respect to y,

$$\frac{1}{p} = 2y + \frac{1dp}{pdy}$$

$\frac{dp}{dy} + 2py = 1.$ This is linear in p and hence.

$$p\,e^{y^2} = \int e^{y^3}dy + c. \quad \text{….(2)}$$

(It must be noted that the integral on the R.H.S. cannot be integrated in finite terms.)

The eliminant of p between (1) and (2) gives the solution.

Note. In the above problem, the solution has not been got explicity by eliminating p. But we have x and y expressed in terms parameter p. This will do.

1.2 Clairaut's form.

The equation known as Clairaut's is of the form

y= px + f (p) ….. (1)

Differentiating with respect to x, p = p + $\{x + f(p)\}\frac{dp}{dx}$

$\frac{dp}{dx} = 0$ or x + f (p) = 0.

$\frac{dp}{dx} = 0$, integrating on, p = c, a constant.

$\therefore$ The solution of (1) is y = cx + f (c).

We have to replace p in Clairaut's equation by c. The other factor y + f (p) = 0 taken along with (1) give, on eliminating of p, a solution of (1). But this solution is not included in the general solution (2). Such a solution as this is called a singular solution.

Examples.

1) Solve y = (x-a) p-p^2

Soln:

This is Clairaut's equation; hence the solution is

y= (x-a) c- c^2

2) Solve y = 2px + y^2p^2

Soln:

Putting X =2x and Y =$y^{2,}$

dX=2dx: dY = 2ydy

$$\therefore P = \frac{dY}{dX} = yp$$

The equation transforms into Y = XP + P^2

This is Clairaut's equation; hence Y=cX+$c^{2.}$

The solution is y^2 = 2xc + c.

We have an extended form of Clairaut's equation of the type

y = x f (p) + φ (p).

Differentiating with respect to x … (1)

$$p = f(p) + [xf'(p) + \varphi'(p)]\frac{dp}{dx}$$

$$\frac{dx}{dp} + \frac{xf'(p)}{f(p) - p} = \frac{\varphi'(p)}{p - f(p)}$$

This is linear in X and hence gives F (x, p , c) = 0

The eliminant of P between this equation and (1) give the solution of (1).

Example.

1) Solve y = xp+x(1+$p^{2)1/2}$

Soln:

Given y = xp+x(1+$p^{2)1/2}$

Differentiating with respect to x.

$$p = p + (1 + p^{2)})^{1/2} + \frac{dp}{dx}\left[x + \frac{xp}{\sqrt{1+p^2}}\right]$$

Hence dp $\dfrac{\sqrt{1+p^2}+p}{(1+p^2)}+\dfrac{dx}{x}=0.$

Integrating, $\displaystyle\int\frac{dp}{\sqrt{1+p^2}}+\int\frac{p\,dp}{1+p^2}+\int\frac{dx}{x}=\log c$

$i.e., \log(p+\sqrt{1+p^2})+\dfrac{1}{2}\log(1+p^2)+\log x=\log c.$

$\log(p+\sqrt{1+p^2}+1+p^2)x=\log c$

$\left\{p\sqrt{1+p^2}+1+p^2\right\}x=c$ …(1)

Eliminating p between (1) and (2) the solution is got.

1.3 LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

A typical linear equation of the second order with constant coefficients is

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=X \qquad \text{…… (1)}$$

where a,b ,c are constants and X is a function of x.

Let us consider (1) without the second number,

$$i.e., a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0 \qquad \text{…..(2)}$$

The solution of this equation (2) is called the complementary function of (1).

To solve (2), assume as a trial solution $y = e^{mx}$ for some value of m. Now $\frac{dy}{dx} = m\,e^{mx}$ and $\frac{d^2y}{dx^2} = m^2 e^{mx}$. Substituting these values in (2), we get

$e^{mx}(am^2+bm+c = 0$ ….. (3)

Hence m satisfies $am^2+bm+c=0$. This equation in m is called the auxiliary equation.

Three cases can arise in the solution of the auxiliary equation.

Case(i). Let the auxiliary equation (3) has two real and distinct roots m_1 and m_2.

$\therefore$ $y=e^{m1x}$ and $y = e^{m_2 x}$ are solutions of (2).

Hence $A\,e^{m_1 x}$, $B\,e^{m_2 x}$ are solutions of (2), where A and B are arbitrary constants. Thus $y= A\,e^{m_1 x} + B\,e^{m_2 x}$ is the most general solution of (2) as the number of constants occuring in this solution is two, equal to the order of the differential equation.

Case (ii). Let the auxiliary equation (3) has two roots equal and real.

Let $m_2=m_1$. The solution $y= A\,e^{m_1 x} + B\,e^{m_2 x}$ becomes.

$(A+B)\,e^{m_1 x} = c\,e^{m_1 x}$ ……4

where c is a single arbitrary constant equal to A+B. Thus the number of constants is reduced to one which is one short of the order of the differential equation (2) and therefore (4) ceases to represent the general solution. Hence we proceed as follows :

Let us put $m_2 = m_1 + \epsilon$ and allow ϵ to tend to zero.

The solution is

$$y = A\,e^{m_1 x} + B\,e^{(m_1+\epsilon)\,x}$$
$$= e^{m_1 x} + (A + B\,e^{\epsilon\,x})$$

$$= \quad e^{m_1 x}\left[A + B\left(1 + \in x + \frac{\in^2 x^2}{2} + ...\right)\right]$$

by the exponential theorem

$= e^{m_1 x}$ (A+B+ϵ B x) the other terms tending to zero as ϵ ⟶ 0.

We can choose B sufficiently big so as to make ϵ B finite asϵ ⟶ 0.

and A large with opposite sign to B so that A + B is finite.

If A + B + C and ϵB = D, the solution corresponding to two equal roots m_1 is $e^{m_1 x}$ (C + D x).

Case (iii). Let the auxiliary equation has imaginary roots.

As imaginary roots ocuur in pairs, let $m_1 = \alpha + i\beta$ where α and β are real; then $m_2 = \alpha - i\beta$.

The solution is y $= A\,e^{(\alpha+i\beta)x} + B\,e^{\alpha-i\beta)x}$

$= e^{\alpha x}\,[A\,e^{i\beta x +} B\text{-}^{i\beta x}]$

$= e^{\alpha x}$ {A cos β x + i sin β x + B (cosβ x – isin βx)} by Euer's formula.

$= e^{\alpha x}$ (c cos β x + Dsin βx), where C and D are arbitrary constants.

This can also be written as y = A $e^{\alpha x}$ cos (βx+B), where A and B arbitrary constants.

Examples.

1) Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 4y = 0$

Soln:

The auxiliary equation is $m^2 - 5m + 4 = 0$

(m -1) (m-4)=0

m=1 and m=4.

$\therefore$ complimentary function y = A e^x + B e^{4x}

2)Solve $\frac{d^2y}{dx^2} - 9y = 0$

Soln:

The auxiliary equation is m^2 -9= 0

m^2-3^2 = 0 ; (m-3)(m+3)=0

m=3 and m=-3

$\therefore$ C.F y = A e^{3x} + B e^{3x}

3)Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$

Soln:

The auxiliary equation is m^2 + 2m + 1 = 0, i.e., $(m+1)^2$ = 0.

m = - 1 twice.

$\therefore$ C.F y = e^{-x} (A + B x).

4)Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4 = 0$

Soln:

The auxiliary equation is m^2 + 4m + 2^2 = 0, i.e., $(m+2)^2$ = 0.

m = - 2 twice.

$\therefore$ y = e^{-2x} (A + B x).

5) Solve $\frac{d^2}{dx^2} - 3\frac{dy}{dx} + 5y = 0$

Soln:

The auxiliary equation is $m^2 - 3m + 5 = 0$.

Solving,(quadratic eqn form) $m = \frac{3 \pm \sqrt{11}i}{2}$

$$\therefore y = e^{\frac{3x}{2}}\left\{ASin\left(\frac{\sqrt{11}}{2}x\right) + B\sin\left(\frac{\sqrt{11}}{2}x\right)\right\}$$

6)Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

Soln:

The auxiliary equation is $m^2 + 4m\ 13 = 0$.

Solving,(quadratic eqn form) $m = \frac{-4 \pm 6i}{2}$

$m = -2 \pm 3i$

$$\therefore y = e^{-2x}\{ASin(3x) + B\sin(3x)\}$$

Exercises:

Solve the following equations :-

1. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$
2. $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 4y = 0$
3. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$
4. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 8y = 0$

The operators D and D^{-1}

Let D stand for the operator $\frac{d}{dx}$ and D^2 for $\frac{d^2}{dx^2}$

This symbol D satisfies the commutative, associative, and distributive laws; for $(D^m+D^n)\ u = (D^n+D^m)u = D^n u + D^m u$

$D^m.D^n u = D^n.D^m u = D^{m+n}u$

and D (u+v) = D (v+u).

We can define the inverse operator D^{-1} as one such that when it operates on any function of x and subsequently the operation by D is performed, the function is left unaltered. Thus D^{-1} represents integration.

We shall define the operator $\frac{1}{f(D)}$ as the inverse of the operator f (D). i.e., $\frac{1}{f(D)}X$ is that function of x which, when operated upon by f(D) yields X.

We shall assume that the order of the operators f(D) and $\frac{1}{f(D)}$ can be interchanged.

The f (D) $\left\{\frac{1}{f(D)}x\right\} = \frac{1}{f(D)} f(D)x = x$

1.4 Particular integral.

Consider equation (1) which can be written symbolically as

$(aD^2 + bD + c)\, y = x$

or shortly f (D) y = X, where f (D) = a D^2 + b D + c .

Let y = u be a particular solution of this equation.

Let Y be the complementary function of (1)

Then y = Y + u is the general solution of (1).

u is called the particular integral of (1).

In symbolic form, it is written as $\frac{1}{f(D)}X$

$$i.e., P.I = \frac{1}{aD^2 + bD + c} X$$

Special methods of finding P.I.

(a) Let X be of the form $e^{\alpha x}$

$D\, e^{\alpha x} = \alpha\, e^{\alpha x}$

More generally, $D^n e^{\alpha x} = (\alpha^n)\, e^{\alpha x}$

$\therefore f(D)e^{\alpha x} = f(\alpha)e^{\alpha x}$ *as* $f(D)$ is a quadratic in D in our case.

Operation on both sides by $\frac{1}{f(D)}, e^{\alpha x} = f(\alpha)\frac{1}{f(D)}e^{\alpha x}$

If $f(\alpha) \neq 0$, $\frac{1}{f(D)}e^{\alpha x} = \frac{1}{f(\alpha)}e^{\alpha x}$

Case (i). Hence the rule is:

In $\frac{1}{f(D),}e^{\alpha x}$, replace D by α if $f(\alpha) \neq 0$

Case (ii) If $f(\alpha) = 0$, α satisfies the auxiliary equation $f(m) = 0$. Then we proceed as follows:

(i) Let the auxiliary equation have two distinct roots m_1 and m_2 and let $\alpha = m_1$.

Then $f(m) = a(m - m_1)(m - m_2)$

$= a(m-\alpha)(m - m_2)$

P.I. $= \frac{1}{a(D-\alpha)(D-m_2)}e^{\alpha x}$

$$= \frac{1}{a(D-\alpha)} \frac{1}{(\alpha - m_2)} e^{\alpha x} by case (i) above$$

To find $\frac{1}{(D-\alpha)} e^{\alpha x}$,let us put z =

Operating on both sides by D – α,$\frac{dz}{dx}$ – α z = $e^{\alpha x}$.

This is linear equation of the first order; hence

$z e^{-\alpha x} = \int e^{-\alpha x} dx = x.$

(It must be noted that no constant of integration is added as we are evaluating only a particular integral.

If the constant be added there will occur in the general solution 3 constants as there are already in the C.F. and thus one constant will be too many).

$\therefore$ **Z = x $e^{\alpha x}$**

Hence P.I. = $\frac{1}{a(\alpha - m_2)} xe^{\alpha x}$.

(ii) Let the auxiliary equation have two equal rootes each equal to α.

i.e., $m_2 = m_1 = \alpha$.

$\therefore f(m) = a(m-\alpha)^2$

$$P.I. = \frac{1}{a(D-\alpha)^2} e^{\alpha x} = \frac{1}{a(D-\alpha)(D-\alpha)} e^{\alpha x}$$

$$= \frac{1}{a} \frac{X e^{\alpha x}}{D-\alpha}$$

If $z = \frac{1}{D-\alpha} xe^{\alpha x}, \frac{dz}{dx} - \alpha z = xe^{\alpha x}$

Sloving, $ze^{-\alpha x} = \int x dx = \frac{x^2}{2}$ (no constant is added).

$$\therefore z = \frac{x^2}{2} e^{\alpha x}$$

$$\therefore P.I. = \frac{x^2}{2} e^{\alpha x}$$

Examples.

Ex.1.Slove $(D^2+5D+6)y = e^x$.

Soln:

To find the C.F of $(D^2+5D+6)y = 0$.

The auxillary equation is $m^2 + 5m + 6 = 0$.

$(m+2)(m+3)=0$

$m = -2$ and -3.

C.F.= $A e^{-2x} + B e^{-3x}$.

$$P.I. = \frac{1}{D^2+5D+6} e^x$$

$= \frac{1}{12} e^x$ on replacing D by 1

$$y = Ae^{-2x} + Be^{-3x} + \frac{e^x}{12}$$

2) Slove $(3D^2+D -14)y = 13 e^{2x}$.

Soln:

To find the C.F of $(3D^2+D-14)y =0$.

The Auxillary equation is $3m^2 + m - 14 = 0$.

Sloving, $m = 2$ and $-7/3$

$\therefore y = A e^{-2x} + B e^{-7x/3}$

$$P.I. = \frac{1}{(D-2)(3D+7)} 13e^{2X}$$

$$=\frac{13}{13}\frac{1}{D-2}e^{2x}$$

$$=\frac{1}{D-2}e^{2x}=xe^{2x}$$ by 4 case (ii)

$\therefore$ y = $Ae^{2x} + Be^{-7x/3} + xe^{2x}$

3) Slove (D^2- 2m D + m^2)y = e^{mx}.

Soln:

To find the C.F of (D^2 -2mD + m^2) = 0.

The auxillary equation is K^2 - 2mk + m^2 = 0.

(Note. K is used here instead of the usual m as there is already another m).

i.e., $(k - m)^2 = 0$

$\therefore$ k = m twice

C.F. = e^{mx} (A + Bx).

$$P.I.=\frac{1}{(D-m)^2}e^{mx}=\frac{x^2}{2}e^{mx}$$ by 4 case (ii)

$$\therefore y=e^{mx}\left(A+Bx+\frac{x^2}{2}\right)$$

Excersice

Solve the following equations:-

1. (D^2 - 5D + 6) y = e^{4x}.

2. (D^2 - 6D + 13) y = $5e^{2x}$.

3. (D^2 - 4D + 6) y = $5e^{-2x}$.

4. (D^2 - 2D + 1) y = $2e^{3x}$.

(b) Let X be of the form cosαx or sin αx, where α is a constant.

D sin αx = cosαx.

D^2 sin αx = $-\alpha^2$ sin αx.

$\therefore \phi(D^2)$ sin αx= $\varphi(-\alpha^2)$ sin dx, as $\phi(D^2)$ is a rational integral function of D^2.

Operating on both sides by $\frac{1}{\phi\,(D^2)}$, $\sin\alpha x = \frac{\phi(-\alpha^2)}{\phi\,(D^2)}\sin\alpha x.$

Case (i). If $\phi(-\alpha^2) \neq 0, \frac{1}{\phi\,(D^2)}\sin\alpha x = \frac{1}{\phi\,(-\alpha^2)}\sin\alpha x$

Hence the rule is :

Replace D^2 by $-\alpha^2$, **Provided** $\phi(-\alpha^2) \neq 0$.

The same rule applies if sin αx be replaced by cosα x

i.e., $\frac{1}{\phi\,(D^2)}\cos\alpha x = \frac{1}{\phi\,(-\alpha^2)}\cos\alpha x.$

Case (ii). If $\phi(-\alpha^2) = 0$, $D^2 + \alpha^2$ is a factor of $\phi(-D^2)$.

To evaluate $\frac{1}{D^2+\alpha^2}\sin\alpha x$, the above rule fails. Hence the following procedure is adopted.

$\frac{1}{D^2+\alpha^2}\sin\alpha x = \frac{1}{D^2+\alpha^2}$. Imaginary part of $e^{\alpha ix}$ as

$e^{i\alpha x} = \cos\alpha x + i\sin\alpha x$ by Euler's formula;

$$= \text{imaginary part of } \frac{1}{D^2+\alpha^2}e^{i\alpha x}$$

$$= \quad '' \quad \frac{1}{(D-\alpha i)(D+\alpha i)}e^{i\alpha x}$$

$$= \quad '' \quad \frac{1}{(D-\alpha i)\,2\alpha i}e^{i\alpha x} \text{ by 4 (a)}$$

$$= \quad '' \quad \frac{xe^{i\alpha x}}{2\alpha i} \text{ by 4(a)}$$

$$= \quad '' \quad -\frac{xi}{2\alpha}(\cos\alpha x + i\sin\alpha x)$$

$$= -\frac{x\cos\alpha x}{2\alpha}$$

Similarly, $\frac{1}{D^2+a^2}\cos ax = \frac{x\sin ax}{2a}$.

Examples.

Ex. 1. Solve $(D^2 - 3D + 2)\, y = \sin 3x$.

Soln:

To find the C.F of $(D^2 - 3D + 2)\, y = 0$.

The auxiliary equation is $m^2 - 3m + 2 = 0$.

(m-2)(m-1)=0

m = 1 and 2.

C.F. = $A\, e^x + B\, e^{2x}$.

$$P.I. = \frac{1}{D^2+3D+2}\sin 3x$$

$$= \frac{1}{-9-3D+2}\sin 3x$$ on replacing D^2 by -9 by 4(b)

$$= \frac{-1}{3D+7}\sin 3x$$

In order to apply the above rule, we must aim at getting D^2 terms only in the denominator; hence we write

$$\frac{1}{3D+7} = \frac{3D-7}{(3D-7)(3D+7)} = \frac{3D-7}{9D^2-49}$$

and proceed.

$$P.I. = \frac{-(3D-7)}{9D^2-49}\sin 3x.$$

$$= \frac{-3\frac{d}{dx}(\sin 3x) + 7\sin 3x}{9(-9)-49}$$ by 4(b)

$$= \frac{-9\cos 3x + 7\sin 3x}{-130}$$

y = C.F. + P.I.

Ex. 2 Show that the solution of the differential equation $\frac{d^2y}{dt^2}+4y=A\sin pt$ which is such that y = o and $\frac{dy}{dt}=0$ when t = 0, is $y=A\frac{(\sin pt-\frac{1}{2}p\sin 2t)}{4-p^2}$ if p ≠2.If p =2, show that $y=\frac{A(\sin 2t-2t\cos 2t)}{8}$

Soln:

Let D stand for $\frac{d}{dt}$ here.

To find the C.F of (D^2+4) y = 0.

The auxillary equation is $m^2 + 4 = 0$.

$m^2=-4$

$m = \pm 2i$.

∴ C.F. = λ cos 2t + μ sin 2t , where λ and μ are arbitrary constants.

(Note that the independent variable is t.)

$$P.I.=\frac{1}{D^2+4}a\sin pt$$

$$=\frac{1}{-p^2+4}A\sin pt \text{ if } p^2\neq 4 \text{ by 4(b).}$$

∴ y = λ cos2t+ μ sin 2t+(A/4-p^2) sin pt

To determine the values of λ and μ, we note that when

t = 0, y = 0 and $\frac{dy}{dt}=0$.

∴ 0 = λ

$\frac{dy}{dt}$ = -2λ sin2t+ 2μcos 2t+ $\frac{Ap}{4-P^2}$cos pt.

$$\therefore 0=2\mu+\frac{Ap}{4-p^2}$$

$$\mu=-\frac{Ap}{2(4-p^2)} \qquad \text{.... (2)}$$

Hence, $y=\frac{A(\sin pt-\frac{1}{2}\sin 2t)}{(4-p^2)}$

If p = 2, $P.I.=\frac{1}{D^2+4}A\sin 2t$

$$= \text{ Imaginary part of } \frac{Ap}{D^2+4}e^{2it}$$

$$= \text{ Imaginary part of } \frac{A}{(D+2i)(D-2i)}e^{2it}$$

$$= \text{ Imaginary part of } \frac{A}{4i}te^{2it}$$

$$= \frac{-At\cos 2t}{4}$$

$$y = \lambda\cos 2t + \mu\sin 2t - \frac{At}{4}\cos 2t\,.$$

When t = 0, y = 0 $\therefore$ λ = 0.

$$\frac{dy}{dt} = -2\lambda\sin 2t + 2\mu\cos 2t - \frac{A}{4}(\cos 2t - 2t\sin 2t)$$

$0 = 2\mu - \frac{A}{4}$, $\mu = \frac{A}{8}$.

$$y = \frac{A(\sin 2t - 2t\cos 2t)}{8}\,.$$

3)Slove (D^2 - 4D + 3)y = sin 3x cos 2x.

Soln:

To find the C.F of (D^2 - 4D +3)y = 0.

To auxillary equation is m^2 - 4m + 3 = 0.

(m-3)(m-1)=0

m =1 and 3.

C.F. = A e^x + B e^{3x}

$$P..I. = \frac{1}{D^2-4D+3}\sin 3x\cos 2x$$

$$= \frac{1}{D^2-4D+3}\frac{\sin 5x+\sin x}{2}$$

$$= \frac{1}{-25-4D+3}\frac{\sin 5x}{2} + \frac{1}{-1-4D+3}\frac{\sin x}{2}$$

$$= \frac{2D-11}{-4(4D^2-121)}\sin 5x + \frac{1+2D}{4(1-4D^2)}\sin x$$

$$= \frac{10\cos 5x - 11\sin 5x}{884} + \frac{\sin x + 2\cos x}{20}$$

y = C.F. + P.I.

$$y = A e^{x} + B e^{3x} + \frac{10\cos 5x - 11\sin 5x}{884} + \frac{\sin x + 2\cos x}{20}$$

4) Slove $(D^2 + 16)\, y = 2e^{-3X} + \cos 4x$.

Soln:

To find the C.F of $(D^2 + 16)y = 0$.

The auxiliary equation is $m^2 + 16 = 0$.

$m^2+4^2=0$; $m = \pm 4i$.

C.F. is = Acos 4x + B sin 4x.

$$Now\ 2e^{-3x} = \frac{2}{D^2+16}e^{-3x}$$

= $(2/7)\, e^{-3x}$

P.I.$_2$ corresponding to $\cos 4x = \frac{1}{D^2+16}\cos 4x$

$$= \frac{1}{D^2+16} \text{Real part of } e^{4ix}$$

$$= \text{Real part of } \frac{1}{(D+4i)(D-4i)}e^{4ix}$$

$$= \text{Real part of } \frac{1}{8i}xe^{4ix}$$

$$= \text{Real part of } -\frac{xi}{8}(\cos 4x + i\sin 4x)$$

$$= \frac{x}{8}\sin 4x.$$

$$\therefore y = A\cos 4x + B\sin 4x + \frac{2}{7}e^{-3x} + \frac{x}{8}\sin 4x.$$

Excersices

Solve the following equations:-

1. $(D^2 + 4)\, y = \sin 3x$.
2. $(D^2 + D + 1)y = \sin 2x$.
3. $(D^2-8D+9)y = 8\cos 5x$.
4. $(D^2-2D-8)y = 4 \cos 2x$.

(c) Let X be of the form x^m (a power of x), m being a positive integer.

To evaluate $\frac{1}{f(D)} x^m$, raise f(D) to power -1 and expand in ascending powers of D as far as Dm. (The higher powers of D operating on X^m give Zero and hence are omitted.) Thesse terms in the expansion of $f\{(D)\}^{-1}$ operating on x^m give the particular integral required.

Examples.

1)Solve $(D^2 + D + 1)\, y = x^2$.

Soln:

To find the C.F of $(D^2+D+1)\, y = 0$:

The auxillary equation is $m^2 + m + 1 = 0$.

Solving, $m = \frac{-1 \pm \sqrt{3}i}{2}$

$$C.F. = e^{-x/2}\left(A\cos\frac{\sqrt{3}x}{2} + B\sin\frac{\sqrt{3}x}{2}\right).$$

$$P.I. = \frac{1}{1+D+D^2}x^2$$

$$.=(1+D+D^2)^{-1}x^2$$

$=\{1-(D+D^2)+(D+D^2)^2\}x^2$, the powers of D higher than 2 are dropped.

$$=(1+D)x^2 = x^2 - 2x.$$

y = C.F.+ P.L.

$$y = e^{-x/2}\left(A\cos\frac{\sqrt{3}x}{2} + B\sin\frac{\sqrt{3}x}{2}\right) + x2 - 2x$$

2) Solve $(D^2+4D+5)y = e^x+x^3+\cos 2x$.

Soln:

To find the C.F of $(D^2+4D+5)y = 0$.

The auxiliary equation is $(m^2+4m+5) = 0$.

Solving, $m = -2\pm i$.

C.F. $= e^{-2x}(A \cos x + B \sin x)$.

P.I. Corresponding to $e^x = \frac{1}{D^2+4D+5}e^x$

$$= \frac{1}{1+4+5}e^x = \frac{1}{10}e^x$$

P.I. Corresponding to $x^3 = \frac{1}{5+4D+D^2}x^3$

$$= \frac{1}{5}\left(1+\frac{4D+D^2}{5}\right)^{-1}x^3$$

$$= \frac{1}{5}\left[1-\left(\frac{4D+D^2}{5}\right)+\left(\frac{4D+D^2}{5}\right)^2-\left(\frac{4D+D^2}{5}\right)^3\right]x^3$$

$$= \frac{1}{5}\left\{1-\frac{4D}{5}+\frac{11D^2}{25}-\frac{24}{125}D^3\right\}x^3$$

$$= \frac{1}{5}\left\{x^3-\frac{12x^2}{5}+\frac{66x}{25}-\frac{144}{125}\right\}$$

P.I.$_2$ can also be found by assuming

$$y = A x^3 + B x^2 + C x + D.$$

P.I.$_3$ corresponding to $\cos 2x = \frac{1}{D^2+4D+5}\cos 2x$

$$= \frac{1}{1+4D}\cos 2x \text{ on putting -4 for } D^2$$

$$= \frac{1-4D}{1-16D}\cos 2x = \frac{\cos 2x + 8\sin 2x}{65}$$

$$y = C.F. + P.I._1 + P.I._2 + P.I._3$$

$$y = e^{-2x}(A\cos x + B\sin x) + \frac{1}{10}e^x + \frac{1}{5}\left\{x^3 - \frac{12x^2}{5} + \frac{66x}{25} - \frac{144}{125}\right\} + \frac{\cos 2x + 8\sin 2x}{65}$$

Exercise:

Solve the following equations : -

1. $(D^2-1)\,y = 2 + 5x.$
2. $(D - 1)^2\, y = x.$
3. $(D^2 + D + 1)\, y = x + \sin x.$

(d) X is of the form $e^{ax} V$, where V is any function of x.

$$D^2(e^{ax} V) = a e^{ax} V + e^{ax} Dv = e^{ax} (D+a) V.$$

$$D^2(e^{ax} V) = D\{e^{ax}(D+a) V\}$$
$$= ae^{ax} (D + a) V + e^{ax} (D^2 + aD) V$$
$$= e^{ax}(D^2 + 2aD + a^2)V = e^{ax} (D+a)^2 V.$$

It follows by induction that $D^n(e^{ax} V) = e^{ax}(D+a)^n V$.

$$\therefore f(D)\, e^{ax}V = e^{ax} f(D+a) V.$$

Operating on both sides by $\frac{1}{f(D)}$

$$e^{ax}V = \frac{1}{f(D)} e^{ax} f(D+a)V .$$

If we set $f(D+a)\, V = V_1$, then this result gives,

$$e^{ax}\frac{1}{f(D+a)}V_1 = \frac{1}{f(D)}e^{ax}V_1$$

Hence $\frac{1}{f(D)} e^{ax} X = e^{ax}\frac{1}{f(D+a)} X$.

Examples.

Slove $(D^2 - 4D + 3)\, y = e^{-x}\sin x$.

Soln:

To find the C.F of Slove $(D^2-4D+3)y = 0$.

The auxiilary equation is m^2 -4m + 3 = 0.

(m-1)(m-3)=0

$\therefore$ m = 1 and 3

C.F. = A e^x + B e^{3x}.

$$P.I. = \frac{1}{D^2-4D+3} e^{-x} Sinx$$

$$= e^{-x} \frac{1}{(D-1)^2-4(D-1)+3} Sinx \text{ by the above rule}$$

$$= e^{-x} \frac{1}{D^2-6D+8} Sinx$$

$$= e^{-x} \frac{1}{7-6D} \sin x \text{ on putting -1 for } D^2$$

$$= e^{-x} \frac{7+6D}{49-36D^2} Sinx$$

$$= e^{-x} \frac{7\sin x + 6\cos x}{85}$$

y = C.F. + P.I.

$$y = A e^x + B e^{3x} + e^{-x} \frac{7\sin x + 6\cos x}{85}$$

Solve (D^2+2D+5)y = xe^x.

Soln:

To find the C.F of (D^2+2D+5) y = 0.

The auxiliary equation is m^2 + 2m + 5 = 0.

Solving, m = -1±2.

C.F. = e^{-x} (A cos2x + B sin 2x).

$$P.I. = \frac{1}{D^2+2D+5} xe^x$$

$$= e^x \frac{1}{(D+1)^2+2(D+1)+5} x$$

$$= e^x \frac{1}{D^2+4D+8} x$$

$$= e^x \frac{1}{8+4D} x$$ (as D^2 can be omitted in thr dinomiator for only x occurs in numerator)

$$= \frac{e^x}{8}\left(1 - \frac{D}{2}\right)x$$

$$= \frac{e^x}{9}\left(x - \frac{1}{2}\right)$$

y = C.F. + P.I.

$$y = e^{-x} (A \cos 2x + B \sin 2x) + \frac{e^x}{9}\left(x - \frac{1}{2}\right)$$

Exercises

Solve the following equations : -

1. $(D^2 + 1)\, y = (x^2 + 1)\, e^x$.
2. $(D^2+4)\, y = x\, e^{2x}$.
3. $(D^2 - 4D + 3)\, y = e^x \cos 2x$.
4. $(D^2 - 2D + 2)y = e^x \cos x$.

1.5 Linear equations with variable coefficients.

We shall first consider the homogeneous linear equartion. A homogeneous linear equation of the second order is of the form

$$a\,x^2 \frac{d^2 y}{dx^2} + bx\frac{dx}{dy} + cy = X$$

Where a, b, c are constrants and X is a function of X.

Method 1 By putting z = logx or $x = e^z$, this equation can be transformed into one with constant coeffieients.

We introduce here an operator $\theta = x\frac{d}{dx}$. Now,

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{d}{dz}; x\frac{dy}{dx} = \frac{dy}{dz} = D\,y \text{ if D stands for } \frac{d}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{1}{x}\frac{d^2 z}{dx^2}\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz} = \frac{1}{x^2}(\frac{d^2 y}{dz^2} - \frac{dy}{dz}) = \frac{D}{x^2}(D-1)y$$

$$\therefore x^2 \frac{d^2 y}{dx^2} = D(D-1)y$$

We note that $D = \frac{d}{dz} = x\frac{dy}{dx} = \theta$

So, putting $x = e^z$ in (1) the equation (1) becomes {a D(D-1)+ bD + c } y = Z ...... (2) where Z is a functiuon of z into which X has been transformed. This equation (2) is a linear equation with constant coefficients and hence the foregoing method can be adopted.

Method 2. Without transforming (1) into a linear equation with constant coeffieients, an independent method may be given.

To find the complementary function of (1), we have to slove

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0$$

If x^m, for some value of m, be taken as a tentative solution, then, on subsitution, we get

am (m-1) + bm + c = 0.

This, being an equation of the seocnd degree in m, has two roots m_1, m_2. Hence the complementry function of (1) is $C_1 x^{m1} + C_2 x^{m2}$, taking the two roots to be distinct.

If however, a root m_1 be repeated twice putting $m_2 = m_1 + \epsilon$ where $\epsilon \to 0$, the corresponding of the C.F. is

$$x^{m_1}(C_1 + C_2\, x^{\in}) = x^{m_1}(C_1 + C_2\; e^{\in \log x})$$

$$= x^{m_1}\left\{C_1 + C_2\,(1 + \in .\log x + \frac{\in^2 (\log x)^2}{2} etc.,)\right\}$$

ϵ^2 being neglected as $\epsilon \to 0$. Putting $C_2\epsilon = B$ and $C_1 + C_2 = A$, the part of the C.F. arising from the two equal roots $m_{1,}$ is $x^{m_1}(A + B\log x)$

To find the Particular Integral.

The P.I. of $ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = X$ .... (1) is now found.

Using $\theta = \frac{d}{dx}$ the first member of (1) can be symbolically written as f(θ)y, where f(θ) = aθ(θ-1) + bθ +c.

$\therefore$(1) can be written as f(θ) y = X.

The P.I. is $\frac{1}{f(\theta)}X$, Where $\frac{1}{f(\theta)}$ is the inverse operator defined as in 3

If f(θ) = (θ - α_1) (θ - α_2), the P.I. can be put either as

$$\frac{1}{(\theta-\alpha_1)}\frac{1}{(\theta-\alpha_2)}X$$

Or $\left(\frac{A_1}{\theta-\alpha_1}+\frac{A_2}{\theta-\alpha_2}\right)X$

by the method of partial fractions.

It must be noted that in the first form, the order of the operators is not commutative. Here, the operations indicated by the factors are to be taken in succession, beginning with the first on the right. Thus the general method og finding the P.I. ultimately depends on the evaluation of $\frac{1}{\theta-\alpha}X$.

To find $\frac{1}{\theta-\alpha}X$.

Let $u=\frac{1}{\theta-\alpha}X$.

By definition of inverse operator, $x\frac{du}{dy}-\alpha u = x$

$$i.e., \frac{du}{dx}-\frac{\alpha}{x}u=\frac{X}{x}.$$

This equation is linear in u and hence its solution is

$$u\,x^{-\alpha}=\int x^{-\alpha-1}X\,dx$$

no constant being added as this is a particular integral.

$$\therefore u=x^{\alpha}\int x^{-\alpha-1}X\,dx$$

(It is advisable for the student to commit this result to memory).

Special method of evaluating the P.I. when X is of the form x^m.

$$\theta\,x^m = x\frac{d}{dx}(x^m)=m\,x^m.$$

$$\theta^2\,x^m = x\frac{d}{dx}(m\,x^m)=m^2\,x^m$$

Generally, $f(\boldsymbol{\theta})\,x^m = f(m)\,x^m$.

Operating on both sides by $\frac{1}{f(\theta)}, x^m = f(m)\frac{1}{f(\theta)}x^m.$

If, $f(m) \neq 0, \frac{1}{f(\theta)}x^m = \frac{1}{f(m)}x^m.$

If, however, f(m) = 0, then f($\boldsymbol{\theta}$) = ($\boldsymbol{\theta}$-m) ϕ ($\boldsymbol{\theta}$) when ϕ (m) ≠ O.P.I. becomes $\frac{1}{\phi(m)}\frac{1}{\theta - m}x^m$

$=\frac{1}{\phi(m)}x^m \int x^{-m-1} x^m$ dx by the above general method

$=\frac{1}{\phi(m)}x^m \log x.$

If m be repeated two times in f(m) = 0,

the P.I. is $\frac{x^m (\log x)}{2!}$, where f(m) = ($\boldsymbol{\theta}$-m)^2.

Examples:

Ex.1. Solve $3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x$

Soln:

Putting z = log x and $D = \frac{d}{dz}$ the equation becomes

[3D (D-1) + D+1] = e^z.

The auxiliary equation is 3m^2 – 2m + 1 = 0

$m = 1 \pm \sqrt{2}\, i$

$C.F. = e^z (A\cos\sqrt{2}\, z + B\sin\sqrt{2}\, z)$

$$= x\{A\cos(\sqrt{2}\log x) + (B\sin(\sqrt{2}\log x)\}$$

$$P.I. = \frac{1}{3D^2 - 2D + 1} e^z$$

$$= \frac{1}{3 - 2 + 1} e^z \ by \S 4(a)$$

$$= \frac{e^z}{2} = \frac{x}{2}.$$

y = C.F. + P.I.

$$= x[A\cos(\sqrt{2}\log x) + B\sin(\sqrt{2}\log x) + 1/2]$$

Ex.2. Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = \log x.$

Soln:

Putting z = log x and $D = \frac{d}{dz}$ the equation becomes

[D (D-1) + (D+1)] y = z

i.e., (D^2+1) y = z

The auxiliary equation is $m^2 + 1 = 0$

C.F. = A cos z + B sin z

= A cos (log x) + B sin (log x).

$$P.I. = \frac{1}{D^2 + 1} z = (1 + D^2)^{-1} z$$

= $(1 – D^2 + ...)$ z = z.

$\therefore$ y = A cos (log x) + B sin (log x) + log x

Solve $x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$

Soln:

Putting z = log x and $D = \frac{d}{dz}$ the equation becomes

$$[D(D-1) + 3D + 1]\, y = \frac{1}{(1-x)^2}$$

$$\text{i.e.,} (D^2 + 2D + 1)\, y = \frac{1}{(1-x)^2}$$

The auxiliary equation is $(m + 1)^2 = 0$

$\therefore$ m = - 1 twice

$$C.F. = e^{-z}(A + Bz) = \frac{1}{x}(A + B\log x)$$

$P.I. = \frac{1}{(\theta+1)^2}\frac{1}{(1-x)^2}$ Changing D to the operator $\theta = x\frac{d}{dx}$

$$= \frac{1}{(\theta+1)} x^{-1}\int \frac{dx}{(1-x)^2}\, by\, \S 8.2$$

$$= \frac{1}{(\theta+1)}\frac{1}{x}\frac{1}{1-x}$$

$$= x^{-1}\int \frac{dx}{x(1-x)} = x^{-1}\int\left(\frac{1}{x} + \frac{1}{1-x}\right)dx$$

$$=\frac{1}{x}\log\frac{x}{1-x}.$$

y = C.F. + P.I.

Solve $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=e^x.$

Soln:

Putting z = log x and $D=\frac{d}{dz}$ the equation becomes

$(D^2 + 3D + 2)\, y = e^x$

The auxiliary is $m^2 + 3m + 2 = 0$.

$\therefore$ m = -1 or -2.

C.F. = $A\, e^{-z} + B\, e^{-z} = A\, x^{-1} + B\, x^{-2}$.

$$P.I.=\frac{1}{(\theta+1)(\theta+2)}e^x, where\,\theta = x\frac{d}{dx}$$

$$=\left[\frac{1}{\theta+1}-\frac{1}{\theta+2}\right]e^x$$

$$=x^{-1}\int e^x\,dx - x^{-2}\int x\,e^x\,dx\,by\,\S 8.2.$$

$= x^{-1} e^x - x^{-2} (xe^x - e^x)$

$= x^{-2} e^x$.

$y = A\, x^{-1} + B\, x^{-2} e^x + x^{-2} e^x$.

Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \frac{\log x \sin(\log x) + 1}{x}$

Soln:

Putting z = log x and $D = \frac{d}{dz}$ the equation becomes

$(D-1)^2\ y = \frac{(z\sin z + 1)}{e^z}.$

The auxiliary equation is $(m - 1)^2 = 0$; m = 1 twice.

C.F. = e^{-z} (A + B z) = x (A + B log x).

$$P.I. = \frac{1}{(D-1)^2}[z\sin z + 1]e^{-z}$$

$$= e^{-z}\ \frac{1}{(D-2)^2}[z\sin z + 1] by\ \S\ 4(\mathrm{d})$$

$$= x^{-1}\left[\mathrm{Im}aginary\ part\ of\ \frac{1}{(D-2)^2}\ z\, e^{iz} + \frac{1}{4}\right]$$

$$= x^{-1}\left[I.P.of\ e^{iz}\ \frac{1}{(D+i-2)^2}\ z + \frac{1}{4}\right]$$

$$= x^{-1}\left[I.P.of\ e^{iz}\ \frac{1}{(i-2)^2}\left\{1 - \frac{2D}{i-2}\right\} z + \frac{1}{4}\right]$$

$$= x^{-1}\left[I.P.of\ \frac{1}{3-4i}\left\{z + \frac{2}{5}(i+2)\right\} + \frac{1}{4}\right]$$

$$= x^{-1}\left[\frac{1}{25}\left\{\left(3z + \frac{4}{5}\right)\sin z + \left(4z + \frac{6}{5}\right)\cos z\right\} + \frac{1}{4}\right]$$

$$= \frac{1}{100x} + \frac{6}{125x}\cos(\log x) + \frac{4}{125x}\sin\log x + \frac{1}{25x}\log x\{4\cos(\log x) + 3\sin(\log x)\}$$

y = C.F.+ P.I.

UNIT – 4 : LAPLACE TRANSFORM

Laplace transform – Inverse transform – Properties-Solving differential equations. Simultaneous equations of first order using Laplace transform.

THE LAPLACE TRANSFORMS

1.1. **Definition:** If a function f(t) is defined for all positive values of the variable t and if $\int_0^\infty e^{-st} f(t)\, dt$ exists and is equal to F(s), then F(s) is called the Laplace transform of f(t) and is denoted by the symbol L{f(t)}.

Hence $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$. The operator L that transforms f(t) into F(s) is called the Laplace transform operator.

Note: $\underset{S\to\infty}{Lt} F(s) = 0.$

Definitions. Piecewise continuity.

A function f (t) is to be piecewise continuous in a closed interval [a,b]if it is defined on that interval and is such that the interval can be broken up into a finite number of sub-intervals in each of which f (t) can have only ordinary finite discontinuities in the interval.

Exponential order.

A function f(*t*) is said to be of exponential order if $\underset{S\to\infty}{Lt} e^{-st} f(t) = 0,$ or if for some number s_0, the product $|f(t)| < M$ for *t*>T,i.e.,$e^{-s_0 t} |f(t)|$ is bounded for large value of *t*,say for *t* > T.

Sufficient conditions for the existence of the Laplace transform.

(i) f(t) is continuous or piecewise continuous in the closed interval [a,b], where a>0

(ii) it is of exponential order

(iii) t^n f(t) is bounded near t = 0 for some number n>1.

From the definition the following results can easily be proved:-

(i) L {f(t) + ϕ (t)} = L{f(t)} + L{ϕ(t)}

Proof:

We have $L\{f(t)+\phi(t)\}=\int_0^\infty e^{-st}[f(t)+\phi(t)]dt$

$$=\int_0^\infty e^{-st}f(t)\,dt+\int_0^\infty e^{-st}\phi(t)\,dt$$

$$L=\{f(t)\}+L\{\phi(t)\}.$$

(ii) L{c f(t)} = cL {f(t)}, where c is a constant

Proof:

We have $L\{c\,f(t)\}=\int_0^\infty e^{-st}c\,f(t)\,dt$

$$=c\int_0^\infty e^{-st}f(t)\,dt$$

$$=cL\{f(t)\}$$

(iii) L {f' (t)} = sL{f(t)}– f(0).

Proof:

We have $L\{f'(t)\}=\int_0^\infty e^{-st}f'(t)\,dt$

$$= f(t)\left[e^{-st}\right]_0^{\infty} - \int_0^{\infty} f(t)(-s)e^{-st}\,dt \quad \text{(on integration by parts)}$$

$$= -f(0) + s\int_0^{\infty} f(t)e^{-st}\,dt$$

$$= s\,L\{f(t)\} - f(0)$$

(iv) $L\{f'(t)\} = s^2 L\{f(t)\} - s\,f(0) - s\,f'(0)$

Poof:

$$L\{f''(t)\} = L\{F'(t)\}, \text{ where } F(t) = f'(t)$$

$$= s\,L\{F(t)\} - F(0) = s\,L\{f'(t)\} - f'(0)$$

$$= s[s\,L\{f(t)\} - f(0)] - f'(0)$$

$$= s^2 L\{f(t)\} - s\,f(0) - f'(0).$$

(v) By extending the previous result, we get

$$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) \ldots - f^{n-1}(0)$$

(vi) If $L\{f(t)\} = F(s)$, then

(a) $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s).$

(b) $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s).$

Proof:

$$L\{f'(t)\} = s\,L\{f(t)\} - f(0)$$

$$= s\,F(s) - f(0)$$

Taking limits as s →∞ on both sides, we get

$$\lim_{s\to\infty}[s\,F(s) - f(0)] = \lim_{s\to\infty} L\{f'(t)\}$$

$$= \lim_{s\to\infty}\int_0^\infty e^{-st}\, f'(t)\,dt$$

$$= 0.$$

$$\therefore \lim_{s\to\infty} s\,F(s) = f(0)$$

$$= \lim_{t\to 0} f(t)$$

This result is known **as Initial value theorem.**

Taking limits as s →0 on both sides of L {f' (t)}, we get

$$\lim_{s\to 0}[s\,F(s) - f(0)] = \lim_{s\to 0}\int_0^\infty e^{-st}\, f'(t)\,dt$$

$$= \int_0^\infty f'(t)\,dt$$

$$= [f(t)]_0^\infty$$

$$= \lim_{t\to\infty} f(t) - f(0).$$

$$\therefore \lim_{s\to 0} s\,F(s) = \lim_{t\to 0} f(t)$$

This result is known as **final value theorem**.

(Vii) $\quad L(e^{-at}) = \dfrac{1}{s+a}$ *provided* $s+a>0.$

Proof:

$$L(e^{-at}) = \int_0^\infty e^{-st}\,e^{-at}\,dt$$

$$= \int_0^\infty e^{-(s+a)t}\,dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a}.$$

(Vii) $L(e^{at}) = \dfrac{1}{s-a}$ *provided* $s-a>0.$

Proof:

$$L(e^{at}) = \int_0^\infty e^{-st}\,e^{at}\,dt$$

$$= \int_0^\infty e^{-(s-a)t}\,dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty = \frac{1}{s-a}.$$

Corolary:. $L(\cosh at) \quad = \dfrac{s}{s^2-a^2}$

Proof:

We have

$$L(\cosh at) = L\left(\frac{e^{at}+e^{-at}}{2}\right)$$

$$=\frac{1}{2}L(e^{at})+\frac{1}{2}L(e^{-at})$$

$$=\frac{1}{2}\frac{1}{s-a}+\frac{1}{2}\frac{1}{s+a}$$

$$=\frac{s}{s^2-a^2}$$

Corolary:. $L(\sinh at) \quad =\frac{a}{s^2-a^2}$

Proof:

We have

$$L(\sinh at) = L\left(\frac{e^{at}-e^{-at}}{2}\right)$$

$$=\frac{1}{2}L(e^{at})-\frac{1}{2}L(e^{-at})$$

$$=\frac{1}{2}\frac{1}{s-a}-\frac{1}{2}\frac{1}{s+a}$$

$$=\frac{a}{s^2-a^2}$$

$$L(\sinh at) = \frac{a}{s^2 - a^2}$$

(Viii) $L(\cos at) = \frac{s}{s^2+a^2}$

Method1.

Proof:

We have

$$L(\cos at) = \int_0^\infty e^{-st} \cos at \, dt$$

$$= \left[\frac{e^{-st}(-s\cos at + a\sin at)}{s^2+a^2} \right]_0^\infty$$

$$= \frac{s}{s^2+a^2}$$

Method 2.

$$L(\cos at) = \text{real part of} \int_0^\infty e^{-st} e^{ait} \, dt$$

$$= \text{real part of } L(e^{ait})$$

$$= \text{real part of } \frac{1}{s - ai}$$

$$= \text{real part of } \frac{s+ai}{s^2+a^2}$$

$$= \frac{s}{s^2+a^2}$$

Method 3. Let f(t) be cos at.

Then $f'(t) = a \sin at$, $f''(t) = -a^2 \cos at$.

We have $L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - f'(0)$.

$\therefore L\{-a^2 \cos at\} = s^2 L\{f(t)\} - s f(0) - f'(0)$.

We have $f(0) = 1$, $f'(0) = 0$.

$\therefore -a^2 L(\cos at) = s^2 L(\cos at) - s$

i.e., $(s^2 + a^2)$ L (cos at) = s.

$$\therefore L(\cos at) = \frac{s}{s^2 + a^2}$$

(ix) $(L \sin at) = \frac{a}{s^2 + a^2}$

Proof:

We have $L(\sin at) = \int_0^\infty e^{-st} \sin at \, dt$

$$= \left[\frac{e^{-st}(-s \sin at - a \cos at)}{s^2 + a^2} \right]_0^\infty$$

$$= \frac{a}{s^2 + a^2}$$

Aliter. Let f(t) be sin at, then f' (t) = a cos at.

We have L {f'(t)} = s L{f(t)} – f(0)

i.e., L (a cos at) = s L (sin at) – 0

i.e., $a \frac{s}{s^2 + a^2} = s L(\sin at) - 0$

$$\therefore L(\sin at) = \frac{a}{s^2 + a^2}$$

(x) $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$.

Proof: We have $L(t^n) = \int_0^\infty e^{-st} t^n \, dt$

Put st $= x, then\ dt = \frac{1}{s} dx.$

$$\therefore L(t^n) = \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x} \frac{1}{s} dx$$

$$=\frac{1}{s^{n+1}}\int_0^\infty x^n e^{-x}\,dx$$

$$=\frac{\Gamma(n+1)}{s^{n+1}}$$

When n is a positive integer $\Gamma(n+1) = n!$

$\therefore L(t^n)=\frac{n!}{s^{n+1}}$ when n is a + ve integer

Cor. $L(1)=\frac{1}{s}$

$$L(t)=\frac{1}{s^2}$$

$$L(t^2)=\frac{2}{s^3}$$

$$L(t^{1/2})=\frac{\Gamma(3/2)}{s^{3/2}}=\frac{\frac{1}{2}\Gamma(\frac{1}{2})}{s^{3/2}}=\frac{\sqrt{\pi}}{2s^{3/2}}$$

$$L(t^{-1/2})=\frac{\Gamma(\frac{1}{2})}{s^{1/2}}=\frac{\sqrt{\pi}}{2s^{1/2}}\,.$$

Examples:

Ex.1. Find L (t^2 + 2t + 3)

Soln:

L (t^2+2t+3) = L(t^2) + 2L (t) + 3L (1)

$$=\frac{2}{s^3}+\frac{2}{s^2}+\frac{3}{s}\,.$$

Ex.2. Find L($\sin^2 2t$).

Soln:

$$Since, \sin^2 2t = \left(\frac{1-\cos 4t}{2}\right), we\, have$$

$$L(\sin^2 2t) = L\left(\frac{1-\cos 4t}{2}\right)$$

$$= \frac{1}{2}L(1) - \frac{1}{2}L(\cos 4t)$$

$$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+4^2}$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+16}\right)$$

$$= \frac{8}{s(s^2+16)}$$

Ex.3. Find L($\sin^3 2t$).

Soln:

Since sin 6t = 3 sin 2t – 4 $\sin^3$ 2t, we have

$$L(\sin^3 2t) = L\left(\frac{3\sin 2t - \sin 6t}{4}\right)$$

$$= \frac{3}{4}L(\sin 2t) - \frac{1}{4}L(\sin 6t)$$

$$= \frac{3}{4}\frac{2}{s^2+2^2} - \frac{1}{4}\frac{6}{s^2+6^2}$$

$$= \frac{48}{(s^2+4)(s^2+36)} \quad .$$

Ex.4. Find L {f(t)}, where

F(t) = 0 when $0 < t \le 2$

= 3 when $t > 2$.

Soln:

We have

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st}\, f(t)\,dt$$

$$= \int_0^2 e^{-st} (0)\,dt + \int_2^\infty e^{-st} (3)\,dt$$

$$= 3\int_2^\infty e^{-st} dt$$

$$= \frac{3}{s} e^{-2s}$$

1.2 The inverse transforms.

Let the symbol L^{-1} {F (s)} denote a function, whose Laplace transform is F(s).Thus if L {f(t)} = F(s) then f(t) = L^{-1} {F(s)}.

The most obvious way of finding the inverse transform of a given function is to look into the table of transforms and get the function whose Laplace transform is the given function.

We can compile the table of transforms from the known results.

s.no	f(t)	F(s)
1.	e^{at}	$\frac{1}{s-a}$
2.	cosh at	$\frac{s}{s^2-a^2}$
3.	sinh at	$\frac{a}{s^2-a^2}$
4.	cos at	$\frac{s}{s^2+a^2}$
5.	Sin at	$\frac{a}{s^2+a^2}$
6.	1	$\frac{1}{s}$
7.	T	$\frac{1}{s^2}$
8.	t^n	$\frac{n!}{s^{n+1}}(n \, is \, a + ve \, \text{int} \, eger)$
9.	$t\,e^{at}$	$\frac{1}{(s-a)^2}$
10.	$t^2\,e^{at}$	$\frac{2}{(s-a)^3}$
11.	$t^n\,e^{at}$	$\frac{n!}{(s-a)^{n+1}}(n \, is \, a + ve \, \text{int} \, eger)$
12.	e^{-at} sin bt	$\frac{b}{(s+a)^2+b^2}$
13.	e^{-at}cosbt	$\frac{s+a}{(s+a)^2+b^2}$
14.	t sin at	$\frac{2as}{(s^2+a^2)^2}$
15.	t cos at	$\frac{s^2-a^2}{(s^2+a^2)^2}$

We can modify the results we have obtained in finding the Laplace transforms of functions to get the inverse transforms of functions.

(i) If L {f(t)} = F(s), then L {e^{-at}f(t)} = F (s+a).

Hence we get the result

$$L^{-1}\{F(s+a)\} = e^{-at} f(t)\}$$

$$= e^{-at} L^{-1} F(s).$$

Thus for example

$$1. L^{-1}\left[\frac{1}{(s+a)^2}\right] = e^{-at} L^{-1}\left(\frac{1}{s^2}\right) = e^{-at}\ t.$$

$$2. L^{-1}\frac{1}{(s+2)^2+16} = e^{-at} L^{-1}\frac{1}{s^2+4^2} = \frac{e^{-2t}\sin 4t}{4}.$$

$$3. L^{-1}\frac{s-3}{(s-3)^2+4} = e^{3t} L^{-1}\left(\frac{s}{s^2+4}\right) = e^{3t}\cos 2t.$$

$$4. L^{-1}\left(\frac{s}{s+2s+5}\right) = L^{-1}\left[\frac{s}{(s+1)^2+2^2}\right]$$

$$= L^{-1}\left[\frac{(s+1)-1}{(s+1)^2+2^2}\right]$$

$$= L^{-1}\left[\frac{s+1}{(s+1)^2+2^2}\right] - L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= e^{-t} L^{-1}\left(\frac{s}{s^2+2^2}\right) - e^{-t} L^{-1}\left(\frac{1}{s^2+2^2}\right)$$

$$= e^{-t}\cos 2t - e^{-t}\frac{\sin 2t}{2}$$

$$=\frac{e^{-t}}{2}(2\cos 2t-\sin 2t).$$

(ii) $If\ L\{f(t)\}=F(s), then\ L[f(at)]=\frac{1}{a}F\left(\frac{s}{a}\right).$

This result can be written in the form

$$L^{-1}\left[\frac{1}{a}F\left(\frac{s}{a}\right)\right]=f\ (at), where\ f(t)=L^{-1}F(s)$$

Putting $\frac{1}{a}=k, we\,have$

$$L^{-1}F[(ks)]=\frac{1}{k}f\left(\frac{t}{k}\right)where\ f(t)=L^{-1}F(s).$$

Examples:

Find $L^{-1}\left[\frac{s}{s^2a^2+b^2}\right]$

Soln:

$$\frac{s}{s^2a^2+b^2}=\frac{1}{a}\frac{sa}{s^2a^2+b^2}=\frac{1}{a}F(sa)$$

Where $\mathrm{F(sa)}=\frac{sa}{s^2a^2+b^2}$

$$\therefore F(s)=\frac{s}{s^2+b^2}$$

$$L^{-1}\left[\frac{s}{s^2a^2+b^2}\right]=\frac{1}{a}L^{-1}\left[\frac{sa}{s^2a^2+b^2}\right]$$

$$=\frac{1}{a}L^{-1}[F(as)]$$

$$=\frac{1}{a}.\frac{1}{a}f\left(\frac{t}{a}\right),$$

Where $f(t)=L^{-1}(F(s))$

$$=L^{-1}.\frac{s}{(s^2+b^2)}$$

= cosbt.

$$\therefore f\left(\frac{t}{a}\right)=\cos\left(\frac{bt}{a}\right).$$

$$Hence\, L^{-1}=\left[\frac{s}{s^2a^2+b^2}\right]=\frac{1}{a^2}\cos\left(\frac{bt}{a}\right).$$

(iii) If L {f(t)} = F (s), then L {tf(t)}= -F' (s)

Hence we get the result

L^{-1} {F'(s)} = - tf(t) = tL^{-1} {F(s)}.

Examples.

Ex.1. Find $L^{-1}\left[\frac{s}{s^2+a^2)^2}\right]$

Soln:

$$F'(s)=\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$\therefore F(s)=\int \frac{s\,ds}{(s^2+a^2)^2}$$

$$=-\frac{1}{2(s^2+a^2)}$$

$$\therefore L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]=-tL^{-1}\left[-\frac{1}{2(s^2+a^2)}\right]$$

$$=\frac{t}{2}L^{-1}\left(\frac{1}{s^2+a^2}\right)$$

$$=\frac{t}{2a}\sin at$$

Ex.2. Find $L^{-1}\left[\frac{s}{(s^2-1)^2}\right]$

Soln:

$$Here\,F'(s)=\frac{s}{(s^2-1)^2}\;\therefore F(s)=\int \frac{s}{(s^2-1)^2}\,ds.$$

$$=-\frac{1}{2(s^2-1)}.$$

$$\therefore L^{-1}\left[\frac{s}{(s^2-1)^2}\right]=-tL^{-1}[-\frac{1}{2(s^2-1)}].$$

$$=\frac{t}{2}L^{-1}\left(\frac{1}{s^2-1}\right)$$

$$=\frac{t}{2}\sinh t.$$

Ex.3. Find $L^{-1}\left[\frac{s+2}{(s^2+4s+5)^5}\right]$

Soln:

$$Here\, F'(s)=\frac{s+2}{(s^2+4s+5)^2}$$

$$\therefore F(s)=\frac{1}{2(s^2+4s+5)}$$

$$\therefore L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\}=-tL^{-1}\left[\frac{1}{-2(s^2+4s+5)}\right]$$

$$=\frac{t}{2}L^{-1}\left[\frac{1}{s^2+4s+5}\right]$$

$$=\frac{t}{2}L^{-1}\left[\frac{1}{(s+2)^2+1^2}\right]$$

$$=\frac{t}{2}e^{-2t}L^{-1}\left[\frac{1}{s^2+1^2}\right]$$

$$=\frac{t\,e^{-2t}\sin t}{2} \quad .$$

(iv) If L {f(t)} = F (s), then L {tf(t)} = - F' (s).

This theorem can be used in the following way to get inverse transforms of certain functions:

-

example $L^{-1}\left[\log\frac{s+1}{s-1}\right]$.

Soln:

Let this be equal to f(t).

$$\text{Then L }\{f(t)\} = \log \frac{s+1}{s-2}.$$

$$\therefore L\{tf(t)\} = -\frac{d}{ds}\log\frac{s+1}{s-1}$$

$$= -\frac{d}{ds}[\log(s+1) - \log(s-1)]$$

$$= -\frac{1}{s+1} + \frac{1}{s-1}.$$

$$\therefore tf(t) = L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s+1}\right)$$

$$= e^{t} - e^{-t}$$

$$= 2 \sinh t.$$

$$\therefore f(t) = \frac{2\sinh t}{t}.$$

(v) $L[\int_0^t f(x)\,dx] = \frac{1}{s} L[f(t)].$

Soln:

Let $\int_0^t f(x)\,dx \; be \; F(t)$

Then F' (t) = f(t) and F (0) = 0

$$\therefore L\{F'(t)\} = sL\{F(t)\} - F(0)$$

$$= sL\{F(t)\}$$

i.e., $L\{f(t)\} = sL\{\int_0^t f(x)\,dx\}$

Hence $L\int_0^t f(x)\,dx = \frac{1}{s}L\{f(t)\}$.

This result can also be used to find the inverse transforms of certain funcitons.

$$\int_0^t f(x)\,dx = L^{-1}\left[\frac{1}{s}\{f(t)\}\right]$$

If L {f(t)} = F(s), then

$$L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t f(x)\,dx,$$

Where f(t) = L^{-1} F(s).

$$\therefore L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t L^{-1}\{F(s)\}\,dt.$$

Examples.

Ex.1. Find $L^{-1}\left[\frac{1}{s(s+a)}\right]$

Soln:

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t \left(\frac{1}{s+a}\right)dt$$

$$= \int_0^t e^{-at}\,dt$$

$$=\left[\frac{e^{-at}}{-a}\right]_0^t$$

$$=\frac{1}{a}(1-e^{-at}).$$

Ex.2. Find $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

Soln:

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]=\int_0^t L^{-1}\left(\frac{1}{s^2+a^2}\right)dt$$

$$=\int_0^t \frac{\sin at}{a}dt$$

$$=\frac{1}{a}\left[\frac{-\cos at}{a}\right]_0^t$$

$$=\frac{1-\cos at}{a^2}$$

Ex.3. Find $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$

Soln:

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]=L^{-1}\left[\frac{1}{s}\cdot\frac{s}{(s^2+a^2)^2}\right]$$

$$=\int_0^t L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]dt$$

$$=\int_0^t \frac{t\sin at}{2a}\,dt$$

$$=\frac{1}{2a}\left[\frac{-t\cos at}{a}+\frac{\sin at}{a^2}\right]_0^t$$

$$=\frac{1}{2a^3}(\sin at-at\cos at).$$

(vii)The method of partial fractions can be used to find the inverse transform of certain functions.

The method is illustrated in the following examples.

Examples.

Ex.1. Find $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$

Soln:

We can split $\frac{1}{s(s+1)(s+2)}$ into partial fractions as

$$\frac{1}{2}\frac{1}{s}-\frac{1}{s+1}+\frac{1}{2}\frac{1}{s+2}.$$

$$\therefore L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\}=\frac{1}{2}L^{-1}\left(\frac{1}{s}\right)-L^{-1}\left(\frac{1}{s+1}\right)$$

$$+\frac{1}{2}L^{-1}\left(\frac{1}{s+2}\right)=\frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t}$$

Ex.2. Find $L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right]$.

Soln:

Splitting into partial fractions, we have

$$\frac{1}{(s+1)(s^2+2s+2)}=\frac{1}{s+1}-\frac{s+1}{s^2+2s+2}$$

$$\therefore L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right]$$

$$=L^{-1}\left(\frac{1}{s+1}\right)-L^{-1}\left(\frac{s+1}{s^2+2s+2}\right)$$

$$=e^{-t}-L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right]$$

$$=e^{-t}-e^{-t}L^{-1}\left(\frac{s}{s^2+1}\right)$$

$= e^{-t} – e^{-t}\cos t$

$= e^{-t} (1- \cos t)$

Ex.3. Find $L^{-1}\left[\frac{1+2s}{(s+2)^2+(s-1)^2}\right]$.

Soln:

$$\frac{1+2s}{(s+2)^2(s-1)^2}=\frac{1}{3}.\frac{(s+2)^2-(s-1)^2}{(s+2)^2(s-1)^2}$$

$$=\frac{1}{3}\left[\frac{1}{(s-1)^2}-\frac{1}{(s+2)^2}\right].$$

Hence $L^{-1}\left[\dfrac{1+2s}{(s+2)^2(s-1)^2}\right]$

$$=\frac{1}{3}L^{-1}\left[\frac{1}{(s-1)^2}\right]-\frac{1}{3}L^{-1}\left[\frac{1}{(s+2)^2}\right].$$

$$=\frac{1}{3}(e^t\,t)-\frac{1}{3}(e^{-2t}\,t)$$

$$=\frac{t}{3}(e^t-e^{-2t}).$$

Exercises

Find the inverse transforms of :

1. $\dfrac{1}{(s-3)^5}$ **2 .** $\dfrac{s}{(s-b)^{2+a^2}}$.

3. $\dfrac{cs+d}{(s+a)^{2+b^2}}$ **4.** $\dfrac{s}{(s+3)^5}$

Laplace transformation can be used to solve ordinary differential equations with constant coefficients.

Examples.

Ex.1. Solve the equation $\dfrac{d^2y}{dt^2}+2\dfrac{dy}{dt}$ - 3y = sin t given that $y=\dfrac{dy}{dt}=0\, when\, t=0$

Soln: The equation can be written in the form

y″ + 2y' – 3y = sin t.

Applying Laplace transforms to both sides, we have

L (y″ + 2y' – 3y) = L (sin t)

$$\text{i.e., } L(y'')+2L(y')-3L(y)=\frac{1}{s^2+1}$$

$$S^2L(y)-sy(0)\,y'(0)+2\{sL(y)-y(0)\}-3L(y)=\frac{1}{s^2+1}$$

Substituting the values of y(0) and y' (0) in the equations

$$s^2\,\bar{y}+2s\,\bar{y}-3\bar{y}=\frac{1}{s^2+1},\, where\,\bar{y}=L(y).$$

$$(s^2+2s-3)\,\bar{y}=\frac{1}{s^2+1}$$

$$\bar{y}=\frac{1}{(s^2+2s-3)(s^2+1)}$$

$$=\frac{1}{(s+3)(s-1)(s^2+1)}$$

$$\therefore y=L^{-1}=\frac{1}{(s-1)(s+3)(s^2+1)}$$

On splitting into partial fractions, we get

$$y=L^{-1}\left(\frac{-\frac{1}{40}}{s+3}+\frac{\frac{1}{8}}{s-1}+\frac{-\frac{1}{10}s-\frac{1}{5}}{s^2+1}\right)$$

$$=-\frac{1}{40}L^{-1}\left(\frac{1}{s+3}\right)+\frac{1}{8}L^{-1}\left(\frac{1}{s-1}\right)$$

$$-\frac{1}{10}L^{-1}\left(\frac{s}{s^2+1}\right)-\frac{1}{5}L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= -\frac{1}{40}e^{-3t} + \frac{1}{8}e^{t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t.$$

Ex.2. Show the solution of the differential equation $\frac{d^2y}{dt^2} + 4y = A\sin kt$ which is such that y = 0 and $\frac{dy}{dt} = 0$ when t = 0 is $y = A\frac{\sin kt - \frac{k}{2}\sin 2t}{4-k^2}$ *if* $k \neq 2$. If k=2,

show that $y = \frac{A(\sin 2t - 2t\cos 2t)}{8}$

Soln:

y″ + 4y = A sin kt

L (y″) + 4L (y) = A L (sin kt)

$$s^2\,\bar{y} - sy(0) - y'(0) + 4\,\bar{y} = A\frac{k}{s^2+k^2}, \textit{ where } L(y) = \bar{y}.$$

Since y (0) = 0, y' (0) = 0, we have

$$(s^2+4)\,\bar{y} = A.\frac{k}{s^2+k^2}.$$

$$\therefore \bar{y} = A.\frac{k}{(s^2+4)(s^2+k^2)}$$

$$\therefore y = AKL^{-1}\frac{1}{(s^2+4)(s^2+k^2)}$$

Case(i). If $k \neq 2$,

$$y = AKL^{-1}\left(\frac{\frac{1}{s^2+4} - \frac{1}{s^2+k^2}}{(k^2-4)}\right)$$

$$=\frac{AK}{k^2-4}\left\{L^{-1}\frac{1}{s^2+4}-L^{-1}.\frac{1}{s^2+k^2}\right\}$$

$$=\frac{Ak}{k^2-4}\left\{\frac{\sin 2t}{2}-\frac{\sin kt}{k}\right\}$$

$$=\frac{A}{4-k^2}\left(\sin kt-\frac{k}{2}\sin 2t\right)$$

Case (ii). K = 2. Then

$$\bar{y}=2AL^{-1}\left\{\frac{1}{(s^2+4)(s^2+4)}\right\}$$

$$=2AL^{-1}\left\{\frac{1}{(s^2+2^2)^2}\right\}$$

$$=2A.\frac{1}{2(2)^3}(\sin 2t-2t\cos 2t)$$

$$=\frac{A}{8}(\sin 2t-2t\cos 2t).$$

Note:-The special advantage of this method in solving differential equations is that the initial conditions are satisfied automatically.It is unnecessary to find the general solution and determine the constsnts using the initial conditions.

1.3 The Laplace transform can also be used to solve systems of differential equations.

Ex.1. Solve the simultaneous equations.

$$3\frac{dx}{dt}+\frac{dy}{dt}+2x=1 \qquad \text{... (1)}$$

$$\frac{dx}{dt}+4\frac{dy}{dt}+3y=0 \qquad \text{...(2)}$$

given x = 0 = y at t = 0.

Soln:

Applying Laplace transforms to both the equations, (1) becomes

3L (x') + L (y') + 2 L (x) = L (1)

$$\text{i.e.,} \; 3\{s\bar{x} - x(0)\} + s\bar{y} - y(0) + 2\bar{x} = \frac{1}{s}$$

where $\bar{x} = L(x), \bar{y} = L(y)$.

Since x(0) = 0, y(0) = 0, we have

$$3s\bar{x} + s\bar{y} + 2\bar{x} = \frac{1}{s}.$$

$$\text{i.e.,} \; (3s+2)\bar{x} + s\bar{y} = \frac{1}{s}$$

Equation (2) becomes

L (x') + 4 L (y') + 3L (y) = 0

$$\text{i.e.,} \; s\bar{x} - x(0) + 4\{s\bar{y} - y(0)\} + 3\bar{y} = 0$$

$$\text{i.e.,} \; s\bar{x} + (4s+3)\bar{y} = 0$$

Solving (3) and (4), we get

$$\bar{x} = \frac{4s+3}{s(s+1)(11s+6)}$$

$$\bar{y} = -\frac{1}{(11s+6)(s+1)}$$

$$\therefore x = L^{-1} = \left\{\frac{4s+3}{s(s+1)(11s+6)}\right\}$$

$$=L^{-1}\left(\frac{\frac{1}{2}}{s}-\frac{\frac{1}{5}}{s+1}-\frac{\frac{33}{10}}{11s+6)}\right)$$

$$=\frac{1}{2}L^{-1}\left(\frac{1}{s}\right)-\frac{1}{5}L^{-1}\left(\frac{1}{s+1}\right)-\frac{33}{10}L^{-1}\left(\frac{1}{11s+6)}\right)$$

$$=\frac{1}{2}-\frac{1}{5}e^{-t}-\frac{3}{10}e^{-\frac{6t}{11}}$$

$$y=L^{-1}\left\{\frac{-1}{(11s+6)(s+1)}\right\}$$

$$=L^{-1}\left\{\frac{1}{5}.\frac{1}{s+1}-\frac{11}{5}.\frac{1}{11s+6}\right\}$$

$$=\frac{1}{5}L^{-1}\left(\frac{1}{s+1}\right)-\frac{11}{5}.\frac{1}{11}L^{-1}\left(\frac{1}{s+\frac{6}{11}}\right)$$

$$=\frac{1}{5}e^{-t}-\frac{1}{5}e^{-\frac{6t}{11}}.$$

Ex.2. Solve the simultaneous equations.

$$\frac{dx}{dt}-\frac{dy}{dt}-2x+2y=1-2t \qquad \text{... (1)}$$

$$\frac{d^2x}{dt^2}+2\frac{dy}{dt}+x=0 \qquad \text{…(2)}$$

With the conditions x = 0, y = 0, $\frac{dx}{dt}=0$ when t = 0

Soln:

Applying Laplace transforms to both the equations, equation(1) becomes

$L(x'')-L(y')-2L(x)+2L(y)=L(1)-L(2t)$

i.e., $s\bar{x}-x(0)-s\bar{y}+y(0)-2\bar{x}+2\bar{y}=\frac{1}{s}-\frac{2}{s^2}$

i.e., $s\bar{x}-s\bar{y}-2\bar{x}+2\bar{y}=\frac{1}{s}-\frac{2}{s^2}$

i.e., $(s-2)\bar{x}-(s-2)\bar{y}=\frac{s-2}{s^2}$

$$\bar{x}-\bar{y}=\frac{1}{s^2} \qquad \ldots (3)$$

Equation (2) becomes

L (x″) + 2L (y')+ L (x) = 0

i.e., $s^2\bar{x}-sx(0)-x'(0)+2s\bar{y}-2y(0)+\bar{x}=0$

i.e., $s^2\bar{x}+2s\bar{y}+\bar{x}=0$

i.e., $(s^2+1)\bar{x}+2s\bar{y}=0 \qquad \ldots (4)$

Solving equations (3) and (4) for $\bar{x}$ and $\bar{y}$, we get

$$\bar{x}=\frac{2}{s(s+1)^2},\ \bar{y}=-\frac{s^2+1}{s^2(s+1)^2}$$

$$\therefore x=L^{-1}\left[\frac{2}{s(s+1)^2}\right]$$

$$=2L^{-1}\left[\frac{1}{s}-\frac{1}{s+1}-\frac{1}{(s+1)^2}\right]$$

$$=2\left\{L^{-1}\left(\frac{1}{s}\right)-L^{-1}\left(\frac{1}{s+1}\right)-L^{-1}\left[\frac{1}{(s+1)^2}\right]\right\}$$

= 2 (1 – e^{-t} – t e^{-t}).

$$y=-L^{-1}.\left\{\frac{s^2+1}{s^2(s+1)^2}\right\}$$

$$=-\int_0^t L^{-1}\left\{\frac{s^2+1}{s(s+1)^2}\right\}dt$$

$$=-\int_0^t L^{-1}.\left\{\frac{1}{s}-\frac{2}{(s+1)^2}\right\}dt$$

$$=-\int_0^t (1-2te^{-t})\,dt$$

$= 2 - t - 2\,(t+1)\,e^{-t}$.

Laplace transform can be used to solve differential equations with variable coefficients.

We have shown that

$$L\{t\,f(t)\}=-\frac{d}{ds}L\{f(t)\}$$

and $\{t^2\,f(t)\}=(-1)^2\dfrac{d^2}{ds^2}L\{f(t)\}$.

These results are used to solve equations containing variable coefficients.

The following worked out examples will illustrate the method.

Examples.

Ex.1. Solve the equation

$$t\frac{d^2y}{dt^2}-(2+t)\frac{dy}{dt}+3y=t-1\,when\,y(0)=0.$$

Soln:

Taking Laplace transforms on both sides, we have

L (ty″) – L {(2+t) y'} + 3L (y) = L (t-1)

i.e., $-\frac{d}{ds}\{s^2 L(y) - sy(0) - y'(0)\}$

$$-2\{sL(y) - y(0)\} + \frac{d}{ds}\{sL(y) - y(0)\} + 3L(y) = \frac{1}{s^2} - \frac{1}{s}.$$

Let $L(y)$ *be* $\bar{y}$.

Putting y(0) = 0,

$$-\frac{d}{ds}\{s^2\,\bar{y} - y'(0)\} - 2(s\,\bar{y}) + \frac{d}{ds}(s\,\bar{y}) + 3\bar{y} = \frac{1-s}{s^2}$$

i.e., $-s^2\frac{d\,\bar{y}}{ds} - 2s\,\bar{y} - 2s\bar{y} + s\frac{d\,\bar{y}}{ds} + \bar{y} + 3\,\bar{y} = \frac{1-s}{s^2}$

i.e., $-(s^2 - s)\frac{d\,\bar{y}}{ds} - 4(s-1)\,\bar{y} = -\frac{s-1}{s^2}$

i.e., $\frac{d\,\bar{y}}{ds} + \frac{4\,\bar{y}}{s} = \frac{1}{s^3}$

Solving this equation, $\bar{y} = \frac{1}{2}.\frac{1}{s^2} + \frac{c}{s^4}.$

$$\therefore y = \frac{1}{s}L^{-1}\left(\frac{1}{s^2}\right) + c\,L^{-1}\left(\frac{1}{s^4}\right)$$

$$= \frac{t}{2} + \frac{c\,t^3}{6}.$$

Hence $y = \frac{t}{2} + At^3$, Where A is an arbitrary constant.

Ex.2. Solve the equation. $\frac{d^2y}{dt^2} + t\frac{dy}{dt} - y = 0$ *if* $y(0) = 0$ *and* $y'(0) = 1.$

Soln:

Taking Laplace transforms on both sides, we have

L (y″) + L (t y') – L (y) = 0

i.e., $s^2 L(y) - sy(0) - y'(0) - \frac{d}{ds}\{sL(y) - y(0)\} - L(y) = 0$

y(0) = 0, y' (0) = 1.

Putting $L(y) = \bar{y}$, we get $s^2\,\bar{y} - 1 - \frac{d}{ds}(s\,\bar{y}) - \bar{y} = 0$

i.e., $s^2\,\bar{y} - 1 - s\frac{d\,\bar{y}}{ds} - \bar{y} - \bar{y} = 0$

i.e., $s\frac{d\,\bar{y}}{ds} - (s^2 - 2)\,\bar{y} + 1 = 0$

Solving this equation $\bar{y}\,s^2\,e^{-s^2/2} = e^{-s^2/2} + c.$

$\therefore \bar{y} = \frac{1}{s^2} + c\frac{e^{-s^2/2}}{s^2}.$

$\bar{y}$ is a Laplace transform.

$\therefore \lim_{s\to\infty} \bar{y} = 0, \therefore c = 0.$

Hence $\bar{y} = \frac{1}{s^2}$

Taking inverse transform y = t.

Certain equations involving integrals can also be solved by Laplace transform

Example. Determine y which satisfies the equation

$$\frac{dy}{dt} + 3y + 2\int_0^t y\,dt = t \text{ for which } y(0) = 0.$$

Soln:

Taking Laplace transforms on both sides, we get

$$L(y')+3L(y)+2L(\int_0^t y\,dt)=L(t)$$

i.e., $sL(y)-y(0)+3L(y)+\frac{2}{s}L(y)=\frac{1}{s^2}$

Putting L (y) = $\bar{y}$ and substituting y(0) = 0, we have

$$s\bar{y}+3\bar{y}+\frac{2}{s}\bar{y}=\frac{1}{s^2}$$

i.e., $\bar{y}\left(s+3+\frac{2}{s}\right)=\frac{1}{s^2}$

i.e., $\bar{y}\left(s+3+\frac{2}{s}\right)=\frac{1}{s^2}$

$$\therefore \bar{y}=\frac{1}{s(s+1)(s+2)}$$

$$\therefore \bar{y}=L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\}$$

$$y=L^{-1}\left\{\frac{1}{2}.\frac{1}{s}-\frac{1}{s+1}+\frac{1}{2}.\frac{1}{s+2}\right\}$$

$$=\frac{1}{2}L^{-1}\left(\frac{1}{s}\right)-L^{-1}\left(\frac{1}{s+1}\right)+\frac{1}{2}L^{-1}\left(\frac{1}{s+2}\right)$$

$$=\frac{1}{2}-e^{-t}+\frac{1}{2}e^{-2t}.$$

UNIT – 5 : PARTIAL DIFFERNTIAL EQUATIONS

Partial differential equations of first order – formation – different kinds of solution – four standard forms- Lagranges method.

1.1 PARTIAL DIFFERNTIAL EQUATIONS

Partial differential equations are those which involve one or more parital derivatives. The order of a partial differnential equation is determined by the highest order of the partial derivativeoccuring in it. For the present, we shall restrict ourselves to partial differential equations involving one dependent variable z and only two independant variables x and y. In what follows, we shall donote

$$\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q,\ \frac{\partial^2 z}{\partial x^2} = r,\ \frac{\partial^2 z}{\partial x \partial y} = s \text{ and } \frac{\partial^2 z}{\partial y^2} = t.$$

Derivation of partial differential equations.

Parrial differential equations can be derived either by the elimination of (1) arbitrary constants from a relation between x, y, z

Or (2) of arbitrary function of these variables.

By elimination of arbitrary constants.

Consider the function

$$f(x,y,z,a,b)=0 \qquad .... (1)$$

Containing two independent arbitrary constants a and b. To eliminate two constants, we require three equations. Differentiating equation (1) partially with respect to x and y in turn, we obtain

$$\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z} = 0 \qquad .... (2)$$

$$\frac{\partial f}{\partial y} + p\frac{\partial f}{\partial z} = 0 \qquad .... (3)$$

Eliminating a and b, we get a partial differential equation of the first order of the form F(x, y,z,p,q) = 0.

Examples.

Ex. 1. Eliminate a and b form z = (x+a) (y+b).

Soln:

Differentiating partially with repect to x

$$\frac{dz}{dx} = y + b$$

Ie) p=y+b

Again ,Differentiating partially with repect to y

$$\frac{dz}{dy} = x + a$$

p = y+b, q = x+a.

Eliminating Differentiating partially with repect to x a and b , we get z = pq

Ex.2. Obtain the partial differential equation of all spheres whose centres lie on the plane z =0 and whose radius is constant and equal to r.

Soln:

The Cartesian equation of all such spheres can be written in the form

$$(X- a)^2 + (y-b)^2 + z^2 = r^2 \quad \ldots\ldots\ldots\ldots..(1)$$

Where a and b are independent arbitrary constants and r is the fixed given constant.

Differentiating (1) partially with respect to x and y in turn,we obtain

2(x-a)+2z (dz/dx)=0

$$(x-a) + pz = 0 \quad \ldots\ldots\ldots\ldots(2)$$

$$(y-b) + qz = 0 \quad \ldots\ldots\ldots\ldots(3)$$

Eliminating a and b between equations (1), (2) and (3), we obtain

$$Z^2 (p^2 + q^2 + 1) = r^2$$

By the elimination of arbitrary functions.

Let u and v be any two functions of x,y,z and connected by arbitrary relation

$$\phi(u,v) = 0 \quad \ldots\ldots\ldots\ldots.(1)$$

Differentiating equation (1) partially with respect to x and y in turn,we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0 \qquad \ldots\ldots\ldots..(2)$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0 \qquad \ldots\ldots\ldots..(3)$$

Eliminating the ratio $\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}\right)$ between equations (2) and (3),we get a partial differential equation of the first order, viz.,

Pp + Qq = R ……….(4)

Where

$$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\equiv\frac{\partial(u,v)}{\partial(y,z)}.$$

$$Q=\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\equiv\frac{\partial(u,v)}{\partial(z,x)}.$$

$$R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\equiv\frac{\partial(u,v)}{\partial(x,y)}.$$

Equation (4) is known as Lagrange's linear equation.

Examples:

Ex. 1. Elimiate the arbitrary function from $z = f(x^2+y^2)$

Soln:

$z = f(x^2+y^2)$ .... (1)

Differentiating partially with respect to x and y

$p = f'(x^2+y^2)2x$ .... (2)

$q = f'(x^2+y^2)2y$ .... (3)

Eliminating $f'(x^2+y^2)$ from (2) and (3), we get py = qx.

Ex. 2 Eliminate the arbitrary functions f and ϕ from the relation Z = f(x+ay) + ϕ (x-ay).

Soln:

Given Z = f(x+ay) + ϕ (x-ay). …………………..(1)

Differentiating partially with respect to x and y

$p = f'(x+ay)+\phi'(x-ay)$ .... (2)

$q = af'(x+ay) - a\phi'(x-ay)$ .... (3)

Differentiating these again, with respect to x an y

$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay)$ .... (4)

$\frac{\partial^2 z}{\partial x^2} = a^2 f''(x+ay) + a^2\phi''(x-ay)$ .... (5)

From (4) and (5), we get $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$

Excersises

1. Obtain a partial differential equation by eliminating a,b from each of the following :-

i) Z = ax +by +a

ii) $\frac{x^2+y^2}{a^2} + \frac{z^2}{b^2} = 1$

Differential integrals of partial differential equations.

A solution or integral of a partial differential equation is a relation between the dependent and the independent variables that satisfies the differential equation. It will be noted that two types of solutions may occur as solutions of the same equation. For example, consider the equations

Z = ax + by …..(1)

and $z = xf(\frac{y}{x})$ …..(2)

If we eliminate the arbitrary constants a and b from the equation (1) and the arbitrary function from the equation (2),we get the same differential equation xp + yq =z

Hence z = ax + by and $z = xf(\frac{y}{x})$ are solutions of the equation

Xp + yq= z.

A solution containing as many arbitrary constants as there are independent variables is called a **complete integral**.

A solution obtained by giving particular values to the arbitrary constants in a complete integral is called **particular integral.**

Singular Integral.

Let F (x, y, z, p, q) = 0 ……(1)

be the partial equation whose complete integral is

ϕ(x, y, z, a,b)= 0 …..(2)

The eliminant of a, between

ϕ (x, y, z,a, b) = 0

$$\frac{\partial\phi}{\partial a} = 0$$

$$\frac{\partial\phi}{\partial b} = 0$$

When it exists ,is called the singular integral.

Geometrically, this represents the envelope of the two parameter surfaces represented by the complete integral (2) of (1).The two parameters occurring in (2) are a and b.

General Integral

In (2), we shall assume an arbitrary relation of the form b= f(a). Then (2) becomes

f[x, y, z, a f(a) = 0 …….(3)

Differentiating (2), partial with respect to a,

$$\frac{\partial\phi}{\partial a} + \frac{\partial\phi}{\partial b} f'(a) = 0 \qquad \text{………(4)}$$

The eliminant of a between these two equations (3) and(4), if it exists, is called the general integral of (1).

Solution of partial differential equation in some simple cases.

We shall consider a number of simple exaples, the solutions of which depend only on the meaning of the partial differentiation.

Examples.

Ex. 1 Slove $\dfrac{\partial^2 z}{\partial y\,\partial y} = 0.$

Soln:

$$\frac{\partial^2 z}{\partial y\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

Hence $\frac{\partial z}{\partial y} = f(y)$ whenf is an arbitrary function.

$$\therefore z = \int f(y)\,dy + \varphi(x)$$

$$= F(y) + \phi(x).$$

Here F(y) and ϕ(x) are arbitrary functions.

Ex. 2 Solve $\frac{\partial^2 z}{\partial y^2} = \sin y$.

Soln:

$$\frac{\partial^2 z}{\partial y^2} = \sin y$$

Integrating on,

$$\frac{\partial z}{\partial y} = -\cos y + f(x).$$

Again integrating,

$\therefore$ Z = -siny + yf(x) + ϕ(x), where f and ϕ are arbitrary functions.

Ex.3 Slove $x + y\frac{\partial z}{\partial x} = 0$.

Soln: $x + y\frac{\partial z}{\partial x} = 0$

$$\frac{\partial z}{\partial x} = -\frac{x}{y}$$

Integrating on,

$$\therefore\ z = -\frac{x^2}{2y} + \phi(y).$$

Ex.4 Solve $\frac{\partial^2 z}{\partial x\,\partial y} = x^2 + y^2$

Soln:

$$\frac{\partial^2 z}{\partial x \partial y} = x^2 + y^2$$

Integrating on,

$$\frac{\partial z}{\partial x} = x^2 y + \frac{y^3}{3} + \phi(x).$$

Again Integrating on,

$$z = \frac{x^3 y}{3} + \frac{xy^2}{3} + F(x) + f(y)$$

Ex. 5 Solve $x\frac{\partial \mathrm{z}}{\partial \mathrm{x}} = 2\mathrm{x} + \mathrm{y} + 3\mathrm{z}$

Soln:

$$x\frac{\partial \mathrm{z}}{\partial \mathrm{x}} = 2\mathrm{x} + \mathrm{y} + 3\mathrm{z}$$

Dividing by x,

$$\frac{\partial \mathrm{z}}{\partial \mathrm{x}} = 2 + \mathrm{y/x} + 3\mathrm{z/x}$$

The equation can be written in the forms

$$\frac{\partial z}{\partial x} - 3\frac{z}{x} = 2 + \frac{y}{x}$$

This is a linear equation.

The integration factor is $1/x^3$

$$\text{Hence } \frac{1}{x^3}\left(\frac{\partial z}{\partial x} - 3\frac{z}{x}\right) = \frac{2}{x^3} + \frac{y}{x^4}$$

$$\text{i.e., } \frac{\partial}{\partial x}\left(\frac{z}{x^3}\right) = \frac{2}{x^3} + \frac{y}{x^4}.$$

$$\therefore \frac{z}{x^3} = -\frac{1}{x^2} - \frac{y}{3x^3} + \phi(y)$$

$$i.e., z = -x - \frac{y}{3} + x^3 f(y)$$

Ex.6 Solve $\frac{\partial^2 z}{\partial x^2} = a^2 z$ given that when x = 0.

$\frac{\partial z}{\partial x} = a \sin y$ and $\frac{\partial z}{\partial y} = 0$

If Z is a function of x alone, the solution would be

$Z = Ae^{ax} + Be^{-ax}$,where A and B constants.

Soln:

Here z is a function x and y; hence the solution of the equation is

$$z = f(y)e^{ax} + \phi(y)e^{-ax}$$

$$\frac{\partial z}{\partial x} = f(y)a\, e^{ax} - \phi(y)a\, e^{-ax}$$

$$\frac{\partial z}{\partial y} = f'(y)e^{ax} + \phi'(y)e^{-ax}$$

When x = 0, $\frac{\partial z}{\partial x} = a \sin y$

$\therefore$ a f(y) - a ϕ (y) = a siny

i.e., f(y) - ϕ(y) = sin y .... (1)

When x = 0 , $\frac{\partial z}{\partial y} = 0$

$\therefore$f′(y) + ϕ′(y) = 0 .... (2)

Differentiating (1), we get

f′(y) - ϕ′ (y) = cos y .... (3)

From (2) and (3), $f'(y) = \frac{1}{2}cosy, \phi'(y) = -\frac{1}{2}cosy.$

$\therefore f(y) = \frac{1}{2}siny + A,\ \phi'(y) = -\frac{1}{2}siny + B.$

But from (1), A = B.

Hence $z = \frac{1}{2}\sin y\, e^{ax} - \frac{1}{2}\sin y\, e^{-ax} + Ae^{ax} + Ae^{-ax}$

= sinysinhax + 2A coshax.

1.2 Standard types of first order equations

Many of the important equations of the order that occur in practice are one or the other of the following standard forms.

Hence $\frac{1}{x^3}\left(\frac{\partial z}{\partial x}-3\frac{z}{x}\right)=\frac{2}{x^3}+\frac{y}{x^4}$

Standard 1. The variable x,y,z do not occur explicitly. Such equations are of the form f(p,q) = 0 where

$$p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y}$$

We can easily verify that z = ax+ by + c is a solution of the equationf(p,q)= 0 provided f(a,b) = 0.

Solving this for b, b = F (a)

Hence the complete integral is z = ax + y F(a) +c.

This singular integral is obtained by eliminating a and c between

z = ax + y F(a) + c

0 = x + y F′ (a)

0 = 1.

The last equation is absurd and shows that there is no singular integral in this case.

To obtain the general integral, we assume an orbitrary relation c = ϕ (a). Then z = ax + y F (a) + ϕ (a).

Differentiating partially with respect to a,

0 = x + y F′ (a) + ϕ′ (a).

The eliminant of a between these equations is the general integral.

Note :- The singular and general integral must be indicated besides the complete integral in every equation. Then only it is said to be completely solved.

Example. Solve $p^2+q^2 = npq$.

The solution is z = ax + by + c, where $a^2+b^2 = nab$.

Solving $b=\frac{a(n\pm\sqrt{(n^2-4)}}{2}$

The complete integral is

$$y = ax + \frac{ay}{2}(n \pm \sqrt{(n^2 - 4)} + c$$

Differentiating partially with respect to c, we see that there is no singular integral, as we get an absurd result.

To find the general integral put c = f (a), where f is arbitrary.

$$z = ax + \frac{ay}{2}(n \pm \sqrt{(n^2 - 4)}) + f(a).$$

Differentiating partially with respect to a,

$$0 = x + \frac{y}{2}(n \pm \sqrt{(n^2 - 4)} + f'(a)$$

The eliminant of a between these equations gives the general integral.

Excercies

Solve the following equations :-

1. $p^2+q^2 = 4$.
2. $p = q^2$.
3. $pq = 1$.
4. $pq+p+q = 0$.
5. $q^2-3q+p = 2$.
6. $3p^2-2q^2 = 4pq$
7. $p^3-q^3 = 0$.
8. $p + q = 3/a$

Standard form 2. Only one of the variables x,y,z occurs explicity. Such equations can be written in one the forms

F(x, p, q) = 0, F(y, p, q) = 0, F(z, p, q)= 0

(i) Let us consider the form F (x, p, q) = 0

Since z is a function x and y.

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

= p dx + qdy.

Let us assume that q = a

The equation becomes F(x, p, a) = 0

Sloving this for p, we get $p = \phi(x, a)$

$\therefore dz = \phi(x,a)dx + a\,dy$.

$$z = \int \phi(x,a)dx + ay + b.$$

This consists of two arbitrary constants a and b and hence it is a complete integral.

(ii) Let us consider the form F (y, p,q) = 0

Let us assume that p = a

$\therefore$F(y, a, q) = 0.

$\therefore$q = ϕ(y,a)

Hence dz = adx + ϕ(y,a) dy

$\therefore z = ax + \int \phi(y\ a)dy + b$, Which is complete integral.

(iii) Let us consider the equation F(z,p,q) = 0

Let us assume that q = ap.

Then the equation become F (z,p,ap) = 0

i.e, p = ϕ (z,a)

Hence dz = ϕ (z,a) dx + a ϕ (z,a) dy

i.e, $\frac{dz}{\phi(z,a)} = x + ay + ady$

i.e., $\int \frac{dz}{\phi(z,a)} = x + ay + b$ which is a complete integral.

Examples.

Solve

(i) $q = xp + p^2$

(ii) $p = y^2q^2$

(iii) $p(1+q^2) = q(z-1)$

(i) $q = xp + p^2$

Soln:

Let q = a

Then $a = xp + p^2$

i.e. $p^2 + xp - a = 0$.

$$p = \frac{-x \pm \sqrt{(x^2 + 4a)}}{2}$$

Hence $dz = \frac{-x \pm \sqrt{(x^2 + 4a)}}{2} dx + ay + b$

$$\therefore \int = \frac{-x \pm \sqrt{(x^2 + 4a)}}{2} dx + ay + b$$

$$= \frac{-x^2}{4} \pm \left\{ \frac{x}{4}\sqrt{(4a+x^2)} + a\sinh^{-1}\left(\frac{x}{2\sqrt{a}}\right)\right\} + ay + b.$$

(ii) $p = y^2q^2$

Soln:

Let $p = a^2$

$\therefore$ $q = \pm a/y$

Hence $dz = a^2\, dx \pm (a/y)\, dy$

$\therefore$ $z = a^2x \pm a \log y + b.$

(iii) $p\,(1+q^2) = q\,(z-1).$

Soln:

Let $q = ap$.

Then $p(1 + a^2p^2) = ap\,(z-1)$

ie., $1+a^2p^2 = a\,(z-1)$

$$p = \pm\frac{\sqrt{(az-a-1)}}{a}.$$

Hence $dz = \pm\dfrac{\sqrt{(az-a-1)}}{a}dx \pm \sqrt{az-a-1}\,dy$

i.e., $\pm\dfrac{a\,dz}{\sqrt{(az-a-1)}} = dx + a\,dy$

i.e., $\pm\displaystyle\int \frac{a\,dz}{\sqrt{(az-a-1)}} = x + a\,y + b$

i.e., $\pm 2\sqrt{(az-a-1)} = x + a\,y + b.$

Excersies

Solve the following equations :-

1.p = 2qx

2. q = 2yp2

3. $9(p^2z + p^2) = 4.$

4.p (1+q) = qz

Standard form 3

Equations of the form $f_1(x,p) = f_2(y, q)$.

In this form the equations is of the first order and the variables are separable. In the equations z does not appear. We shall assume as a tentative solution that each of these quantities is equal to a.

$f_1(x,p) = a$, Solving $p = \phi_1(a, x)$.

$f_2(y,q) = a$, Solving $q = \phi_2(a, y)$.

Hence $dz = \phi_1(a,x)dx + \phi_2(a,y)\,dy$.

$\therefore z = \int \phi_1(a,x)dx + \int \phi_2(a,y)dy + b$ which is a complete integral.

Example. Solving the equation p +q = x + y.

Soln:

We can write the equation in the form p-x = y-q.

Let p-x = a. Then y-q = a.

Hence p = x+a. q = y-a

$\therefore dz = (x+a)\,dx + (y-a)\,dy$.

$$\therefore z = \frac{(x+a)^2}{2} + \frac{(y-a)^2}{2} + b.$$

There is no singular integral and the general integral is found as usual.

Standard 4. Clairant's form

This is of the form z = px + qy + f(p,q).

The solution of the equation is z = ax + by + f(a,b) for p = a and q = b can easily be verified to satisfy the given equations.

Example. Solve $z = px\,qy + \sqrt{(1 + p^2 + q^2)}$

The complete integral is obviously

$$z = ax + by\sqrt{(1 + a^2 + b^2)}\,.$$

To find the singular integral, differentiating partially with respect to a and b,

$x+\frac{a}{\sqrt{1+a^2+b^2}}=0$ and $y+\frac{b}{\sqrt{1+a^2+b^2}}=0$.

Eliminating a and b the singular integral is $x^2+y^2+z^2 = 1$.

To find the general integral, assume b = f (a), where f is arbitrary.

Then $z = ax+ f(a) y + \{1+ a^2 + (f (a)^2)\}^{1/2}$

Differntiating partially with respect to a and eliminate a between the two eqations.

Excersies 26

Solve the following equations:-

1. z = px +qx + pq
2. $z = px + qy + 2\sqrt{pq}$
3. $z = px + qy + \frac{p}{q} - p$.

1.3 LAGRANGE'S EQUATION.

We have shown that, if we eliminate the arbitrary function F from the relation F(u,v) = 0, where u and v are functions of x,y,z. we get the equations

$$\frac{\partial(u,v)}{\partial(y,z)}p+\frac{\partial(u,v)}{\partial(z,x)}q=\frac{\partial(u,v)}{\partial(x,y)}.$$

This is expressed in the form Pp + Qq = R, where P,Q and R are function of x,y,z.

This partial differential equation is known as Lagrange's equation.

In the following article we shall try to find the solution of the equation Pp + Qq = R.

The general solution of the partial differential equation Pp + Qq = R is F (u,v) = 0 where F is an arbitrary function and u (x,y,z) = C_1 and v (x,y,z) = C_2 from two independent solutions of the equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

Taking total differential on the equation u (x,y,z) = C_1 we get

$$\frac{\partial u}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial u}{\partial z}dz=0.$$

Since u(x,y,z) = C_1 is a solution of the equation

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$P\frac{\partial u}{\partial x}+Q\frac{\partial u}{\partial y}+R\frac{\partial u}{\partial z}=0. \qquad \text{............ (1)}$$

Similarly, $v(x,y,z) = C_2$ is a solution of the equation

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$P\frac{\partial v}{\partial x}+Q\frac{\partial v}{\partial y}+R\frac{\partial v}{\partial z}=0. \qquad \text{............ (2)}$$

From equation (1) and (2), we get

$$\frac{P}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}=\frac{Q}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}=\frac{R}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}} \qquad \text{.............(3)}$$

$$i.e., \frac{P}{\frac{\partial(u,v)}{\partial(y,z)}}=\frac{Q}{\frac{\partial(u,v)}{\partial(z,x)}}=\frac{R}{\frac{\partial(u,v)}{\partial(x,y)}}$$

We have shown that the elimination of the arbitrary function F from the equation F(u,v) = 0, where u and v are functions of x,y,z leads to the partial differential equation

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)}=\frac{\partial(u,v)}{\partial(x,y)} \qquad \text{.......... (4)}$$

Substituting from equation (3) in (4), we get the equation

$pP + qQ = R$ .... (5)

Hence we see that F(u,v) = 0 is a solution of the equation (5).

If $u = c_1$ and $v = c_2$ are the solutions of $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

Examples.

Ex. 1 Solve $(y^2 + z^2)p - xyq = -xz$

Soln:

This equation can be written as the form

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

Where P=$(y^2 + z^2)$ **, Q = -xy and R = -xz.**

The auxiliary equations are

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = -\frac{dz}{xz} \qquad \text{.......... (1)}$$

Taking the last two equations, we get $\frac{dy}{y} = \frac{dz}{z}$

Integrating we get log y = log z + constant.

$$\therefore \frac{y}{z} = c_1$$

Each of the equations (1) is equal to

$$\frac{xdx + ydy + zdz}{x(y^2+z^2) - xy^2 - xz^2}$$

$$i.e., \frac{xdx + ydy + zdz}{0}.$$

$$\therefore xdx + ydy + zdz = 0.$$

Hence after integration this reduces $x^2 + y^2 + z^2 = c_2$.

Hence the general solution of the equation is

$$F\left(\frac{y}{z}, x^2 + y^2 + z^2\right) = 0.$$

Ex.2 Find the general solution of (y+z)p + (z+x)q = x+y

Soln:

This equation can be written as the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Where P= y + z **, Q = z+x and R = x+y**

The auxiliary equations are $\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$

Each is equal to $\frac{dx+dy+dz}{2(x+y+z)} = \frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$

Taking the first two, we get after integration

$\frac{1}{2}$ log(x+y+z) = -log (y-x) + constant

$\therefore$ $(x+y+z)(y-x)^2 = C_1$

By taking the last two, we get

- log (y-x) = -log (z-y) + constant.

$$\frac{z-y}{y-x} = c_2$$

Hence the general solution of the equation is

$$F\left\{(x+y+z)(y-x)^2, \frac{z-y}{y-x}\right\} = 0$$

Ex. 3 Solve $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = (x+y)z$

Soln: This equation can be written as the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Where P= x^2 **, Q =** y^2 **and R =(x+y)z.**

The auxiliary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

$$i.e., \frac{dx-dy}{x^2-y^2} = \frac{dz}{(x+y)z}$$

$$i.e., \frac{dx-dy}{x-y} = \frac{dz}{z}$$

i.e., log(x-y) = log z + constant.

$$\therefore \frac{x-y}{z} = c_1$$

Also $\frac{dx}{x^2} = \frac{dy}{y^2}$

Hence -(1/x) = -(1/y) + constant.

(1/y)-(1/x) = c_2

Hence the solution is $F\left(\frac{1}{y} - \frac{1}{x}, \frac{x-y}{z}\right) = 0$.

Ex.4 Find the equation of the cone satisfying the equation xp + yq = z and passing through the circle $x^2 + y^2 + z^2 = 4$,

Soln: Soln:

This equation can be written as the form

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Where P= x **, Q = y and R = z.**

The auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Hence two independent solution of the equations are

$\frac{x}{y} = a$ ----------- (i) and $\frac{y}{z} = b$ ------------ (ii)

$\therefore$ The general solution of the equation is $F\left(\frac{x}{y}, \frac{y}{z}\right) = 0$.

Here we have to find a functional relation between $\frac{x}{y}$ and $\frac{y}{z}$ such that they also satisfy the equations

$x^2+y^2+z^2 = 4$ ------- (iii) and x+y+z = 2 ---------- (iv)

Hence eliminate x,y,z from (i), (ii), (iii) and (iv)

$$y = \frac{x}{a}, z = \frac{y}{b} = \frac{x}{ab}$$

Substituting values of y,z in terms of x in (iii) and (iv), we get

$$x^2 + \frac{x^2}{a^2} + \frac{x^2}{a^2b^2} = 4 \quad \text{i.e., } x^2\left(1 + \frac{1}{a^2} + \frac{1}{a^2b^2}\right) = 4.$$

$$x + \frac{x}{a} + \frac{x}{ab} = 2, \qquad \text{i.e., } x\left(1 + \frac{1}{a} + \frac{1}{ab}\right) = 2.$$

$$\therefore 1 + \frac{1}{a^2} + \frac{x}{a^2b^2} = \left(1 + \frac{1}{a} + \frac{1}{ab}\right)^2$$

$$\text{i.e., } \frac{2}{a} + \frac{2}{ab} + \frac{2}{a^2b} = 0$$

i.e., ab + a+ 1 = 0

Now if we replace a by (x/y) and b by (y/z), we get the required surface

xy + yz + zx = 0.

Ex. 5 Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.

Soln:

The subsidiary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

$$\frac{dx - dy}{x^2 - yz - (y^2 - zx)} = \frac{d(x-y)}{(x-y)(x+y+z)}$$

$$= \frac{d(y-z)}{(y-z)(x+y+z)}$$

$$\therefore \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

$$\therefore \frac{x-y}{y-z} = c_1 \qquad \text{............ (1)}$$

Using multipliers x,y,z each of the subsidiary equations

$$= \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\text{and is also equal to } \frac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - xy}$$

Taking the last two ratios $\sum x\,dx = (\sum x) d (\sum x)$.

Integrating, $\frac{(x+y+z)^2}{2} - \frac{(x^2+y^2+z^2)}{2} = c_2$

$\therefore$ xy + yz +zx = c_2 …………..(2)

From (1) and (2), $= (\frac{x-y}{y-z}, xy + yz + zx) = 0$,where f is arbitrary.

Note. We should not take the second solution as $\frac{x-z}{z-z} = c_2$ since each of the subsidiary equations.

$$= \frac{d(z-x)}{(z-x)(x+y+z)} -$$

But this solution is not independent of the first solution $\frac{x-y}{z\ -x} = c_1$, since $\frac{x-y}{y-z} + 1 = c_1 + 1$

gives $\frac{x-y}{y-z} = c_1 + 1$ which is merely the second solution.

Excersices 28

Solve given $p = \frac{\partial z}{\partial x}$; $q = \frac{\partial z}{\partial y}$:

1. xp + yq = z.
2. xp - yq = xy
3. ap + bq + cz = 0

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