UNIT-I

Survey of the Elementary Principles

1.1 Mechanics of a Particle

Let m be the mass of the given particle, o be the fixed origin and $\vec{r}$ be the radius vector of m with respect to 'O' at any instant of time't'.

The ***velocity*** of m with respect to 'O' is the rate of change of displacement.

(ie) velocity$\vec{v}=\frac{d\vec{r}}{dt}=\dot{\vec{r}}$

Acceleration is the rate of change of velocity.

(ie) acceleration a = $\frac{d\vec{v}}{dt}=\frac{d\dot{\vec{r}})}{dt}=\ddot{\vec{r}}$

Linear momentum of a particle

The ***linear momentum*** of a particle of mass 'm' with velocity 'v' is mv and it is denoted by $\vec{p}=m\vec{v}=m\dot{\vec{r}})$.

By Newton's second law of motion the rate of change of momentum of a body is proportional to the impressed force.

If $\vec{F}$ is the ***impressed force*** acting on a particle of mass 'm' then $\vec{F}$ is proportional to the rate of change of momentum.

(ie) $\vec{F}\ \alpha\ \frac{d\vec{p}}{dt}$

$\vec{F}$=k $\frac{d\vec{p}}{dt}$ where k is a constant.

By choosing suitable mass and time we get the constant k=1.

$\therefore \vec{F}=\frac{d\vec{p}}{dt}$

$=\frac{d}{dt}m\vec{v}$

$=m\frac{d\vec{v}}{dt}$

$=m\vec{a}.$

Conservation Theorem for the linear momentum of a particle

Statement:

If the total force, $\vec{F}$, is zero then, $\vec{p}$ is conserved.

Proof:

We know that,

$\vec{F} = \frac{d\vec{p}}{dt}$

If $\vec{F} = 0$, $\qquad \frac{d\vec{p}}{dt} = 0$

(ie) P is constant.

(ie) The linear momentum is conserved.

Angular momentum of a particle

The Angular momentum of a particle of mass ‘m’ with respect to a fixed point ‘O’ is defined to be $\vec{L} = \vec{r}$ x $\vec{p}$ where $\vec{p}$ is the linear momentum of the particle.

Moment of force (or) torque

The moment of a force $\vec{F}$ with respect to a fixed origin is defined as $\vec{N} = \vec{r}$ x$\vec{F}$

Prove that moment of force is the rate of change of angular momentum.

(ie) To Prove $\vec{N} = \frac{d\vec{L}}{dt}$.

Proof:

We have,

$\vec{L} = \vec{r}$ x $\vec{p}$

$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r}$ x $\vec{p})$

$= \frac{d\vec{r}}{dt}$ x $\vec{p} + \vec{r}$ x $\frac{d\vec{p}}{dt}$

$= \vec{v}$ x $\vec{p} + \vec{r}$ x $\vec{F}$

$= \vec{v}$ x m$\vec{v} + \vec{r}$ x $\vec{F}$

$=m(\vec{v} \times \vec{v}) + \vec{r} \times \vec{F}$

$= 0 + \vec{r} \times \vec{F}$

$= \vec{r} \times \vec{F}$

$= \vec{N}$

Hence $\vec{N} = \frac{d\vec{L}}{dt}$.

Conservation Theorem for the Angular Momentum of a Particle

Statement:

If the total torque, $\vec{N}$, is zero then $\dot{\vec{L}} = 0$, and the angular momentum $\vec{L}$ is conserved.

Proof:

We Know that,

$\vec{N} = \frac{d\vec{L}}{dt}$

If $\vec{N} = 0$.

$\frac{d\vec{L}}{dt} = 0$

$\vec{L}$ is constant

(ie) $\vec{L}$ is conserved.

Hence the angular momentum is conserved.

Work done

The work done by the external force $\vec{F}$ upon the particle in going from point 1 to point 2 is defined by $W_{12} = \int_1^2 \vec{F}\, d\vec{s}$ where $d\vec{s}$ corresponds to an infinitesimal displacement.

Prove that work done is equal to the change in the kinetic energy

Proof:

We Know that,

$$W_{12}=\int_1^2 \vec{F} d\vec{s}$$

$$= \int_1^2 \frac{d\vec{p}}{dt} d\vec{s}$$

$$=\int_1^2 \frac{d(m\overrightarrow{v})}{dt}\frac{d\vec{s}}{dt}dt$$

$$= m \int_1^2 \frac{d\overrightarrow{v}}{dt}\overrightarrow{v}\, dt$$

$$= m \int_1^2 dv.v$$

$$= m\int_1^2 \frac{1}{2}(dv^2)$$

$$=\frac{m}{2}\int_1^2 (dv^2)$$

$$= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

= kinetic energy at 2- kinetic energy at 1

Therefore $W_{12} =T_2-T_1$…………………………….. (1)

The scalar quantity $\frac{1}{2}mv^2$ is called the kinetic energy of the particle and its denoted by T.

A necessary and sufficient condition that W_{12} be independent of the physical part taken by the particle is $\vec{F}= -\nabla v(\vec{r})$ where v is called the potential energy.

We know that,

The work done in displacing a particle from point 1 to point 2 is given by

$W_{12}= \int_1^2 \overrightarrow{F}.d\vec{r}$ where r_1 and r_2 are the position vectors of the particle at the point 1 and point 2.

$$W_{12} = \int_{r_1}^{r_2} \vec{F}.\,d\vec{r} + \int_{r_0}^{r_2} \vec{F}.\,d\vec{r}$$

$$= \int_{r_1}^{r_0} \vec{F}.\,d\vec{r} - \int_{r_2}^{r_0} \vec{F}.\,d\vec{r}$$

$$W_{12} = v_1 - v_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

For the conservative system $T_2 - T_1 = v_1 - v_2$

$\therefore T_1 + v_1 = T_2 + v_2$

Hence if the force acting on a particle are conservative, then the total energy of the particle, T+V is conserved.

1.2 MECHANICS OF A SYSTEM OF PARTICLES

Consider a system of particles of masses $m_1, m_2, \ldots m_i \ldots$ with position vectors $\vec{r}_1, \vec{r}_2 \ldots \vec{r}_i \ldots$

Let us consider the forces acting on the particle of mass m_i.

Let $\vec{F}_{ji}$ be the internal force on the i[th] particle due to the j[th] particle.

The total internal force acting on a particle of mass m_i is $\sum_j \vec{F}_{ji}$.

Clearly $\vec{F}_{ii} = 0$ and $\vec{F}_{ji} = -\vec{F}_{ij}$

This implies $\vec{F}_{ij} + \vec{F}_{ji} = 0$

Let $\vec{F}_i^e$ be the external force acting on the particle of mass m_i.

$\therefore$ the total force acting on the particle of mass m_i is $\sum_j \vec{F}_{ji} + \vec{F}_i^e$

We have assumed that $\vec{F}_{ij}$ obey Newton's third law of motion, that the forces of two particles exert on each other are equal and opposite. This assumption is called as weak law of action and reaction.

The total force acting on the system of particles $= \sum_i \left(\sum_j \vec{F}_{ji} + \vec{F}_i^e \right)$

$$= \sum_{i,j} \vec{F}_{ji} + \vec{F}_i^e$$

$$= \sum_i \vec{F}_i^e$$

$$= \vec{F}^e$$

Center of mass

Let $\vec{R}$ be the average of the radii vectors of the particles, weighted in proportion to their mass:

$\vec{R} = \frac{\sum m_{i\vec{r}_i}}{\sum m_i}$

$= \frac{\sum m_{i\vec{r}_i}}{\sum M}$ where M = $\sum m_i$ *is* the total mass of the system.

The vector $\vec{R}$ defines a point known as the center of mass or center of gravity of the system.

Differentiate with respect to 't'

$$M \frac{d\vec{R}}{dt} = \sum m_i \frac{d\vec{r}_i}{dt}$$

$$= \sum m_i v_i$$

$$= \sum p_i$$

$$\therefore M\frac{d\vec{R}}{dt} = \vec{p}$$

$$M\frac{d^2\vec{R}}{dt} = \frac{d\vec{p}}{dt} = \dot{\vec{p}}$$

(ie) $M\frac{d^2\vec{R}}{dt^2} = \vec{F}^e$ where $\vec{F}^e$ is a total external force.

Conservation Theorem for the linear momentum of a system of particles

Statement:

If the total external forces zero, the total linear momentum is conserved.

Proof:

We know that,

$\dot{\vec{p}} = \vec{F}^{e,}$ where $\vec{F}^e$ is the total external force and p is the linear momentum.

If $\vec{F}^e = 0$,then $\dot{\vec{p}} = 0$

Hence p is conserved.

Conservation Theorem for total angular momentum:

Statement:

If $\vec{L}$ is the angular momentum of the system of particles, the $\vec{L}$ is constant in time if the applied torque is zero.

Proof:

The angular momentum of the system of particles is $\vec{L} = \sum_i \vec{r}_i \vec{p}_i$

$= \sum_i \vec{r}_i m_i \vec{v}_i$

$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\sum_i \vec{r}_i m_i \vec{v}_i)$

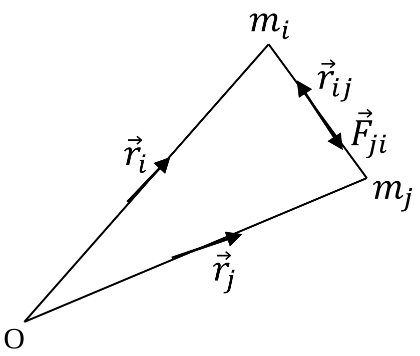
$= \sum_i \frac{d}{dt}\vec{r}_i \text{x} m_i \vec{v}_i + \sum_i \vec{r}_i \text{ x } \frac{d}{dt}(m_i \vec{v}_i)$

$= \sum_i \vec{v}_i \text{x } m_i \vec{v}_i + \sum_i \vec{r}_i \text{x} \frac{d}{dt}\vec{p}_i$

$= \sum_i (\overrightarrow{v_i \text{x} v_i}) + \sum_i \vec{r}_i \text{x} \dot{\vec{p}}_i$

$= \sum_i \vec{r}_i \text{x} \dot{\vec{p}}_i$

$= \sum_i \vec{r}_i \text{x } \vec{F}_i$



$= \sum_i \{\sum_j \vec{F}_{ji} + \vec{F}_i^e\}$ where $\sum_j \vec{F}_{ji}$ is the internal force on the i^th^ particle and $\vec{F}_i^e$ is the external force on the particle of mass m_i.

$= \sum_i \{\vec{r}_i \text{x} \sum_j \vec{F}_{ji}\} + \sum_i (\vec{r}_i \text{x } \vec{F}_i^e)$

$\frac{d\vec{L}}{dt} = \sum_{i,j} \vec{r}_i \text{x } \vec{F}_{ji} + \sum_i (\vec{r}_i \text{x } \vec{F}_i^e)$

Now,

$$\sum_{i,j} \vec{r}_i \text{x} \vec{F}_{ji} = \sum_{i,j} [\vec{r}_i \text{x} \vec{F}_{ji} + \vec{r}_j \text{x} \vec{F}_{ij}]$$

$$= \sum_{i,j} [\vec{r}_i \text{x} \vec{F}_{ji} + \vec{r}_j \text{x}(-\vec{F}_{ji})]$$

$$= \sum_{i,j} [\vec{r}_i \text{x} \vec{F}_{ji} - \vec{r}_j \text{x} \vec{F}_{ji}]$$

$$= \sum_{i,j} [(\vec{r}_i - \vec{r}_j) \text{x} \vec{F}_{ji}]$$

$$= \sum_{i,j} \vec{r}_{ij} \text{x} \vec{F}_{ji} = 0$$

If the internal forces between two particles in addition to being equal and opposite also lie along the line joining the particles, then all these cross products vanish, is known as the strong law of action and reaction.

There for,

$$\frac{d\vec{L}}{dt} = \sum_i(\overrightarrow{r_i} \times \overrightarrow{F}_i^e)$$

$$= \sum_i \overrightarrow{N}_i^e$$

$$= \overrightarrow{N}^e$$

(ie) The time derivative of the total angular momentum is equal to the moment of the external force about the given point.

If $\overrightarrow{N}^e = 0$, then $\frac{d\vec{L}}{dt} = 0$

∴ L is constant.

That is the angular momentum of the system of particles is constant.

Problem

Prove that the total angular momentum about a point 'o' is the angular momentum of the system concentrated at the center of mass plus the angular momentum of the motion about the center of mass.

Solution:

Let G be the center of mass, 'O' be the fixed point and $\vec{r}_i$ be the position vector of the particle mass m_i with respect to 'O'.

Let $\vec{r}_i'$ be the position vector of the particle of mass m_i with respect to the center of mass G and $\vec{R}$ be the position vector of G with respect to o.

We have,

$$\vec{r}_i = \vec{r}_i' + \vec{R}$$

$$\therefore \frac{d\vec{r}_i}{dt} = \frac{d}{dt}(\vec{r}_i' + \vec{R})$$

$$= \frac{d\vec{r}_i'}{dt} + \frac{d\vec{R}}{dt}$$

That is $\vec{v}_i = \vec{v}_i{'}$+V where $\vec{v}_i$is the velocity of the i^{th} mass. $\vec{v}_i{'}$ is the velocity of the i^{th} mass with respect to G and V is the velocity of G with respect to 'o'.

If I denote the total angular momentum of the system of particles then

$\overrightarrow{L} = \sum_i \vec{r}_i \, \vec{p}_i$ where $\vec{r}_i$ is the position vector and $\vec{p}_i$ is the linear momentum of the i^{th} particle.

$$\therefore \overrightarrow{L} = \sum_i \vec{r}_i \, \text{x} \, m_i \, \vec{v}_i$$

$$= \sum_i (\vec{r}_i{'} + \vec{R}) \text{ x } m_i(\vec{v}_i{'} + V)$$

$$= \sum_i m_i(\vec{r}_i{'} + \vec{R}) \text{ x } (\vec{v}_i{'} + V)$$

$$= \sum_i m_i(\vec{r}_i{'} \text{ x } \vec{v}_i{'} + \vec{r}_i{'} \text{ x v} + \vec{R} \text{ x } \vec{v}_i{'} + \vec{R}\text{x}V)$$

$$= \sum_i m_i(\vec{r}_i{'} \text{ x } \vec{v}_i{'}) + \sum_i m_i(\vec{r}_i{'} \text{ x v}) + \sum_i m_i(\vec{R} \text{ x } \vec{v}_i{'}) + \sum_i m_i(\vec{R} \text{ x } V)$$

$$= \sum_i m_i\vec{r}_i{'} \text{ x } \vec{v}_i{'} + \sum_i m_i\vec{r}_i{'} \text{ x v} + \vec{R}\sum_i m_i\vec{v}_i{'} + \vec{R}\text{x}V\sum_i m_i$$

$$= \sum_i \vec{r}_i{'} \text{ x } (m_i\vec{v}_i{'}) + \sum_i(m_i\vec{r}_i{'})\text{x v} + \vec{R}\sum_i m_i \frac{d\vec{r}_i{'}}{dt} + \overrightarrow{(R\text{x}V)}M$$

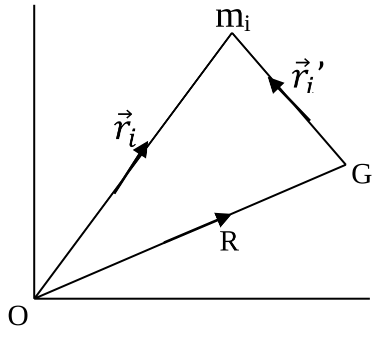
where $M = \sum_i m_i$

$$= \sum_i \vec{r}_i{'} \text{ x } (m_i\vec{v}_i{'}) + \sum_i(m_i\vec{r}_i{'})\text{x v} + \vec{R}\text{x}\frac{d}{dt}\sum_i m_i \, \vec{r}_i{'} + \overrightarrow{(R\text{x}V)}M$$

We Know That,

The radius vector of the center of mass with respect to the center of mass m=0.

$\therefore \frac{\sum m_{i\vec{r}_i{'}}}{\sum m_i} = 0$ this implies $\sum m_{i\vec{r}_i{'}} = 0$



$L = \sum_i \vec{r}_i{'} \text{ x } (m_i\vec{v}_i{'}) + \vec{R}\text{x } MV$

$= \sum_i \vec{r}_i{'} \text{ x } (\vec{p}_i{'}) + \vec{R}\text{x p}$

The first term in the R.H.S is the angular momentum of the system about the center of mass G, which is known as the angular momentum of the motion about the center of mass. The second term in the R.H.S is the angular momentum of the single mass m concentrated at the center of mass.

Find an expression for kinetic energy of the system of particles

(OR)

Prove that $T=\frac{1}{2}mv^2+\frac{1}{2}\sum_i m_i \vec{v}_i^{\,2}$

Solution:

Consider a particle of mass m_i with position vector $\vec{r}_i$ about 'O'.

Let G be center of mass.

∴We have,

$\vec{r}_i = \vec{r}_i{}' + \vec{R}$

$\therefore \frac{d\vec{r}_i}{dt} = \frac{d}{dt}(\vec{r}_i{}' + \vec{R})$

$= \frac{d\vec{r}_i{}'}{dt} + \frac{d\vec{R}}{dt}$

(ie) $\vec{v}_i = \vec{v}_i{}' + V$

Kinetic energy of m_i with respect to 'O' = $\frac{1}{2}m_i v_i^2$

∴ kinetic energy of the system of particle $= \frac{1}{2}\sum_i m_i \vec{v}_i^{\,2}$

$= \frac{1}{2}\sum_i m_i (\vec{v}_i{}' + V)(\vec{v}_i{}' + V)$

$= \frac{1}{2}(\vec{v}_i{}'2 + \vec{v}_i{}'V + V\vec{v}_i{}' + V^2)$

$= \frac{1}{2}\sum_i m_i \vec{v}_i{}'^2 + \frac{1}{2}\sum_i m_i\ 2\vec{v}_i{}'V + \frac{1}{2}\sum_i m_i V^2)$

$= \frac{1}{2}\sum_i m_i \vec{v}_i{}'^2 + V\sum_i m_i \frac{d\vec{r}_i{}'}{dt} + \frac{1}{2}MV^2)$

We Know that,

The radius vector of the center of mass with respect to the center of mass = 0

$\therefore \frac{\sum m_{i\vec{r}_i'}}{\sum m_i} = 0$

$\Rightarrow \sum m_{i\vec{r}_i'} = 0$

$\therefore$ T= $\frac{1}{2}$mv^2+$\frac{1}{2}\sum_i m_i \vec{v}_i'^2$ where $\frac{1}{2}\sum_i m_i \vec{v}_i'^2$ is the kinetic energy of the motion about the center of mass and $\frac{1}{2}$mv^2 is the kinetic energy obtained all the mass where concentrated at the center of mass.

Note

A force F is said to be conservative force if the work done by the force F in displaying a particle from a position A to a position B is independent of the path joining A and B. But dependent only on the position of A and B.

That is,

$\int \overrightarrow{F}.d\vec{r}$ is independent of c.

We Know that,

A necessary and sufficient condition for a force F is said to be conservative force if ∇xF=0

If ∇xF=0 then there exist a scalar potential φ such that F=-∇ φ

This φ is nothing but v where v is the potential energy.

$\therefore$F =-∇ V

In mechanics if F is a force, then $\int_A^B \overrightarrow{F}.d\vec{r}$ is the work done by the force F. In displacing a particle fro, position A to position B.

If the above integral is independent of the path joining A and B, Then F is said to be a conservative force.

If F is the conservative force, then we can find a function v such that

F =-∇ V where v is the potential energy.

Energy Conservation theorem for a system of particles

Statement:

The total energy of a system of particles at configuration 1 is equal to the total energy of a system of particles at configuration 2.

Proof:

We Know That,

The equation of motion for a system of particles is $\vec{F}_i = \dot{\vec{P}}_i$

∴The total force acting on the system is $\overrightarrow{F} = \sum_i \vec{F}_i$

The work done by $\vec{F}_i$ on the elementary mass m_i is $\vec{F}_i.\vec{d}s_i$

Now,

Work done by the forces acting on the system of particles in bringing the system from configuration 1 to configuration 2 is given by

$$w_{12} = \sum_i \int_1^2 \vec{F}\,\vec{d}s_i$$

$$= \sum_i \dot{\vec{P}}i\,\vec{d}s_i$$

$$= \sum_i \int_1^2 \frac{\overrightarrow{dp_i}}{dt}\vec{d}s_i$$

$$= \sum_i \int_1^2 \frac{\vec{d}}{dt}(\overrightarrow{m_i v_i)}ds_i$$

$$= \sum_i m_i \int_1^2 \frac{\overrightarrow{dv_i}}{dt}\overrightarrow{ds_i}$$

$$= \sum_i m_i \int_1^2 \frac{\overrightarrow{dv_i}}{dt}\frac{\overrightarrow{ds_i}}{dt}dt$$

$=\sum_i m_i \int_1^2 \frac{d\overrightarrow{v_i}}{dt} v_i \, dt$

$=\sum_i m_i \int_1^2 d v_i \,.\, v_i$

$=\sum_i m_i \int_1^2 \frac{1}{2} d \, v_i^2$

$=\sum_i m_i \frac{1}{2} [v_i^2]_1^2$

$=\sum_i [\frac{1}{2} m_i \, v_i^2]^2{}_1$

$=[\sum_i T_i]^2{}_1$

$w_{12} = T_2 - T_1$

Again ,

$w_{12} = \sum_i \vec{F}i \, \vec{d}s_i$

$= \sum_i \int_1^2 [\sum_j \overrightarrow{F}_{ji} + \overrightarrow{F}_i^e] d\overrightarrow{s_i}$

$= \sum_i \int_1^2 [\sum_j \overrightarrow{F}_{ji}] d\overrightarrow{s_i} + \sum_i \int_1^2 [\overrightarrow{F}_i^e] \, d\overrightarrow{s_i}$

=Work done by all the internal forces in bringing the system from configuration 1 to configuration 2 + Work done by all the external forces in bringing the system from configuration 1 to configuration 2.

Suppose,

The external forces acting on the system is conservative. Then $\overrightarrow{F}_i^e = -\nabla_i \overrightarrow{v}_i^e$

∴the work done by the external force $= \sum_i \int_1^2 [\overrightarrow{F}_i^e] \, d\overrightarrow{s_i} = \sum_i \int_1^2 [-\nabla_i \overrightarrow{v}_i^e] d\overrightarrow{s_i}$

$= - \sum_i \int_1^2 \frac{\partial}{\partial r_i} \overrightarrow{v}_i^e \, d\overrightarrow{r_i}$ [since for elementary displacement $d\overrightarrow{s_i} = d\overrightarrow{r_i}$]

$= - \sum_i \int_1^2 (i \frac{\partial}{\partial x_i} \overrightarrow{v}_i^e + j \frac{\partial}{\partial y_i} \overrightarrow{v}_i^e + k \frac{\partial}{\partial z_i} \overrightarrow{v}_i^e)(i \, dx_i + j \, dy_i + k \, dz_i)$

$= - \sum_i \int_1^2 (\frac{\partial}{\partial x_i} \overrightarrow{v}_i^e + \frac{\partial}{\partial y_i} \overrightarrow{v}_i^e + \frac{\partial}{\partial z_i} \overrightarrow{v}_i^e)$

$=- \sum_i \int_1^2 d \, \overrightarrow{v}_i^e$

$=-[\sum_i \overrightarrow{v_i}^e]^2{}_1$

$=-[\overrightarrow{v}_i^e]^2{}_1$

$= \overrightarrow{v}_1^e - \overrightarrow{v}_2^e$

Suppose the internal force is acting on the system of particles are conservative then they can be derived from a potential constant.

∴the internal force is $\vec{F}_{ij}$ and $\vec{F}_{ji}$between the i[th] and j[th] particles can be derived from the same potential function $\vec{v}_{ij}$.

∴The work done by the (i,j) th pair = $-\{\int_1^2[\nabla_i\vec{v}_{ij}\, d\vec{s_i} + \nabla_j\vec{v}_{ij}\, d\vec{s_j}\,]\}$

We Know That,

$\nabla_i\vec{v}_{ij} = \nabla_{ij}\vec{v}_{ij} = -\nabla_j\vec{v}_{ij}$

∴The work done by the (i,j) th pair = $-\{\int_1^2[\nabla_{ij}\vec{v}_{ij}\, d\vec{s_i} - \nabla_{ij}\vec{v}_{ij}\, d\vec{s_j}\,]\}$

$$=-\{\int_1^2 \nabla_{ij}\vec{v}_{ij}\,(d\vec{s_i} - d\vec{s_j})\}$$

$$=-\int_1^2 \nabla_{ij}\vec{v}_{ij}\,(d\vec{r_i} - d\vec{r_j})$$

$$=-\int_1^2 \nabla_{ij}\vec{v}_{ij}\, d(\vec{r_i} - \vec{r_j})$$

$$=-\int_1^2 \nabla_{ij}\vec{v}_{ij} d\vec{r_{ij}}$$

∴the total work done by the internal force $=-\sum_{i,j}\int_1^2[\nabla_{ij}\vec{v}_{ij} d\vec{r_{ij}}]$

The factor $\frac{1}{2}$ occurs in the above expression, when we take the summation over i,j.

The total work done by the internal force = $-\frac{1}{2}\sum_{i,j}\int_1^2 \frac{\partial}{\partial r_{ij}}\vec{v}_{ij} d\vec{r_{ij}}$

$$=-\frac{1}{2}\sum_{i,j}\int_1^2(\frac{\partial vij}{\partial x_{ij}} + j\,\frac{\partial vij}{\partial y_{ij}} + k\,\frac{\partial ij}{\partial z_ij})(i\, dx_{ij} + j\, dy_{ij} + k\, dz_{ij})$$

$$=-\frac{1}{2}\sum_{i,j}\int_1^2 d(v_{ij})$$

$$=-[\frac{1}{2}\sum_{i,j} v_{ij}]$$

$$= -[v_i]^2{}_1$$

$$=v_1{}^i - v_2{}^i$$

∴$W_{12} = \sum_i\int_1^2\sum_j F_{ji} ds_i + \sum_i\int_1^2 F_i{}^e\, ds_i$

$=(v_1^i - v_2^i) + (v_1^e - v_2^e)$

$=(v_1^i + v_1^e) – (v_2^e + v_2^i)$

$= v_1 - v_2$ where $v = v^e + v^i$

But we have already prove the work done by the forces in displacing the system of particles from configuration 1 to configuration 2 is $w_{12} = T_2 - T_1$.

We get,

$V_1 - v_2 = T_2 - T_1$.

That is,

$T_1 + v_1 = T_2 + v_2$ where T+V is called the total energy.

This shows that the total energy is conserved in shifting system from configuration 1 to configuration 2.

1.3. Constraints..

The limitations or the geometrical restrictions on the motion of a particles or system of particles are known as constraints.

The equation of motion for i^th mass is $\sum_j F_{ji} + \vec{F}_i^e = \dot{\vec{P}}_i$

$$= \frac{d}{dt}(m_i \vec{v}_i)$$

$$= m_i \ddot{\vec{r}}_i$$

Here we consider only the internal as well as the external forces but there are constraints which limit the motion of the particle or system.

Examples of constraints

1. Consider a rigid body, which is a system involving constraints, where the constraints on the motion of the particles keep the distance $\vec{r}_{ij}$ between the i^{th} and j^{th} particle remains unchanged. Here the particles are not moving inside.

2. The beads of an abacus are constraint to one dimensional motion by the supporting wires.

3. Gas molecules with in a container are constraint by the walls of the vessel to move only inside the container.

4. The motion of a particle placed on the surface of the solid sphere is subject to the constraint that it can move only on the surface or outside the surface.

Classification of constraints

1. Holonomic constraint and non holonomic constraint

Let $\overrightarrow{r_1}$,$\overrightarrow{r_2}$…$\overrightarrow{r_n}$ be the position coordinates of the system of particles. If the conditions of the constraints can be expressed as the equations connecting the coordinates of the particle having the form f($\overrightarrow{r_1}$,$\overrightarrow{r_2}$…$\overrightarrow{r_n}$,t)=0, then the constraints are said to be holonomic constraints.

If the conditions of the constraints are not expressed in the above form, then they are called non holonomic constraints.

Examples of holonomic constraints

1.The constraints involved in the rigid body in which the distance between any two particles is always fixed are holonomic. Since the condition of constraints are expressed as $(\vec{r}_i-\vec{r}_j)^2=c_{ij}^2$.

2.The constraints involved when a particle is restricted to move along a curve are holonomic, suppose a particle moves in a plane along the line (x+y) = 7, the condition of constraints is xy-7=0.

Examples of non holonomic constraints

1.The constraints involved in the motion of the molecules in the gas container are non holonomic. The condition of constraints in this case are expressed as $r^2-a^2\leq 0$.

2. The constraints involving in the motion of the particle placed on the surface of the sphere is non holonomic. The condition of constraints in this case are expressed as $r^2-a^2 \geq 0$ where a is the radius of the sphere and r is the distance of the particle from the center of the sphere.

scleronomous constraints

If the equation of constraints is independent of time, then the constraint is known as scleronomous constraints.

Example

A bead sliding on a rigid curve wire fixed in space is an example scleronomus constraints.

Rheonomus constraints

Constraints which contains time explicitly is known as Rheonomus constraints.

Example

In the above example if the wire itself is moving in some prescribed fashion then the constraint is Rheonomus.

Generalized Coordinates

The minimum possible number of independent coordinates required to specify the configurations of a system at any intent of time is known as the generalized coordinates.

It is denoted by the letters, $q_1, q_2, \ldots q_n$.

If $q_1, q_2, \ldots q_n$ are the generalized coordinates of the system then $\dot{q}_1, \dot{q}_2 \ldots \dot{q}_n$ are the components of the velocities corresponding to the above coordinates. The generalized coordinates must satisfy the following two conditions.

1. The values of the coordinates determine the configuration of the system.

2. They may be varied arbitrarily and independently of each other, without violating the constraints of the system.

Examples

1. Consider a particle which moves in space, we can fixed in the position of the particle by using the coordinates x,y,z. Hence we require 3 generalized coordinates to fix the particles which moves in space.

2. When a particle moves in a plane it may be described by Cartesian coordinates x and y or the polar coordinate $r_1\theta_.$ So the generalized coordinates are two.

3. Consider a particle which is constraint to move only on a sphere of radius a. Then the generalized coordinates required are 2 namely $\theta\ and\ \varphi$(longitude and latitude).

4. The beats of an abacus has the generalized coordinate x (the Cartesian coordinate along the horizontal wire)

Degrees of freedom

The number of independent ways in which a mechanical system can move without vialing any constraint is called the number of degrees of freedom of the system. It is indicated by the least possible number of coordinates to describe the system.

Example

1.The degrees of freedom for a particle which moves freely in the space is 3, if it is constrained to move along a certain space curve is the degrees of freedom is 1.

The degrees of freedom for a system containing n particles is 3N- k where k is the number of constraints on the system.

Transformation equations

The old coordinates$\vec{r_1}$,$\vec{r_2}$,…$\vec{r_n}$ are expressed as functions of the generalized coordinates $q_1,q_2,\ldots q_n$. and possibly functions of time. These equations are called as the Transformation equations.

Example

1.If we consider a particle with the spherical polar coordinates as the generalized coordinates, then the transformation equations are,

x=rsin$\theta cos\varphi$ ≡x(r, θ,φ)

Y= rsin$\theta sin\varphi$ ≡y(r, θ,φ)

z= rcosθ ≡z(r, θ,φ)

In the same way if (x_i,y_i,z_i) are the cartition coordinates of the i^{th} particle of the system whose generalized coordinates are $q_1,q_2,\ldots q_n$. then the transformation equation are $x_i=x_i\,(q_1,q_2,\ldots q_n,t)$

$y_i=y_i\,(q_1,q_2,\ldots q_n,t)$

$z_i=z_i\,(q_1,q_2,\ldots q_n,t)$ where t denote the time.

If $\vec{r_i}$ = x_i i+ y_ij+ z_ik denote the position vector of the i^{th} particle then the transformation equation can be defined as $\vec{r_i}=r_i\,(q_1,q_2,\ldots q_n,t)$.

Unit –II

(1.4) D'Alemberts Principle and Lagranges Equations

Virtual displacement

A virtual displacement of a system refers to a change in the configuration of the system as the result of any arbitrary infinitesimal change of the coordinates $\delta\vec{r}_i$ consistent with the forces and constraints imposed on the system at the given instant t.

Virtual work done

The work done by the forces of the system during the virtual displacement is known as Virtual work done.

Principle of Virtual work done

For a system which is in equilibrium the virtual work done of the applied force is zero.

Proof

Consider a mechanical system which is in equilibrium.

Hence the total force on each particle vanishes.

(ie) $\vec{F}_i = 0$ for all i.

Clearly the dot product of the force $\vec{F}_i$ on the virtual displacement $\delta\vec{r}_i$ must vanishes.

(ie) $\vec{F}_i\, \delta\vec{r}_i = 0$

$\sum_i \vec{F}_i\, \delta\vec{r}_i = 0$ .... (1)

If there are constraints then the force $\vec{F}_i$ acting on the i^{th} particle can be written as the sum of the constraint force f_i and the applied force $\vec{F}_i^a$.

$$\vec{F}_i = \vec{F}_i^a + f_i$$

$$\sum_i(\vec{F}_i \,.\, \delta\vec{r}_i) = \sum_i(\vec{F}_i^a + f_i)\, \delta\vec{r}_i$$

$$\sum_i(\vec{F}_i \,.\, \delta\vec{r}_i) = \sum_i(\vec{F}_i^a\, \delta\vec{r}_i + f_i\delta\vec{r}_i) = 0$$

$\sum_i(\vec{F}_i^a + \vec{f}_i - \dot{\vec{P}}_i).\,\delta\vec{r}_i = 0$ ………………………….. (2)

From (1) and (2)

$\sum_i(\vec{F}_i^a\,\delta\vec{r}_i + f_i\delta\vec{r}_i) \quad = 0$

Suppose we restrict ourselves to such mechanical system where the virtual work done by the forces of constraint is zero.

(ie)$\sum_i(f_i\delta\vec{r}_i) = 0$

We get,

$\sum_i(\vec{F}_i^a\,\delta\vec{r}_i) = 0$

Hence this is known as principle of virtual work done.

D'Alemberts Principle

Statement:

The virtual work done by the applied force together with the reverse effective force of the system vanishes.

Proof:

Equation of motion for the ith particle of the system is $\vec{F}_i = \dot{\vec{P}}_i$ where $\dot{\vec{P}}_i$ is the linear momentum of the ith particle due to the force $\vec{F}_i$ acting on the ith particle.

$\therefore \vec{F}_i + (-\dot{\vec{P}}_i) = 0$

This shows that the mechanical system can be considered to be at rest under the force $\vec{F}_i + (-\dot{\vec{P}}_i)$ where $\vec{F}_i$ is the total force and $(-\dot{\vec{P}}_i)$ is the reverse effective force acting on the particle.

If $\delta\vec{r}_i$ is the virtual displacement of the ith particle , then $(\vec{F}_i - \dot{\vec{P}}_i)\,\delta\vec{r}_i = 0$………(1)

If forces of constraints are present, then $\vec{F}_i = \vec{F}_i^a + \vec{f}_i$ where $\vec{F}_i^a$ is the applied force and $\vec{f}_i$ is the force of constraint on the ith particle.

Equation (1) implies $\sum_i(\vec{F}_i^a + \vec{f}_i - \dot{\vec{P}}_i) . \delta\vec{r}_i = 0$

(ie) $\sum_i(\vec{F}_i^a - \dot{\vec{P}}_i) . \delta\vec{r}_i - \sum_i(\vec{f}_i . \delta\vec{r}_i) = 0$ …………………(2)

If we restrict our self to the cases where the forces of constraint have no work then $\sum_i(\vec{f}_i . \delta\vec{r}_i) = 0$.

Equation (2) becomes $\sum_i(\vec{F}_i^a - \dot{\vec{P}}_i) . \delta\vec{r}_i = 0$ ………………(3)

This is called D'Alemberts Principle.

We have restricted ourselves to the system where the virtual work done by the forces of constraints disappeared. So that we can drop the superscript a in (3).

∴D'Alemberts Principle becomes $\sum_i(\vec{F}_i^a - \dot{\vec{P}}_i) . \delta\vec{r}_i = 0$.

Note

If the system is at rest, then $\sum_i(\vec{f}_i . \delta\vec{r}_i) = 0$

If the system is in motion, then $\sum_i(\vec{F}_i^a - \dot{\vec{P}}_i) . \delta\vec{r}_i = 0$.

Derive Lagranges equation from D'Alemberts Principle

Consider a holonomic system with n generalized coordinates $q_1, q_2, \ldots q_n$.

Let $\vec{r}_i$ be the position vector of the i^{th} mass.

By using the transformation equation, we can write $\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots q_n, t)$

Let $\vec{v}_i$ be the velocity of the i^{th} mass.

$$\begin{aligned}\vec{v}_i &= \frac{d}{dt}\vec{r}_i \\ &= \frac{\partial\vec{r}_i}{\partial q_1}.\frac{dq_1}{dt} + \frac{\partial\vec{r}_i}{\partial q_2}.\frac{dq_2}{dt} + \ldots + \frac{\partial\vec{r}_i}{\partial q_n}.\frac{dq_n}{dt} + \frac{\partial\vec{r}_i}{\partial t} \\ &= \frac{\partial\vec{r}_i}{\partial q_1}\dot{q}_1 + \ldots + \frac{\partial\vec{r}_i}{\partial q_n}\dot{q}_n + \frac{\partial\vec{r}_i}{\partial t}\end{aligned}$$

$$\text{(ie)}\frac{d\vec{r}_i}{dt} = \sum_{k=1}^{n}\frac{\partial\vec{r}_i}{\partial q_1}\dot{q}_k + \frac{\partial\vec{r}_i}{\partial t} \quad \ldots\ldots\ldots\ldots\ldots\ldots..(1)$$

Similarly,

The virtual displacement $\delta\vec{r}_i$ can be connected with the virtual displacement δq_i.

$$\therefore \delta\vec{r}_i = \frac{\partial\vec{r}_i}{\partial q_1}\delta q_1 + \ldots + \frac{\partial\vec{r}_i}{\partial q_n}\delta q_n$$

$$\delta\vec{r}_i = \sum_{j=1}^{n}\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots..(2)$$

Now,

Consider the virtual work done by the forces acting on the system.

$$\text{Virtual work done} = \sum_i \vec{F}_i \, \delta\vec{r}_i$$

$$= \sum_i \vec{F}_i \sum_{j=1}^{n}\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j$$

$$= \sum_{i,j} \vec{F}_i \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j$$

$$= \sum_j \left[\sum_i \vec{F}_i \frac{\partial\vec{r}_i}{\partial q_j}\right]\delta q_j$$

$= \sum_j Q_j \, \delta q_j$ where the Q_j are called the components of generalized force defined as $Q_j = \sum_i \vec{F}_i \frac{\partial\vec{r}_i}{\partial q_j}$

The Virtual work done by the force acting on the system is equal to $\sum_i \vec{F}_i \, \delta\vec{r}_i = \sum_j Q_j \, \delta q_j$ ……………………………..(3)

The D'Alemberts Principle is $\sum_i(\vec{F}_i^a - \dot{\vec{P}}_i).\delta\vec{r}_i = 0$

$$(3) \Rightarrow \sum_j Q_j \, \delta q_j = \sum_i(\dot{\vec{P}}_i).\delta\vec{r}_i$$

$$= \sum_i(\dot{\vec{P}}_i).\sum_j\left[\frac{\partial\vec{r}_i}{\partial q_j}\right]\delta q_j$$

$$= \sum_i(m_i\ddot{\vec{r}}_i).\sum_j\left[\frac{\partial\vec{r}_i}{\partial q_j}\right]\delta q_j$$

$=\sum_{i,j} m_i\ddot{\vec{r}}_i\frac{\partial \vec{r}_i}{\partial q_j}\ \delta q_j$

(ie) $\sum_j Q_j\,\delta q_j \quad = \sum_j[\sum_i(m_i\ddot{\vec{r}}_i)\frac{\partial \vec{r}_i}{\partial q_j}]\delta q_j$……………………………..(I)

Now,

$\sum_i(m_i\ddot{\vec{r}}_i).\frac{\partial \vec{r}_i}{\partial q_j} \quad =\sum_i m_i\frac{d}{dt}\dot{\vec{r}}_i\frac{\partial \vec{r}_i}{\partial q_j}$

But $\frac{d}{dt}(\dot{\vec{r}}_i\frac{\partial \vec{r}_i}{\partial q_j}) \quad = \frac{d}{dt}\dot{\vec{r}}_i\frac{\partial \vec{r}_i}{\partial q_j} + \dot{\vec{r}}_i\frac{d}{dt}(\frac{\partial \vec{r}_i}{\partial q_j})$

$\frac{d}{dt}(\dot{\vec{r}}_i)\frac{\partial \vec{r}_i}{\partial q_j} \quad = \frac{d}{dt}(\dot{\vec{r}}_i\frac{\partial \vec{r}_i}{\partial q_j}) + \dot{\vec{r}}_i\frac{d}{dt}(\frac{\partial \vec{r}_i}{\partial q_j})$

$\sum_i(m_i\ddot{\vec{r}}_i).\frac{\partial \vec{r}_i}{\partial q_j} \quad = \sum_i[\frac{d}{dt}(m_i\dot{\vec{r}}_i\frac{\partial \vec{r}_i}{\partial q_j}) - m_i\dot{\vec{r}}_i\frac{d}{dt}(\frac{\partial \vec{r}_i}{\partial q_j})]$

$= \frac{d}{dt}(\sum_i m_i\dot{\vec{r}}_i\frac{\partial \vec{r}_i}{\partial q_j}) - \sum_i m_i\dot{\vec{r}}_i\frac{d}{dt}(\frac{\partial \vec{r}_i}{\partial q_j})$

$\therefore$Equation(I)

$\Rightarrow\sum_j Q_j\delta q_j \ =\sum_j[\frac{d}{dt}(\sum_i m_i\dot{\vec{r}}_i\frac{\partial \vec{r}_i}{\partial q_j})\sum_i m_i\dot{\vec{r}}\frac{d}{dt}(\frac{\partial \vec{r}_i}{\partial q_j})]\ \delta q_j$……………………(II)

Claim:(i)

$\frac{\partial \dot{\vec{r}}_i}{\partial q_j} = \frac{\partial \vec{r}_i}{\partial q_j}$

We Know That,

$\vec{r}_i = \vec{r}_i(q_1,q_2,\ldots q_n,t)$

Similarly,

$\frac{\partial \vec{r}_i}{\partial q_j}$ is a function of $(q_1,q_2,\ldots q_n,t)$.

Now

$\frac{\partial \dot{\vec{r}}_i}{\partial q_j} = \frac{\partial}{\partial \dot{q}_j}(\dot{\vec{r}}_i)$

$= \frac{\partial}{\partial \dot{q}_j}(\sum_{k=1}^{n} \frac{\partial \vec{r}_i}{\partial q_1} \dot{q}_k + \frac{\partial \vec{r}_i}{\partial t})$ [since (1)]

$= \frac{\partial}{\partial \dot{q}_j} [\frac{\partial \vec{r}_i}{\partial q_1}.\frac{dq_1}{dt} + \frac{\partial \vec{r}_i}{\partial q_2}.\frac{dq_2}{dt} + \ldots + \frac{\partial \vec{r}_i}{\partial q_n}.\frac{dq_n}{dt} + \frac{\partial \vec{r}_i}{\partial t}]$

$= \frac{\partial}{\partial \dot{q}_j} [\frac{\partial \vec{r}_i}{\partial q_1} \dot{q}_1 + \ldots + \frac{\partial \vec{r}_i}{\partial q_n} \dot{q}_n + \frac{\partial \vec{r}_i}{\partial t}]$

(ie) $\frac{\partial \dot{\vec{r}}_i}{\partial q_j} = \frac{\partial \vec{r}_i}{\partial q_j}$

Claim:(ii)

$\frac{d}{dt}$ and $\frac{\partial}{\partial q_j}$ are interchangeable.

$\frac{d}{dt}(\frac{\partial \vec{r}_i}{\partial q_j}) = \frac{\partial}{\partial q_1}(\frac{\partial \vec{r}_i}{\partial q_j})\frac{dq_1}{dt} + \ldots + \frac{\partial}{\partial q_n}(\frac{\partial \vec{r}_i}{\partial q_j})\frac{dq_n}{dt} + \frac{\partial}{\partial t}(\frac{\partial \vec{r}_i}{\partial q_j})$

$= \frac{\partial}{\partial q_1}(\frac{\partial \vec{r}_i}{\partial q_j})\dot{q}_1 + \ldots + \frac{\partial}{\partial q_n}(\frac{\partial \vec{r}_i}{\partial q_j})\dot{q}_n + \frac{\partial}{\partial t}(\frac{\partial \vec{r}_i}{\partial q_j})$

$= (\sum_{k=1}^{n} \frac{\partial}{\partial q_k}(\frac{\partial \vec{r}_i}{\partial q_j})\dot{q}_k + \frac{\partial}{\partial t}(\frac{\partial \vec{r}_i}{\partial q_j})$ ……………………….(A)

Also,

$\frac{\partial}{\partial q_j}\frac{d\vec{r}_i}{\partial t} = \frac{\partial}{\partial q_j}[\sum_{k=1}^{n}(\frac{\partial \vec{r}_i}{\partial q_j})\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}]$ (since (1)) ………………………….(B)

From (A) and (B) We get,

$\frac{d}{dt}(\frac{\partial \vec{r}_i}{\partial q_j}) = \frac{\partial}{\partial q_j}\frac{d\vec{r}_i}{\partial t}$

Hence $\frac{d}{dt}$and $\frac{\partial}{\partial q_j}$ are interchangeable.

(II) $\Rightarrow \sum_j Q_j \delta q_j = \sum_j[\frac{d}{dt}(\sum_i m_i\dot{\vec{r}}_i\frac{\partial \dot{\vec{r}}_i}{\partial q_j}) - \sum_i m_i\dot{\vec{r}}_i\frac{d}{dt}(\frac{\partial \vec{r}_i}{\partial q_j})]\, \delta q_j$

$= \sum_j[\frac{d}{dt}(\sum_i m_i\dot{\vec{r}}_i\frac{\partial \dot{\vec{r}}_i}{\partial q_j}) - \sum_i m_i\dot{\vec{r}}_i\frac{\partial}{\partial q_j}\frac{d\vec{r}_i}{dt}]\, \delta q_j$

$$= \sum_j [\frac{d}{dt}(\sum_i m_i \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_j}) - \sum_i m_i \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial q_j}]\, \delta q_j$$

$$= \sum_j [\frac{d}{dt}(\frac{\partial}{\partial \dot{q}_j} \sum_i m_i \frac{1}{2} v_i^2) - \frac{\partial}{\partial q_j} \sum_i m_i \frac{1}{2} v_i^2]\, \delta q_j$$

$$= \sum_j [\frac{d}{dt}(\frac{\partial}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j}]\, \delta q_j$$

(ie) $\sum_j [\frac{d}{dt}(\frac{\partial}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j}]\, \delta q_j - \sum_j Q_j \,\delta q_j = 0$

$\sum_j [\frac{d}{dt}(\frac{\partial}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} - Q_j]\, \delta q_j = 0$

Since the constraints are holonomic the generalized coordinates q_j's are all independent of each other.

Hence the above equation will be true only if each coefficient of δq_j is zero.

(ie) $\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - (\frac{\partial T}{\partial q_j}) = Q_j$

These n equations are called as Lagranges equation from D'Alemberts Principle or Lagranges equation of motion for a holonomic system.

Note

The above equation is also known as the Lagranges equation of motion for a holonomic non conservative system.

Derive Lagranges equation of motion for a holonomic conservative system

Proof:

We know the Lagranges equation of motion for a holonomic conservative system with n generalized coordinates ($q_1, q_2, \ldots q_n$) is $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$, j = 1,2,3,…n where $Q_j = \sum_i \vec{F}_i \frac{\partial r_i}{\partial q_j}$

Suppose the system is conservative,

$\vec{F}_i = -\nabla_i v_i$ for some potential v_i, where the suffix i of the ∇ operator indicates the differentiation must be done with respect to the components of $\vec{r}_i$.

We have,

$$Q_j = \sum_i \vec{F}_i \frac{\partial r_i}{\partial q_j}$$

$$= \sum_i (-\nabla_i v_i)\frac{\partial r_i}{\partial q_j}$$

$$= -\sum_i \frac{\partial}{\partial r_i} v_i \frac{\partial r_i}{\partial q_j}$$

$$= -\sum_i \frac{\partial v_i}{\partial q_j}$$

$$= -\frac{\partial}{\partial q_j}\left(\sum_i v_i\right)$$

$Q_j = -\frac{\partial v}{\partial q_j}$ where $v = (\sum_i v_i)$ is the potential energy.

We have,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = -\frac{\partial v}{\partial q_j}$$

(ie) $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial (T-V)}{\partial q_j} = 0$

We know that,

V depends upon $(q_1, q_2, \ldots q_n)$ and v does not depend upon $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_n$.

There fore $\frac{\partial v}{\partial \dot{q}_j} = 0$, j=1,2,…n.

Lagranges equation can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial v}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$$

$$\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}(T-V)\right] - \frac{\partial (T-V)}{\partial q_j} = 0$$

Taking L = T-V, the Lagrangian for the conservative system, the Lagranges equation becomes $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j}$ =0, j=1,2,…n.

This equation is known as Lagranges equation of motion for a holonomic conservative system.

(1.5) velocity –Dependent potentials and the dissipation functions

Lagranges equation can be put in the form $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j}) - \frac{\partial L}{\partial q_j}$ =0, event if there is no potential function V, in the usual sense, providing the generalized forces are obtained from a function U($q_j, \dot{q}_j$) by the prescription $Q_j = -\frac{\partial u}{\partial q_j} + \frac{d}{dt}(\frac{\partial u}{\partial \dot{q}_j})$

Proof:

We know that,

The Lagranges equation is,

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} = Q_j$$

(ie) $\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} = -\frac{\partial u}{\partial q_j} + \frac{d}{dt}(\frac{\partial u}{\partial \dot{q}_j})$

(ie) $\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial u}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} + (\frac{\partial u}{\partial q_j}) = 0$

(ie) $\frac{d}{dt}[\frac{\partial (T-u)}{\partial \dot{q}_j}] = \frac{\partial}{\partial q_j}(T - U) = 0.$

Take L = T –U, where U is the generalized potential or velocity dependent potential .

$$\frac{d}{dt}(\frac{\partial T}{\partial \dot{q}_j}) - \frac{\partial T}{\partial q_j} = 0$$

This is of the same form as the Lagranges equation of motion for the conservative holonomic system.

Application of velocity dependent potential in the electro magnetic forces on moving charges

Proof:

The force experience by a particle of charge 'q' at rest in an electric field of intensity E is given by F_1 =qE. But if the particle is in motion the particle of charge q will experience and additional force which is linearly proportional to the velocity of charge this additional force is called magnetic force and it is given by $F_2 = \frac{q}{c}(\overrightarrow{\vec{v}B})$ where V is the velocity of the particle of charge q and B is the magnetic field of intensity.

∴the total force of a uniformly moving particle of charge q is F= $F_1 + F_2$

$= qE + \frac{q}{c}(\overrightarrow{\vec{v}B})$

$= q[E + \frac{1}{c}(\overrightarrow{\vec{v}B})]$ …………………….. (1)

The above equation is called the Lorentz force equation or Lorentz formula.

In Gaussian units the Maxwell's equations are curl $\vec{E} = \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ ………. (a)

curl $\vec{H} = \nabla \times \vec{H} = -\frac{1}{c}\frac{\partial \vec{D}}{\partial t}$ ………. (b)

div $\vec{D} = \nabla \times \vec{D} = 4\pi\rho$…………...(c)

div $\vec{B} = \nabla \times \vec{B} = 0$ ……………… (d)

……………………………(2)

where ρ is the density of electric charge c is the velocity of light B.

From (2)(d) we have $\nabla \times \vec{B} = 0$ it follows that there is a magnetic vector potential A such that B = curl A.

(ie) B $= \nabla \times \vec{A}$……………(3)

(2)(a) $\Rightarrow$ curl $\vec{E}$ $= -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$

$= -\frac{1}{c}\frac{\partial}{\partial t}$(curl A) (since 3)

$= -\frac{1}{c}\frac{\partial}{\partial t}$ curl $\vec{A}$ – curl grad φ where φ *is a scalar* function

$= -\frac{1}{c}$curl $\frac{\partial \vec{A}}{\partial t}$ - curl grad φ

E $= -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$- curl grad φ …………………(4)

The Lorentz force in terms of the potential φ and A is,

$$F = q\,[\,-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \text{grad}\,\varphi + \frac{1}{c}\{\overrightarrow{V\ \text{x}}\nabla \text{ x } \vec{A}\}] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

To find $\vec{V}$ x(∇ x $\vec{A}$)

$$\nabla \text{ x } \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \vec{A}_x & \vec{A}_y & \vec{A}_z \end{vmatrix}$$

$$= i\left(\frac{\partial \vec{A}_z}{\partial y} - \frac{\partial \vec{A}_y}{\partial z}\right) - j\left(\frac{\partial \vec{A}_x}{\partial z} - \frac{\partial \vec{A}_z}{\partial x}\right) + k\left(\frac{\partial \vec{A}_y}{\partial x} - \frac{\partial \vec{A}_x}{\partial y}\right)$$

$$\vec{V} \text{ x}(\nabla \text{ x } \vec{A}) = \begin{vmatrix} i & j & k \\ \vec{V}_x & \vec{V}_y & \vec{V}_z \\ \frac{\partial \vec{A}_z}{\partial y} - \frac{\partial \vec{A}_y}{\partial z} & \frac{\partial \vec{A}_x}{\partial z} - \frac{\partial \vec{A}_z}{\partial x} & \frac{\partial \vec{A}_y}{\partial x} - \frac{\partial \vec{A}_x}{\partial y} \end{vmatrix}$$

$$= i[\vec{V}_y \frac{\partial \vec{A}_y}{\partial x} - \frac{\partial \vec{A}_x}{\partial y}] - \vec{V}_z\left(\frac{\partial \vec{A}_x}{\partial z} - \frac{\partial \vec{A}_z}{\partial x}\right) - j[\vec{V}_x \frac{\partial \vec{A}_y}{\partial x} - \frac{\partial \vec{A}_x}{\partial y}] - \vec{V}_z\left(\frac{\partial \vec{A}_z}{\partial y} - \frac{\partial \vec{A}_y}{\partial z}\right)] + k$$
$$[\vec{V}_x \frac{\partial \vec{A}_x}{\partial z} - \frac{\partial \vec{A}_z}{\partial x} - \vec{V}_y \frac{\partial \vec{A}_z}{\partial y} - \frac{\partial \vec{A}_y}{\partial z}]$$

$= \text{i}[\vec{V}_y \frac{\partial \vec{A}_y}{\partial x} + \vec{V}_x \frac{\partial \vec{A}_z}{\partial x}] + \text{j} [\vec{V}_x \frac{\partial \vec{A}_x}{\partial y} + \vec{V}_z \frac{\partial \vec{A}_z}{\partial y}] + \text{k} [\vec{V}_x \frac{\partial \vec{A}_x}{\partial z} + \vec{V}_y \frac{\partial \vec{A}_y}{\partial z}] - \{\text{i}(\vec{V}_y \frac{\partial \vec{A}_x}{\partial y} + \vec{V}_z \frac{\partial \vec{A}_x}{\partial z})() + \text{j} (\vec{V}_x \frac{\partial \vec{A}_y}{\partial x} + \vec{V}_z \frac{\partial \vec{A}_y}{\partial z}) + \text{k} (\vec{V}_x \frac{\partial \vec{A}_z}{\partial x} + \vec{V}_y \frac{\partial \vec{A}_z}{\partial y})\}$

Adding subtracting $\vec{V}_x \frac{\partial \vec{A}_x}{\partial x}, \vec{V}_y \frac{\partial \vec{A}_y}{\partial y}, \vec{V}_z \frac{\partial \vec{A}_z}{\partial z}$ in the i^{th}, j^{th}, k^{th} components respectively.

We get,

$\vec{V} \text{ x}(\nabla \text{ x } \vec{A}) = \text{i}(\vec{V}_x \frac{\partial \vec{A}_x}{\partial x} + \vec{V}_y \frac{\partial \vec{A}_y}{\partial x} + \vec{V}_z \frac{\partial \vec{A}_z}{\partial x}) + \text{j}(\vec{V}_x \frac{\partial \vec{A}_x}{\partial y} + \vec{V}_y \frac{\partial \vec{A}_y}{\partial y} + \vec{V}_z \frac{\partial \vec{A}_z}{\partial y}) + \text{k}(\vec{V}_x \frac{\partial \vec{A}_x}{\partial z} + \vec{V}_y \frac{\partial \vec{A}_y}{\partial z} + \vec{V}_z \frac{\partial \vec{A}_z}{\partial z}) - \{\text{i}(\vec{V}_x \frac{\partial \vec{A}_x}{\partial x} + \vec{V}_y \frac{\partial \vec{A}_x}{\partial y} + \vec{V}_z \frac{\partial \vec{A}_x}{\partial z}) + \text{j}(\vec{V}_x \frac{\partial \vec{A}_y}{\partial x} + \vec{V}_y \frac{\partial \vec{A}_y}{\partial y} + \vec{V}_z \frac{\partial \vec{A}_y}{\partial z}) + \text{k}(\vec{V}_x \frac{\partial \vec{A}_z}{\partial x} + \vec{V}_y \frac{\partial \vec{A}_z}{\partial y} + \vec{V}_z \frac{\partial \vec{A}_z}{\partial z})\}$

$= (\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k)(\vec{A}_x\vec{V}_x + \vec{A}_y\vec{V}_y + \overrightarrow{A}_z\vec{V}_z) - (\vec{V}_x \frac{\partial}{\partial x} + \vec{V}_y \frac{\partial}{\partial y} + \vec{V}_z \frac{\partial}{\partial z})(\vec{A}_x \text{i} + \vec{A}_y \text{j} + \vec{A}_z \text{k})$

$\vec{V} \text{ x}(\nabla \text{ x } \vec{A}) = \nabla (\vec{A}.\vec{V}) - (\vec{V}.\nabla)\vec{A}$ …………………(6)

Also A =A(x,y,z,t)

$\frac{dA}{dt} = \frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial y}\dot{y} + \frac{\partial A}{\partial z}\dot{z} + \frac{\partial A}{\partial t}$

$= \frac{\partial A}{\partial x}\vec{V}_x + \frac{\partial A}{\partial y}\vec{V}_y + \frac{\partial A}{\partial z}\vec{V}_z + \frac{\partial A}{\partial t}$

$=(\vec{V}_x \text{ i} + \vec{V}_y \text{ j} + \vec{V}_z \text{ k}) (\frac{\partial A}{\partial x} i + \frac{\partial A}{\partial y} j + \frac{\partial A}{\partial z} k + \frac{\partial A}{\partial t})$

$= \vec{V} \text{ x}(\nabla \text{ x } \vec{A}) + \frac{\partial A}{\partial t}$ ………………………….. (7)

(6) $\Rightarrow \vec{V} \text{ x}(\nabla \text{ x } \vec{A}) = \nabla (\vec{A} . \vec{V}) - (\vec{V} . \nabla\vec{A})$

$= \nabla (\vec{A} . \vec{V}) - \frac{dA}{dt} + \frac{\partial A}{\partial t}$

(5) implies

$$F = q[-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\varphi + \frac{1}{c}\{\vec{V} \times (\nabla \times \vec{A})\}]$$

$$= q[-\frac{1}{c}\frac{\partial \vec{A}}{\partial t} - \nabla\varphi + \frac{1}{c}\{\nabla\overrightarrow{A}.\vec{V} - \frac{dA}{dt} + \frac{\partial A}{\partial t}\}]$$

$=q[-\nabla\varphi + \frac{1}{c}\{\nabla \overrightarrow{A}.\vec{V} - \frac{dA}{dt}\}]$ The x components of the force is given by ,

$$F_x = q[-\frac{\partial}{\partial x}\{\varphi - \frac{1}{c}(\overrightarrow{A}.\vec{V})\} - \frac{1}{c}\frac{d}{dx}A_x]$$

$$= q[-\frac{\partial}{\partial x}\{\varphi - \frac{1}{c}(\overrightarrow{A}.\vec{V})\} - \frac{1}{c}\frac{d}{dt}\frac{\partial}{\partial v_x}\vec{A}.\vec{V}]$$

Since the scalar product potential q φ is independent of the velocity $\frac{\partial}{\partial v_x}(q\,\varphi) = 0$

$$\therefore F_x = -\frac{\partial}{\partial x}(q\,\varphi - \frac{q}{c}(\vec{A}.\vec{V}) - \frac{q}{c}\frac{d}{dt}\frac{\partial}{\partial v_x}(\vec{A}.\vec{V}) + \frac{\partial}{\partial v_x}q\,\varphi$$

$$= -\frac{\partial}{\partial x}(q\,\varphi - \frac{q}{c}(\vec{A}.\vec{V}) - \frac{q}{c}\frac{d}{dt}\frac{\partial}{\partial v_x}(\vec{A}.\vec{V}) + \frac{d}{dt}\frac{\partial}{\partial v_x}q\,\varphi$$

$$= -\frac{\partial}{\partial x}(q\,\varphi - \frac{q}{c}(\vec{A}.\vec{V}) + \frac{d}{dt}\frac{\partial}{\partial v_x}(q\,\varphi - \frac{q}{c}(\vec{A}.\vec{V}))$$

(ie) $F_x = -\frac{\partial v}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial v_x}$ where $U = q\,\varphi - \frac{q}{c}(\vec{A}.\vec{V})$ ………………….. (8)

Here U is known as the generalized potential or velocity dependent 1 potential.

The Lagrangian for a charged particle in an electromagnetic field is L =T –U

$$L = T - (q\,\varphi - \frac{q}{c}(\vec{A}.\vec{V}))$$

Derive Lagranges equation for the case when there are frictional forces.

Proof:

We know that,

The frictional force is proportional to the velocity of the particle.

The x component of the frictional forces can be written as $F_{fx} = -K_xV_x$

Where k_x is the constant.

Frictional forces of this type may be derived in 0 terms of a function F known as Rayleigh's dissipation function and defined as F = $\frac{1}{2}\sum_i \ (k_x V_i x^2 + k_y V_i y^2 + k_z V_i z^2)$ where the summation is over the particles of the system.

From this definition it is clear that $F_{fx} = -K_x V_x$

$$= -\frac{\partial \mathrm{F}}{\partial v_x}$$

In symbolically $\vec{F}_f = F_{fx^i} + F_{fy^j} + F_{fz^k}$

$$= -\frac{\partial \mathrm{F}}{\partial v_x}\,\mathrm{i} - \frac{\partial \mathrm{F}}{\partial v_y}\,\mathrm{j} - \frac{\partial \mathrm{F}}{\partial v_z}$$

$$= -[\frac{\partial \mathrm{F}}{\partial v_x}\,\mathrm{i} + \frac{\partial \mathrm{F}}{\partial v_y}\,\mathrm{j} + \frac{\partial \mathrm{F}}{\partial v_z}\,\mathrm{k}]$$

$$= -\nabla_v \mathrm{F}$$

$$Q_j = \sum_i F_{fi}\frac{\partial r_i}{\partial q_j}$$

$$= -\sum_i(\frac{\partial \mathcal{F}_i}{\partial V_i x}\,\mathrm{i} + \frac{\partial \mathcal{F}_i}{\partial V_i y}\,\mathrm{j} + \frac{\partial \mathcal{F}_i}{\partial V_i z}\,\mathrm{k})\frac{\partial \dot{r}_i}{\partial \dot{q}_j}$$

$$= -\sum_i(\frac{\partial \mathcal{F}_i}{\partial \vec{V_i}}\frac{\partial \vec{V_i}}{\partial \dot{q}_j})$$

$$= -\sum_i(\frac{\partial \mathcal{F}_i}{\partial \dot{q}_j})$$

$$Q_j = -\frac{\partial \mathcal{F}}{\partial \dot{q}_j}$$

We know the lagranges equation for a holonomic system are $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j$

Let $Q_j = Q_j' + Q_j''$ where Q_j' corresponds to conservative forces and Q_j'' corresponds to frictional forces.

Then,

$Q_{j'} = -\frac{\partial V}{\partial q_j}$ and $Q_{j''} = -\frac{\partial F}{\partial \dot{q}_j}$ where V is the potential of the conservative forces and $\mathcal{F}$ is the Rayleigh's dissipation function.

$\therefore$The lagranges equation becomes,

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}-\frac{\partial F}{\partial \dot{q}_j}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}-\frac{\partial V}{\partial q_j}\right)-\frac{\partial}{\partial q_j}(T - V) + \frac{\partial F}{\partial \dot{q}_j} = 0$$

$$\frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial}{\partial q_j}(T - V) + \frac{\partial F}{\partial \dot{q}_j} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} + \frac{\partial F}{\partial \dot{q}_j} = 0 \text{ where } L=T-V$$

(1.6) Simple application of the Lagranges formulation

Obtain the kinetic energy in terms of the generalized velocities.

Consider a mechanical system be the n generalized coordinates $q_1, q_2, \ldots q_n$.

Let m_i be the mass of the i^{th} particle and r_i be the position vector of the particle.

The position vector $\vec{r}_i$ can be expressed in terms of generalized coordinates.

(ie) $\vec{r}_i = \vec{r}_i(q_1, q_2, \ldots q_n, t)$

Let T denote the kinetic energy of the system.

$$T = \frac{1}{2}\sum_i m_i v_i^2$$

We know that,

$$\vec{v}_i = \frac{d\vec{r}_i}{dt}$$

$$= \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}$$

$$T = \frac{1}{2}\sum_i m_i\left\{\sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}\right\}^2$$

$= \frac{1}{2}\sum_i m_i [\sum_j \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}] [\sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}]$

$= \frac{1}{2}\sum_i m_i [\sum_{j,k} \frac{\partial \vec{r}_i}{\partial q_j \partial q_k}\dot{q}_j\, \dot{q}_k + \sum_j \frac{\partial \vec{r}_i}{\partial q_j}\frac{\partial \vec{r}_i}{\partial t}\dot{q}_j + \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\frac{\partial \vec{r}_i}{\partial t}\, \dot{q}_k + (\frac{\partial \vec{r}_i}{\partial t})^2]$

$= \frac{1}{2}\sum_{j,k}[\sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j \partial q_k}]\dot{q}_j\, \dot{q}_k + \sum_j[\frac{1}{2}\sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j}\frac{\partial \vec{r}_i}{\partial t} + \frac{1}{2}\sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j}\frac{\partial \vec{r}_i}{\partial t}]\, \dot{q}_j$ $+\frac{1}{2}\sum_i m_i (\frac{\partial \vec{r}_i}{\partial t})^2$

(ie) $T = M_0 + \sum_j M_j\dot{q}_j + \frac{1}{2}\sum_{j,k}\dot{q}_j\, \dot{q}_k (M_{jk})$ where $M_0 = \sum_i \frac{1}{2} m_i (\frac{\partial \vec{r}_i}{\partial t})^2$, $M_j = \sum_i \frac{1}{2} m_i \frac{\partial \vec{r}_i}{\partial q_j}\frac{\partial \vec{r}_i}{\partial t}$ and $M_{jk} = \sum_i m_i \frac{\partial \vec{r}_i}{\partial q_j \partial q_k}$

If the given system is independent of time, then $\frac{\partial \vec{r}_i}{\partial t} = 0$

$M_0 = M_j = 0$

$T = \frac{1}{2}\sum_{j,k}\dot{q}_j\, \dot{q}_k$

This shows that T is a homogenous quadratic form in these generalized velocities.

Motion of a single particle in space using coordinates

(Derive Lagrange's equation of motion for a single particle moving freely in space).

Proof:

Let x,y,z be the cartition coordinates of the particle.

Let m be the mass and T be the kinetic energy of the particle.

$T = \frac{1}{2}mv^2$

$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial y} = \frac{\partial T}{\partial z} = 0$ …………………….. (1)

$$\left.\begin{aligned} \frac{\partial T}{\partial \dot{x}} &= m\dot{x} \\ \frac{\partial T}{\partial \dot{y}} &= m\dot{y} \\ \frac{\partial T}{\partial \dot{z}} &= m\dot{z} \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

We know that,

The Lagranges equation of motion for a non conservative system m is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j.$$

Let F_x, F_y, F_z be the generalized force along the x,y,z axis respectively.

Then the Lagranges equation becomes,

$$\left.\begin{aligned} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} &= F_x \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) - \frac{\partial T}{\partial y} &= F_y \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{z}}\right) - \frac{\partial T}{\partial z} &= F_z \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

By using (1),(2),(3)

$$\frac{d}{dt}(m\dot{x}) = F_x$$

$$\frac{d}{dt}(m\dot{y}) = F_y$$

$$\frac{d}{dt}(m\dot{z}) = F_z$$

$$\text{(ie)} \left.\begin{aligned} m\ddot{x} &= F_x \\ m\ddot{y} &= F_y \\ m\ddot{z} &= F_z \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

In vector form eqn (4) can be written as $m(\ddot{x}_i + \ddot{y}_j + \ddot{z}_k) = F_x i + F_y j + F_z k$

$m\,\ddot{r} = \vec{F}.$

Motion of a single particle in space using plane polar coordinate

Proof:

Let r,θ be the plane polar coordinate of the particle.

If x,y be the Cartesion coordinates of the particle, then the transformation equation is x=r cosθ and y= r sinθ

$T = \frac{1}{2}mv^2$

$= \frac{1}{2}m\dot{x}^2+\dot{y}^2)$

$\dot{x}= -r \sin\theta\dot{\theta}+\dot{r} \cos\theta$

$\dot{y}= r \cos\theta\dot{\theta}+\dot{r}\sin\theta$

$(\dot{x}^2+\dot{y}^2) \quad =(-r \sin\theta\dot{\theta}+\dot{r} \cos\theta)^2 +(r \cos\theta\dot{\theta}+\dot{r}\sin\theta)^2$

$= \dot{r}^2(\cos^2\theta +\sin^2\theta)+ r^2(\dot{\theta}^2 \sin^2\theta + \cos^2\theta\dot{\theta}^2)$

$=\dot{r}^2+ r^2\dot{\theta}^2(\sin^2\theta + \cos^2\theta)$

$=\dot{r}^2+ r^2\dot{\theta}^2$

The kinetic energy $T = \frac{1}{2}m(\dot{r}^2+ r^2\dot{\theta}^2)$

$\frac{\partial T}{\partial r} =mr\dot{\theta}^2$

$\frac{\partial T}{\partial \dot{r}} =m\dot{r}$

$\frac{\partial T}{\partial \theta} =0$

$\frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta}$

Let Q_r and Q_θ be the generalized forces along the radial and transverse directions.

Then $Q_r = F.\frac{\partial \vec{r}}{\partial r}$

$Q_\theta \quad = F.\frac{\partial \vec{r}}{\partial \theta}$

Now,

r $\quad = x_i + y_j$

r $\quad = r\cos\theta\ i + r\sin\theta\ i$

$\frac{\partial \vec{r}}{\partial r} \quad = \cos\theta\ i + \sin\theta\ j$

$\frac{\partial \vec{r}}{\partial \theta} \quad = -r\sin\theta\ i + r\cos\theta j$

$\quad = r(-\sin\theta\ i + \cos\theta j)$

$\quad = r\hat{n}$

$Q_r = F.\hat{r} = F_r$

The Lagranges equation are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = Q_\theta$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = F_r$$

$$\frac{d}{dt}(m\dot{r}) - mr\dot{\theta}^2 = F_r$$

$$m\ddot{r} - mr\dot{\theta}^2 = F_r$$

$$m(\ddot{r} - r\dot{\theta}^2) = F_r$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = F_\theta r$$

$$\frac{d}{dt}(mr^2\dot{\theta}) - 0 = F_\theta r$$

$m\frac{d}{dt}(r^2\dot{\theta}) = F_\theta r$

$F_\theta = \frac{1}{r} m\frac{d}{dt}(r^2\dot{\theta})$

Derive Lagrange's equation of motion for Atwood's machine.

A string passes over a smooth fixed pulley and carries at the ends masses m_1 and m_2.

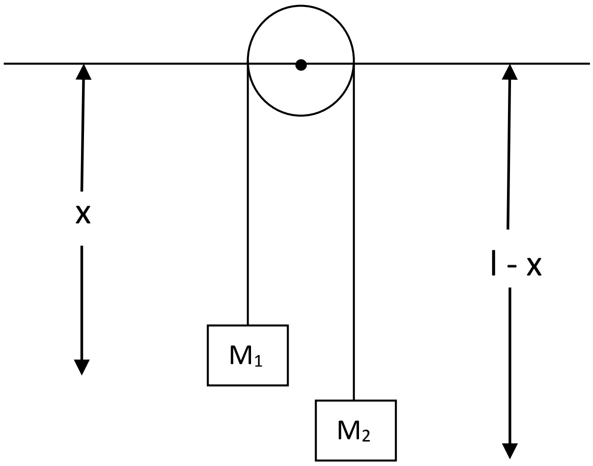
Let x be the depth of m_1 below the axis pulley. The depth of m_2 below AB is

l –x where l is the length of the string between the two weights.

The kinetic energy $T = \frac{1}{2} m_1(\frac{d}{dt}x)^2 + \frac{1}{2} m_2(\frac{d}{dt}(l-x))^2$

$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2$$

Let AB be the standard level of potential energy.



Potential energy of $m_1 = -m_1gx$ and

potential energy of $m_2 = - m_2 g(l-x)$

The Lagrangian L =T-V

(ie) $L = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + m_1gx + m_2 g(l-x)$

We know that,

The Lagrangian equation of motion for a holonomic conservative system is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

Since the pulley is assumed frictionless and mass less it is clear that there is only one independent coordinate 'x'

The Lagrange's equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

Now,

$$\frac{\partial L}{\partial \dot{x}} = (m_1+m_2)\dot{x}$$

$$\frac{\partial L}{\partial x} = 0 + m_1 g + m_2 g(0-1)$$

$$= m_1 g - m_2 g$$

$$= (m_1 - m_2)g$$

There fore,

The Lagrange's equation becomes,

$$\frac{d}{dt}((m_1+m_2)\dot{x}) - (m_1 - m_2)g = 0$$

$$(m_1+m_2)\ddot{x} - (m_1 - m_2)g = 0$$

$$(m_1+m_2)\ddot{x} = (m_1 - m_2)g$$

$$\ddot{x} = \frac{(m1 - m2)g}{(m1+m2)}$$

Obtain the equation of motion of a bead sliding on a uniformly rotating wire in a force free space.

Proof:

Consider a bead sliding on a uniformly rotating wire in a force free space.

The wire is straight and is rotated uniformly about some fixed axis perpendicular to the wire.

Since the wire is rotating uniformly, the angular velocity w is constant.

Consider a bead as a point.

Let the coordinate of the bead be (r, θ)

The transformation equation of the bead is x=r cosθ and y=rsinθ

We can expressed the constraint by the relation θ =ωt

Here θ is not independent and r is only independent object.

Generalized coordinate is r.

The transformation equation becomes x=r cos ωt, y=rsin ωt

$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$

Now,

X =r cos ωt

$\dot{x}$ = - rsin ωt. $\omega + \dot{r}$ cos ωt

$\dot{x}^2 = \omega^2 r^2 \sin^2 \omega t + \dot{r}^2 \cos^2 \omega t - 2r\, \omega \dot{r} \sin \omega t \cos \omega t$

y=r sin ωt

$\dot{y}$= rcos ωt. $\omega + \dot{r}$ sin ωt

$\dot{y}^2 = \omega^2 r^2 \cos^2 \omega t + \dot{r}^2 \sin^2 \omega t + 2r\, \omega \dot{r} \sin \omega t \cos \omega t$

$\dot{x}^2 + \dot{y}^2 = \omega^2 r^2 \sin^2 \omega t + \dot{r}^2 \cos^2 \omega t - 2r\, \omega \dot{r} \sin \omega t \cos \omega t + \omega^2 r^2 \cos^2 \omega t + \dot{r}^2 \sin^2 \omega t + 2r\, \omega \dot{r} \sin \omega t \cos \omega t$

$= \omega^2 r^2 + \dot{r}^2$

$T = \frac{1}{2} m(\omega^2 r^2 + \dot{r}^2)$

We know that,

The Lagranges equation of motion is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = Q_r$$

Since we consider the force free space $Q_r = 0$

$$\frac{\partial T}{\partial \dot{r}} = \frac{1}{2} m(2\dot{r})$$

$=m\dot{r}$

$\frac{\partial T}{\partial r}=\frac{1}{2}$ m(0+2r ω^2)

$=\frac{1}{2}$ m2r ω^2

$=$ mr ω^2

$\frac{d}{dt}$ m$\dot{r}$ $-$ mr ω^2=0

m$\ddot{r}$ $-$ mr ω^2=0

m$\ddot{r}$= mr ω^2

$\ddot{r}$=r ω^2

The bead moves out wise due to the contrapuntal acceleration.

Problem

1.Show that Lagranges equation in the form of equation $\frac{d}{dt}(\frac{\partial \vec{T}}{\partial \dot{q}_j}) - \frac{\partial \vec{T}}{\partial q_j} = \vec{Q}_j$ can also be written as $\frac{d}{dt}\frac{\overrightarrow{\partial \dot{T}}}{\partial q_j}$ -2$\frac{\partial \vec{T}}{\partial q_j} = \vec{Q}_j$. These are sometimes known as the Nielsen form of the Lagranges equations.

Proof:

We know that,

T is a kinetic energy.

$\vec{\dot{T}} = \frac{d}{dt}\vec{T}$

$= \frac{d\vec{T}}{dt}$

$\frac{\partial \vec{T}}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\frac{d\vec{T}}{dt}$

$$= \quad \frac{\partial}{\partial q_j}\left(\frac{d\vec{T}}{dq_j}\dot{q}_j\right)$$

$$=\frac{\partial}{\partial q_j}\frac{d\vec{T}}{dq_j}\dot{q}_j + \frac{d\vec{T}}{dq_j}\frac{\partial\dot{q}_j}{\partial\dot{q}_j}$$

$$= \frac{\partial}{\partial q_j}\left(\frac{d\vec{T}}{dq_j}\right)\dot{q}_j + \frac{d\vec{T}}{dq_j}\,(1)$$

$$= \frac{d}{dq_j}\left(\frac{\partial\vec{T}}{\partial\dot{q}_j}\right)\dot{q}_j + \frac{d\vec{T}}{dq_j}$$

$$= \frac{d}{dt}\left[\left(\frac{\partial\vec{T}}{\partial\dot{q}_j}\right)\frac{1}{\frac{dq_j}{dt}}\right]\dot{q}_j + \frac{d\vec{T}}{dq_j}$$

$$\frac{\partial\vec{T}}{\partial\dot{q}_j} = \frac{d}{dt}\left[\left(\frac{\partial\vec{T}}{\partial\dot{q}_j}\right) + \frac{\partial\vec{T}}{\partial q_j}\right]$$

Given,

$$\frac{d}{dt}\left[\left(\frac{\partial\vec{T}}{\partial\dot{q}_j}\right) - \frac{\partial\vec{T}}{\partial q_j} = Q_j\right.$$

$$\frac{d}{dt}\left[\left(\frac{\partial\vec{T}}{\partial\dot{q}_j}\right) = Q_j + \frac{\partial\vec{T}}{\partial q_j}\right.$$

$$\frac{\partial\vec{T}}{\partial\dot{q}_j} = Q_j + \frac{\partial\vec{T}}{\partial q_j} + \frac{\partial\vec{T}}{\partial q_j}$$

$$= 2\frac{\partial\vec{T}}{\partial q_j} + Q_j$$

$$\frac{\partial\vec{T}}{\partial\dot{q}_j} - 2\frac{\partial\vec{T}}{\partial q_j} = Q_j \text{ for all j}$$

This is Nielsen form of the Lagranges equation.

Unit-III

Variational principles and Lagrange's Equations

2.1 Hamilton Principle:

Configuration space and motion of the system:

Definition

Consider a system with n generalized coordinates $q_1, q_2, \ldots, q_n$. The instantaneous configuration of a system can be specified by a particular point in a Cartesian hyper triangle space of n dimensional with q's form the n coordinate axes. This n dimensional space is known as configuration space. As time goes in the state of the system changes and the system point moves in configuration space tracing out a curve described as "The path of the motion of the system''.

Hence the path of motion of the system refers to a curve in the configuration space along with the system point moves. The time t is consider as a parameter of the curve, to each point on the path there is associated one are more values of the time.

Hamilton's Principle

For monogenic system Hamilton's Principlestates that "The motion of the system from time t_1 to time t_2 is such that the line integral I = $\int_{t_1}^{t_2} L\, dt$ where L = T-V, has a stationary value for the correct path of themotion. We have summarized Hamilton's Principle that the motion is such that the variation of the line integral I for fixed time t_1 and t_2 is zero.

(ie)$\delta I = \delta \int_{t_1}^{t_2} L\,(q_1, q_2, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots \dot{q}_n, t)dt = 0$

(2.2) Some techniques of the calculus of variations:

Find the path y=y(x) between two values x_1 and x_2 such that the line integral J of some function f(y,y',x) is an extremum.

Solution:

We have to consider only the varied path for which y_1=y(x_1) and y_2=y(x_2)

(ie)We consider all paths passes through (x_1,y_1) and (x_2,y_2).

Now we label all possible paths we have fix under consideration with different values of a parameter α. Such that α=0 the curve would coincide with the path or the paths giving on extremum for the integral.

Then y would be a function of both x and α can be represented by y(x, α)=y(x,0)+ $\alpha\eta$(x) which vanishes at x=x_1 and x=x_2.

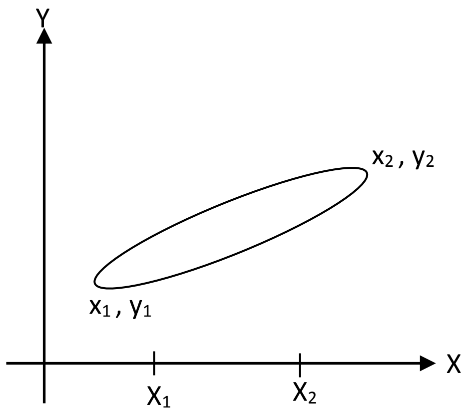
Then J=$\int_{x_1}^{x_2} f$ (y,$\frac{dy}{dx}$,x) dx is also a function of α.

We get,

J(α) = $\int_{x_1}^{x_2} f$ (y(x, α),$\frac{dy}{dx}$(x, α),x) dx

(ie) J(α) = $\int_{x_1}^{x_2} f$ (y,$\dot{y}$,x) dx

Differentiate with respect to 'α'



$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right)dx$$

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right)dx + \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha}\right)dx \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots(*)$$

Now,

$$\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right)dx = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial \dot{y}}\frac{\partial}{\partial \alpha}\left(\frac{dy}{dx}\right)\right)dx$$

$$= \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial \dot{y}}\left(\frac{d}{dx}\right)\frac{\partial y}{\partial \alpha}\right)dx$$

$$= \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial \dot{y}}\, d\,\frac{\partial y}{\partial \alpha}\right)$$

Since all the varied paths passes through (x_1,y_1) and (x_2,y_2)

$(\frac{\partial y}{\partial \alpha})_{x1}$ =0 and$(\frac{\partial \dot{y}}{\partial \alpha})_{x2}$ = 0

We have,

$\int_{x_1}^{x_2}(\frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha})dx = -\int_{x_1}^{x_2}(\frac{\partial y}{\partial \alpha} (\frac{d}{dx}) \frac{\partial f}{\partial \dot{y}})dx$

Sub *

$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2}(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha})dx - \int_{x_1}^{x_2}(\frac{\partial y}{\partial \alpha} (\frac{d}{dx}) \frac{\partial f}{\partial \dot{y}})dx$

Multiply both side by dα and evaluating the derivative $\alpha = 0$

$(\frac{\partial J}{\partial \alpha})_{\alpha=0}d\alpha = \int_{x_1}^{x_2}(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}})](\frac{\partial y}{\partial \alpha})_{\alpha=0}dxd\alpha$

$(\frac{\partial J}{\partial \alpha})_{\alpha=0}d\alpha = \delta J$

$(\frac{\partial y}{\partial \alpha})_{\alpha=0}d\alpha = \delta y$

$(\frac{\partial \dot{y}}{\partial \alpha})_{\alpha=0}d\alpha = \delta \dot{y}$

$\delta J = \int_{x_1}^{x_2}(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}})] \delta y \, dx$

Now the integral J will be an extremum only if δJ =0

Here δJ =0

Here δy represents some arbitrary variation of α its o value and it is arbitrary.

Hence δJ =0 only if $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0$

Thus J is an extremum only for these curve y=y(x) for which $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}} = 0$.

Application of calculus of variation

Prove that the shortest distance between two points in a plane as well as in space is a straight line.

Proof:

Case (i)

We know that,

An arc element of arc length in a plane is ds = $\sqrt{dx^2 + dy^2}$

$$=\sqrt{dx^2(1 + \frac{dy^2}{dx^2})}$$

$$ds = \sqrt{(1 + \frac{dy^2}{dx^2})}\ dx$$

The total length of any curve between points 1 and 2 is S = $\int_1^2 ds$

$$= \int_{x_1}^{x_2} \sqrt{(1 + \frac{dy^2}{dx^2})}\ dx$$

$$=\int_{x_1}^{x_2} \sqrt{1 + \dot{y}^2}$$

S = $\int_{x_1}^{x_2} f(y, \dot{y}, x)$dx where $f(y, \dot{y}, x) = \sqrt{1 + \dot{y}^2}$

From calculus of variation $\frac{\partial f}{\partial y} = \frac{d}{dx}(\frac{\partial f}{\partial \dot{y}})=0$.

We find S will be least for the path joining x_1 and x_2 for which

$\frac{\partial f}{\partial y} - \frac{d}{dx}(\frac{\partial f}{\partial \dot{y}})=0$………………………….. (1)

$f(y, \dot{y}, x) = \sqrt{1 + \dot{y}^2}$

$\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial \dot{y}} = \frac{1}{2}\frac{2\dot{y}}{\sqrt{1+\dot{y}^2}}$

$= \frac{\dot{y}}{\sqrt{1+\dot{y}^2}}$

(1) ⇒s 0 - $\frac{d}{dx}(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}})$ =0

$\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = 0$

By Squaring on both side

We get,

$\frac{\dot{y}^2}{1+\dot{y}^2} = C^2$

$\dot{y}^2 = C^2(1 + \dot{y}^2)$

$\dot{y}^2 = C^2 + \dot{y}^2 C^2$

$\dot{y}^2(1-C^2) = C^2$

$\dot{y}^2 = \frac{C^2}{(1-C^2)}$

$\dot{y} = \frac{C}{\sqrt{1-C^2}} = a$

$(ie)\ \frac{dy}{dx} = a$

$y = ax + b$

S is least along the straight line joining the two points. So the shortest distance between two points in a plane is a straight line.

The constants ‘a’ and ‘b’ can be found from the initial condition that the curve passes through the two end points (x_1,y_1) and (x_2,y_2).

Case:(ii)

If there are two points in space ds $= \sqrt{dx^2 + dy^2 + dz^2}$

$= \sqrt{dx^2(1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}}$

$= \sqrt{(1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}}$ dx

$$S = \int_1^2 ds$$

$$= \int_{x_1}^{x_2} \sqrt{(1 + (\frac{dy}{dx})^2 + (\frac{dz}{dx})^2}\ dx$$

$$= \int_{x_1}^{x_2} \sqrt{1 + \dot{y}^2 + \dot{z}^2}\ dx$$

S = $\int_{x_1}^{x_2} f(y, \dot{y}, z, \dot{z}, x)$dx where $f(y, \dot{y}, z, \dot{z}, x)$ = $\sqrt{1 + \dot{y}^2 + \dot{z}^2}$

S is an extremum if f satisfies the condition

$\frac{\partial f}{\partial y}$ - $\frac{d}{dx}$ ($\frac{\partial f}{\partial \dot{y}}$)=0............................... (1)

$\frac{\partial f}{\partial z}$ - $\frac{d}{dx}$ ($\frac{\partial f}{\partial \dot{z}}$)=0............................... (2)

$\frac{\partial f}{\partial y} = 0$

$\frac{\partial f}{\partial \dot{y}} = \frac{1}{2} \frac{2\dot{y}}{\sqrt{1+\dot{y}^2+\dot{z}^2}}$

$= \frac{\dot{y}}{\sqrt{1+\dot{y}^2+\dot{z}^2}}$

$\frac{\partial f}{\partial z} = 0$

$\frac{\partial f}{\partial \dot{z}} = \frac{1}{2} \frac{2\dot{z}}{\sqrt{1+\dot{y}^2+\dot{z}^2}}$

$= \frac{\dot{z}}{\sqrt{1+\dot{y}^2+\dot{z}^2}}$

(1) Implies 0 - $\frac{d}{dx}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2+\dot{z}^2}}\right)$ =0

$\frac{\dot{y}}{\sqrt{1+\dot{y}^2+\dot{z}^2}} = C_1$

By squaring we get,

$\frac{\dot{y}^2}{1+\dot{y}^2+\dot{z}^2} = {C_1}^2$

$\dot{y}^2 = {C_1}^2(1 + \dot{y}^2 + \dot{z}^2)$

$\dot{y}^2 = {C_1}^2 + \dot{y}^2{C_1}^2 + {C_1}^2\dot{z}^2$

$\dot{y}^2(1-{C_1}^2) = {C_1}^2 + {C_1}^2\dot{z}^2$

$\dot{y}^2 = \frac{{C_1}^2(1+\dot{z}^2)}{(1-{C_1}^2)}$ .... (3)

$(2) \Rightarrow \; 0 - \frac{d}{dx}\left(\frac{\dot{z}}{\sqrt{1+\dot{y}^2+\dot{z}^2}}\right) \qquad = 0$

$\frac{\dot{z}}{\sqrt{1+\dot{y}^2+\dot{z}^2}} \qquad = C_2$

By squaring we get,

$\frac{\dot{z}^2}{1+\dot{y}^2+\dot{z}^2} \qquad = {C_2}^2$

$\dot{z}^2 \qquad = {C_2}^2(1 + \dot{y}^2 + \dot{z}^2)$

$\dot{z}^2 \qquad = {C_2}^2 + \dot{y}^2{C_2}^2 + {C_2}^2\dot{z}^2$

$\dot{y}^2{C_2}^2 \qquad = \dot{z}^2 - {C_2}^2 - {C_2}^2\dot{z}^2$

$\dot{y}^2 \qquad = \frac{\dot{z}^2 - {C_2}^2(1+\dot{z}^2)}{({C_2}^2)}$ ………………………… (4)

Equating (3) and (4)

$\frac{{C_1}^2(1+\dot{z}^2)}{(1-{C_1}^2)} = \frac{\dot{z}^2 - {C_2}^2(1+\dot{z}^2)}{({C_2}^2)}$

$0 = \dot{z}^2\,(1 - {C_2}^2 - {C_1}^2) - {C_2}^2$

$\dot{z}^2\,(1 - {C_2}^2 - {C_1}^2) = {C_2}^2$

$\dot{z}^2 = \frac{{C_2}^2}{(1 - {C_2}^2 - {C_1}^2)}$

$\dot{z} = \frac{C_2}{\left(\sqrt{1 - {C_2}^2 - {C_1}^2}\right)}$

$\dot{z} = a$

(ie) $\frac{dz}{dx} = a$

Z = ax+b

Similarly,

y=cx+d

These two equations represents two planes, the line of intersection of the above two planes will be the shortest path of two points in space.

The shortest distance between two points in space is a straight line.

Geodesics

Curves that give shortest distance between two points on a given surface are called the Geodesics of the surface.

Show that the surface of revolution obtained by revolving a curve between two fixed points about the y axis is minimum if the curve is a centenary.

Proof:

Let (x_1,y_1) and (x_2,y_2) be two points in a plane.

Take any curve joining these points and rotate the curve about the y axis.

Then we get a surface of revolution. Let ds be the length of the elementary arc which is at distance x from the y axis.

Now,

The surface area of the strip of length ds is $2\pi x\, ds$.

Let I be the total surface area I = $\int_{x_1}^{x_2} 2\pi x\, ds$

ds $= \sqrt{1+\dot{y}^2}\, dx$

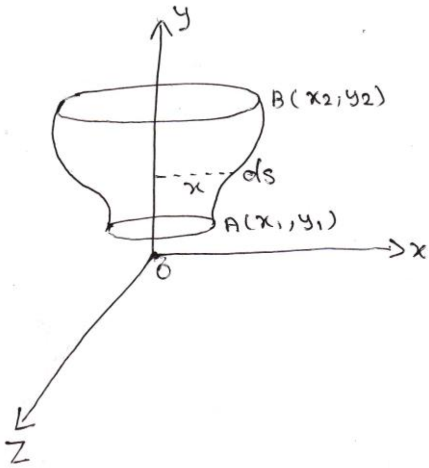
$=\int_{x_1}^{x_2} 2\pi x \sqrt{1+\dot{y}^2}\, dx$

$= \int_{x_1}^{x_2} f(y,\dot{y},x)dx$ where $f(y,\dot{y},x) = 2\pi x\sqrt{1+\dot{y}^2}$

By calculus of variation it will be least if f satisfies the equation.

$\frac{\partial f}{\partial y}$ - $\frac{d}{dx}$ ($\frac{\partial f}{\partial \dot{y}}$)=0…………………………… (1)

$\frac{\partial f}{\partial y} = 0$



$\frac{\partial f}{\partial \dot{y}} = 2\pi x \frac{1}{2}(1+\dot{y}^2)^{-\frac{1}{2}} 2\dot{y}$

$= \frac{2\pi x\dot{y}}{\sqrt{1+\dot{y}^2}}$

(1) $\Rightarrow$ $0 - \frac{d}{dx}\frac{2\pi x\dot{y}}{\sqrt{1+\dot{y}^2}} = 0$

$\frac{x\dot{y}}{\sqrt{1+\dot{y}^2}} = \frac{c}{2\pi}$

$x\dot{y} \quad = C_1(\sqrt{1+\dot{y}^2})$

Squaring on both sides

$x^2\dot{y}^2 = C_1^2(1+\dot{y}^2)$

$x^2\dot{y}^2 = C_1^2 + C_1^2\dot{y}^2$

$(x^2 - C_1^2)\dot{y}^2 \quad = C_1^2$

$\dot{y}^2 \quad = \frac{C_1^2}{(x^2 - C_1^2)}$

$\dot{y} \quad = \frac{C_1}{\sqrt{(x^2 - C_1^2)}}$

$\frac{dy}{dx} \quad = \frac{C_1}{\sqrt{(x^2 - C_1^2)}}$

$dy \quad = \frac{C_1}{\sqrt{(x^2 - C_1^2)}}dx$

Integrating on both sides,

$$\int dy \ = \int \frac{C_1}{\sqrt{(x^2 - {C_1}^2)}} \text{dx}$$

$$\text{Y} \quad = C_1 \int \frac{1}{\sqrt{(x^2 - {C_1}^2)}} \text{dx}$$

$$\text{Y} \quad = C_1(\cosh^{-1}(\frac{x}{c_1}) + b$$

$$\text{y-b} \quad = C_1(\cosh^{-1}(\frac{x}{c_1}))$$

$$\frac{\text{y}-\text{b}}{C_1} = \cosh^{-1}(\frac{x}{c_1})$$

$$\text{Cosh}\ (\frac{\text{y}-\text{b}}{C_1}) = (\frac{x}{c_1})$$

$$\text{X} = c_1\text{Cosh}\ (\frac{\text{y}-\text{b}}{C_1})$$

The values of b and c_1 can be determined by the condition that the curve passes through (x_1,y_1) and (x_2,y_2)

The required curve is a centenary passing through two points.

Show that the surface of revolution obtained by revolving a curve between two fixed points about the x axis is minimum if the curve is a centenary.

Proof:

Let (x_1,y_1) and (x_2,y_2) be two points in a plane.

Take any curve joining these points and rotate the curve about the x axis.

Then we get a surface of revolution . Let ds be the length of the elementary arc which is at distance y from the x axis.

Now,

The surface area of the strip of length ds is 2πy ds.

Let I be the total surface area I = $\int_{y_1}^{y_2} 2\pi y \, ds$

ds $= \sqrt{1+\dot{x}^2}\, dy$

$= \int_{y_1}^{y_2} 2\pi y \sqrt{1+\dot{x}^2}\, dy$

$= \int_{y_1}^{y_2} f(x,\dot{x},y) dy$ where $f(x,\dot{x},y) = 2\pi y\sqrt{1+\dot{x}^2}$

By calculus of variation it will be least if f satisfies the equation.

$\frac{\partial f}{\partial x} - \frac{d}{dy}\left(\frac{\partial f}{\partial \dot{x}}\right)=0$………………………….. (1)

$\frac{\partial f}{\partial x}= 0$

$\frac{\partial f}{\partial \dot{x}} = 2\pi y \frac{1}{2}(1+\dot{x}^2)^{-\frac{1}{2}} 2\dot{x}$

$= \frac{2\pi y\dot{x}}{\sqrt{1+\dot{x}^2}}$

(1) Implies $0- \frac{d}{dy}\frac{2\pi y\dot{x}}{\sqrt{1+\dot{x}^2}}=0$

$\frac{y\dot{x}}{\sqrt{1+\dot{x}^2}} = \frac{c}{2\pi}$

$y\dot{x} = C_1(\sqrt{1+\dot{x}^2})$

Squaring on both sides

$y^2\dot{x}^2 = C_1^2(1+\dot{x}^2)$

$y^2\dot{x}^2 = C_1^2 + C_1^2\dot{x}^2$

$(y^2 - C_1^2)\dot{x}^2 = C_1^2$

$\dot{x}^2 = \frac{C_1^2}{(y^2 - C_1^2)}$

$\dot{x} = \frac{C_1}{\sqrt{(y^2 - C_1^2)}}$

$$\frac{dx}{dy} = \frac{C_1}{\sqrt{(y^2 - C_1{}^2)}}$$

$$dx = \frac{C_1}{\sqrt{(y^2 - C_1{}^2)}} dx$$

integrating on both sides,

$$\int dx \;\; = \int \frac{C_1}{\sqrt{(y^2 - C_1{}^2)}} dy$$

$$x \;\; = C_1 \int \frac{1}{\sqrt{(y^2 - C_1{}^2)}} dy$$

$$x \;\; = C_1(\cosh^{-1}(\frac{y}{c_1}) + b$$

$$x\text{-}b \;\; = C_1(\cosh^{-1}(\frac{y}{c_1}))$$

$$\frac{x-b}{C_1} \;\; = \cosh^{-1}(\frac{y}{c_1})$$

$$\text{Cosh}\,(\frac{x-b}{C_1}) = (\frac{y}{c_1})$$

$$Y \;\; = c_1 \text{Cosh}\,(\frac{x-b}{C_1})$$

The values of b and c_1 can be determined by the condition that the curve passes through (x_1,y_1) and (x_2,y_2)

The required curve is a catenary passing through two points.

State and prove Brachistohrone problem

Statement:

Show that the paths followed by a particle in sliding from one point to another in the absence of friction, in the shortest time in a cycloid.

(OR)

Using the variational principle find the equation the curve joining two points along, which a particle falling from rest under the influence of gravity travelling from the higher to the lower point in the least time.

Solution:

Let V be the speed along the curve.

Then the time required to fall a distance ds is $\frac{ds}{v}$

The time taking to travel from the point one to point 2 is $t_{12} = \int_1^2 \frac{ds}{v}$ ……energy at point 2……… (1)

Suppose y is measured from initial point of release, now the total energy at point 1 =Kinetic energy at point 1 +Potential energy at point 2

$=0 + 0$

$= 0$

Total energy at point 2 = Kinetic energy at point 1 +Potential energy at point 2

$=\frac{1}{2}mv^2 - mgh$

$=\frac{1}{2}mv^2 - mgy$

Where y is the vertical distance.

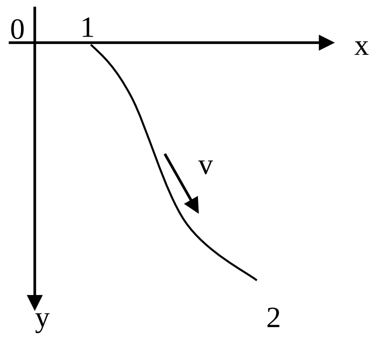
By the principle of conservation of energy, the total energy at point 1 = Total energy at point 2.

$0 = \frac{1}{2}mv^2 - mgy$

$v^2 = 2gy$ ………………………………… (2)

(1) Implies $t_{12} = \int_1^2 \frac{ds}{\sqrt{2gy}}$

$= \int_1^2 \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}} dx$



$= \int_{x_1}^{x_2} f(y, \dot{y}, x) dx$ where $f(y, \dot{y}, x) = \frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}}$

Now,

$df = \frac{\partial f}{\partial x}.dx + \frac{\partial f}{\partial y}.dy + \frac{\partial f}{\partial \dot{y}}.d\dot{y}$

$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}.\frac{dy}{dx} + \frac{\partial f}{\partial \dot{y}}.\frac{d\dot{y}}{dx}$………………………….. (3)

Since t_{12} is minimum by calculus of variation we have,

$\frac{\partial f}{\partial y} - \frac{d}{dx}(\frac{\partial f}{\partial \dot{y}}) = 0$

$\frac{\partial f}{\partial y} = \frac{d}{dx}(\frac{\partial f}{\partial \dot{y}})$ …………………. (4)

(2) implies $\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{d}{dx}\frac{\partial f}{\partial y}.\frac{dy}{dx} + \frac{\partial f}{\partial \dot{y}}.\frac{d}{dx}\frac{d\dot{y}}{dx}$

$= \frac{\partial f}{\partial x} + \frac{d}{dx}\frac{\partial f}{\partial y}.\dot{y}$

$\Rightarrow \frac{df}{dx} - \frac{d}{dx}(\frac{\partial f}{\partial y}.\dot{y}) = \frac{\partial f}{\partial x}$

$$\frac{d}{dx}(f - \left(\frac{\partial f}{\partial y}.\dot{y}\right)) = \frac{\partial f}{\partial x}$$

$$\frac{d}{dx}(f - \left(\frac{\partial f}{\partial y}.\dot{y}\right)) = 0$$

$$\left(f - \left(\frac{\partial f}{\partial y}.\dot{y}\right)\right) = \text{constant}$$

$$\frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}} - \frac{\dot{y}}{\sqrt{2gy\sqrt{1+\dot{y}^2}}}.\dot{y} = \text{constant}$$

$$\frac{\sqrt{1+\dot{y}^2}}{\sqrt{2gy}} - \frac{\dot{y}^2}{\sqrt{2gy\sqrt{1+\dot{y}^2}}} = \text{constant}$$

$$\frac{\sqrt{1+\dot{y}^2}\sqrt{1+\dot{y}^2} - \dot{y}^2}{\sqrt{2gy}\sqrt{1+\dot{y}^2}} = \text{constant}$$

$$\frac{1+\dot{y}^2-\dot{y}^2}{\sqrt{2gy}\sqrt{1+\dot{y}^2}} = \text{constant}$$

$$\frac{1}{\sqrt{2gy}\sqrt{1+\dot{y}^2}} = \text{constant}$$

$$\frac{1}{\sqrt{2g}\sqrt{1+\dot{y}^2}}\frac{1}{\sqrt{y}} = \text{constant}$$

$$\frac{1}{\sqrt{y(1+\dot{y}^2)}} = \text{constant}$$

$$\sqrt{y(1+\dot{y}^2)} = \frac{1}{\sqrt{2g}}$$

$$y(1+\dot{y}^2)^2 = \frac{1}{2g}$$

$y(1+\dot{y}^2)^2 = 2a$ (since $2a = \frac{1}{2g}$) ……………………………. (5)

Now,

$\dot{y} = \frac{dy}{dx}$ =tan ψ

(5) implies

Y(1 +tan$^2\psi$) =2a

Y(sec$^2\psi$) =2a

Y = 2a$\frac{1}{\sec 2\,\psi}$

= 2a cos$^2\psi$

Y = a(1+cos 2ψ) …………………………..(6)

To find x

We have,

$\frac{dy}{dx}$ =tan ψ

dx = $\frac{dy}{\tan\psi}$

dx =cot ψ dy

=$\frac{\cos\psi}{\sin\psi}$d[(a(1+cos 2$\psi$)]

=$\frac{\cos\psi}{\sin\psi}$ (-asin2ψ 2$d\psi$)

= -2a $\frac{\cos\psi}{\sin\psi}$ 2cosψsin ψ

=-2a.2cos$^2\psi$ d ψ

dx = 2a(1+cos 2ψ)d ψ

Integrating,

$\int dx$ = -2a $\int (1 + \cos 2\psi)$ d ψ

X= -2a$[\psi + \frac{\sin 2\psi}{2}]$ + c

X = -a (2 ψ +sin2ψ) + c …………………. (7)

x-c = -a (2 ψ +sin2ψ)

Thus the required curve has a parametric equation x-c = -a (2 ψ +sin2ψ) and y = a(1 +cos 2ψ)

Hence the required curve is a cycloid.

(2.3)Derivation of Lagranges equation from Hamiltonian principle

(or)

Derive Euler Lagrangian differential equation

Solution:

The calculus of variation principle can be extended to many number of independent variables.

Let $y_1, y_2, \ldots y_n$ be independent variables and x be any dependent variable.

Consider the integral,

J =$\int_1^2 f[y_1(x), y_2(x), \ldots, y_n(x), \dot{y}_1(x), \dot{y}_2(x), \ldots \dot{y}_n(x), x]dx$

Now,

Label all parts of $y_1(x), y_2(x), \ldots, y_n(x)$ with different values of the parameter α.

In this case we can write

$y_1(x,\alpha) = y_1(x,0) + \alpha\, \eta_1(x)$

$y_2(x,\alpha) = y_2(x,0) + \alpha\, \eta_2(x)$

…………………………

…………………………

………………………….

$Y_n(x,\alpha) = y_n(x,0) + \alpha\, \eta_n(x)$

Suppose, when $\alpha = 0$, these parts $y_1(x,0), y_2(x,0), \ldots y_n(x,0)$ are the required parts so as to give an extreme for J.

Also,

$\eta_1(x), \eta_2(x), \ldots, \eta_n(x)$ vanishes at the end points.

Now, substituting the value of y Now, substituting the value of y_i's in terms of α in J we get,

$$J = \int_1^2 f(y_1(x,\alpha), y_2(x,\alpha), \ldots, Y_n(x,\alpha), \dot{y}_1(x,\alpha), \dot{y}_2(x,\alpha), \ldots, \dot{y}_n(x,\alpha), x)dx$$

$$J = \int_1^2 f(\dot{y}_1(x,\alpha), \dot{y}_2(x,\alpha), \ldots, \dot{y}_n(x,\alpha), x)dx$$

The condition for J to be an extremum is $\left(\frac{\partial J}{\partial \alpha}\right)_{\alpha=0} = 0$

$$\frac{\partial J}{\partial \alpha} = \int_1^2 \left(\sum_{i=1}^{n} \left[\frac{\partial f}{\partial y_i} . \frac{\partial y_i}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}_i} . \frac{\partial \dot{y}_i}{\partial \alpha} \right] \right) dx$$

$$= \int_1^2 \left(\sum_{i=1}^{n} \left[\frac{\partial f}{\partial y_i} . \frac{\partial y_i}{\partial \alpha} \right] \right) dx + \int_1^2 \left(\sum_{i=1}^{n} \left[\frac{\partial f}{\partial \dot{y}_i} . \frac{\partial \dot{y}_i}{\partial \alpha} \right] \right) dx$$

Consider,

$$\int_1^2 \left(\sum_{i=1}^{n} \left[\frac{\partial f}{\partial \dot{y}_i} . \frac{\partial \dot{y}_i}{\partial \alpha} \right] \right) dx$$

$$= \int_1^2 \left(\sum_{i=1}^{n} \left[\frac{\partial f}{\partial \dot{y}_i} . \frac{d}{dx} \frac{\partial y_i}{\partial \alpha} \right] \right) dx$$

$=\int_1^2 (\sum_{i=1}^{n} [\frac{\partial f}{\partial \dot{y}_i} . d \frac{\partial y_i}{\partial \alpha}])$

$= \sum_{i=1}^{n} [\frac{\partial f}{\partial \dot{y}_i} . \frac{\partial \dot{y}_i}{\partial \alpha}]_1^2$ - $\int_1^2 (\frac{\partial y_i}{\partial \alpha}) \frac{d}{dx} \frac{\partial f}{\partial \dot{y}_i}$

But we know that y_i(x,α) =y_i(x,0) +αη_i(x)

$\frac{\partial y_i}{\partial \alpha} = \eta_i(x)$

But we have assumed that η vanishes at $[\frac{\partial y_i}{\partial \alpha}]_1^2 = [\eta_i(x)]_1^2$

$\int_1^2 (\sum_{i=1}^{n} [\frac{\partial f}{\partial \dot{y}_i} . \frac{\partial \dot{y}_i}{\partial \alpha}])dx = 0 - \int_1^2 (\sum_{i=1}^{n} [\frac{\partial y_i}{\partial \alpha} \frac{d}{dx} \frac{\partial f}{\partial \dot{y}_i}])$

$\frac{\partial J}{\partial \alpha} = \int_1^2 (\sum_{i=1}^{n} [\frac{\partial f}{\partial \dot{y}_i} . \frac{\partial \dot{y}_i}{\partial \alpha}])dx - \int_1^2 (\sum_{i=1}^{n} [\frac{\partial y_i}{\partial \alpha} \frac{d}{dx} \frac{\partial f}{\partial \dot{y}_i}])$

$= \int_1^2 (\sum_{i=1}^{n} [\frac{\partial y_i}{\partial \alpha}] \{ \frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}_i} \})dx$

Multiply both sides by dα and evaluating the derivatives at α =0,

we get,

$(\frac{\partial f}{\partial \alpha})_{\alpha=0}\, d\alpha = \int_1^2 (\sum_{i=1}^{n} [\frac{\partial y_i}{\partial \alpha}]_{\alpha=0}\, d\alpha\,] \{ \frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}_i} \})dx$

Denote $(\frac{\partial f}{\partial \alpha})_{\alpha=0}\, d\alpha = \delta J$

And $[\frac{\partial y_i}{\partial \alpha}]_{\alpha=0}\, d\alpha = \delta y_i$

$\delta J = \int_1^2 (\sum_{i=1}^{n} [\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}_i}])\, \delta y_i dx$

Now, the integral J will be an extremum only if $\delta J = 0$

(ie) only if $\int_1^2 (\sum_{i=1}^{n} [\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial \dot{y}_i}])\, \delta y_i dx = 0$

Since $y_i{}'s$ are independent $\delta y_i{}'$s are also independent.

The above equation is true only if $\sum_{i=1}^{n}[\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}_i}] = 0$

(ie) $\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}_i} = 0$

(or) $\frac{d}{dx}\frac{\partial f}{\partial \dot{y}_i} - \frac{\partial f}{\partial y_i} = 0$

These equations are called as the Euler Lagrangian differential equation.

(2.4) Extension of Hamiltons principle to non-holonomic system:

Derive Lagranges equation of motion for a non holonomic system

Proof:

Let us consider a non holonomic system in which the constraints are expressed by m equations of the form $\sum_k a_{lk} d_{qk} + a_{lt} dt = 0$ …………………(1)

We know that,

The Lagranges equation for a non holonomic system is,

$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_k}) - (\frac{\partial L}{\partial q_k}) = \sum_{l=1}^{m} \lambda_l a_{lk}$ ………………………….. (2)

If the system is holonomic, then the equation of the constraint can be written as $f(q_1, q_2, \dots, q_n, t) = 0$

By differentiation we get,

$\frac{\partial f}{\partial q_1} dq_1 + \frac{\partial f}{\partial q_2} dq_2 + \dots + \frac{\partial f}{\partial q_n} dq_n + \frac{\partial f}{\partial t} dt = 0$

(ie) $\sum_{k=1}^{n}[\frac{\partial f}{\partial q_k} dq_k + \frac{\partial f}{\partial t} dt] = 0$

This is of the (1) with its coefficients

$a_{lk} = \frac{\partial f}{\partial q_k}$, $a_{lt} = \frac{\partial f}{\partial t}$

sub in (2)

We get the Lagranges equation for a holonomic system.

$$(2) \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \left(\frac{\partial L}{\partial q_k}\right) = \sum_{l=1}^{m} \lambda_l \frac{\partial f}{\partial q_k}$$

A hoop is rolling down an inclined plane without slipping. Discuss its motion, using the undetermined multipliers.

Proof:

Let α be the inclination of the inclined plane and l be the length of the inclined plane.

Let A be the fixed point on the hoop.

Let BP = x and ΔPCA $=\theta$ then x and θ are the two generalized coordinates.

Let 'a' be the radius of the hoop

The rolling constraint can be expressed as $\dot{x} = a\dot{\theta}$

(ie)dx = a d θ

a d $\theta - dx$ =0

a d $\theta + (-1)dx$ =0 ……………………(1)

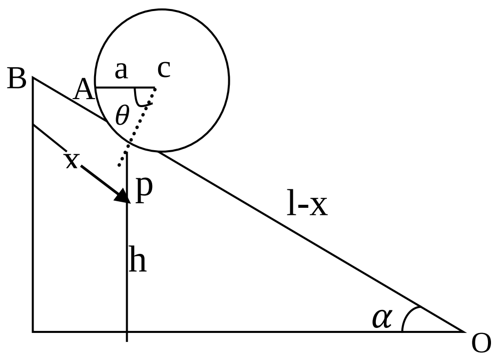
The Kinetic energy of T = Kinetic energy of motion of the center of mass + Kinetic energy of motion about the center of mass

(ie) T=$\frac{1}{2}$m(velocity of the center of mass)^2 + $\frac{1}{2}$m(velocity of the particle at the center of mass)^2

= $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}ma^2\dot{\theta}^2$

Potential energy of the hoop is v= mgh

= mg(l -x)sinα

$L = T - V$ 

$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}ma^2\dot{\theta}^2 - mg(l - x)\sin\alpha$

$\frac{dL}{dx} = (-mg)(-1)\sin\alpha = mg\sin\alpha$

$\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}m2\dot{x} = m\dot{x}$

$\frac{\partial L}{\partial \theta} = 0$

$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2}ma^2 2\dot{\theta} = ma^2\dot{\theta}$

Since there is one equation of the constraints, only one Lagranges multiplier λ is need.

There fore,

*La*granges equation for x and θ are $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = \lambda a x$ …………………….. (2)

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \left(\frac{\partial L}{\partial \theta}\right) = \lambda a\theta$ …………………………(3)

The coefficient appearing in the constraint equation are $a_x = -1$ and $a_\theta = a$.

(2) ⇒

$\frac{d}{dt} m\dot{x} - mg\sin\alpha = -\lambda$

$m\ddot{x} - mg\sin\alpha = -\lambda$ ………………………. (4)

(3) implies

$\frac{d}{dt} ma^2\dot{\theta} - 0 = \lambda_a$

$ma^2\ddot{\theta} - 0 = \lambda_a$

$ma\ddot{\theta} = \lambda$ ………………………. (5)

5 $\Rightarrow$

$m\ddot{x} = \lambda$ ……………………..(6)

(4) implies

$\lambda – mg \sin\alpha = -\lambda$

$2\lambda = mg \sin\alpha$

$\lambda = \frac{mg \sin\alpha}{2}$ ………………………. (7)

By (6) and (7)

$M\ddot{x} = \frac{mg \sin\alpha}{2}$

$\ddot{x} = \frac{g \sin\alpha}{2}$

The acceleration down the plane $= \frac{g}{2} \sin\alpha$

We have,

$\ddot{\theta} = \frac{\ddot{x}}{a} = \frac{g \sin\alpha}{2a}$

Instead of rolling if the hoop slips down the plane then its equation of motion is,

(4)$\Rightarrow$implies $M\ddot{x} - mg \sin\alpha = 0$

$M\ddot{x} = mg \sin\alpha$

$\ddot{x} = g \sin\alpha$

$This$ shows that the hoop rolls down the inclined plane with only one half the acceleration it would have slipped down a friction less plane and the frictional force of constraint is given by $\lambda = \frac{\text{mg sin}\alpha}{2}$

To find the velocity of the hoop at the bottom of the inclined plane.

$We\ have,$

$$\ddot{x} = \frac{\text{g sin}\alpha}{2}$$

$$But\ \ddot{x} = \frac{d}{dt}(\dot{x})$$

$$= \frac{d\dot{x}}{dx}\frac{dx}{dt}$$

$$= \frac{dv}{dx}.\text{v}$$

$$\text{V dv} = \ddot{x}\ \text{dx}$$

$$= \frac{\text{g sin}\alpha}{2}\ \text{dx}$$

By integrating, We get,

$$\frac{v^2}{2} = \frac{\text{g sin}\alpha\text{x}}{2} + \text{C}$$

At the top x =0, dv =0

$c = 0$

$$\frac{v^2}{2} = \frac{\text{g sin}\alpha\text{x}}{2}$$

$$v^2 = \text{g sin}\alpha\text{x}$$

$$v = \sqrt{\text{g sin}\alpha\text{x}}$$

At the bottom x=l

$$v = \sqrt{\text{gl sin}\alpha}$$

Velocity of hoop at the bottom is $v = \sqrt{\text{gl sin}\alpha}$

(2.6)Conservation theorem and symmetric properties

Definition

If the Lagrangian of a system does not contain a particular generalized coordinate $q_{j.}$

(ie) i $+\frac{\partial L}{\partial q_j} = 0$

Then $q_{j.}$ is called a cyclic coordinate or ignorable coordinate.

Example

In the hoop rolling problem we have L = $\frac{1}{2}m(\dot{r}^2 - a^2\dot{\theta}^2)$ – mg(l -x) sinα where x and θ are the generalized coordinates but L doe's not contain θ.

(ie) $\frac{\partial L}{\partial \theta} = 0$

θ is a cyclic coordinates or ignorable coordinate.

Conservation Theorem for generalized momentum

Statement:

The generalized momentum conjugate to cyclic coordinate is conserved.

Proof:

Let q_j be the cyclic coordinate and P_j be the generalized momentum conjugate to $q_{j.}$

We know that the Lagrangian equation of motion is

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$ ……………….. (1)

Since q_j is cyclic coordinates.

$$\frac{\partial L}{\partial q_j} = 0$$

(1) implies $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0$

$$\frac{d}{dt}(P_j) = 0$$

$$P_j = \text{constant}$$

Hence the generalized momentum conjugate to cyclic coordinate is conserved.

State and prove Conservation theorem for linear momentum

Statement:

If a given component of the total applied force vanishes, the corresponding component of the linear momentum is conserved.

Proof:

Consider a conservative system which is not a function of velocities.

Let q_j be the generalized coordinates such that the change dq_j represents a translation of the system as a whole is some given direction.

Then T will be independent of position. $\frac{\partial L}{\partial \dot{q}_j} = 0$ and $\frac{\partial T}{\partial q_j} = 0$ …………..(1)

The Lagrangian relation for such a system becomes

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

$$\frac{d}{dt}(P_j) = \frac{\partial L}{\partial q_j}$$

$\dot{P_J} = \frac{\partial}{\partial q_j}$(T-V)

$= \frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j}$

$=0 - \frac{\partial V}{\partial q_j}$

$= \frac{\partial V}{\partial q_j}$

$\dot{P_J}$ = Q_j ………………………… (2)

If we show that the generalized force Q_j represents the component of the total force along the direction of q_j and Pjis the component of the total linear momentum along this direction.

Then equation (2) will be give the equation of motion of the total linear momentum.

We know that,

$Q_j = \sum_i F_i \frac{\partial \vec{r_i}}{\partial q_j}$

If $\hat{n}$ is the unit vector along the direction of translation of the system along same axis.

Then $\frac{\partial \vec{r_i}}{\partial q_j}$ $= \lim_{dq_{j \to 0}} \frac{r_i(q_j + dq_j) - r_{iq_j}}{dq_j}$

$= \lim_{dq_{j \to 0}} \frac{r_i(q_j) + r_i(dq_j) - r_{iq_j}}{dq_j}$

$= \lim_{dq_{j \to 0}} \frac{r_{idq_j}}{dq_j}$

$= \lim_{dq_{j \to 0} \hat{n}} \frac{d_{q_j}}{dq_j}$

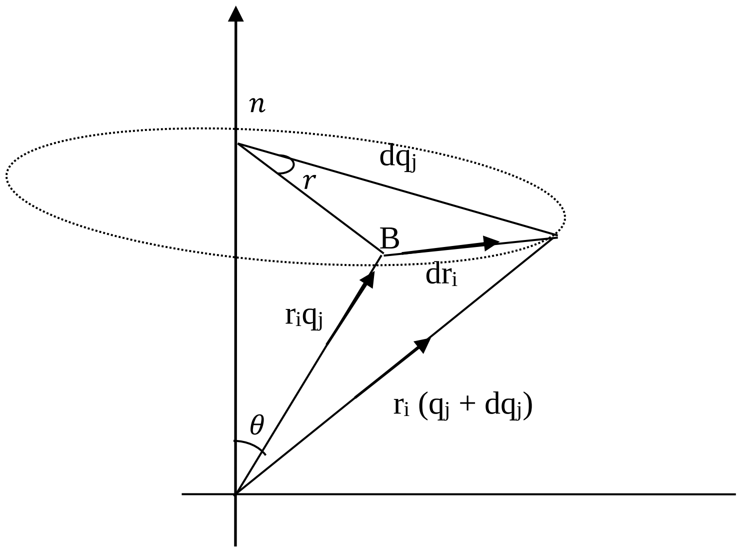
$=\hat{n}$

$Q_j \quad = \sum_i F_i \, \hat{n}$

$= F\hat{n}$

Where F is the total force acting on the system.

Thus Q_j represents the component of the total force along the direction of translation of q_j.



We know that,

$P_j \quad = \frac{\partial L}{\partial \dot{q}_j}$

$= \frac{\partial (T-V)}{\partial \dot{q}_j}$

$= \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial V}{\partial \dot{q}_j}$

$= \frac{\partial T}{\partial \dot{q}_j} - 0$

$= \frac{\partial T}{\partial \dot{q}_j}$

$= \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\sum_i m_i {v_i}^2\right)$

$= \frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\sum_i m_i \, {\dot{r}_i}^2\right)$

$= \sum_i m_i \, \dot{r}_i \hat{n}$

$= \hat{n} \sum_i m_i \, \dot{r}_i$

$= \hat{n}$

There fore p_j represents the component of the total linear momentum along $\hat{n}$.

Thus we have shows that,

$\dot{p}_j = Q_j$

Suppose $Q_j = 0$

The p_j will be a constant

Suppose the translation coordinate q_j is cyclic.

$$\frac{\partial L}{\partial q_j} = 0$$

$$\frac{\partial (T-V)}{\partial q_j} = 0$$

$$\frac{\partial T}{\partial q_j} - \frac{\partial V}{\partial q_j} = 0$$

$$0 - \frac{\partial V}{\partial q_j} = 0$$

$$-\frac{\partial V}{\partial q_j} = 0$$

$$Q_j = 0$$

$$\dot{p}_j = 0$$

P_j is conserved.

Thus if a given component of the total applied force vanishes then the corresponding total linear momentum is conserved.

(ie) If a translation q_j is cyclic.

Then $Q_j = 0$

∴ P_j is conserved.

Derive the energy function which is a function of independent variable q_j and the time derivative $\dot{q}_j$ along with the time and show that the total time derivative of h is given by $\frac{dh}{dt}=-\frac{\partial L}{\partial T}$

Proof:

Consider a conservative medical system .

Let $q_1, q_2, \dots, q_n$ be the n generalized coordinates.

We know the Lagrangian L=L(q,$\dot{q}$,t)

$$\frac{dL}{dt} = \frac{\partial L}{\partial q1}\frac{d\dot{q}_1}{dt} + \dots + \frac{\partial L}{\partial qn}\frac{d\dot{q}_n}{dt} + \frac{\partial L}{\partial \dot{q}_1}\frac{d\dot{q}_1}{dt} + \dots + \frac{\partial L}{\partial \dot{q}_n}\frac{d\dot{q}_n}{dt} + \frac{\partial L}{\partial t}$$

$$= \sum_j \frac{\partial L}{\partial qj}\frac{dqj}{dt} + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_J}{dt} + \frac{\partial L}{\partial t}$$

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial qj}\dot{q}_J + + \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_J}{dt} + \frac{\partial L}{\partial t}$$

The Lagrangian equation for this system is

$$\frac{dL}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial qj} = 0$$

$$\frac{\partial L}{\partial qj} = \frac{dL}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)$$

$$(1) \Rightarrow \frac{dL}{dt} = \sum_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_J}{dt} + \sum_j\left(\frac{\partial L}{\partial \dot{q}_j}\frac{d\dot{q}_J}{dt}\right) + \frac{\partial L}{\partial t}$$

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\left[\sum_j\left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) - L\right] + \frac{\partial L}{\partial t} = 0 \quad \dots\dots\dots\dots\dots\dots (2)$$

The quantity $\sum_j\left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) - L]$ is called the energy function and it is denoted by h.

$(2) \Rightarrow \frac{d}{dt}(h) + \frac{\partial L}{\partial t} = 0$

$\frac{dh}{dt} + \frac{\partial L}{\partial t} = 0$

$\frac{dh}{dt} = - \frac{\partial L}{\partial t}$

Conservation theorem and Symmetry property

Let (x_i, y_i, z_i) be the Cartesian coordinate of the i[th] mass m_i, then T = $\frac{1}{2}\sum_i m_i(\dot{x_i}^2 + \dot{y_i}^2 + \dot{z_i}^2)$

We know that,

L =T-V

$\frac{\partial L}{\partial \dot{x_i}} = \frac{\partial (T-V)}{\partial \dot{x_i}}$

$= \frac{\partial (T)}{\partial \dot{x_i}} - \frac{\partial (V)}{\partial \dot{x_i}}$

$= \frac{\partial (T)}{\partial \dot{x_i}}$

$= \frac{\partial (\frac{1}{2}\sum_i m_i(\dot{x_i}^2 + \dot{y_i}^2 + \dot{z_i}^2))}{\partial \dot{x_i}}$

$= \frac{1}{2}\sum_i m_i \, 2\dot{x_i}$

$= \sum_i m_i \, \dot{x_i}$

= p_ix where p_ix denote linear momentum of the i[th] particle in the direction of the x axis.

The generalized momentum p_i associated with the generalized coordinate q_j is defined as $p_j = \frac{\partial L}{\partial \dot{q_j}}$

If the transformation equation defining the generalized coordinates do not involve the time explicitly and if the potential does not depend on the generalized velocities then show that h = E. Also prove that $\frac{dE}{dt}$ = -2$\mathcal{F}$ where 2$\mathcal{F}$ is the rate of energy dissipation.

Proof:

We know that,

The total kinetic energy of a system can be written as T =T_0+T_1+T_2 ……………..(1) where T_0 is a function of generalized coordinates.

$T_1(q,\dot{q})$ is a linear in the generalized velocities and $T_2(q,\dot{q})$ is a quadratic function of the $\dot{q}$'s.

Then the Lagrangian can be decomposed as $L(q,\dot{q},t) = L_0(q,t) + L_1(q,\dot{q},t) + L_2(q,\dot{q},t)$ …………………………… (2)

Here L_2 is a homogeneous function of the second degree in $\dot{q}$ while L_1 is homogeneous of the first degree in $\dot{q}$.

Now,

By Euler's theorem iff is a homogeneous function of degree n in the variable x, then $\sum_i x_i \frac{\partial F}{\partial x_i}$ = nf

We have,

$$h = \sum_j \dot{q}_j \frac{\partial F}{\partial \dot{q}_j} - L$$

using equation (2) in above we get,

$$h = 2L_2 + L_1 - L$$

$$= 2L_2 - L_0 - L_2$$

$$= L_2 - L_0$$

In the transformation equation defining the generalized coordinate do not involve the time explicitly the T =T_2.

If further,

The potential does not depend on the generalized velocities then L_2 = T and L_0 =-V

So that h=T +V

h = E where E is the total energy.

We have,

$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_j})-\frac{\partial L}{\partial \text{qj}} +(\frac{\partial \mathcal{F}}{\partial \dot{q}_j}) = 0$

Then the equation $\frac{d}{dt}[\sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L] +\frac{\partial L}{\partial \text{t}} = 0$

Can be written as

$\frac{dh}{dt} +\frac{\partial L}{\partial \text{t}} = \sum_j \dot{q}_j(\frac{\partial \mathcal{F}}{\partial \dot{q}_j})$

By definition of $\mathcal{F}$ it is a homogeneous function of the q's of degree 2.

$\frac{dh}{dt} = -2\mathcal{F}-\frac{\partial L}{\partial \text{t}}$

If L is not an explicit function of time and the system is such that h is the same as the energy that is h =E.

Then equation becomes,

$\frac{dE}{dt} = -2\mathcal{F}$

Unit –IV

The Two Body central force problem

Reduction to the equivalent one body

Show that the central force motion of the two bodies about the center of mass Can always be reduced to an equivalent one body problem.

Solution:

Consider a monogenic system of two mass points m_1 andm_2 where the only force are those due to an interaction potential 'U'.

Let $\vec{r}_1$and $\vec{r}_2$ be the position vector of m_1 andm_2 with respect to the fixed point o. Let G be the center of mass and it's position vector be $\vec{R}$.

Denote $\overrightarrow{AB}$ by $\vec{r}$ from the figure $\vec{r}$ = $\vec{r}_2$ -$\vec{r}_1$

Suppose V =U($\vec{r}$)

Now, the system of two mass points has sixth degrees of freedom and hence sixth generalized coordinates. We can take the 6 generalized coordinates to the 3 components of $\vec{r}$.

The Lagrangian L = T($\overrightarrow{\dot{R}}, \dot{\vec{r}}$) – U($\vec{r}$)

Now,

T =T_G + T' where T_G denote the kinetic energy of the motion of the center of mass and T' denote the kinetic energy of motion about the center of mass.

Let the position vector of m_1 with respect to G is $\overrightarrow{r_1{}'}$ and the position vector of m_2 with respect to G is $\overrightarrow{r_2{}'}$.

$\therefore$T = $\frac{1}{2}(m_1 + m_2)\overrightarrow{\dot{R}^2}$ + $\frac{1}{2}m_1\overrightarrow{\dot{r}_1{}^{2\prime}}$ +$\frac{1}{2}m_2\overrightarrow{\dot{r}_2{}^{2\prime}}$ from the figure $\vec{R}$ = $\vec{r}_1$ - $\overrightarrow{r_1{}'}$

$$\therefore \overrightarrow{r_1{}'} = \vec{r}_1 - \vec{R}$$

But we know that,

$$\vec{R} = \frac{\sum m_i r_i}{\sum m_i}$$

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

$$\overrightarrow{r_1{}'} = \vec{r}_1 - \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

$$= \frac{m_1\vec{r}_1 + m_2\vec{r}_2 - m_1\vec{r}_1 - m_2\vec{r}_2}{m_1 + m_2}$$

$$= \frac{m_{2(}\vec{r}_1 - \vec{r}_2)}{m_1 + m_2}$$

$$= \frac{-m_{2(}\vec{r}_2 - \vec{r}_1)}{m_1 + m_2}$$

$$= \frac{-m_{2(\vec{r}})}{m_1 + m_2}$$

$$\overrightarrow{\dot{r}_1{}'} = \frac{-m_{2(\vec{r}})}{m_1 + m_2}$$

Similarly,

$$\overrightarrow{\dot{r}_2{}'} = \frac{m_{1(\vec{r}})}{m_1 + m_2}$$

$$T = \frac{1}{2}(m_1 + m_{2)}\overrightarrow{\dot{R}^2} \quad + \frac{1}{2}m_1\left(\frac{-m_2}{m_1 + m_2}\right)^2\overrightarrow{\dot{r}^2} + \frac{1}{2}m_2\frac{m_1}{m_1 + m_2}\overrightarrow{\dot{r}^2}$$

$$T = \frac{1}{2}(m_1 + m_{2)}\overrightarrow{\dot{R}^2} \quad + \frac{1}{2}\frac{m_{1m_2}}{m_1 + m_2}\left(\frac{m_2}{m_1 + m_2}\right)^2\overrightarrow{\dot{r}^2} + \frac{1}{2}\frac{m_{1m_2}}{m_1 + m_2}\frac{m_1}{m_1 + m_2}\overrightarrow{\dot{r}^2}$$

Take $\mu = \frac{m_{1m_2}}{m_1 + m_2}$ is called the reduced mass.

$T = \frac{1}{2}(m_1 + m_{2)}\overrightarrow{\dot{R}^2} + \frac{1}{2}\ \mu\overrightarrow{\dot{r}^2}$

$L = \frac{1}{2}(m_1 + m_{2)}\overrightarrow{\dot{R}^2} + \frac{1}{2}\ \mu\overrightarrow{\dot{r}^2} - U(\vec{r})$

The above expression for L shows that the 3 coordinates of R are cyclic.

The Lagranges equation of motion corresponding to $\vec{R}$ is $\frac{d}{dt}(\frac{\partial L}{\partial \dot{R}}) - \frac{\partial f}{\partial R} = 0$

$\frac{d}{dt}[\frac{1}{2}(m_1 + m_{2)}2\vec{\dot{R}}] - 0 = 0$

$(m_1 + m_{2)}\vec{\dot{R}} = \text{constant}$

$\vec{\dot{R}}$ = a constant

The center of mass is either at rest or moving uniformly.

The Lagranges equation of motion corresponding to $\vec{r}$ is $\frac{d}{dt}(\frac{\partial L}{\partial \dot{r}}) - \frac{\partial f}{\partial r} = 0$

$\frac{d}{dt}(\frac{1}{2}\mu 2\vec{\dot{r}}) - (\frac{-\partial Ur}{\partial r}) = 0$

$\mu\vec{\ddot{r}} = (\frac{-\partial Ur}{\partial r}) = f(r)$

Thus non of the equation of motion for $\vec{r}$ will contained terms involving $\vec{R}$ or $\vec{\dot{R}}$].

Since $\vec{R}$ is cyclic we can ignore $\vec{R}$in the expression of L.

Dropping the first term in the above value of Lagrangian, we get

$L = (\frac{1}{2}\mu\vec{\dot{r}}) - U(r)$

This is the Lagrangian for the motion of the single particle with a fixed center of mass, where $\vec{r}$ is the position vector of the particle from the center of force.

Thus, the two body force is equivalent to one body problem with mass $\mu = \frac{m_{1}m_{2}}{m_1+m_2}$ and position vector $\vec{r} = \vec{r}_2 - \vec{r}_1$.

Show that the central force motion of a particle is always planer and that in such a motion the time derivative of the Arial velocity vanishes.

Proof:

Central force is that force which is always directed away or towards a fixed center and magnitude of which is a function of the distance from the fixed center.

Let $\vec{r}$ be the position vector of the particle.

$\dot{\vec{r}}$ is also a vector.

Consider the product $\vec{r}$ x $\dot{\vec{r}}$ it is a vector perpendicular to both r and $\dot{r}$.

$$\frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = \frac{d\vec{r}}{dt} \times \dot{\vec{r}} + \vec{r} \times \frac{d\dot{\vec{r}}}{dt}$$

$$= \dot{\vec{r}} \times \ddot{\vec{r}}$$

For a central force motion the force is always a function of $\vec{r}$ and it is acting in the direction away or towards the center of force.

The acceleration is $\ddot{\vec{r}}$ is taken as a function of $\vec{r}$ in the direction of $\ddot{r}$ = f(r) $\vec{r}$

Where f(r) represents the magnitude of central force.

$$\frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = \vec{r} \times f(r)\,\vec{r}$$

$$= f(r)\,\vec{r} \times \vec{r}$$

$$= 0$$

($\vec{r}$ x $\dot{\vec{r}}$) is constant.

$\vec{r}$ and $\dot{\vec{r}}$ should always perpendicular to the constant vector ($\vec{r}$ x $\dot{\vec{r}}$)

Hence $\vec{r}$ and $\dot{\vec{r}}$ should lie in a plane for which $\vec{r}$ x$\dot{\vec{r}}$ is normal.

The central force motion is always motion in a plane.

Since the part of motion of central force is in a plane, we take (r,θ) be the position of the particle.

X= r cos θ and y = r sin θ

$\dot{x}$ = -r sin$\theta\dot{\theta}$ +$\dot{r}$ cos θ

$\dot{y}$ = r cos$\theta\dot{\theta}$ +$\dot{r}$ sin θ

$(\dot{x})^2 = r^2\sin^2\theta(\dot{\theta})^2 + \dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta$

$(\dot{y})^2 = r^2\cos^2\theta(\dot{\theta})^2 + \dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\cos\theta\sin\theta$

$(\dot{x})^2 + (\dot{y})^2 = r^2\sin^2\theta(\dot{\theta})^2 + \dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta + r^2\cos^2\theta(\dot{\theta})^2 + \dot{r}^2\sin^2\theta$
$+ 2r\dot{r}\dot{\theta}\cos\theta\sin\theta$

$(\dot{x})^2 + (\dot{y})^2 = \dot{r}^2 + r^2(\dot{\theta})^2$

$T = \frac{1}{2}m((\dot{x})^2 + (\dot{y})^2)$

$= \frac{1}{2}m\,\dot{r}^2 + r^2(\dot{\theta})^2$

V =V(r)

L =T- V

$= \frac{1}{2}m((\dot{x})^2 + (\dot{y})^2) - V(r)$

$\frac{\partial L}{\partial \theta} = 0$

$\frac{\partial L}{\partial \dot{\theta}} = \text{mr}^{2}(\dot{\theta})$

The Lagranges equation of motion corresponding to θ is

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$\frac{d}{dt}(\text{mr}^{2}(\dot{\theta})) - 0 = 0$

$\frac{d}{dt}(\text{mr}^{2}(\dot{\theta})) = 0$

$\frac{d}{dt}(\frac{1}{2}\text{r}^{2}(\dot{\theta})) = 0$

This shows that the time derivative of the Arial velocity is vanishes.

For two body central force problem obtains the equation of motion and the first integral

(OR)

For two body central force problem show that $\frac{1}{2}m\,\dot{r}^2 + \frac{1}{2}\frac{l^2}{mr^2}$ +v constant.

Solution:

Let P be the position of the particle.

Since the path of motion of a central force is in a plane , consider the coordinate of p as (r,θ).

We have,

X= r cos θ and y = r sin θ

$\dot{x}$ $= -r\sin\theta\dot{\theta} + \dot{r}\cos\theta$

$\dot{y}$ $= r\cos\theta\dot{\theta} + \dot{r}\sin\theta$

$(\dot{x})^2 = r^2\sin^2\theta(\dot{\theta})^2 + \dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta$

$(\dot{y})^2 = r^2\cos^2\theta(\dot{\theta})^2 + \dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\cos\theta\sin\theta$

$(\dot{x})^2 + (\dot{y})^2 = r^2\sin^2\theta(\dot{\theta})^2 + \dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\cos\theta\sin\theta + r^2\cos^2\theta(\dot{\theta})^2 + \dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\cos\theta\sin\theta$

$(\dot{x})^2 + (\dot{y})^2 = \dot{r}^2 + r^2(\dot{\theta})^2$

$T = \frac{1}{2}m((\dot{x})^2 + (\dot{y})^2)$

$T = \frac{1}{2}m\,\dot{r}^2 + r^2(\dot{\theta})^2$

$V = V(r)$

$L = T - V$

$= \frac{1}{2}m((\dot{x})^2 + (\dot{y})^2) - V(r)$

$\frac{\partial L}{\partial \theta} = 0$

$\frac{\partial L}{\partial \dot{\theta}} = mr^2(\dot{\theta})$

$\frac{\partial L}{\partial r} = mr^2(\dot{\theta}) - \frac{\partial v}{\partial r}$

$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$

The Lagranges equation of motion corresponding to θ is

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$\frac{d}{dt}$ (mr $^2(\dot{\theta})$) -0 = 0

$\frac{d}{dt}$ (mr $^2(\dot{\theta})$) = 0

mr $^2(\dot{\theta})$ = constant

mr $^2(\dot{\theta}) =$ l (say)

r $^2(\dot{\theta})$ = constant

$\frac{1}{2}$r $^2(\dot{\theta})$ = constant

Hence for a central of motion the Arial velocity of the moving particle is conserved.

The conservation of the angular momentum is equivalent to saying the Arial velocity is constant.

The Lagranges equation of motion corresponding to r is ,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = 0$$

$$\frac{d}{dt}m\dot{r} - mr^2(\dot{\theta}) - \frac{\partial v}{\partial r} = 0$$

$$m\ddot{r} - mr^2(\dot{\theta}) - \frac{\partial v}{\partial r} = 0$$

$$m\ddot{r} - \frac{l^2}{mr^3} + \frac{\partial v}{\partial r} = 0$$

This is a second order differential equation in r,

By solving this we get the value of r in terms of v.

$$m\ddot{r} = \frac{l^2}{mr^3} - \frac{\partial v}{\partial r}$$

$= \frac{\partial}{\partial r}$ - + $\frac{1}{2}\frac{l^2}{mr^2}$- $\frac{\partial v}{\partial r}$

= - $\frac{\partial}{\partial r}\{\frac{1}{2m}\frac{l^2}{r^2}$ + v}

Multiply both sides by $\frac{dr}{dt}$

(ie) m$\ddot{r}\,\frac{dr}{dt}$ = [- $\frac{\partial}{\partial r}\{\frac{1}{2m}\frac{l^2}{r^2}$ + v}]$\frac{dr}{dt}$

$\frac{d}{dt}$ [$\frac{1}{2}$ m $\dot{r}^2$] =- $\frac{d}{dt}$ {$\frac{1}{2m}\frac{l^2}{r^2}$ + v}]

$\frac{d}{dt}$\{ [$\frac{1}{2}$ m $\dot{r}^2$] + {$\frac{1}{2m}\frac{l^2}{r^2}$ + v}]}

{ [$\frac{1}{2}$ m $\dot{r}^2$] + {$\frac{1}{2m}\frac{l^2}{r^2}$ + v}]} = constant.

(ie) [$\frac{1}{2}$ m $\dot{r}^2$] + $\frac{m^2r^4\dot{\theta}^2}{2mr^2}$ +v = constant

(ie) [$\frac{1}{2}$ m $\dot{r}^2$] +$\frac{1}{2}mr^2\dot{\theta}$ +v = constant.

(ie) T +V is constant

(3.3)The equivalent one dimensional problem and classification of orbits

Find the magnitude and direction of the velocity of the central orbit

Proof:

Let v be the velocity of the particle at p its components along the radius vector is $\dot{r}$ and perpendicular to the radius velocity is r$\dot{\theta}$

Let r_n be the unit vector along the radius vector and θ_n be the unit vector in the transverse direction

$\therefore$ v = $\dot{r}r_n + r\dot{\theta}\theta_n$

$\therefore$ magnitude of v is given by ,

$V^2 = \dot{r}^2 + (r\dot{\theta})^2$

We have,

$\dot{\theta} = \frac{l}{mr^2}$ and $\dot{r} = \sqrt{\frac{2}{m}\left[E - V - \frac{l^2}{2mr^2}\right]}$

$$V^2 = \frac{2}{m}\left[E - V - \frac{l^2}{2mr^2}\right] + r^2\frac{l^2}{m^2r^4}$$

$$= \frac{2}{m}\left[E - V - \frac{l^2}{2mr^2}\right] + \frac{l^2}{m^2r^2}$$

$$= \frac{1}{m^2r^2}\left[2mr^2E - 2mr^2V + l^2 - l^2\right]$$

$$= \frac{1}{m^2r^2}\left[2mr^2(E - V)\right]$$

$$V^2 = \frac{2}{m}(E-V)$$

$$V = \sqrt{\frac{2}{m}[E - V]}$$

If the direction of the velocity v makes an angle.

Then $\tan\varphi = r\frac{d\theta}{dr}$

$$= r\frac{\frac{d\theta}{dt}}{\frac{dr}{d\theta}}$$

$$= \frac{\frac{l}{mr}}{\sqrt{\frac{2}{m}\left[E - V - \frac{l^2}{2mr^2}\right]}}$$ is the direction of the velocity.

The virial theorem

Statement:

Suppose the motion of the system is periodic or not periodic, then $\overline{T} = -\frac{1}{2}\overline{\sum_i \vec{F}_i \vec{r}_i}$ where $\vec{F}_i$ is the applied force including the force of constraint and → denote the time average over the interval o to τ. The equation $\overline{T} = -\frac{1}{2}\overline{\sum_i \vec{F}_i \vec{r}_i}$ is known as the Virial theorem and the right hand side is called the virial of Claudius.

Proof:

Consider a general system of mass points with position vector $\vec{r}_i$ and applied force $\vec{F}_i$

Then the fundamental equation of motion are $\dot{\vec{p}}_i = \vec{F}_i$ …………………….. (1)

Consider the quantity G =$\sum_i \vec{p}_i \, \vec{r}_i$ …………………………………(2)

Where the summation is over all particles in the system.

$$\frac{dG}{dt} = \sum_i \vec{p}_i \frac{d\vec{r}_i}{dt} + \sum_i \vec{r}_i . \frac{d\vec{p}_i}{dt}$$

$$= \sum_i \vec{p}_i \dot{\vec{r}}_i + \sum_i \vec{r}_i . \dot{\vec{p}}_i$$

$$= \sum_i m_i \dot{\vec{r}}_i \dot{\vec{r}}_i + \sum_i \vec{r}_i . \vec{F}_i$$

$$= \sum_i m_i \vec{v}_i + \sum_i \vec{r}_i . \vec{F}_i$$

$$\frac{dG}{dt} = \sum_i 2T + \sum_i \vec{r}_i . \vec{F}_i \quad \text{……………………………….(3)}$$

Where T is the kinetic energy of the system take the time average of the above equation (3) over a time interval τ, which is obtained by integration both sides with respect to t from o to τ and dividing by τ.

We get,

$$\frac{1}{\tau}\int_0^{\tau}\frac{dG}{dt}dt \quad = \frac{1}{\tau}\int_0^{\tau}\sum_i 2T + \sum_i \vec{r}_i . \vec{F}_i \text{ dt}$$

$$= \frac{1}{\tau}\int_0^{\tau}\sum_i 2Tdt + \frac{1}{\tau}\int_0^{\tau}\sum_i \vec{r}_i . \vec{F}_i \text{ dt}$$

$$\therefore \frac{d\vec{G}}{dt} \quad = \overline{\sum_i \vec{F}_i\vec{r}_i} + \overline{2T}$$

(ie) $\overline{2\vec{T}} + \overline{\sum_i \vec{F}_i\vec{r}_i} = \frac{1}{\tau}[G(\tau - G(0)]$ …………….. (4)

Case (i)

If the motion is periodic

(ie) all coordinate repeat after a certain time.

If τ is chosen to be the period of time, then G(τ) = G(0)

(4)$\Rightarrow \overline{2\vec{T}} + \overline{\sum_i \vec{F}_i\vec{r}_i} = 0$

$\vec{T} = -\frac{1}{2}\overline{\sum_i \vec{F}_i\vec{r}_i}$ ………………………… (5)

Case(ii)

If the motion is non-periodic

Let us consider the coordinates and velocities for all particles remain finite. So that there is an upper bound to G. By choosing τsufficiently long, the R.H.S

Of equation (4) can be made as small as derived.

For a proper choice of τ R.H.S

Of equation (4) reduces to zero.

$$(4)\Rightarrow \overline{2\vec{T}} + \overrightarrow{\sum_i \vec{F}_i\vec{r}_i} = 0$$

$$\vec{T} = -\frac{1}{2}\overrightarrow{\sum_i \vec{F}_i\vec{r}_i}$$

Thus when the motion is finite or if the coordinates and velocities of all the particle remains finite, then $\vec{T} = -\frac{1}{2}\overrightarrow{\sum_i \vec{F}_i\vec{r}_i}$, where $\vec{F}_i$ is the applied force including the force of constraints. The above equation as the virial theorem and R.H.S is called the virial of clausius.

(3.5) The Differential equation for the orbit and integral power law potentials:

Find the differential equation of the orbits for a central force motion

Proof:

Let m be the mass of the particle which moves under central force. Let P(r,θ) be the position of the particle at time t.

We know that,

In the case of motion under the action of central force L = [$\frac{1}{2}$ m $\dot{r}^2$] +$\frac{1}{2}mr^2\dot{\theta}$ +v(r)

The Lagranges equation for r and θ are

$\mathrm{M}(\ddot{r} - r\dot{\theta}^{2}) + \frac{\partial v}{\partial r} = 0$ ………………….. (1)

$\mathrm{Mr}^2\dot{\theta}$ = l (say)

From equation (2)

$\mathrm{Mr}^2 \frac{d\theta}{\partial t} = \mathrm{l}$

$\mathrm{l\,dt} = \mathrm{Mr}^2\, d\theta$

$\mathrm{dt} = \frac{\mathrm{Mr2}}{l}\, d\theta$

$\frac{d}{dt} = \frac{\mathrm{l}}{\mathrm{Mr2}} \frac{d}{d\theta}$

$\frac{d^2}{dt^2} = \frac{d}{dt}(\frac{d}{dt})$

$= \frac{d}{dt} \frac{\mathrm{l}}{\mathrm{M}r^2} \frac{d}{d\theta}$

$= \frac{\mathrm{l}}{\mathrm{Mr2}} \frac{d}{dt}(\frac{\mathrm{l}}{\mathrm{M}r^2} \frac{d}{d\theta})$

$\therefore \ddot{r} = \frac{\mathrm{l}}{\mathrm{M}r^2} \frac{d}{dt}(\frac{\mathrm{l}}{\mathrm{M}r^2} \frac{d}{d\theta})$

Also,

$r\dot{\theta}^2 = \mathrm{r}(\frac{\mathrm{l}}{\mathrm{M}r^2})^2$

$= \mathrm{r}\, \frac{l^2}{m^2 r^4}$ and

$-\frac{\partial v}{\partial r} = \mathrm{f(r)}$

$\therefore$ equation (1) $\Rightarrow \mathrm{m}[\frac{\mathrm{l}}{\mathrm{M}r^2} \frac{d}{d\theta} \frac{\mathrm{l}}{\mathrm{M}r^2} \frac{dr}{d\theta} - \frac{l^2}{m^2 r^3}]\text{-f(r)} = 0$

(ie)$\frac{\mathrm{l}}{r^2} \frac{d}{d\theta} (\frac{\mathrm{l}}{\mathrm{M}r^2} \frac{dr}{d\theta})\text{-}\ \frac{l^2}{m r^3} = \mathrm{f(r)}$

(ie)$\frac{d}{d\theta}(\frac{l}{r^2}\frac{dr}{d\theta})$-$\frac{l}{r}$ = $\frac{Mr^2}{l^2}$ f(r) ……………….. (3)

Which is the equation of the central orbit.

Put u = $\frac{l}{r}$

r = $\frac{l}{u}$

$\frac{dr}{d\theta}$ = $\frac{dr}{du}\frac{du}{d\theta}$

= - $\frac{1}{u^2}\frac{du}{d\theta}$

$u^2\frac{dr}{d\theta}$ = -$\frac{du}{d\theta}$

(3) ⇒$\frac{d}{d\theta}$-$\frac{du}{d\theta}$ –u = $\frac{M}{l^2}\frac{1}{U^2}$f($\frac{l}{u}$)

This is the differential equation for the orbit if the force law f is known.

We know that,

$\frac{dv}{du} = \frac{dv}{dr}\frac{dr}{du}$

= -f(r)- $\frac{1}{u^2}$

= f($\frac{l}{u}$)$\frac{1}{u^2}$

$\frac{d[v(r)]}{du}$ = f($\frac{l}{u}$)$\frac{1}{u^2}$

$\frac{d^2u}{d\theta^2}$ + u = $\frac{-m}{l^2}\frac{d}{du}$[v($\frac{l}{u}$)] ……………………….. (II)

This is the differential equation for the orbit if the potential v is known.

Prove that the central orbit is symmetrical about the apsidal vectors.

Proof:

We know that,

$$\frac{d^2u}{d\theta^2} + u = \frac{-m}{l^2} f(\frac{1}{u})\frac{1}{u^2}$$

$$\frac{d^2u}{d\theta^2} + u = \frac{-m}{l^2}\frac{d}{du}[v(\frac{1}{u})]$$

Let A be an apse

So the radius vector of A is an extreme.

Hence $\frac{dr}{d\theta} = 0$

$$\frac{dr}{du}\frac{du}{d\theta} = 0$$

$$\frac{du}{d\theta} = 0$$

This gives a turning point for the central orbit.

Hence an apse is a turning point for the central orbit.

Now,

Choose the initial line along the apsidal vector.

To prove the central orbit is symmetrical about the apsidal vector or apsidal line, it is enough if we prove the equation is invariant by putting ($-\theta$) in the place of θ

Clearly the equation (I) and (II) is unchanged by putting $-\theta$ in the place of θ.

The central orbit is symmetrical about the apsidal vectors.

Discuss the nature of the central orbit for a specific force laws which are power law function of r.

Proof:

Let m be the mass of the particle and (r, θ) be the coordinates.

$T = \frac{1}{2}$ m $\dot{r}^2$ + r $^2(\dot{\theta})^2$ and v = v(r)

$\frac{1}{2}$ m $\dot{r}^2$ + r $^2(\dot{\theta})^2$ +V =E

$\frac{1}{2}$ m $\dot{r}^2$ + r $^2(\dot{\theta})^2$ = E –V

m $\dot{r}^2$ + r $^2(\dot{\theta})^2$ $=\frac{2}{m}$(E –V)

$\dot{r}^2$ +r 2 $\frac{l^2}{m^2r^4}$ = $\frac{2}{m}$(E –V)

$\dot{r}^2 = \frac{2}{m}$(E –V)-$\frac{l^2}{m^2r^2}$

$(\frac{dr}{dt})^2 = \left[E - V - \frac{l^2}{2mr^2}\right]$

$$\frac{dr}{d\theta}\frac{d\theta}{dt} = \sqrt{\frac{2}{m}\left[E - V - \frac{l^2}{2mr^2}\right]}$$

$$\frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{l}{mr^2}\sqrt{\frac{2}{m}\left[E - V - \frac{l^2}{2mr^2}\right]}$$

$$d\theta \quad = \frac{l}{mr^2}\frac{dr}{\sqrt{\frac{2}{m}\left[E-V-\frac{l^2}{2mr^2}\right]}}$$

$$d\theta \quad = \frac{dr}{r^2\sqrt{\frac{2mE}{l^2}-\frac{2mV}{l^2}-\frac{l}{r^2}}}$$

θ varies from θ_0 to θ and r variesfrom r_0 to r

$$\int_{\theta_0}^{\theta} d\theta = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{l}{r^2}}}$$

$$\theta - \theta_0 = \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{l}{r^2}}}$$

$$\theta = \theta_0 + \int_{r_0}^{r} \frac{dr}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - \frac{l}{r^2}}}$$

Put u = $\frac{1}{r}$

du = -$\frac{1}{r^2}$dr

dr =-r^2 du

$$\theta = \theta_0 + \int_{u_0}^{u} \frac{-du}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - du}}$$

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - du}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Consider the force law, which are power law functions of r.

V =ar^{n+1}

r = $\frac{1}{u}$

v =a$\frac{1}{u^{n+1}}$

$$(1) \Rightarrow \theta = \theta_0 - \int_{u_0}^{u} \frac{du}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mau^{n+1}}{l^2} - u^2}}$$

If the quantity in the radical is at most quadratic then the denominator has the form $\sqrt{\alpha u^2 + \beta u + \gamma}$ and the integration can be directly obtained in terms of circular functions.

The required condition is –n-1 = 0,1,2

n+1 = 0,-1,-2

n = -1,-2,-3

Excluding the case n = -1 we get, n=a -2,-3 corresponding to the inverse square r inverse cube force laws.

For n =1 the solution can be obtained in terms of simple functions.

$$\theta = \theta_0 - \int_{u_0}^{u} \frac{du}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2mau^{-2}}{l^2} - u^2}}$$

Put u^2 = x

$2u\, du =$ dx

$$du = \frac{dx}{2u} = \frac{1}{2\sqrt{x}}$$

$$\theta = \theta_0 - \int_{x_0}^{x} \frac{\frac{1}{2\sqrt{x}}}{r^2\sqrt{\frac{2mE}{l^2} - \frac{2ma}{l^2}\frac{1}{x} - x}}$$

$$\theta = \theta_0 - \int_{x_0}^{x} \frac{\frac{1}{2\sqrt{x}}}{\frac{1}{\sqrt{x}}\sqrt{\frac{2mE}{l^2} - \frac{2ma}{l^2} - x^2}}$$

$$\theta = \theta_0 - \frac{1}{2}\int_{x_0}^{x} \frac{dx}{\sqrt{\frac{2mE}{l^2} - \frac{2ma}{l^2} - x^2}}$$

$Thus\ we\ find\ the\ terms$ is the radical is quadratic in x

The above integral can be integrated by using the circular functions.

$Thus\ if$ n =1,-2,-3 the solution will be of circular functions.

Elliptic Integrals

Definition

An elliptic integral is $\int R(x,\omega)dx$ where R is any rational function of x and ω, ω is defined as $\omega = \sqrt{\alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \eta}$. If α and β

Are not simultaneously zero, then the integral will be evaluated in terms of circular functions.

Find when the solutions for the orbit can be obtained in terms of elliptic functions.

Solution:

The solution of the problem depends on the integral $\int \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mau^{-n-1}}{l^2} - u^2}}$ ……………(1)

The solution could be reduced to elliptic integrals if the term in the radical is of the form $\alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \eta$, where α and $\beta$are not simultaneously zero.

The required condition will be

-n-1 =3,4

n+1 =-3,-4

n=-4,-5 ………………………(1)

Put $u^2 = x$

$2u\,du = dx$

$du = \frac{dx}{2u} = \frac{1}{2\sqrt{x}}$

$$\int \frac{\frac{1}{2\sqrt{x}}dx}{\sqrt{\frac{2mE}{l^2} - \frac{2max^{\frac{-n-1}{2}}}{l^2} - x}}$$

(ie) $\frac{1}{2}\int \frac{dx}{\sqrt{\frac{2mEx}{l^2} - \frac{2max^{\frac{-n-1}{2}}}{l^2} - x^2}}$

The solution could be reduced to elliptic integrals if $\frac{-n+1}{2}$ =3,4

-n+1 =6,8

-n =5,7

n=-5,-7 …………………………. (2)

multiply both denominator and numerator by x

we get,

$$\frac{1}{2}\int \frac{xdx}{\sqrt{\frac{2mEx^3}{l^2} - \frac{2max^{\frac{-n+5}{2}}}{l^2} - x^2}}$$

Hence the solution reduces to the elliptic integrals if $\frac{-n+5}{2}$ =0,1,2,3,4

$-n + 5$ =0,2,4,6,8

$-n$ = -5,-3,-1,1,3

n =5,3,1,-1.-3

For $n = -3$,1 leads to the solution interms of circular functions.

The case n =-1 is not possible.

Thus the new possibilities for obtaining elliptic integral is n =3,5

…………….(3)

multiply both numerator and denominator of equation (I)

By u^{ρ} where ρ is some undetermined exponent

$$\int \frac{u^{\rho}du}{u^{\rho}\sqrt{\frac{2mE}{l^2} - \frac{2mEu^{-n-1+2\rho}}{l^2} - u^2}}$$

$The\ expression\ in\ the\ radical\ will\ be\ a\ polynomial\ of\ higher\ order\ then$ the quadratic except6 if $\rho = 1$

-n-1+2 =0,1,2,3,4

-n+1 =0,1,2,3,4

-n = -1,0,1,2,3

n=1,0,-1,-2,-3

For n=-1,-2,-3 the solution reduced to circular functions

The case n = -1 has already being eliminated so that the procedure leads to elliptics functions only for n =0

Hence the integral is exponents which result in eliptics functions are 5,3,0,-4,-5,-7.

Unit –V

(3.6)Conditions for closed orbits:

Statement:

State and prove Bertsans's theorem closed orbits for all bound particles are the inverse square law and the hoop's law.

Proof:

We know that,

For any given l, if the potential V'(r) has a minimum or maximum at some distance r_0 and if the energy E = V'(r_0), then the orbit is a circle about the central of force.

Since V'(r) has an extremum at r_0 .

Then $(\frac{dv'}{dr})_{r=r_0}$

(ie) f'(r_0) = 0

But f'(r) = $f(r) + \frac{l^2}{mr^3}$ ………………………….. (1)

For extremum at r_0 we get,

f'(r_0) = $f(r_0) + \frac{l^2}{m{r_0}^3}$ …………………………… (2)

0 = $f(r_0) + \frac{l^2}{m{r_0}^3}$

(ie) $f(r_0) = -\frac{l^2}{m{r_0}^3}$ ……………………………. (3)

The energy of the particle

E =V(r_0) +$\frac{l^2}{2m{r_0}^2}$ ……………………………….(4)

If the energy is raised to a little above that required for a circular orbit, then the path is also bounded and not circular.

But when V' is maximum if the energy E is raised to a little above that for a circular motion, the orbit is unbounded.

Hence in the first case we say that the circular orbit is stable and is the second case the circular orbit is unstable.

V' is minimum or maximum according as $(\frac{d^2V'}{dr^2})$ >0 (or) $(\frac{d^2V'}{dr^2})$ <0

∴ The circular orbit is stable or unstable according as $(\frac{d^2V'}{dr^2})$ >0 (or) $(\frac{d^2V'}{dr^2})$ <0

∴ The circular orbit occurs when $(\frac{d^2V'}{dr^2})_{r=r_0} = (\frac{-\partial f}{\partial r})_{r=0} + \frac{3l^2}{m{r_0}^4} > 0$

- $(\frac{\partial f}{\partial r})_{r=r_0} > \frac{-3l^2}{m{r_0}^4}$

$(\frac{\partial f}{\partial r})_{r=r_0} < (\frac{l^2}{m{r_0}^3})\frac{3}{r_0}$

$\Rightarrow (\frac{\partial f}{\partial r})_{r=r_0} < \frac{3}{r_0}(-f(r_0))$ [∵ using (3)]

$\Rightarrow (\frac{\partial f}{\partial r})_{r=r_0} < -3\,\frac{f(r_0)}{r_0}$ .... (5)

$\Rightarrow (\frac{\partial f}{\partial r_0})\frac{r_0}{f(r_0)} < -3$

$\Rightarrow \frac{f'(r_0)r_0}{f(r_0)} < -3$

$\Rightarrow \frac{\frac{1}{f(r_0)}f'(r_0)}{\frac{1}{r_0}} < -3$

$\Rightarrow \frac{d\,(\log f(r_0))}{d\,log(r_0)} < -3$ .... (6)

Consider the power law of force

$f = \frac{-k}{r^{n+1}}$

$\frac{\partial f}{\partial r} = \frac{k(n+1)}{r^{n+2}}$

Then the stability condition equation (5) becomes

$(\frac{k(n+1)}{r^{n+2}})_{r=r_0} < \frac{3}{r_0}(-f(r_0))$

$(\frac{k(n+1)}{{r_0}^{n+2}}) < \frac{-3}{r_0}(\frac{-k}{{r_0}^{n+1}})$

$(\frac{(n+1)}{{r_0}^{n+2}}) < (\frac{3}{{r_0}^{n+2}})$

$\Rightarrow$n +1 < 3

$\Rightarrow$n < 2

Thus the power law attractive potential varying more slowly than $\frac{1}{r^2}$ is capable of stable circular orbit for all value of r_0.

The orbit equation can be written as

$\frac{d^2u}{d\theta^2}$ + u = J(u) …………………………… (7) where J(u) = $\frac{-m}{l^2u^2}\, f(\frac{1}{u})$ …………….. (8)

The condition for circular orbit of radius $r_0 = \left(\frac{1}{u_0}\right)$

(8)$\Rightarrow$J(u_0) $= \frac{-m}{l^2 u_0}$ f$\left(\frac{1}{u_0}\right)$

$\Rightarrow$J(u_0) $= \frac{-m}{l^2 u_0}$ f(r_0)

$\Rightarrow$J(u_0) $= \frac{-m}{l^2 u_0}$ f$\left(-\frac{l^2}{m{r_0}^3}\right)$[∵ using (3)]

$\Rightarrow$J(u_0) $= \frac{1}{{u_0}^2{r_0}^3} = \frac{{u_0}^3}{{u_0}^2}$

$\Rightarrow$J(u_0) $= u_0$ ………………………….. (9)

If the energy is slightly greater than the needed for a circular orbit and if the potential is such that the motion is a stable, then u will be remained bounded any vary slightly from the circular orbit. In this case the Taylor's Series is

$$J(u) = J(u_0) + \frac{u - u_0}{1!}\frac{dJ}{du_0} + \frac{(u-u_0)^2}{2!}\frac{d^2J}{du_0{}^2} + \ldots\ldots\ldots\ldots$$

Neglect the higher power of $\frac{dJ}{du_0}$

$$\therefore J(u) = J(u_0) + \frac{u - u_0}{1!}\frac{dJ}{du_0} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Put u - u_0=x

$$\frac{d^2u}{d\theta^2} = \frac{d^2x}{d\theta^2}$$

$$(7) \Rightarrow \frac{d^2u}{d\theta^2} + u = J(u)$$

$$= J(u_0) + \frac{u - u_0}{1!}\frac{dJ}{du_0}$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + x = x\,\frac{dJ}{du_0}$$

$$\Rightarrow \frac{d^2x}{d\theta^2} + x = x\,\frac{dJ}{du_0}$$

(ie) $\frac{d^2x}{d\theta^2} + x\left(1 - \frac{dJ}{du_0}\right) = 0$

(ie) $\frac{d^2x}{d\theta^2} + \beta^2 x = 0$ where $\beta^2 = \left(1 - \frac{dJ}{du_0}\right)$

By a suitable choice of origin, the solution of the above equation can be written as

X = a cos $\beta\theta$

$u - u_0$ = a cos $\beta\theta$

$$u = u_0 + a\cos\beta\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

which is the equation of simple Harmonic motion about u_0.

$(8) \Rightarrow J = \frac{-m}{l^2 u_0} f\left(\frac{1}{u}\right)$

$\frac{dJ}{du} = \frac{2m}{l^2u^2} f(\frac{1}{u}) \frac{-m}{l^2 u_0} \frac{d}{du} f\left(\frac{1}{u}\right)$

$= \frac{-2}{u} J - \frac{m}{l^2u^2} \frac{d}{du} [f\left(\frac{1}{u}\right)]$

For circular orbit the condition is

$\frac{dJ}{du_0} = -2 + \frac{u_0}{f_0} \frac{df}{du_0}$

$Here \beta^2 = (1 - \frac{dJ}{du_0})$

$= 1 - [-2 + \frac{u_0}{f_0} \frac{df}{du_0}]$

$= 3 + \frac{u_0}{f_0} \frac{df}{du_0}$

$= 3 - \frac{r_0}{f_0} \frac{df}{dr_0}$

$\therefore \beta^2 = 3 - \frac{r}{f} \frac{df}{dr}$ ………………………………. (12)

When r_0 sweeps around the plane once θ varies from 0 to 2π and these are β Cycles for simple Harmonic motion given by equation (11)

If β is a rational number $\frac{p}{q}$ then the q revolution of the radius vector would begin to retrace itself.

(ie) the orbit would be closed.

For an initial energy the angular momentum is given by $f(r_0) \left(-\frac{l^2}{m{r_0}^3}\right)$ and

$E = V(r_0) + \frac{l^2}{2m{r_0}^2}$

It is possible to establish the stable circular orbit at each r_0 satisfying $\frac{df}{dr}/r = r_0 < -3 \frac{f(r_0)}{r_0}$

Then we have

(12) $\Rightarrow \beta^2 = 3 + \frac{r}{f}\frac{df}{dr}$

$\Rightarrow \beta^2 - 3 = \frac{r}{f}\frac{df}{dr}$

$\Rightarrow \beta^2 - 3 = \frac{d(\log f)}{d(logr)}$

$\Rightarrow\ d(\log f) = (\beta^2 - 3)\ d(logr)$

Integrating both sides,

log f $= (\beta^2 - 3)\ logr$ + log k

$\Rightarrow$ log f $= \log r^{(\beta^2 - 3)} + logk$

$\Rightarrow$ log f $= \log r^{(\beta^2 - 3)}.k$

$\Rightarrow$ f(r) $= kr^{(\beta^2 - 3)}$

The force law is f(r) = $\frac{-k}{(r^{3-\beta^2})}$ ………………………. (13)

Any force law of this form where β is rational number will have a closed stable orbits.

When β = 1, suppose the initial condition deviate more than slightly from the requirement for the circular orbit.

Considering one more form in the Taylor's series for J(u) we can say that the orbit are closed only for β^2 = 1 and β^2 = 4. When β^2 = 1 we get the inverse square law f(r) = $\frac{-k}{r^2}$ and β^2 = 4 we get the hoop's law f(r) = -kr

Hence the proof.

Obtain the equation of orbit in the form $\frac{1}{r} = \frac{mk}{l^2}[1+\sqrt{1 + \frac{2El^2}{mk^2}}\cos(\theta - \dot{\theta})$***.***

Also discuss the nature of the conic for varies value of e and E.

(Or)

Derive the equation of the central orbit under inverse square law and classify the nature of conic in terms of total energy E .

(Or)

Discuss the inverse square law of force.

Solution:

We know that,

The differential equation of the central orbit

$$\frac{d^2u}{d\theta^2} + u = \frac{-m}{l^2u^2} f(\frac{1}{u}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

In the inverse square law of force

$$f(r) = \frac{-k}{r^2} \text{ and } V(r) = \frac{-k}{r}$$

Put $r = \frac{1}{u}$

$$f(\frac{1}{u}) = \frac{-k}{(\frac{1}{u})^2} = -ku^2$$

$$(1) \Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{-m}{l^2u^2}(-ku^2)$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{mk}{l^2}$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u - \frac{mk}{l^2} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Take y = u - $\frac{mk}{l^2}$

$\frac{dy}{d\theta} = \frac{du}{d\theta}$

(2)$\Rightarrow \frac{d^2u}{d\theta^2}$ +y = 0

This equation gives the solution y = B $\cos(\theta - \dot{\theta})$

$\Rightarrow$ u = $\frac{mk}{l^2}$ + B $\cos(\theta - \dot{\theta})$

$\Rightarrow \frac{1}{r} = \frac{mk}{l^2} [1+\frac{Bl^2}{mk}\cos(\theta - \dot{\theta})]$

$\Rightarrow \frac{1}{r} = \frac{mk}{l^2} [1+e\cos(\theta - \dot{\theta})]$ Where e = $\frac{Bl^2}{mk}$

This is the orbit equation with eccentricity $\frac{Bl^2}{mk}$

We know that,

$$\theta = \dot{\theta} - \int \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2mV}{l^2} - u^2}}$$

$In\ this$ case the inverse square law of force V = $\frac{-k}{r}$

$$\therefore \theta = \dot{\theta} - \int \frac{du}{\sqrt{\frac{2mE}{l^2} - \frac{2m}{l^2}(\frac{-k}{r}) - u^2}}$$

$$= \dot{\theta} - \int \frac{du}{\sqrt{\frac{2mE}{l^2} + \frac{2mk}{l^2}(u) - u^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Where $\dot{\theta}$ is a constant of integration determined by the initial condition. The indefinite integral is of the standard form $\int \frac{dx}{\sqrt{\alpha + \beta x + \gamma x^2}}$

$$\int \frac{dx}{\sqrt{\alpha+\beta x+\gamma x^2}} = \frac{1}{\sqrt{-\gamma}} \cos^{-1}(-\frac{\beta+2\gamma x}{\sqrt{q}}) \quad \text{……………….. (4)}$$

$q = \beta^2$-4 $\alpha\,\gamma$

To apply this to equation (3) we get,

$\alpha = \frac{2mE}{l^2}$, $\beta = \frac{2mk}{l^2}$, $\gamma = -1$

q = $(\frac{2mk}{l^2})^2$ -4$(\frac{2mE}{l^2})$(-1)

= $(\frac{2mk}{l^2})^2[1 + 4(\frac{2mE}{l^2})$ x$\frac{l^4}{4m^2k^2})$]

= $(\frac{2mk}{l^2})^2[1 + (\frac{2El^2}{l^2mk^2})$]

$$(4) \Rightarrow \int \frac{dx}{\sqrt{\alpha+\beta x+\gamma x^2}} = \frac{1}{\sqrt{-(-1)}} \cos^{-1}(-\frac{(\frac{2mk}{l^2}+2(-1)x)}{\sqrt{(\frac{2mk}{l^2})^2[1+\frac{2El^2}{mk^2}]}})$$

$$\therefore (3) \Rightarrow \theta = \dot{\theta} - \frac{1}{\sqrt{1}} \cos^{-1}(-\frac{(\frac{2mk}{l^2}+2(u)}{\frac{2mk}{l^2}\sqrt{[1+\frac{2El^2}{mk^2}]}})$$

$$= \dot{\theta} - \cos^{-1}(-\frac{(\frac{2mk}{l^2}+2(u)}{\frac{2mk}{l^2}\sqrt{[1+\frac{2El^2}{mk^2}]}})$$

$$= \dot{\theta} - \cos^{-1}(\frac{-1+(\frac{ul^2}{mk})}{\sqrt{[1+\frac{2El^2}{mk^2}]}})$$

$$\therefore \theta - \dot{\theta} = \cos^{-1}(\frac{-1+(\frac{ul^2}{mk})}{\sqrt{[1+\frac{2El^2}{mk^2}]}})$$

$$\Rightarrow \cos(\theta - \dot{\theta}) \sqrt{[1+\frac{2El^2}{mk^2}]} = -1 + \frac{ul^2}{mk}$$

$$\sqrt{[1+\frac{2El^2}{mk^2}]} \cos(\theta - \dot{\theta}) + 1 = \frac{\frac{1}{r}l^2}{mk}$$

$\frac{1}{r} = \frac{mk}{l^2}[1 + \sqrt{[1 + \frac{2El^2}{mk^2}]} \cos(\theta - \dot{\theta})]$

$= \frac{mk}{l^2}[1 + e \cos(\theta - \dot{\theta})]$ …………………… (5)

This is the equation of conic at one focus $\frac{1}{r} = c[1 + e \cos(\theta - \dot{\theta})]$

∴ Equation (5) represents a conic with eccentricity e = $\sqrt{[1 + \frac{2El^2}{mk^2}]}$

Discuss the nature of orbit $\frac{1}{r} = c[1 + e\cos(\theta - \dot{\theta})]$ and prove that the axidal distance are the roots are $r^2 + \frac{K}{E}\ r - \frac{a(1-e^2)}{1+e\cos(\theta - \dot{\theta})}$. Also derive the elliptic equation r = $\frac{a(1 - e^2)}{[1 + e\cos(\theta - \dot{\theta})]}$

Proof:

Given $\frac{1}{r} = c[1 + e \cos(\theta - \dot{\theta})]$ ………………….. (1)

Where e is the eccentricity of the conic section by comparison with $\frac{1}{r} = \frac{mk}{l^2}[1 + \sqrt{[1 + \frac{2El^2}{mk^2}]} \cos(\theta - \dot{\theta})]$

It follows that the orbit is always a conic section, with eccentricity e = $\sqrt{[1 + \frac{2El^2}{mk^2}]}$ ……………………………. (2)

The nature of orbit depends on the magnitude e according to the following,

For a hyperbola e >1 ⇒E>0

For a parabola e < 1 ⇒E =0

For an ellipse e < 1 ⇒E >0

For a circle e = 0 ⇒$\sqrt{[1+\frac{2El^2}{mk^2}]}$ = 0

⇒$\frac{2El^2}{mk^2}$ = -1

⇒E = $\frac{mk^2}{2l^2}$ ………………………… (3)

For a circular orbit T and V are constant in time and from the Viral theorem E =T + V

⇒E = $\frac{-V}{2}$ +V

⇒E = $\frac{V}{2}$

⇒E =$\frac{-k}{2r_0}$

The statement of equilibrium between the central force and effective force can be written as f(r_0) = $\frac{l^2}{m{r_0}^3}$

⇒ $r_0 = \frac{l^2}{km}$ ………………….. (4)

The equation (4) ⇒ E =$\frac{-k}{2}\frac{km}{l^2}$

⇒ E =$\frac{-mk^2}{2l^2}$ is the condition for circular motion.

In the case of elliptic orbits it can be shown the major axis depends on the energy.

The semi major axis is one half the sum of two apsidal distance r_1 and r_2.

Let SA = r_1 and SA' = r_2

Let AA' = 2a and BB' =2b

AA' = 2a be the major axis and BB' =2b is the minor axis.

Clearly A and A' are apses, where r attains maximum values say r_1 and r_2.

$\therefore 2a = r_1 + r_2$ ……………………….(6)

We have one equation $\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2) + V = E$

$\frac{1}{2}m(\dot{r}^2) + \frac{1}{2}m(r^2\dot{\theta}^2) + V = E$

$\frac{1}{2}m(\dot{r}^2) = E - \frac{1}{2}m(r^2\dot{\theta}^2) - V$

$\dot{r}^2 = \frac{2}{m}(E - V - \frac{m(r^2\dot{\theta}^2)}{2})$

$= \frac{2}{m}((E - V) - \frac{m(r^2\dot{\theta}^2)}{2})$

$= \frac{2}{m}((E - V) - \frac{2}{m}\frac{m(r^2\dot{\theta}^2)}{2})$

$= \frac{2}{m}((E - V) - r^2\dot{\theta}^2)$

$= \frac{2}{m}((E - V) - r^2(\frac{l^2}{mr^2})^2$

$= \frac{2}{m}((E - V) - \frac{l^2}{m^2r^2}$

$\dot{r} = \sqrt{\frac{2}{m}((E - V) - \frac{l^2}{m^2r^2}}$

$Here \dot{r} = \frac{dr}{dt} = 0$ then $\sqrt{\frac{2}{m}((E - V) - \frac{l^2}{m^2r^2}} = 0$

$\Rightarrow \frac{2E}{m} - \frac{2V}{m} - \frac{l^2}{m^2r^2} = 0$

$\Rightarrow \frac{2E}{m} + \frac{2k}{rm} - \frac{l^2}{m^2r^2} = 0$ [since $V = -\frac{k}{r}$] ……………. (7)

Divided by $\frac{2}{m}$

$\Rightarrow$E $+\frac{k}{r} - \frac{l^2}{2mr^2} = 0$ …………………….. (8)

(7) divided by E

(7) $\Rightarrow \frac{2}{m} + \frac{2k}{rmE} - \frac{l^2}{Em^2r^2} = 0$

Multiply by r^2

$\Rightarrow \frac{2}{m}r^2 + \frac{2k}{rmE}r - \frac{l^2}{Em^2} = 0$

Clearly, this has roots r_1 and r_2 and multiply by $\frac{m}{2}$,

$\Rightarrow r^2 + \frac{k}{E}r - \frac{l^2}{Em^2} = 0$ …………. (9)

$r_1 + r_2 = -\frac{k}{E}$

$2a = -\frac{k}{E}$

$\Rightarrow a = -\frac{k}{2E}$ …………………………… (10)

$\Rightarrow E = -\frac{k}{2a}$

Substituting this value in equation (2)

$$e = \sqrt{1 + \frac{2(\frac{-k}{2a})l^2}{mk^2}}$$

$$e = \sqrt{1 - \frac{l^2}{amk}}$$

$$e^2 = 1 - \frac{l^2}{amk}$$

$\Rightarrow 1 - e^2 = \frac{l^2}{amk}$

$\Rightarrow a(1 - e^2) = \frac{l^2}{mk}$ …………………….. (11)

Substituting in equation (1)

$\Rightarrow \frac{1}{r} = \frac{mk}{l^2}[1 + e\cos(\theta - \dot{\theta})]$

$r = \frac{l^2}{mk}\frac{1}{[1 + e\cos(\theta - \dot{\theta})]}$

$r = \frac{a(1 - e^2)}{[1 + e\cos(\theta - \dot{\theta})]}$

when $\theta - \dot{\theta} = 0$

$r_1 = \frac{a(1 - e^2)}{[1 + e\cos 0]}$

$r_1 = \frac{a(1 - e^2)}{1 + e}$

$r_1 = \frac{a(1 + e)(1 - e)}{1 + e}$

$= a(1 - e)$

when $\theta - \dot{\theta} = \pi$

$r_2 = \frac{a(1 - e^2)}{[1 + e\cos \pi]}$

$r_2 = \frac{a(1 - e^2)}{1 + e(-1)}$

$r_2 = \frac{a(1 + e)(1 - e)}{1 - e}$

$r_2 = a(1 + e)$

Explain the motion in time in the Kepler problem

Proof:

The motion of the particle in time as the orbit is the relation between the radial distance of the particle r and the time is given by $t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}((E - V) - \frac{l^2}{m^2 r^2}}}$

$$= \frac{\sqrt{m}}{2} \int_{r_0}^{r} \frac{dr}{\sqrt{(E - V) - \frac{l^2}{2mr^2}}}$$

In the inverse square law force $f(r) = \frac{-k}{r^2} = \frac{-\partial v}{\partial r}$

$$\Rightarrow \frac{\partial v}{\partial r} = \frac{k}{r^2}$$

$$\Rightarrow V = -\frac{k}{r}$$

$$t = \frac{\sqrt{m}}{2} \int_{r_0}^{r} \frac{dr}{\sqrt{E + \frac{k}{r} - \frac{l^2}{2mr^2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

we have,

$$mr^2\dot{\theta} = l$$

$$\Rightarrow mr^2 \frac{d\theta}{dt} = l$$

$$dt = \frac{mr^2}{d\theta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

which complain with the orbit equation,

$$\frac{1}{r} = \frac{mk}{l^2}[1 + e\cos(\theta - \dot{\theta})]$$

$$r^2 = \frac{l^4}{m^2k^2} \frac{1}{[1 + e\cos(\theta - \dot{\theta})]^2}$$

(2)⇒$dt = \frac{m[\frac{l^4}{m^2k^2[1+e\cos(\theta-\dot{\theta})]^2}]}{l} d\theta$

Let us suppose that when the time t =0, $\theta = \theta_0$ and θ =0,t=t

Integrating the above limit we get,

$$t = \frac{l^3}{mk^2}\int_{\theta_0}^{\theta} \frac{d\theta}{[1+e\cos(\theta-\dot{\theta})]^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Let us consider the parabolic motion e =1. To measure the plane polar angle from the radius vector at the point of closes approach. A point most usually designated as perihelion.

This convention corresponds to setting θ' in the orbit equation = 0

$$(3) \Rightarrow t = \frac{l^3}{mk^2}\int_0^{\theta} \frac{d\theta}{[1+\cos\theta]^2}$$

$$= \frac{l^3}{mk^2}\int_0^{\theta} \frac{d\theta}{[2\cos\frac{\theta}{2}]^2}$$

$$= \frac{l^3}{4mk^2}\int_0^{\theta} sec^4\frac{\theta}{2}\, d\theta$$

$$= \frac{l^3}{4mk^2}\int_0^{\theta} sec^2\frac{\theta}{2}\, sec^2\frac{\theta}{2}\, d\theta$$

$$= \frac{l^3}{4mk^2}\int_0^{\theta} (1+tan^2\frac{\theta}{2})\, sec^2\frac{\theta}{2}\, d\theta$$

$$= \frac{l^3}{2mk^2}\int_0^{\theta} (1+tan^2\frac{\theta}{2})\, sec^2\frac{\theta}{2}\, d\theta$$

Put $x = \tan\frac{\theta}{2}$

$$dx = \frac{1}{2}\, sec^2\frac{\theta}{2}\, d\theta$$

$$t = \frac{l^3}{2mk^2}\int_0^{\tan\frac{\theta}{2}} (1+x^2).dx$$

$$= \frac{l^3}{2mk^2}[\mathrm{x} + \frac{x^3}{3}]_0^{\tan\frac{\theta}{2}}$$

$$= \frac{l^3}{2mk^2}[\tan\frac{\theta}{2} + \frac{1}{3}tan^3\frac{\theta}{2}]$$

For elliptic motion equation (1)

Through an auxiliary variable ψ denoted as the eccentric anomaly and different from the relation r = a(1-ecosψ)

We have,

$$\mathrm{e} = \sqrt{1 + \frac{2El^2}{mk^2}}$$

$$e^2 = 1 + \frac{2El^2}{mk^2}$$

$$1 - e^2 = 1 - (1 + \frac{2El^2}{mk^2})$$

$$= -\frac{2El^2}{mk^2}$$

$$\frac{l^2}{m} = -\frac{-k^2}{2E}(1 - e^2)$$

$$= -\frac{-k^2}{2(\frac{-k}{2a})}(1 - e^2)$$

$$= \mathrm{ka}\,(1 - e^2)$$

$$(1) \Rightarrow \mathrm{t} = \frac{\sqrt{m}}{2}\int_{r_0}^{r}\frac{dr}{\sqrt{\mathrm{E} + \frac{k}{r} - \frac{1}{2r^2}ka(1-e^2)}}$$

$$= \sqrt{\frac{m}{2}}\int_{r_0}^{r}\frac{dr}{\sqrt{\frac{-k}{2a} + \frac{k}{r} - \frac{1}{2r^2}ka(1-e^2)}}$$

$$= \sqrt{\frac{m}{2k}}\int_{r_0}^{r}\frac{rdr}{\sqrt{\frac{-r^2}{2a} + r - \frac{a}{2}(1-e^2)}}$$

We have r =a(1-e cosψ)

dr =aesinψdψ

$$t = \sqrt{\frac{m}{2k}}\int_{\psi_0}^{\psi}\frac{a(1-ecos\psi)aesin\psi\ d\psi}{\sqrt{a(1-ecos\psi)-\frac{a^2}{2a}(1-ecos\psi)^2-\frac{a(1-e^{2)}}{2}}}$$

$$= \sqrt{\frac{m}{2k}}\frac{a^2e}{\sqrt{a}}\int_{\psi_0}^{\psi}\frac{(1-ecos\psi)sin\psi\ d\psi}{\sqrt{(1-ecos\psi)-\frac{(1-ecos\psi)^2}{2}-\frac{(1-e^{2)}}{2}}}$$

$$=\sqrt{\frac{m}{2k}}\sqrt{2}a^{\frac{3}{2}}e\int_{\psi_0}^{\psi}\frac{(1-ecos\psi)sin\psi\ d\psi}{\sqrt{2(1-ecos\psi)-(1-ecos\psi)^2-(1-e^{2)}}}$$

$$=\sqrt{\frac{m}{2k}}a^{\frac{3}{2}}e\int_{\psi_0}^{\psi}\frac{(1-ecos\psi)sin\psi\ d\psi}{\sqrt{e^2-e^2cos\psi}}$$

$$=\sqrt{\frac{m}{2k}}a^{\frac{3}{2}}e\int_{\psi_0}^{\psi}\frac{(1-ecos\psi)sin\psi\ d\psi}{e\sqrt{1-cos^2\psi}}$$

$$=\sqrt{\frac{m}{2k}}a^{\frac{3}{2}}\int_{\psi_0}^{\psi}\frac{(1-ecos\psi)sin\psi\ d\psi}{sin\psi}$$

$$=\sqrt{\frac{m}{2k}}a^{\frac{3}{2}}\int_{\psi_0}^{\psi}(1-ecos\psi)\ d\psi$$

If it starts from perihelion we have,

$$t = \sqrt{\frac{ma^3}{k}}\int_{0}^{\psi}(1-ecos\psi)\ d\psi$$

Derive $\tau = 2\pi a^{\frac{3}{2}}\sqrt{\frac{m}{k}}$ and prove that $\tau = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{GM}}$. Hence deduce Kepler's third law.

Proof:

From the conservation of angular momentum the Arial velocity is constant.

Given $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta}$

$$\frac{dA}{dt} = \frac{l}{2m}$$

Integrating over a complete

$$\int_0^{\tau} \frac{dA}{dt}\, dt = \int_0^{\tau} \frac{l}{2m}\, dt$$

$\Rightarrow$ A = $\frac{l}{2m}[t]_0^{\tau}$

A = $\frac{l\tau}{2m}$ …………………… (1)

Now the area of an ellipse is A = πab ……………. (2)

By the definition of eccentricity the semi minor axis b is related to a is given by

b = a$\sqrt{1-e^2}$

b = a$\sqrt{1-\left(1-\frac{l^2}{mka}\right)}$

= a $\sqrt{\left(\frac{l^2}{mka}\right)}$

= $\frac{a^{\frac{1}{2}}l}{\sqrt{mk}}$

(2) c A $=\pi$ a.$\frac{a^{\frac{1}{2}}l}{\sqrt{mk}}$

A $=\frac{\pi a^{\frac{3}{2}}l}{\sqrt{mk}}$ ………………….. (3)

From (1) and (3)

we get $\frac{l\tau}{2m} = \frac{\pi a^{\frac{3}{2}}l}{\sqrt{mk}}$

$$\Rightarrow \frac{\tau}{2m} = \frac{\pi a^{\frac{3}{2}}}{\sqrt{mk}}$$

$$\Rightarrow \frac{\tau\sqrt{mk}}{2m} = \pi a^{\frac{3}{2}}$$

$$\Rightarrow \tau = 2\pi a^{\frac{3}{2}} \frac{\sqrt{m}}{\sqrt{k}}$$

Let m_1 and m_2 be the masses of the planet and the sun respectively.

Since the motion of the planet about the sun is a two body problem the reduce mass is given by $\mu = \frac{m_1 m_2}{m_{1+} m_2}$ ……………………… (4)

The gravitational law of attraction is f(r) = $\frac{-Gm_1 m_2}{r^2}$ ………………………. (5)

The inverse square law of force is f(r) = $\frac{-k}{r^2}$ …………………………… (6)

Equation (5) and (6)

We get,

$$\frac{-Gm_1 m_2}{r^2} = \frac{-k}{r^2}$$

$$\Rightarrow Gm_1 m_2 = k$$

The period τ of the elliptic motion is $\tau = 2\pi a^{\frac{3}{2}} \frac{\sqrt{m}}{\sqrt{k}}$

If m is replaced by the reduce mass $\tau = 2\,\pi a^{\frac{3}{2}}\sqrt{\frac{m_1 m_2}{m_{1+}m_2}\frac{1}{Gm_1 m_2}}$

$$\tau = 2\,\pi a^{\frac{3}{2}}\sqrt{\frac{1}{G(m_{1+}m_{2})}}$$

$Since$ the mass of the planet m_1 is very small compared to the mass of the sun m_2 we can neglect the mass of the planet.

$$\tau = \frac{2\,\pi a^{\frac{3}{2}}}{\sqrt{G(m_{2})}}$$

Squaring on both sides,

$$\tau^2 = (\frac{2\,\pi a^{\frac{3}{2}}}{\sqrt{G(m_{2})}})^2 = \frac{4\pi^2\,a^3}{Gm_2}$$

τ^2 = ka^3 where k = $\frac{4\pi^2}{Gm_2}$

$\Rightarrow\ \tau^2 \alpha\ a^3$

This shows that the square of the period is propositional to the cube of the mean distance from the sun.

This prove the Keplers third law of the planetary motion.

Derive the Kepler's equation and define mean anomaly Also define true anomaly and eccentric anomaly.

Proof:

We have,

$$t = \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{2}{m}((E - V) - \frac{l^2}{m^2 r^2}}}$$

$$= \frac{\sqrt{m}}{2} \int_{r_0}^{r} \frac{dr}{\sqrt{(E - V) - \frac{l^2}{2mr^2}}}$$

In the inverse square law force $f(r) = \frac{-k}{r^2} = \frac{-\partial v}{\partial r}$

$$\Rightarrow \frac{\partial v}{\partial r} = \frac{k}{r^2}$$

$$\Rightarrow V = -\frac{k}{r}$$

$$t = \frac{\sqrt{m}}{2} \int_{r_0}^{r} \frac{dr}{\sqrt{E + \frac{k}{r} - \frac{l^2}{2mr^2}}}$$

we have,

$$t = \frac{\sqrt{m}}{2} \int_{r_0}^{r} \frac{dr}{\sqrt{\frac{-k}{2a} + \frac{k}{r} - \frac{l^2}{2mr^2}}}$$

$$= \frac{\sqrt{m}}{2} \int_{r_0}^{r} \frac{r\,dr}{\sqrt{\frac{-kr^2}{2a} + kr - \frac{l^2}{2m}}}$$

We have,

$$e = \sqrt{1 + \frac{2El^2}{mk^2}}$$

$$e^2 = 1 + \frac{2El^2}{mk^2}$$

$$1 - e^2 = 1 - (1 + \frac{2El^2}{mk^2})$$

$$= -\frac{2El^2}{mk^2}$$

$$1 - e^2 = \frac{l^2}{mak}$$

$$(1 - e^2)\text{ak} = \frac{l^2}{m}$$

$$\frac{1}{2}(1 - e^2)\text{ak} = \frac{1}{2}\frac{l^2}{m}$$

There fore t = $\frac{\sqrt{m}}{2}\int_{r_0}^{r}\frac{rdr}{\sqrt{\frac{-kr^2}{2a}+kr-\frac{1}{2}ka(1-e^2)}}$

$$= \sqrt{\frac{m}{2k}}\int_{r_0}^{r}\frac{rdr}{\sqrt{\frac{-r^2}{2a}+r-\frac{a}{2}(1-e^2)}}$$

We have r =a(1-e cosψ)

dr =aesinψdψ

$$t = \sqrt{\frac{m}{2k}}\int_{\psi_0}^{\psi}\frac{a(1-ecos\psi)aesin\psi\, d\psi}{\sqrt{a(1-ecos\psi)-\frac{a^2}{2a}(1-ecos\psi)^2-\frac{a(1-e^2)}{2}}}$$

$$= \sqrt{\frac{m}{2k}}\frac{a^2e}{\sqrt{a}}\int_{\psi_0}^{\psi}\frac{(1-ecos\psi)sin\psi\, d\psi}{\sqrt{(1-ecos\psi)-\frac{(1-ecos\psi)^2}{2}-\frac{(1-e^2)}{2}}}$$

$$= \sqrt{\frac{m}{2k}}\sqrt{2}a^{\frac{3}{2}}e\int_{\psi_0}^{\psi}\frac{(1-ecos\psi)sin\psi\, d\psi}{\sqrt{2(1-ecos\psi)-(1-ecos\psi)^2-(1-e^2)}}$$

$$= \sqrt{\frac{m}{2k}}a^{\frac{3}{2}}e\int_{\psi_0}^{\psi}\frac{(1-ecos\psi)sin\psi\, d\psi}{\sqrt{e^2-e^2cos\psi}}$$

$$= \sqrt{\frac{m}{2k}}a^{\frac{3}{2}}e\int_{\psi_0}^{\psi}\frac{(1-ecos\psi)sin\psi\, d\psi}{e\sqrt{1-cos^2\psi}}$$

$$= \sqrt{\frac{m}{2k}}\, a^{\frac{3}{2}} \int_{\psi_0}^{\psi} \frac{(1-ecos\psi)sin\psi\; d\psi}{sin\psi}$$

$$= \sqrt{\frac{m}{2k}}\, a^{\frac{3}{2}} \int_{\psi_0}^{\psi} (1 - ecos\psi)\; d\psi$$

If it starts from perihelion we have,

$$t = \sqrt{\frac{ma^3}{k}} \int_0^{\psi} (1 - ecos\psi)\; d\psi \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}$$

and the frequency of revaluation force $\omega = \frac{2\pi}{\tau}$ …………………….. (A)

To Find τ

Let τ be the period of the particle in which the particle complete one full revolution.

Hence τ varies from 0 to 2π.

We have,

$$\tau = \sqrt{\frac{ma^3}{k}} \int_0^{2\pi} (1 - ecos\,\psi) d\psi$$

$$\tau = \sqrt{\frac{ma^3}{k}}\, (\psi - esin\psi)_0^{2\pi}$$

$$\tau = \sqrt{\frac{ma^3}{k}} (2\pi)$$

$$\tau = 2\,\pi a^{\frac{3}{2}} \frac{\sqrt{m}}{\sqrt{k}}$$

$$(A) \Rightarrow \omega = \frac{2\pi}{2\,\pi a^{\frac{3}{2}} \frac{\sqrt{m}}{\sqrt{k}}}$$

$$\omega = \frac{1}{a^{\frac{3}{2}} \frac{\sqrt{k}}{\sqrt{m}}}$$

$$\omega = \sqrt{\frac{k}{ma^3}}$$

(I) $\Rightarrow$ t = $\sqrt{\frac{ma^3}{k}}(\int_0^{\psi}(1 - ecos\psi)\ d\psi$

= $\frac{1}{\omega}$ $(\int_0^{\psi}(1 - ecos\psi)\ d\psi$

t = $\frac{1}{\omega}(\psi - esin\psi)_0^{\psi}$

= $\frac{1}{\omega}(\psi - esin\psi)$

$\Rightarrow \omega t\ \ =\ \psi - esin\psi$

This equation is known as Kepler's equation.

Anomaly

The quantity ωt goes through the range 0 to 2π along with ψ and θ in the course of a complete orbital revaluation and is also denoted as anomaly specifically the mean anomaly.

True Anomaly

Let p present the position of the earth at any instant in is elliptic orbit. PL is drawn perpendicular to the axe line AA' and produced to meet that auxiliary circle of the elliptic at p.

The angle PSL(θ) is called the true anomaly.

(ie) $\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\psi}{2}$

Derive $\boldsymbol{\tan\frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{\psi}{2}}$

Proof:

We know that,

The eccentric anomaly ψ is defined by r = a $(1 - ecos\psi)$ ……………………… (1)

Also the keplar equation is $\omega t = \psi - esin\psi$ ……………………. (2)

The equation of the elliptical orbit is r = $\frac{a(1-e^2)}{1+ecos(\theta-\theta')}$ ………………………..(3)

(ie) r = $\frac{a(1-e^2)}{1+ecos\theta}$ where the polar angle $\theta = \theta - \theta'$,

From equation (1) and (3) we get,

$$a\,(1 - ecos\psi) = \frac{a(1-e^2)}{1+ecos(\theta-\theta')}$$

$$a\,(1 - ecos\psi) = \frac{a(1-e^2)}{1+ecos\theta}$$

$$1 - ecos\psi = \frac{(1-e^2)}{1+ecos\theta}$$

$$1 + ecos\theta = \frac{(1-e^2)}{1-ecos\psi}$$

$$1 + ecos\theta - 1 = \frac{(1-e^2)}{1-ecos\psi} - 1$$

$$ecos\theta \quad = \frac{(1-e^2-1+ecos\psi)}{1-ecos\psi} \text{ -1}$$

$$cos\theta \quad = \frac{(cos\psi-e)}{1-ecos\psi}$$

$$1 + cos\theta = \frac{(cos\psi-e)}{1-ecos\psi} + 1$$

$$1 + cos\theta = \frac{(1-ecos\psi+cos\psi-e)}{1-ecos\psi}$$

$$= \frac{(1-e)[1+cos\psi]}{1-ecos\psi}$$

Now,

$$1 - cos\theta \;= 1 - \frac{cos\psi-e}{1-ecos\psi}$$

$$= \frac{1-ecos\psi-cos\psi-e}{1-ecos\psi}$$

$$= \frac{cos\psi(1+\text{e})+(1+e)}{1-ecos\psi}$$

$$\frac{1-cos\theta}{1+cos\theta} \quad = \frac{(1-cos\psi)(1+\text{e})}{1-ecos\psi}$$

$$= \frac{(1-cos\psi)(1+\text{e})}{1-e(1+cos\psi)}$$

$$\frac{2sin^2\frac{\theta}{2}}{2cos^2\frac{\theta}{2}} \quad = \frac{2sin^2\frac{\psi}{2}(1+\text{e})}{1-e2cos^2\frac{\psi}{2}}$$

$$\tan^2\frac{\theta}{2} \quad = \sqrt{\frac{1+e}{1-e}}\tan^2\frac{\psi}{2}$$

$$\tan^2\frac{\theta}{2} \quad = \sqrt{\frac{1+e}{1-e}}\tan\frac{\psi}{2}$$