

OPERATIONS RESEARCH – DKM24

Unit I

Linear Programming – Simplex method – Transportation and its variation.

Unit II

Network Models – CPM – PERT

Unit III

Integer Programming

Unit IV

Inventory models – Decision Analysis and Games.

Unit V

Queuing Models.

UNIT-I

THE SIMPLEX METHOD

Standard LP form and its basic solutions

Standard LP Form

The properties of the standard LP form is :

- i. All the constraints (with the exception of the non negativity restrictions on the variables are equations with non negative right hand side.
- ii. All the variables are non negative.
- iii. The objective function may be of the maximization or minimization type.

1. Conversion of inequalities into equation

An inequality of the type \leq (\geq) can be converted to an equation by adding its L.H.S. with a slack (surplus) variables.

Example : 1 (\leq constraint)

$$x_1 + 2x_2 \leq 3 \text{ is equivalent to } x_1 + 2x_2 + S_1 = 3.$$

Where the slack $S_1 \geq 0$.

Example : 2 (\geq constraint)

$$3x_1 + x_2 \geq 5 \text{ is equivalent to } 3x_1 + x_2 + S_1 = 5, \text{ where the surplus } S_2 \geq 0.$$

If the equation contains negative in the R.H.S. then we multiplying the equation by (-1).

Example : 3

$$x_1 + 3x_2 = -5, \text{ We write the equation by } -x_1 + 3x_2 = 5.$$

An (\leq) inequality can be converted to a (\geq) by multiply both sides of the inequality by (-1).

Example: 4

$$2 < 4.$$

When we multiply both sides by (-1) we get $-2 > -4$.

2. Conversion of unrestricted variable into non negative variables.

An unrestricted variable x_j can be expressed in terms of two non negative variables by

$$\text{Using the substitution } x_j = x_j^+ - x_j^-$$

$$\text{Where } x_j^+, x_j^- \geq 0.$$

For example,

1. $x_j = -5$

$$\text{where } x_j^+ = 0, x_j^- = 5$$

2. $x_j = 5$

$$\text{where } x_j^+ = 5, x_j^- = 0$$

in both cases x_j^+ and x_j^- are non negative.

The substitution is effected in all the construction and in the objective function. After solving the problem in terms of x_j^+ and x_j^- , the value of the original variable is then determined through back substitution.

3. Conversion of Maximization to Minimization

The maximization of the function $f(x_1, x_2, \dots, x_n)$ is equivalent to the minimization of $-f(x_1, x_2, \dots, x_n)$.

$$\text{ie, } \max(f(x)) = -\min(-f(x))$$

Example

Express the following LP model in standard form maximize $z=2x_1+3x_2+5x_3$.

$$\text{Subject to } x_1+x_2-x_3 \geq -5$$

$$-6x_1+7x_2-9x_3 \leq 4$$

$$x_1+x_2+4x_3 = 10$$

$$x_1, x_2 \geq 0$$

x_3 unrestricted.

Solution :

Conversion of the given problem is

- i. Subtract the surplus S_1 from the L.H.S. of the first constraint and then multiply both sides by -1 to obtain a non-negative R.H.S.
- ii. Add the slack S_2 to the L.H.S. of the second constraint.
- iii. Because the third constraint is already in equation form, no slack (or) surplus needed in this case.
- iv. Substitute unrestricted $x_3 = x_3^+ - x_3^-$ in the objective and all the constraints, where x_3^+ and $x_3^- \geq 0$.

The standard L.P. form of the given problem is maximize $x = 2x_1 + 3x_2 + 5x_3^+ - 5x_3^-$.

Subject to

$$-x_1 - x_2 + x_3^+ - x_3^- + S_1 = 5.$$

$$-6x_1 + 7x_2 - 9x_3^+ + 9x_3^- + S_2 = 4.$$

$$x_1 + x_2 - 4x_3^+ + 4x_3^- = 10$$

$$x_1, x_2, x_3^+, x_3^-, S_1, S_2 \geq 0$$

Determination of Basic Solution

The standard LP form includes m simultaneous Linear equation in ' n ' unknowns (or) variables ($m > n$). We divide the ' n ' variables into two sets.

- i. $(n-m)$ variables to which we assign 0 values and
- ii. The remaining ' m ' variables whose values are determined by solving the resulting m equations. If the m equations yield a unique solution then the associated m variables are called basic variables and the remaining $(n-m)$ zero variables are referred to as non basic variables. In this case, the resulting unique solution comprises a basic solution. If all the variables assume non negative values then the basic solution is feasible otherwise, it is infeasible. The maximum number of possible basic solution for the equation in ' n ' unknown is $\binom{n}{m} = \frac{n!}{m!(n-m)!}$

Example : 1

Consider the following set of two eqns. in five unknown ($m=2, n=5$)

$$x_1 + x_2 + 4x_3 + 2x_4 + 3x_5 = 8$$

$$4x_1 + 2x_2 + 2x_3 + x_4 + 6x_5 = 4$$

Solution:

The maximum number of possible basic solutions for $m=2, n=5$ is

$$\begin{aligned}
\binom{n}{m} &= \frac{n!}{m!(n-m)!} \\
&= \frac{5!}{2!(5-2)!} = \frac{5!}{2! \times 3!} \\
&= \frac{5 \times 4 \times 3!}{2 \times 3!} \\
&= 10
\end{aligned}$$

Case : 1 **Basic feasible solution**

Zero (non basic) variable (x_2, x_4, x_5).

Equations :

Solve $x_1 + 4x_3 = 8$
 $4x_1 + 2x_3 = 4$

$$\begin{array}{r}
4x_1 + 16x_3 = 32 \\
4x_1 + 2x_3 = 4 \\
\hline
14x_3 = 28 \\
x_3 = \frac{28}{14} = 2
\end{array}$$

Solutions:

Unique with $x_1=0, x_3=2$

Status :

Feasible basic solution because the basic variables x_1 and $x_3 \geq 0$

Case : 2 **Basic infeasible solution**

Zero (non basic) variable (x_3, x_4, x_5).

Equations: $x_1 + x_2 = 8$
 $4x_1 + 2x_2 = 4$

Solutions :

Unique with $x_1=-6, x_2=14$

Status :

Infeasible basic solution because $x_1 < 0$

Case : 3 Infinity of solution

Zero (non basic) variable (x_1, x_2, x_5)

Equations: $4x_3 + 2x_4 = 8$

$$2x_3 + x_4 = 4$$

Solution:

No unique solutions because the equations are dependent.

Status : Infinity of solutions.

Case : 4 Non existing Solution

Zero variable (x_1, x_3, x_4)

Equations :

$$x_2 + 3x_5 = 8$$

$$2x_2 + 6x_5 = 4$$

Solution

Non solution exist because the equations are inconsistent.

Status

Non existing solution.

Example : 2

Find the all basic feasible solution for the following equations $x_1 + 3x_2 + x_3 = 6$, $3x_1 + 2x_2 + x_4 = 6$.

Solution:

The maximum number of possible basic solutions for m equations in n unknowns is

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} \quad n=4 \quad m=2$$

$$\frac{4!}{2!(4-2)!} = \frac{4!}{2! \times 2!} = \frac{4 \times 3 \times 2}{2 \times 2} = 6$$

$$x_1 + 3x_2 + x_3 + 0x_4 = 6, \quad (1)$$

$$3x_1 + 2x_2 + 0x_3 + x_4 = 6, \quad (2)$$

$$n=4, m=2$$

$$n-m = 4-2=2$$

$$\text{Put } x_1 = 0, \quad x_2 = 0$$

$$(1) \Rightarrow x_3 + 0x_4 = 6$$

$$(2) \Rightarrow 0x_3 + x_4 = 6$$

$$\therefore x_4 = 6 \quad \text{and } x_3 = 6$$

$$\text{Put } x_1 = 0, \quad x_3 = 0$$

$$(1) \Rightarrow 3x_2 + 0x_4 = 6$$

$$(2) \Rightarrow 2x_2 + x_4 = 6$$

$$\therefore x_2 = 2 \quad \text{and } x_4 = 2$$

$$\text{Put } x_1 = 0, \quad x_4 = 0$$

$$(1) \Rightarrow 3x_2 + x_3 = 6$$

$$(2) \Rightarrow 2x_2 + 0x_3 = 6$$

$$\therefore x_2 = 3 \quad \text{and } x_3 = 3$$

$$\text{Put } x_2 = 0, \quad x_3 = 0$$

$$(1) \Rightarrow x_1 + 0x_4 = 6$$

$$(2) \Rightarrow 3x_1 + x_4 = 6$$

$$\therefore x_1 = 6 \quad 3 \times 6 + x_4 = 6$$

$$18 + x_4 = 6$$

$$x_4 = 6 - 18$$

$$x_4 = -12$$

Put $x_2=0$, $x_4=0$

The solution is $x_1=2$, $x_3=4$

Put $x_3=0$, $x_4=0$

The solution is $x_1=6/7$, $x_2=12/7$

The basic feasible solutions are

1. $x_1 = 0, x_2 = 0, x_3 = 6, x_4 = 6$
2. $x_1 = 0, x_2 = 2, x_3 = 0, x_4 = 2$
3. $x_1 = 0, x_2 = 3, x_3 = 3, x_4 = 0$
4. $x_1 = 6, x_2 = 0, x_3 = 0, x_4 = -12$
5. $x_1 = 2, x_2 = 0, x_3 = 4, x_4 = 0$
6. $x_1 = 6/7, x_2 = 12/7, x_3 = 0, x_4 = 0$

THE SIMPLEX ALGORITHM

The simplex method always start at a basic feasible solution. Then it attempts to find another basic feasible solution that will improve the objective value. This is possible only if an increase in a current zero (non basic) variable can lead to an improvement in the objective value. For a current zero (non basic) variable to become positive one of the current basic variables must be removed. The selected non basic variable is called the entering variable and the removed basic variable is called leaving variable.

The new basic Solution is based on Gauss Jordan row operation. The column associated with the entering variable is called the “Pivot Column”. The row associated with the leaving variable is called the “pivot row”. The intersection of the pivot column and pivot row is called the “pivot element”.

The Gauss Jordan method used the following to produce the new basic solution.

- i. new pivot row = current pivot row \div pivot element (or) old pivot row \div pivot element.
- ii. for all other row inducing the z-row.

New row = current row – (its pivot column co-efficient) x new pivot row.

Optimality Condition

The rules for selecting the entering variable is called the optimality condition.

Rules for selecting the entering variable

The z-row is obtained from the objective function by writing $z - c_1x_1 - c_2x_2 \dots - c_nx_n = 0$. In the starting table 1, $-c_1, -c_2, \dots, -c_n$ are the z-row variable co-efficient.

For maximization problem at any stage the variable associated with the most negative coefficient in the z-row enters into the basic.

If there is a tie it may be taken arbitrary. In the LP of minimization type of the variable associated with the most positive co-efficient in the z-row is the entering variable.

Feasibility condition

The rules for selecting the leaving variable is called the feasibility condition.

Leaving variable

For both the maximization and minimization problem compute the ratio between the element in the solution column and the non-negative element in the pivot column. Then the basic variable associated with the smallest non-negative ratio is the leaving variable.

The step of the simplex method

1. Determine a starting basic feasible solution.
2. Select an entering variable using optimality condition. Stop if there is no entering variables.
3. Select a leaving variable using the feasibility condition.
4. Determine the new basic solution by using the appropriate Gauss Jordan computations goto step 1.

1. Maximize $z = 2x_1 + x_2 - 3x_3 + 5x_4$

Subject to

$$x_1 + 2x_2 + 2x_3 + 4x_4 \leq 40$$

$$2x_1 - x_2 + x_3 + 2x_4 \leq 8$$

$$4x_1 - 2x_2 + x_3 - x_4 \leq 10$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Solution

Introducing slack variable, the

$$\text{LP becomes } \max z = 2x_1 + x_2 - 3x_3 + 5x_4 + 0S_1 + 0S_2 + 0S_3$$

Subject to

$$x_1 + 2x_2 + 2x_3 + 4x_4 + S_1 + 0S_2 + 0S_3 = 40$$

$$2x_1 - x_2 + x_3 + 2x_4 + 0S_1 + S_2 + 0S_3 = 8$$

$$4x_1 - 2x_2 + x_3 - x_4 + 0S_1 + 0S_2 + S_3 = 10$$

$$x_1, x_2, x_3, x_4, S_1, S_2, S_3 \geq 0$$

The z-row is obtained

$$z - 2x_1 - x_2 + 3x_3 - 5x_4 - 0S_1 - 0S_2 - 0S_3 = 0$$

Starting table

Basics	Z	x_1	x_2	x_3	x_4	S_1	S_2	S_3	Solution	ratio
Z	1	-2	-1	3	-5	0	0	0	0	-
S_1	0	1	2	2	4	1	0	0	40	$\frac{40}{4} = 10$
S_2	0	2	-1	1	2	0	1	0	8	$\frac{8}{2} = 4$
S_3	0	4	-2	1	-1	0	0	1	10	-



-5 is the most negative co-efficient in the z row.

x_4 is the entering variable.

4 is the minimum ratio.

S_2 is the leaving variable.

2 is the pivot element.

New pivot row = current pivot row \div pivot element

New row = Current row - (Its pivot column co. eff.) \times new pivot row.

New pivot row = Old S_2 row \div 2

New Z row = Old z row – (-5) x new pivot row.

New S₁ row = Old S₁ row – 4 x new pivot row

New S₃ row = Old S₃ row – (-1) x new pivot row

Table I

Basics	Z	x ₁	x ₂	x ₃	x ₄	S ₁	S ₂	S ₃	Solution	ratio
Z	1	3	-7/2	11/2	0	0	5/2	0	20	-
S ₁	0	-3	4	0	0	1	-2	0	24	$\frac{24}{4} = 6$
x ₄	0	1	-1/2	1/2	1	0	1/2	0	4	-
S ₃	0	5	-5/2	3/2	0	0	1/2	1	14	-



$$\text{New pivot row (x}_4\text{)} = \frac{1}{2} (0 \ 2 \ -1 \ 1 \ 2 \ 0 \ 1 \ 0 \ 8)$$

$$= (0 \ 1 \ -\frac{1}{2} \ \frac{1}{2} \ 1 \ 0 \ \frac{1}{2} \ 0 \ 4)$$

New z row = old z row – (-5) x new row

$$= (1 \ -2 \ -1 \ 3 \ -5 \ 0 \ 0 \ 0 \ 0) + 5 (0 \ 1 \ -\frac{1}{2} \ \frac{1}{2} \ 1 \ 0 \ \frac{1}{2} \ 0 \ 4)$$

$$= (1 \ -2 \ -1 \ 3 \ -5 \ 0 \ 0 \ 0 \ 0) + (0 \ 5 \ -\frac{5}{2} \ \frac{5}{2} \ 5 \ 0 \ \frac{5}{2} \ 0 \ 20)$$

$$= (1 \ 3 \ \frac{-7}{2} \ \frac{11}{2} \ 0 \ 0 \ \frac{5}{2} \ 0 \ 20)$$

New S₁ row = old S₁ row – 4x (new pivot row)

$$= (0 \ 1 \ 2 \ 2 \ 4 \ 1 \ 0 \ 0 \ 40) - 4 (0 \ 1 \ -\frac{1}{2} \ \frac{1}{2} \ 1 \ 0 \ \frac{1}{2} \ 0 \ 4)$$

$$= (0 \ 1 \ 2 \ 2 \ 4 \ 1 \ 0 \ 0 \ 40) - (0 \ 4 \ -2 \ 2 \ 4 \ 0 \ 2 \ 0 \ 16)$$

$$= (0 \ -3 \ 4 \ 0 \ 0 \ 1 \ -2 \ 0 \ 24)$$

New S₃ row = (0 4 -2 1 -1 0 0 1 10) - (-1) (0 1 -1/2 1/2 1 0 1/2 0 4)

$$= (0 \ 5 \ -\frac{5}{2} \ \frac{3}{2} \ 0 \ 0 \ \frac{1}{2} \ 1 \ 14)$$

-7/2 is the most negative co-efficient in the z row.

x₂ is the entering variable.

6 is the minimum ratio

S_1 is leaving variable (current pivot row)

4 is the pivot element.

New pivot row = current pivot row \div pivot element.

$$= S_1 \text{ row} \div 4$$

New z row = old z row $- (-7/2) \times$ new pivot row

New x_4 row = old x_4 row $- (-1/2) \times$ new pivot row

New S_3 row = old S_3 row $- (-5/2) \times$ new pivot row

Table 2

Basic	Z	x_1	x_2	x_3	x_4	S_1	S_2	S_3	Solution	ratio
Z	1	$3/8$	0	$11/2$	0	$7/4$	$3/4$	0	41	
x_2	0	$-3/4$	1	0	0	$1/4$	$-1/2$	0	6	
x_4	0	$5/8$	0	$1/2$	0	$1/8$	$1/4$	0	7	
S_3	0	$25/8$	0	$3/2$	0	$5/8$	$-3/4$	1	29	

Since all the z-row coefficient are non-negative the solution is optimal.

The optimal basic feasible Solution is

$$x_1 = 0$$

$$x_2 = 6$$

$$\max z = 41$$

$$2. \text{ Minimize } z = 5x_1 - 4x_2 + 6x_3 - 8x_4$$

Subject to

$$x_1 + 2x_2 + 2x_3 - 4x_4 \leq 40$$

$$2x_1 - x_2 + x_3 + 2x_4 \leq 8$$

$$4x_1 - 2x_2 + x_3 + x_4 \leq 10$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Solution:

Introducing slack variable the L.P.P is

$$\min Z = 5x_1 - 4x_2 + 6x_3 - 8x_4 + 0S_1 + 0S_2 + 0S_3$$

Subject to

$$x_1 + 2x_2 + 2x_3 + 4x_4 + S_1 + 0S_2 + 0S_3 = 40$$

$$2x_1 - x_2 + x_3 + 2x_4 + 0S_1 + 0S_2 + 0S_3 = 8$$

$$4x_1 - 2x_2 + x_3 - x_4 + 0S_1 + 0S_2 + 0S_3 = 10$$

$$x_1, x_2, x_3, x_4, S_1, S_2, S_3 \geq 0$$

The Z row is obtained from

$$Z - 5x_1 - 4x_2 + 6x_3 + 8x_4 - 0S_1 - 0S_2 - 0S_3 = 0$$

Starting table

Basic	Z	x_1	x_2	x_3	x_4	S_1	S_2	S_3	Solution	ratio
Z	1	-5	4	-6	8	0	0	0	0	-
S_1	0	1	2	2	4	1	0	0	40	10
S_2	0	2	-1	1	2	0	1	0	8	4
S_3	0	4	-2	1	-1	0	0	1	10	-

8 is the largest positive co-efficient in the Z row

x_4 is the entering variable

4 is the least ratio

S_2 is the leaving variable

2 is the pivot element

New Pivot row = S_2 row \div 2

New Z row = old z row - 8 x new pivot row

New S_1 row = old S_1 row - 4 x new pivot row

New S_3 row = old S_3 row - (-1) x new pivot row

Table : 1

Basic	Z	x_1	x_2	x_3	x_4	S_1	S_2	S_3	Solution	ratio
Z	1	-13	8	-10	0	0	-4	0	-32	-
S_1	0	-3	4	0	0	1	-2	0	24	6
x_4	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	1	0	$\frac{1}{2}$	0	4	-
S_3	0	5	$-\frac{5}{2}$	$\frac{3}{2}$	0	0	$\frac{1}{2}$	1	14	-

8 is the largest in coefficient in z row

x_2 is the entering variable

6 is the least ratio

S_1 is the leaving variable

4 is the pivot element.

New pivot row = S_1 row \div 4

New Z row = old z row - 8 x (new pivot row)

New x_4 row = old x_4 row - $(-\frac{1}{2})$ x new pivot row

New x_3 row = old x_3 row - $(-\frac{5}{2})$ x new pivot row

Table : ii

Basic	Z	x_1	x_2	x_3	x_4	S_1	S_2	S_3	Solution	ratio
Z	1	-7	0	-10	0	-2	0	0	-80	-
x_2	0	$-\frac{3}{4}$	1	0	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	6	6
x_4	0	$\frac{5}{8}$	0	$\frac{1}{2}$	1	$\frac{1}{8}$	$\frac{1}{4}$	0	7	-
S_3	0	$\frac{25}{8}$	0	$\frac{3}{2}$	0	$\frac{5}{8}$	$-\frac{3}{4}$	1	29	-

Since all the co-efficient in the z row are negative. The solution is optimal.

The optimal basic feasible solution is

$$x_1 = 0 \quad x_2 = 6 \quad x_3 = 0 \quad x_4 = 7$$

$$\text{Min } z = -80$$

Special cases in simplex method Application

Four special cases arise in the application of the simplex method. They are

- i. Degeneracy
- ii. Unbounded solution
- iii. Non Existing solution (or) Infeasible solution.

I. Degeneracy

Degenerate Solution

A solution is said to be degenerate if one or more basic variable of are zero.

Degeneracy

In the application of the condition of simplex method a tie for the minimum ratio may be broken arbitrary for the purpose of determined the leaving variable. When this happens one or more basic variable will be zero in the next iteration. In this case the new Solution is degenerate.

This condition reveals that the LP model has atleast one constraint.

Cycling (or) Circling

Cycling is a implication of degeneracy when ties are broken arbitrarily in the condition the simplex procedure may report the same sequence of iteration without any improvement in the objective value and never terminating the computation. Such a situation is called cycling.

Note:

In some situation the solution may be temporarily degenerate.

A practiced rules to avoid cycling.

- i. Rearrange the column of the initial simple matrix such that the column of the basic variable are written first.
- ii. Divide each number in the tied rows by the corresponding positive number of the pivot column in that row.
- iii. Compare the ratio column by column find in the identity and then the rest.
- iv. The row which first constrain the smallest ratio contains the outgoing variable after selecting the pivot row simplex procedure is applied.

1. Problem

Show that the following LPP is temporarily degenerate

$$\text{Maximize } Z = 3x_1 + 2x_2.$$

Subject

$$4x_1 + 3x_2 \leq 12$$

$$4x_1 + x_2 \leq 8$$

$$4x_1 - x_2 \leq 8$$

$$x_1, x_2 \geq 0$$

Solution:

Introducing slack variable the LPP is $\max Z = 3x_1 + 2x_2 + 0S_1 + 0S_2 + 0S_3$

Subject to

$$4x_1 + 3x_2 + S_1 = 12$$

$$4x_1 + x_2 + S_2 = 8$$

$$4x_1 - x_2 + S_3 = 8$$

$$x_1, x_2, S_1, S_2, S_3 \geq 0$$

The objective functions are obtained by $Z - 3x_1 - 2x_2 = 0$

Initial Table:

Basic	Z	x_1	x_2	S_1	S_2	S_3	Solution	ratio
Z	1	-3	-2	0	0	0	0	-
S_1	0	4	3	1	0	0	12	3
S_2	0	4	1	0	1	0	8	2



S ₃	0	4	-1	0	0	1	8	2
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x_1 is the entering variable, there is a tie for the leaving variable between S_2 and S_3 .

Let S_2 be the leaving variable.

4 is the pivot element.

New pivot row = S_2 row \div 4

New Z row = old z row - (-3) new Pivot row

New S_1 row = old S_1 row - 4 x new Pivot row

New S_3 row = old S_3 row - 4 x new Pivot row

Table: 1

Basic	Z	x_1	x_2	S_1	S_2	S_3	Solution	ratio
Z	1	0	$-\frac{5}{4}$	0	$\frac{3}{4}$	0	6	-
S_1	0	0	2	1	-1	0	4	2
x_1	0	1	$\frac{1}{4}$	0	$\frac{1}{4}$	0	2	8
S_3	0	0	-2	0	-1	1	0	-

New Z row = (1 -3 -2 0 0 0 0) -

$$(0 -3 \frac{-3}{4} 0 \frac{-3}{4} 0 -6)$$

$$= (1 \ 0 \ \frac{-5}{4} \ 0 \ \frac{3}{4} \ 0 \ 6)$$

x_2 is the entering variable

S_1 is the leaving variable

2 is the pivot element.

New pivot row = S_1 row \div 2

New Z row = old z row - ($-\frac{5}{4}$) new P.R.

$$\text{New } x_1 \text{ row} = \text{old } x_1 \text{ row} - \frac{1}{4} \times \text{new P.R.}$$

$$\text{New } S_3 \text{ row} = \text{old } S_3 \text{ row} - (-2) \text{ new P.R.}$$

In table 1 the value of the basic variable S_3 is zero and hence the current Solution is degenerate basic Solution. The reason for the degenerate is the tie acquire in the starting table for the leaving variable S_2 and S_3 .

Table: 2

Basic	Z	x_1	x_2	S_1	S_2	S_3	Solution	ratio
Z	1	0	0	$\frac{5}{8}$	$\frac{1}{8}$	0	$\frac{17}{2}$	-
x_2	0	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	2	2
x_1	0	1	0	$-\frac{1}{8}$	$\frac{3}{8}$	0	$\frac{3}{2}$	8
S_3	0	0	0	1	-2	1	4	-

$$\text{New Z row} = (1 \ 0 \ -\frac{5}{4} \ 0 \ \frac{3}{4} \ 0 \ 6) +$$

$$(0 \ 0 \ \frac{5}{4} \ \frac{5}{8} \ -\frac{5}{8} \ 0 \ \frac{10}{4})$$

$$= (1 \ 0 \ 0 \ \frac{5}{8} \ \frac{1}{8} \ 0 \ \frac{17}{2})$$

$$\text{New } x_1 \text{ row} = (0 \ 1 \ \frac{1}{4} \ 0 \ \frac{1}{4} \ 0 \ 2) -$$

$$(0 \ 0 \ \frac{1}{4} \ \frac{1}{8} \ -\frac{1}{8} \ 0 \ \frac{1}{2})$$

$$= (0 \ 1 \ 0 \ -\frac{1}{8} \ \frac{3}{8} \ 0 \ \frac{3}{2})$$

$$\text{New } S_3 \text{ row} = (0 \ 0 \ -2 \ 0 \ -1 \ 1 \ 0) +$$

$$(0 \ 0 \ 2 \ 1 \ -1 \ 0 \ 4)$$

$$(0 \ 0 \ 0 \ 1 \ -2 \ 1 \ 4)$$

Since all z row co-efficient are positive.

The solution is optimal.

Since all the basic variable are non zero the solution is non degenerate. Also the solution is temporarily degenerate in the second equation iteration.

The optimal basic feasible Solution is $x_1 = 3/2$; $x_2=2$;

$$\begin{aligned}\max z &= 3x_1+2x_2 \\ &= 3x \frac{3}{2} + 2x2 \\ &= \frac{9}{2} + 4 \\ &= \frac{17}{2}\end{aligned}$$

Alternative optima

When the objective function is parallel to finding constraint, the objective function will assume the same optimal value at more than one solution point. This optimal solutions are called alternative optima.

In otherwords if a constraint of an LPP is satisfy an iteration by the optimal Solution. Then the LPP has more than one optimal Solution. This solutions are called alternate optima.

Indication of Alternate optima in simplex table.

If the coefficient in the z-row associated with a non basic variable in the LPP is 0 in the optimal table, then that variable can enter the basic Solution without changing the value of the objective function.

Note:

If $\bar{x} = (x_1, x_2, \dots, x_n)$ and

$\bar{y} = (y_1, y_2, \dots, y_n)$ are 1000 optimal solution then $\bar{x} + (1-\alpha)\bar{y}$, where $0 \leq \alpha \leq 1$ is also an optimal Solution.

$$\min z = -4x_1+6x_2-2x_3+4x_1$$

subject to

$$x_1+2x_2+2x_3+4x_4 \leq 40$$

$$2x_1-x_2+x_3+2x_4 \leq 8$$

$$4x_1 - 2x_2 + x_3 - x_4 \leq 10$$

$$x_1, x_2, x_3, x_4 \geq 0$$

Solution :

Introducing slack variable the LPP is $\min z = -4x_1 + 6x_2 - 2x_3 + 4x_4 + 0S_1 + 0S_2 + 0S_3$.

Subject to

$$x_1 + 2x_2 + 2x_3 + 4x_4 + S_1 + 0S_2 + 0S_3 = 40$$

$$2x_1 - x_2 + x_3 + 2x_4 + 0S_1 + S_2 + 0S_3 = 8$$

$$4x_1 - 2x_2 + x_3 - x_4 + 0S_1 + 0S_2 + S_3 = 10$$

$$x_1, x_2, x_3, x_4, S_1, S_2, S_3 \geq 0$$

The Z row is obtained from

$$Z + 4x_1 - 6x_2 + 2x_3 - 4x_4 - 0S_1 - 0S_2 - 0S_3$$

Starting table

Basic	Z	x_1	x_2	x_3	x_4	S_1	S_2	S_3	Solution	ratio
Z	1	4	-6	2	-4	0	0	0	0	-
S_1	0	1	2	2	4	1	0	0	40	40
S_2	0	2	-1	1	2	0	1	0	8	4
S_3	0	4	-2	1	-1	0	1	1	10	2.5



New pivot row = S_3 row \div 4

New Z row = old z row - 4 x new pivot row

New S_1 row = old S_1 row - 1 x new pivot row

New S_2 row = old S_2 row - 2 x new pivot row

Table - I

Basic	Z	x ₁	x ₂	x ₃	x ₄	S ₁	S ₂	S ₃	Solution	ratio
Z	1	0	-4	1	-3	0	0	-1	-10	-
S ₁	0	0	5/2	7/4	17/4	1	0	-1/4	75/2	
S ₂	0	0	0	1/2	5/2	0	1	-1/2	3	6
x ₁	0	1	-1/2	1/4	-1/4	0	1	1/4	5/2	10



$$\begin{aligned} \text{New Z row} &= (1 \ 4 \ -6 \ 2 \ -4 \ 0 \ 0 \ 0 \ 0) - \\ &\quad (0 \ 4 \ -2 \ 1 \ -1 \ 0 \ 0 \ 1 \ 10) \\ &= (1 \ 0 \ -4 \ 1 \ -3 \ 0 \ 0 \ -1 \ -10) \end{aligned}$$

$$\begin{aligned} \text{New S}_1 \text{ row} &= (0 \ 1 \ 2 \ 2 \ 4 \ 1 \ 0 \ 0 \ 40) - \\ &\quad (0 \ 1 \ -1/2 \ 1/4 \ -1/4 \ 0 \ 0 \ 1/4 \ 5/2) \\ &= (0 \ 0 \ 5/2 \ 7/4 \ 17/4 \ 1 \ 0 \ -1/4 \ 5/2) \end{aligned}$$

$$\begin{aligned} \text{New S}_2 \text{ row} &= (0 \ 2 \ -1 \ 1 \ 2 \ 0 \ 1 \ 0 \ 8) - \\ &\quad (0 \ 2 \ -1 \ 1/2 \ -1/2 \ 0 \ 0 \ 1/2 \ 5) \\ &= (0 \ 0 \ 0 \ 1/2 \ 5/2 \ 0 \ 1 \ -1/2 \ 3) \end{aligned}$$

$$\text{New pivot row} = \text{S}_2 \text{ row} \div 2$$

$$\text{New z row} = \text{old z row} - 1 \times \text{new pivot row}$$

$$\text{New S}_1 \text{ row} = \text{old S}_1 \text{ row} - 7/4 \times \text{new pivot row}$$

Table 2

Basic	Z	x ₁	x ₂	x ₃	x ₄	S ₁	S ₂	S ₃	Solution	ratio
Z	1	0	-4	0	-8	0	-2	-	-16	
S ₁	0	0	5/2	0	-9/2	1	-7/2	3/2	27	
x ₃	0	0	0	1	5	0	2	-1	6	
x ₁	0	1	-1/2	0	-3/2	0	-1/2	1/2	1	

$$\text{New Z row} = (1 \ 0 \ -4 \ 1 \ -3 \ 0 \ 0 \ -1 \ -10) -$$

$$(0 \ 0 \ 0 \ 1 \ 5 \ 0 \ 2 \ -1 \ 6)$$

$$= (1 \ 0 \ -4 \ 0 \ -8 \ 0 \ -2 \ 0 \ -16)$$

$$\text{New S}_1 \text{ row} = (0 \ 0 \ 5/2 \ 7/4 \ 1/4 \ 1 \ 0 \ -1/4 \ 75/2) -$$

$$(0 \ 0 \ 0 \ 7/4 \ 35/4 \ 0 \ 7/2 \ 7/4 \ 21/2)$$

$$= (0 \ 0 \ 5/2 \ 0 \ -18/2 \ -9/2 \ 1 \ -7/2 \ 3/2 \ 27)$$

$$\text{New x}_1 \text{ row} = (0 \ 1 \ -1/2 \ 1/4 \ -1/4 \ 0 \ 0 \ 1/4 \ 5/2) +$$

$$(0 \ 0 \ 0 \ -1/4 \ -5/4 \ 0 \ -1/2 \ 1/4 \ -3/2)$$

$$= (0 \ 1 \ -1/2 \ 0 \ -3/2 \ 0 \ -1/2 \ 1/2 \ 1)$$

Since all the z new coefficients are non negative Solution is optimal.

The optimal basic feasible Solution is

$$x_1=1, x_2=0, x_3=6, x_4=0$$

$$\min z = -4x_1 + 6x_2 - 2x_3 + 4x_4$$

$$= -4(1) + 6(0) - 2(6) + 0$$

$$\text{Min } z = -16$$

If we choose x_1, x_3 as the entering variables then we get another two solutions.

Unbounded Solution

In some LP model the value of the variables may be increased indefinitely without dating any of the constraints. As a result the objective value may increase (maximization cases) (or) decrease (minimization cases) indefinitely. In this case, the LP has unbounded Solution.

Rules for recognizing unboundedness

If at any iteration the constraint coefficient of any non basic variable are non positive then the Solution is unbounded.

If in addition the objective coefficient of that variable is negative in the cases of maximization (or) positive in the case of minimization then the objective value also is unbounded.

1. Show that the following LPP has unbounded solution.

$$\text{Max } z = 2x_1 - x_2 + 3x_3$$

Subject to

$$x_1 - x_2 + 5x_3 \leq 10$$

$$2x_1 - x_2 + 3x_3 \leq 40$$

$$x_1, x_2, x_3 \geq 0$$

Sol:

Introducing slack variables S_1, S_2 the LPP is

$$\text{max } z = 2x_1 - x_2 + 3x_3 + 0S_1 + 0S_2$$

subject to

$$x_1 - x_2 + 5x_3 + 0S_1 + 0S_2 = 10$$

$$2x_1 - x_2 + 3x_3 + 0S_1 + 0S_2 = 40$$

$$x_1, x_2, x_3, S_1, S_2 \geq 0$$

$$z - 2x_1 + x_2 - 3x_3 + 0 + 0 = 0$$

Starting Table

Basic	Z	x ₁	x ₂	x ₃	S ₁	S ₂	Solution	ratio
Z	1	-2	1	-3	0	0	0	-
S ₁	0	1	-1	5	1	0	10	2
S ₂	0	2	-1	-3	0	1	40	-

x₃ – entering variable

S₁ – leaving variable

5 – pivot element

New pivot row = S₁ row ÷ 5

New z row = old z row + 3 (new pivot row)

New S₂ row = old S₂ row + 3 (new pivot row)

Table : 1

Basic	Z	x ₁	x ₂	x ₃	S ₁	S ₂	Solution	ratio
Z	1	$-\frac{7}{5}$	$\frac{2}{5}$	0	$\frac{3}{5}$	0	6	-
x ₃	0	$\frac{1}{5}$	$-\frac{1}{5}$	1	$\frac{1}{5}$	0	2	10
S ₂	0	$\frac{13}{5}$	$-\frac{8}{5}$	0	$\frac{3}{5}$	1	46	$\frac{230}{13}$

$$z = (1 \ -2 \ 1 \ -3 \ 0 \ 0 \ 0) +$$

$$(0 \ \frac{3}{5} \ -\frac{3}{5} \ 3 \ \frac{3}{5} \ 0 \ 6)$$

$$= (1 \ -\frac{7}{5} \ \frac{2}{5} \ 0 \ \frac{3}{5} \ 0 \ 6)$$

$$S_2 = (0 \ 2 \ -1 \ -3 \ 0 \ 1 \ 40) +$$

$$(0 \quad 3/5 \quad -3/5 \quad 3 \quad 3/5 \quad 0 \quad 6)$$

$$= (0 \quad \frac{13}{5} \quad -\frac{8}{5} \quad 0 \quad \frac{3}{5} \quad 1 \quad 46)$$

Table : 2

Basic	Z	x ₁	x ₂	x ₃	S ₁	S ₂	Solution	ratio
Z	1	0	-1	7	2	0	20	-
x ₃	0	1	-1	5	1	0	10	-
S ₂	0	0	1	-13	-2	1	20	20

↑

Table : 3

Basic	Z	x ₁	x ₂	x ₃	S ₁	S ₂	Solution	ratio
Z	1	0	0	-6	0	1	40	-
x ₁	0	1	0	-8	-1	1	30	-
x ₂	0	0	1	-13	-2	1	20	-

↑

x₃ enter the basis all the constraint coefficient of x₃ are non +ve.

x₃ has been increased indefinitely without violating any constraint.

∴ The LPP has unbounded solution.

Infeasible Solution

If the constraint are not satisfied simultaneously the LPP has no feasible solution. This situation can never occur if all the constraint are of the type ≤ (Assuming non negative R.H.S. constraints).

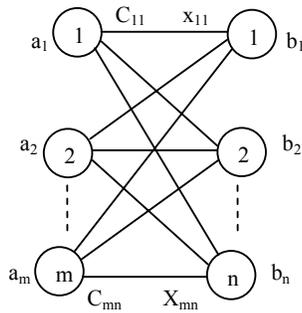
Because the slack provide a feasible solution.

Transportation Model and Variance

The transportation model is a special case of LPP. It deals with situation in which a commodity is shipped from source (eg. Factories) to destinations. (eg. Wase houses). The objective is to determine the amounts shipped from each source to each

destination that minimize the total shipped cost while satisfying both the supply limits and the demand requirements.

The general problem is represented in network diagram. There are ‘m’ sources and ‘n’ destinations each represented by a node. The arcs linking the source and destination represent the routes between the sources and destinations.



Arc (i, j) joining source i to destination carries two pieces of information.

- i. The transportation cost c_{ij} and
- ii. The amount shipped.

The amount of supply at source ‘i’ is a_i and the amount of demand at destination ‘j’ is b_j. The objective of the model is to determine the unknown x_{ij} that will minimize the total transportation cost while satisfying all the supply and demand.

Balanced Transportation problem

The transportation problem which the total supply and total demanded are equal is called a balanced transportation problem.

$$\text{ie, } \sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

Suppose the total supply does not equal the total demand. The T.P. model is said to be unbalanced. In this case the dummy source of destination is added.

$$\text{It } \sum_{i=1}^n b_j = \sum_{i=1}^m a_i \text{ then we add dummy source with its supply, } \sum_{j=1}^n b_j - \sum_{i=1}^m a_i$$

It $\sum_{i=1}^n a_i > \sum_{j=1}^n b_j$ then we add a dummy destination with corresponding demand, $\sum_{i=1}^m a_i - \sum_{j=1}^n b_j$

Ex. Balanced

	D ₁	D ₂	D ₃	D ₄	Supply
O ₁	1	2	3	4	6
O ₂	4	3	2	0	8
O ₃	0	2	2	1	10
Demand	4	6	8	6	

$$\sum a_i = 6+8+10 = 24$$

$$\sum b_j = 4+6+8+6 = 24$$

$\sum a_i = \sum b_j$, the given TP is balanced.

Example : Un Balanced

	D ₁	D ₂	Supply
O ₁	80	215	1000
O ₂	100	108	1300
O ₃	102	68	1200
	2300	1400	

$$\sum a_i = 1000 + 1300 + 1200 = 3500$$

$$\sum b_j = 2300 + 1400 = 3700$$

$$\sum a_i < \sum b_j$$

$$\sum b_j - \sum a_i = 200$$

	D ₁	D ₂	Supply
O ₁	80	215	1000
O ₂	100	108	1300
O ₃	102	68	1200
O ₄	0	0	200
Demand	2300	1400	

In mathematical formulation the total cost of TP is given by

$$z = \sum_{j=1}^n \sum_{i=1}^n c_{ij} x_{ij} \text{ or}$$

equivalently

$$z = \sum_{j=1}^m \sum_{i=1}^n c_{ij} x_{ij}$$

The mathematical formulation of the TP is given by,

$$\text{Minimize } z = \sum_{j=1}^m \sum_{i=1}^n c_{ij} x_{ij}$$

Subject to, $\sum_{j=1}^n x_{ij} \leq a_i, \forall i = 1, 2, \dots, m$ (supply construction)

$$\sum_{i=1}^m x_{ij} \geq b_j, j = 1, 2, \dots, n \text{ (Demand Construction)}$$

$$x_{ij} \geq 0, i=1, 2, \dots, m \text{ and } j=1, 2, \dots, n$$

Feasible solution

A set of values to the variable x_{ij} satisfying the constraints in the transportation problem is called feasible solution.

Optimal feasible solution

A feasible solution which also minimize the total cost is called a optimal feasible Solution. In mathematical formulation of a balanced transportation problem is given by

$$\text{Minimize } z = \sum_{j=1}^n \sum_{i=1}^m c_{ij} x_{ij}$$

Subject to

$$\sum_{j=1}^n x_{ij} = a_i, i = 1, 2, \dots, m$$

$$\sum_{i=1}^m x_{ij} = b_j, j = 1, 2, \dots, n$$

$$x_{ij} \geq 0, i=1, 2, \dots, m \text{ and } j=1, 2, \dots, n.$$

Thus the above transportation problem involves mn unknown x_{ij} . There are $m+n$ constraints of all which are inequation form. Thus $m+n$ equation are not independent. Since we have one condition.

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

Hence there are $m+n-1$ independent constraint.

Non Traditional Transportation Models

The application of the transportation model is not limited to be transport commodity is between sources and destination.

Production Inventory Control

Boralis manufactures backpacks for serious hikers. The demand for its product occurs during march to june of each year. Boralis estimates the demand for the 4 months to be 100, 200, 180 and 300 units respectively. The company uses part time labour to manufacture the backpacks and as such its production capacity varies monthly. It is estimated the Boralis can produce 50, 180, 280 and 270 units for march to june respectively. Because the production capacity and demand for the different months do not match. Accurrent month's demand may be satisfied in one of three ways.

- i. Current month's production
- ii. Surplus production in an earlier month
- iii. Surplus production in a later moth.

In the first case, the production cost per backpack is Rs.40.

The second case an additional holding cost of Rs.0.50 per backpack per month.

In the third case an additional penalty cost of Rs.2 per backpack is incurred for each month delay.

Boralis wishes to determine the optimal production schedule for the 4 months.

The elements of the production inventory problem for the transportation model is given as follows.

Transportation	Production Inventory
1. Source i	Production period i
2. Destination j	Production capacity of period i Demand period j.
3. Supply amount at source i	Production capacity of period i
4. Demand at destination j	Demand for period j
5. Unit transportation cost from Source i to destination j	Unit cost (Production + inventory + Penalty) in period I for period j.

The resulting transportation model is given as follows

	1	2	3	4	Capacity
1	40.00	40.50	41.00	41.50	50
2	42.00	40.00	40.50	41.00	180
3	44.00	42.00	40.00	40.50	280
4	46.00	44.00	42.00	40.00	270
Demand	100	200	180	300	

The unit transportation cost from period i to period j is computed as

$$C_{ij} = \begin{cases} \text{Production cost in } i = j \\ \text{Production cost in } i + \text{holding cost from } i \text{ to } j, i < j. \\ \text{Production cost in } i + \text{penalty cost from } i \text{ to } j, i > j. \end{cases}$$

For example

$$C_{11} = \text{Rs.}4000$$

$$C_{24} = 40.00 + (0.50 + 0.50) = \text{Rs.}41.00$$

$$C_{41} = 40.00 + (2.00 + 2.00 + 8.00) = \text{Rs.}46.00$$

THE TRANSPORTATION ALGORITHM

Steps of the Transportation Algorithm

Step I

Determine a starting basic feasible Solution.

Step II

Use the optimality condition of the simplex method to determine the entering variable from among all the non basic variable. If the optimality condition is satisfied. Stop, otherwise goto step-III.

Step III

Use the feasibility condition of the simplex method to determine the leaving variable from among on the current basic variables and find the new basic Solution return to step II.

Determination of the starting solution

A general transportation model with m -source and n -destination has $m+n$ constraint equation. One for each source and each destination. The model has $m+n-1$ independent constraint equation which means that the starting basic solution consist of $m+n-1$ basic variable.

The starting basic feasible Solution can be formed using one of the following method.

1. North West corner method
2. Least cost method
3. Vogel Approximation method

1. North West corner Method

The method starts as the North-West corner cell (route) of the tableau (variable x_n).

Step I

Allocate as much as possible to the selected cell and adjust the associated amount of supply and demand of by subtracting the allocated amount.

Step II

Cross out the row (or) column with 0. Supply (or) demand to indicate that no further assignment can be made in that row (or) column. If both the row and coloumn net to 0. Simultaneously cross out only one and leave a zero supply or demand in the uncrossed out row (or) column.

Step III

It exactly one row (or) column is left uncrossed out stop. Otherwise move to the all to the right if a column, if a column has just been crossed (or) the one before if a row has be crossed out. Goto stop I.

Problem

Using North west corner method find a basic feasible Solution to the following T.P.

	W ₁	W ₂	W ₃	Supply (a _i)
F ₁	8	10	12	900
F ₂	12	13	12	1000
F ₃	14	10	11	1200
Demand	1200	1000	900	

Solution

	W ₁	W ₂	W ₃	Supply (a _i)
F ₁	8	10	2	900
F ₂	12	13	12	1000
F ₃	14	10	11	1200
Demand (b _j)	1200	1000	900	

$$\begin{aligned} \Sigma a_i &= 900 + 1000 + 1200 \\ &= 3100 \end{aligned}$$

$$\begin{aligned} \Sigma b_j &= 1200 + 1000 + 900 \\ &= 3100 \end{aligned}$$

$$\Sigma a_i = \Sigma b_j$$

The Transportation problem is balanced basic feasible Solution.

	W ₁	W ₂	W ₃	Supply
F ₁	300 8	10	12	900/0
F ₂	12	13	12	1000
F ₃	14	10	11	1200
Demand	1200/300	1000	900	

	W ₁	W ₂	W ₃	Supply
F ₂	300 12	13	12	1000/700
F ₃	14	10	11	1200
Demand	300/0	1000	900	

	W ₂	W ₃	Supply
F ₂	700 13	12	700/0
F ₃	10	11	1200/900
Demand	1000/300	900	

	W ₂	W ₃	Supply
F ₃	300 10	900 11	1200
Demand	300	900	

Initial basic feasible Solution is given in the following table

	W ₁	W ₂	W ₃	Supply
F ₁	900 8	10	12	900
F ₂	300 12	700 13	12	1000
F ₃	14	10	900 11	1200
Demand	1200	1000	900	

$$x_{11}=900, x_{21}=300, x_{22}=700$$

$$x_{32}=300, x_{33}=900$$

The Total transportation cost is

$$z = \sum_{i=1}^n \sum_{j=1}^m C_{ij} x_{ij}$$

$$= 8 \times 900 + 12 \times 300 + 13 \times 700 + 10 \times 300 + 11 \times 900$$

$$= 7200 + 3600 + 9100 + 3000 + 9900$$

$$= 32800$$

2. Least Cost method (Matrix Maxima Method)

Step 1

Identify the cell with the least transportation cost in the entire transportation table and allocate as much as possible to this cell so that either supply or demand is exhausted.

Step 2

Eliminate the row or column in which either supply or demand is exhausted. If both the supply and demand is exhausted then eliminate either the row or column but not both.

Step 3

Reduce the supply or demand units by the amount assigned to the cell in the uncrossed row or column.

Step 4

Repeat step 1 through step 3 until all the rim requirements are exhausted (completed).

Example

Solve the following transportation problem by using least cost method

	W ₁	W ₂	W ₃	Supply
F ₁	8	10	12	900
F ₂	12	13	12	1000
F ₃	14	10	11	1200
Demand	1200	1000	900	

Solution

	W ₁	W ₂	W ₃	Supply
F ₁	900 / 8	10	12	900/0
F ₂	12	13	12	1000
F ₃	14	10	11	1200
Demand	1200/300	1000	900	

	W ₁	W ₂	W ₃	Supply
F ₂	12	13	12	1000
F ₃	14	1000 10	11	1200/200
	300	1000	900	

	W ₁	W ₃	Supply
F ₂	12	12	1000
F ₃	14	200 11	200/200
Demand	300	900/700	

	W ₁	W ₃	Supply
F ₂	300 12	200 12	1000

Demand 300 900/700

Initial basic feasible solution is given in the following table.

	W ₁	W ₂	W ₃	Supply
F ₁	900 8	10	12	900
F ₂	300 12	13	700 12	1000
F ₃	14	1000 10	200 11	1200
Demand	1200	1000	900	

$$x_{11} = 900, x_{21} = 300, x_{23} = 700, x_{32} = 1000, x_{33} = 200$$

∴ The total transportation cost

$$\begin{aligned}
 Z &= \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} \\
 &= 8 \times 900 + 12 \times 300 + 12 \times 700 + 10 \times 1000 + 11 \times 200 \\
 &= 7200 + 3600 + 8400 + 10000 + 3200 \\
 &= 31400
 \end{aligned}$$

3. Vogel Approximation Method

Step I

For each row (column) with strictly positive supply or demand. Determine a penalty measure by subtracting the smallest unit cost element in the row (column) from the next smallest unit cost element in the same row (or) column.

Step II

Identify then row (or) column with the largest penalty. Break ties arbitrary. Allocate as much as possible to the variable with the least unit cost in the selected row (or) column. Adjust the supply and demand and cross out the satisfied row or column. If a row and column are satisfied simultaneously only one of the twice two is cross out and the remaining now (column) is assigned zero supply (demand).

Step III

- a) It exactly one row or column with zero supply (or) demand remains uncrossed out stop.
- b) It one row (column) with positive supply (demand) remains un crossed out. Determine the basic variable in the row (column) if the least cost method. Stop.
- c) If all the crossed out row and column have zero. Supply and demand determine the zero basic variable by the least cost method. Stop.
- d) Otherwise go to step 1.

1. Using Vogel approximation method find the basic solution to the following transportation method.

	W_1	W_2	W_3	Supply
F_1	8	10	12	900
F_2	12	13	12	1000
F_3	14	10	11	1200
Demand	1200	1000	900	

Solution:

	W ₁	W ₂	W ₃	Supply	row penalty
F ₁	900 8	10	12	900/0	(2)
F ₂	12	13	12	1000	(0)
F ₃	14	10	11	1200	(1)
Demand	1200/300	1000	900		

Column Penalty (4)
↑

	W ₁	W ₂	W ₃	Supply	row penalty
F ₂	12	13	12	1000	(0)
F ₃	14	1000 10	11	1200/200	(1)
Demand	300	1000/0	900		

Column Penalty (2) (3)
↑

	W ₁	W ₃	Supply	Row penalty
F ₂	12	12	1000	(0)
F ₃	14	200 11	200	(3) ←

Demand 300 900/700
Column Penalty (2) (1)

	W ₁	W ₃	Supply	Row penalty
F ₂	12	12	1000	(0)

Demand 300 700
Column Penalty (12) (12)

Initial Basic Feasible Solution

	W ₁	W ₂	W ₃	Supply a ₂
F ₁	900 8	10	12	900
F ₂	300 12	13	700 12	1000
F ₃	14	1000 10	200 11	1200
Demand	1200	1000	900	

Total Transportation cost is

$$z = 8 \times 900 + 300 \times 12 + 12 \times 700 + 10 \times 1000 + 11 \times 200$$

$$= 7200 + 3600 + 8400 + 10000 + 2200 = 31400$$

Algorithm to find the optimum basic feasible solution

Step : 1

Obtain the starting basis feasible Solution.

Step : 2

To determine the entering variable

Use the method of multipliers to determine the entering variable from among the current non basic variables.

Method of multipliers

- Associate the multipliers u_i and v_j with row i and column j of the transportation table.
- For each current basic variables x_{ij} , the multipliers are chosen to satisfy the $m+n$ equations.
 $U_i + v_j = c_{ij}$ for each basic x_{ij} .
- Solve the equation. The set of equations can be solved by arbitrarily setting $u_1=0$.
- Compute the net evaluation $u_i+v_j-c_{ij}$ for each non basic variables x_{ij} and with the value in the right-lower corner of each non basic all, $u_i+v_j-c_{ij} < 0 \forall i, j$ then the current solution is optimal otherwise the non basic variable with maximum positive net evaluation is selected as the entering variable.

Step 3

(To determine the leaving variable) suppose x_{ij} is the entering variable.

- a) Construct a closed loop that starts and ends at the entering variable cell (i, j). The Loop must consists of connected horizontal and vertical segment only. Each corner of the closed loop except the entering variable cell must coincide with a current basic variable.
- b) Assign an amount θ to the entering variable cell (i, j), subtract and add alternatively the amount θ at the successive corner's of the loop.
- c) Choose the value ' θ ' so that
 - i) The supply and demand requirement remains satisfied and
 - ii) New values of the variable remain non negative. For the reason stated in (0) we choose θ as the minimum value of the basic variables with -ive sign in the loop.
- d) The entering variable is selected from among the -ive signed corner variables of the loop having smallest value.

Go to Step 2.

1. Solve the following Transportation Problem.

	W_1	W_2	W_3	Supply
F_1	8	10	12	900
F_2	12	13	12	1000
F_3	14	10	11	1200
Demand	1200	1000	900	

Solution:

Obtaining the starting solution by using North west corner method.

	W_1	W_2	W_3	Supply
F_1	900 / 8	10	12	900/0
F_2	300 / 12	700 / 13	12	1000/700/0
F_3	14	300 / 10	900 / 11	1200/900
Demand	1200	1000	900	
	300/0	300		

The above table gives the initial basic feasible Solution of the given transportation problem.

First we have to find the multipliers v_i 's and u_j 's by using the equation.

$u_i + v_j = c_{ij}$ for basic variable.

$u_1=0$

Basic variable	$u_i + v_j = c_{ij}$	Solution
x_{11}	$u_1 + v_1 = 8$	$u_1 = 0 \Rightarrow v_1 = 8$
	$0 + v_1 = 8$	
x_{21}	$u_2 + v_1 = 12$	$v_1 = 8 \Rightarrow u_2 = 4$
x_{22}	$u_2 + v_2 = 13$	$u_2 = 4 \Rightarrow v_2 = 9$
x_{32}	$u_3 + v_2 = 10$	$v_2 = 9 \Rightarrow u_3 = 1$
x_{33}	$u_3 + v_3 = 11$	$u_3 = 1 \Rightarrow v_3 = 10$
$u_1 = 0$	$u_2 = 4$	$u_3 = 1$
$v_1 = 8$	$v_2 = 9$	$v_3 = 10$

non basic variable	$u_i + v_j - c_{ij}$
x_{12}	$u_1 + v_2 - c_{12} = 0 + 9 - 10 = -1$
x_{13}	$u_1 + v_3 - c_{13} = 0 + 10 - 12 = -2$
x_{23}	$u_2 + v_3 - c_{23} = 4 + 10 - 12 = 2$
x_{31}	$u_3 + v_1 - c_{31} = 1 + 8 - 14 = -5$

TP is of minimization type we choose the max ratio as entering variable.

$u_i + v_j - c_{ij} > 0$ some i, j

The current solution is not optimal to find the leaving variable.

Draw a loop which start with the cell (2, 3) the loop has only the horizontal and vertical line and every corner element of the loop is a basic variable (allocated cell) expecting starting point.

$v_1=8 \quad v_2=9 \quad v_3=10$

$u_1=0$	900	8	10	12
$u_2=4$	300	12	700	13 - θ + 12
$u_3=1$	14	300	10 + θ -	900 11

The value with (-) sign are 700, 900

The minimum $\{900, 700\} = 700$

$$\theta = 700.$$

The initial basic feasible solution becomes.

	W_1	W_2	W_3	Supply
F_1	900 8	10	12	900
F_2	300 12	13	700 12	1000
F_3	14	1000 10	200 11	1200
Demand	1200	1000	900	

x_{22} is the leaving variable

Basic variable	$u_i + v_j = c_{ij}$	Solution
x_{11}	$u_1 + v_1 = 8$	$u_1 = 0 \Rightarrow v_1 = 8$
x_{21}	$u_2 + v_1 = 12$	$v_1 = 8 \Rightarrow u_2 = 4$
x_{23}	$u_2 + v_3 = 12$	$u_2 = 4 \Rightarrow v_3 = 8$
x_{32}	$u_3 + v_2 = 10$	$u_3 = 3 \Rightarrow v_2 = 7$
x_{33}	$u_3 + v_3 = 11$	$v_3 = 8 \Rightarrow u_3 = 3$

$$u_1=0 \quad u_2=4 \quad u_3=3 \quad v_1=8 \quad v_2=7 \quad v_3=8$$

non basic variable $u_i + v_j - c_{ij}$

$$x_{12} \quad u_1 + v_2 - c_{12} \\ 0 + 7 - 10 = -3$$

$$x_{13} \quad u_1 + v_3 - c_{13} \\ 0 + 8 - 12 = -4$$

$$x_{22} \quad u_2 + v_2 - c_{22} \\ 4 + 7 - 13 = -2$$

$$x_{31} \quad u_3 + v_1 - c_{31} \\ 3 + 8 - 14 = -3$$

$$u_i + v_j - c_{ij} < 0 \quad \forall i, j$$

∴ The solution is optimal

	W ₁	W ₂	W ₃	Supply
F ₁	900 8	10	12	900
F ₂	300 12	13	700 12	1000
F ₃	14	1000 10	200 11	1200

$$\begin{aligned}
 \text{The transportation cost} &= 8 \times 900 + 12 \times 300 + 12 \times 100 + 10 \times 1000 + 11 \times 200 \\
 &= 7200 + 3600 + 8400 + 10000 + 3200 \\
 &= 31400
 \end{aligned}$$

2. Solve the T.P.

	W ₁	W ₂	W ₃	a _i
F ₁	0	2	1	6
F ₂	2	1	5	9
F ₃	2	4	3	5
b _j	5	5	10	

Solution

Obtaining the starting solution by using North West Corner method.

	W ₁	W ₂	W ₃	a _i
F ₁	0	2	1	6/1/0
F ₂	2	1	5	9/5
F ₃	2	4	3	5
b _j	5/0	5/4/0	10	

The above table gives the initial basic feasible Solution of the given transportation problem first. We have to find the multipliers v_i 's and u_j 's by using the equation $u_i + v_j = c_{ij}$ for basic variable.

$$u_1 = 0$$

Basic variable	$u_i + v_j = c_{ij}$	solution
x_{11}	$u_1 + v_1 = c_{11}$	$u_1 = 0$
	$0 + v_1 = 0$	$\Rightarrow v_1 = 0$
x_{12}	$u_1 + v_2 = c_{12}$	$u_1 = 0 \Rightarrow v_2 = 2$
x_{22}	$u_2 + v_2 = c_{22}$	
	$u_2 + 2 = 1$	$u_2 = -1$
x_{23}	$u_2 + v_3 = 5$	$u_2 = -1 \Rightarrow v_3 = 6$
x_{33}	$u_3 + v_3 = 3$	$u_3 = 3 - 6 = -3$

$$u_1 = 0, \quad u_2 = -1, \quad u_3 = -3$$

$$v_1 = 0, \quad v_2 = 2, \quad v_3 = 6$$

Non basic variable	$u_i + v_j - c_{ij}$
x_{13}	$u_1 + v_3 - c_{13} = 0 + 6 - 1 = 5 \leftarrow$
x_{21}	$u_2 + v_1 - c_{21} = -1 + 0 - 2 = -3$
x_{31}	$u_3 + v_1 - c_{31} = -3 + 0 - 2 = -5$
x_{32}	$u_3 + v_2 - c_{32} = -3 + 2 - 4 = -5$

TP is of minimization type choose the max ratio as entering variable.

$$u_i + v_j - c_{ij} > 0 \text{ for same } i, j$$

The current solution is not optimal.

To find the leaving variable

Draw a loop which start with the cell (1, 3) the loop has only horizontal and vertical line and every corner element of the loop is a basic variable expecting starting point.

$$v_1 = 0 \quad v_2 = 2 \quad v_3 = 6$$

$$u_1 = 0$$

$$u_2 = -1$$

$$u_3 = -3$$

5	0	1	2	-	0	+	1
2	4	1	+		-	5	5
2		4			5		3

The value with (-) sign are {1, 5}

The minimum {1, 5}=1

$$\Theta = 1$$

The initial basic feasible Solution becomes

	W_1	W_2	W_3
F_1	5	0	2
F_2	2	5	1
F_3	2	4	5

x_{12} is the leaving variable

Basic variable	$u_i + v_j = c_{ij}$	Solution
x_{11}	$u_1 + v_1 = 0$	$u_1 = 0 \Rightarrow v_1 = 0$
x_{13}	$u_1 + v_3 = 1$	$u_1 = 0 \Rightarrow v_3 = 1$
x_{22}	$u_2 + v_2 = 1$	$u_2 = 4 \Rightarrow v_2 = -3$
x_{23}	$u_2 + v_3 = 5$	$v_3 = 1 \Rightarrow u_2 = 4$
x_{33}	$u_3 + v_3 = 3$	$v_3 = 1 \Rightarrow u_3 = 2$

$$u_1 = 0, u_2 = 4, u_3 = 2 \quad v_1 = 0, v_2 = -3, v_3 = 1$$

Non basic variable	$u_i + v_j - c_{ij}$
x_{12}	$u_1 + v_2 - 2 = 0 + (-3) - 2 = -5$
x_{21}	$u_2 + v_1 - 2 = 4 + 0 - 2 = 2 \leftarrow$
x_{31}	$u_3 + v_1 - 2 = 2 + 0 - 2 = 0$
x_{32}	$u_3 + v_2 - 4 = 2 - 3 - 4 = -5$

Transformation problem is of minimization type we choose the max. ratio as entering variable $u_i + v_j - c_{ij} > 0 \quad \forall$ some i, j

The current solution is not optimal

To find the leaving variable.

Draw a loop which start with a cell (2, 1). The loop has only horizontal and vertical line and every corner element of the loop is a basic variable expecting starting point.

The initial basic feasible Solution becomes.

5	0	-	+	2	1
2	θ			- 1 4	5
2				4	3

x_{23} is the leaving variable.

Basic variable	$u_i + v_j = c_{ij}$	Solution
x_{11}	$u_1 + v_1 = 0$	$u_1 = 0, v_1 = 0$
x_{13}	$u_1 + v_2 = 1$	$u_1 = 0 \Rightarrow v_2 = 1$
x_{21}	$u_2 + v_1 = 2$	$u_1 = 0 \Rightarrow u_2 = 2$
x_{22}	$u_2 + v_2 = 1$	$u_2 = 2 \Rightarrow v_2 = -1$
x_{33}	$u_3 + v_3 = 3$	$v_3 = 1 \Rightarrow u_3 = 2$
	$u_1 = 0, u_2 = 2, u_3 = 2$	
	$v_1 = 0, v_2 = -1, v_3 = 1$	

Non Basic variable	$u_i + v_j = c_{ij}$	Solution
x_{12}	$0 + 1 - 2$	-1
x_{23}	$2 + 1 - 5$	-2
x_{31}	$2 + 0 - 2$	0
x_{32}	$2 - 1 - 4$	-3
	$u_i + v_j - c_{ij} < 0 \quad \forall i, j$	

∴ The Solution is optimal

	W ₁	W ₂	W ₃
F ₁	1 0	2	5 1
F ₂	4 2	5 1	5
F ₃	2	4	5 3

$$\begin{aligned}
 \text{Total Transportation cost is} &= 1 \times 0 + 5 \times 1 + 2 \times 4 + 5 \times 1 + 5 \times 3 \\
 &= 1 + 5 + 8 + 5 + 15 \\
 &= 33
 \end{aligned}$$

SIMPLEX EXPLANATION OF THE HUNGARIAN METHOD

The assignment problem in which ‘n’ workers are assigned to ‘n’ jobs can be represented as an LP model in the following manner.

Let c_{ij} be the cost of assigning worker i to job j and define

$$x_{ij} = \begin{cases} 1 & \text{if worker } i \text{ is assigned to job } j \\ 0 & \text{otherwise} \end{cases}$$

Then the LP model is given as minimize $z = \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij}$

Subject to

$$\sum_{j=1}^n x_{ij} = 1, i=1, 2, \dots, n$$

$$\sum_{i=1}^n x_{ij} = 1, j=1, 2, \dots, n$$

$$x_{ij} = 0 \text{ or } 1.$$

The optimum solutions of the LP model remains unchanged if a constant is added to (or) subtract from any row (or) column of the cost matrix c_{ij} . To prove this point, let p_i and q_j be constant subtracted from row i and column j , then the cost element c_{ij} is changed to $c_{ij} = c_{ij} - p_i - q_j$.

Next we show that, using the objective coefficient ‘ c_{ij} ’ yield the same optimum value of x_{ij} when c_{ij} is used.

$$c_{ij} = c_{ij} - p_i - q_j$$

$$z = \sum_{i=1}^n \sum_{j=1}^n c_{ij} - x_{ij}$$

$$= \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} - \sum_{i=1}^n \sum_{j=1}^n p_i x_{ij} - \sum_{i=1}^n \sum_{j=1}^n q_j x_{ij}$$

$$= \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} - \sum_{i=1}^n p_i \sum_{j=1}^n x_{ij} - \sum_{j=1}^n q_j \sum_{i=1}^n x_{ij}$$

$$= \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} - \sum_{i=1}^n p_i \sum_{j=1}^n q_j$$

$$Z = \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} - \text{constant}$$

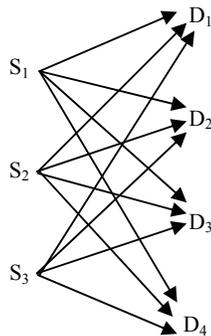
The new objective function differ from the original one by a constant the optimum value of x_{ij} must be same in the both case.

TRANSSHIPMENT MODEL

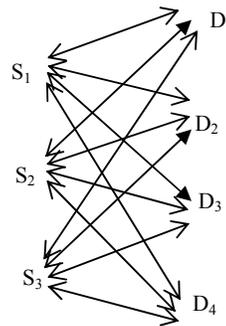
In a transportation problem the commodity is closed from destination to and from source to same where not considered.

A three sources and four destination problem can be given as follows.

Transportation



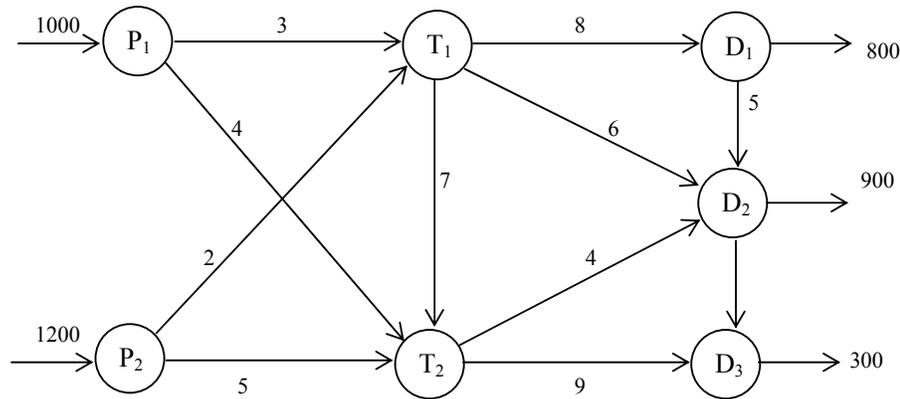
Transshipment



In a transshipment problem the flow can be in both directions. As transshipment problem has more reflexivity than a transportation problem, it can give the better solution than a transportation problem it transshipment is cheaper the T.P. Here the destination and source will play the roles of each other as we can transfer the entire for a part of commodity to one of the sources for destination and then to the required destination.

Example

Two automobile plants P_1 and P_2 are linked to three dealers D_1 , D_2 and D_3 by way of two transit centres T_1 and T_2 according to the network given below.



The supply amount at plants P_1 and P_2 are 1000 and 1200 cars and the demand amounts at dealers D_1 , D_2 and D_3 are 800, 900 and 300 cars. The shipping cost per car (in one hundred of dollars) between pairs of models are shown on the connecting links are of the network.

Transshipment occurs in the entire supply amount of 2200 (1000+1200) cars at nodes P_1 and P_2 could pass through any mode of the network before ultimately reaching their destinations at nodes D_1 , D_2 and D_3 . The mode of the network with both an input and output arcs (T_1 , T_2 , D_1 and D_2) act as both sources and destinations and are referred to as transshipment nodes. The remaining nodes are either pure supply nodes (P_1 and P_2) and pure Demand nodes (D_3).

The transshipment model can be converted into a regular transportation model with 6 sources (P_1 , P_2 , T_1 , T_2 , D_1 , D_2) and 5 destinations (T_1 , T_2 , D_1 , D_2 and D_3). The amount of supply and demand at the different nodes are computed as

supply at a pure supply mode = original supply

Supply at a transshipment mode = original supply + buffer

Demand at a pure demand mode = Original demand

Demand at a transshipment mode = 0.D + Buffer.

The buffer amount should be sufficient large to allow all the original supply (or demand) units to pass through any of the transshipment mode.

Let B be the desired buffer amount then

$$B = \text{Total supply (or demand)}$$

$$= 1000 + 1200 \text{ (or } 800 + 900 + 500)$$

$$B = 2200 \text{ Cars.}$$

Destination

	P ₁	P ₂	T ₁	T ₂	D ₇	D ₂	D ₃
P ₁	-	-	3	4	-	-	-
P ₂	-	-	2	5	-	-	-
T ₁	-	-	-	7	8	6	-
T ₂	-	-	-	-	-	4	9
D ₁	-	-	-	-	-	5	-
D ₂	-	-	-	-	-	-	3
D ₃	-	-	-	-	-	-	-

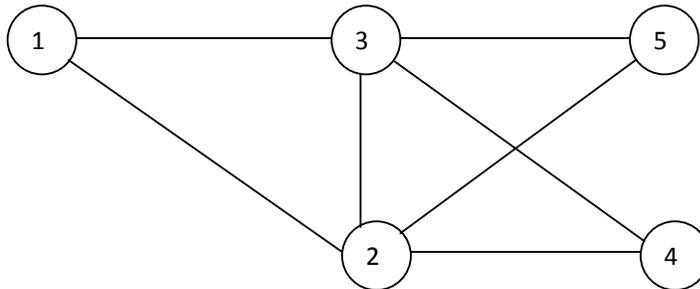
UNIT II

NETWORK MODEL

NETWORK

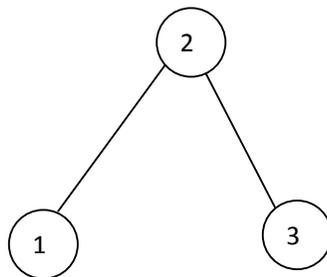
A network consists of set of nodes linked by arc (or) branches. The notation for describing a network is (N, A) where N is the set of nodes and A is the set of arc.

For example



$$N = \{1, 2, 3, 4, 5\}$$

$$A = \{(1,2), (1,3), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)\}$$



$$N = \{1, 2, 3\}$$

$$A = \{(2,1), (2,3)\}$$

Directed or Oriented Network

An arc is said to be directed or oriented if it allows '+ve' flow in one direction and zero flow in opposite direction. The directed network has all directed branches.

Path:

A path is a sequence of distinct branches that join two nodes regardless of the direction of the flow in each branch. A path forms a loop or a cycle if it connects a node to itself.

Path

1-2-3-4-5

1-3-4-5

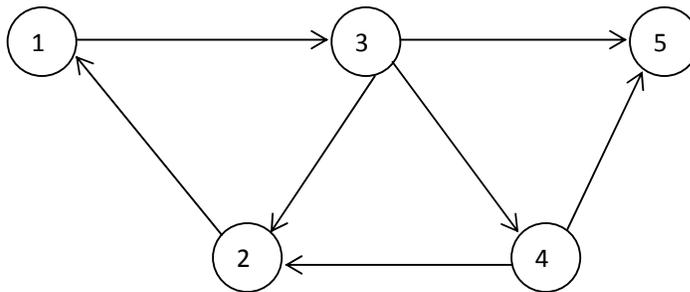
Loop:

1-2-3-1

1-2-4-2-1

1-3-5-2-1

Directed path:

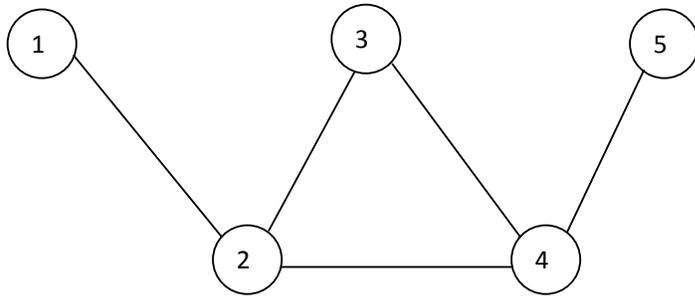


Directed loop

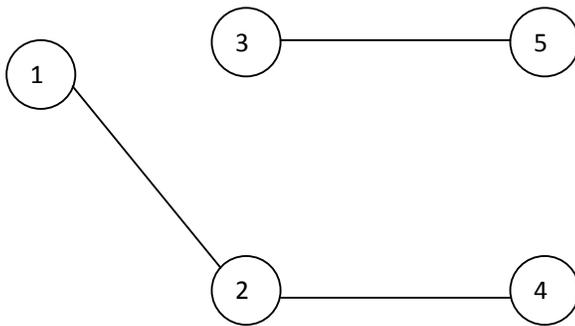
Directed loop (or) circuit is a loop in which all the branches are oriented in the same direction.

Connected Network

A connected network is such that every two distinct nodes are linked by at least one path.

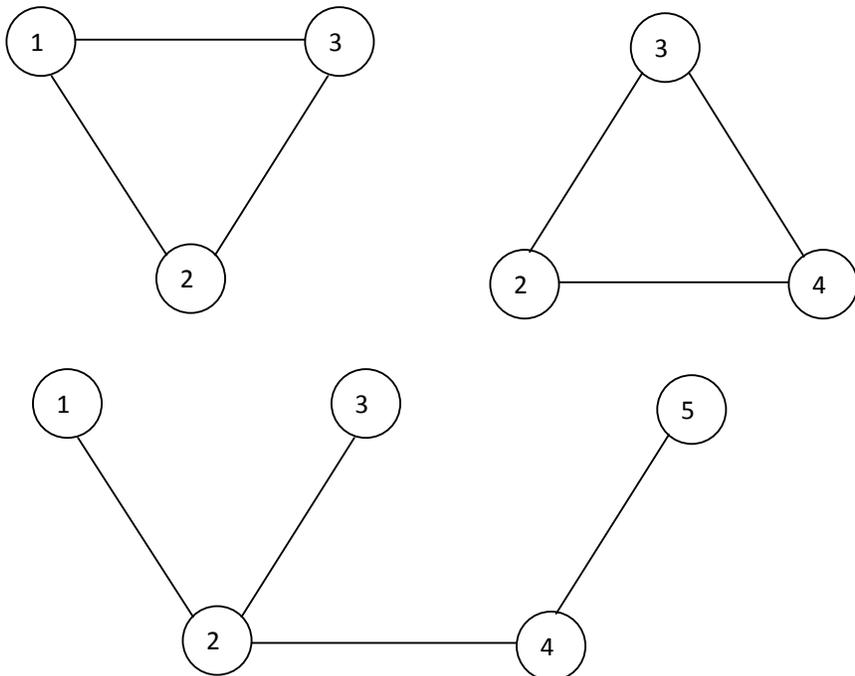


Disconnected



Tree:

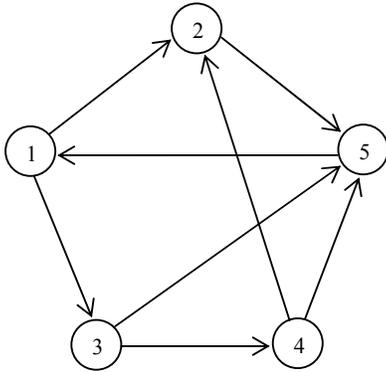
A tree is a connected network that may involve only a subset of all nodes of the network.



Spanning Tree

The spanning tree links all the nodes of the network with no loop allowed.

1. For each network determined.
 - a) Path
 - b) loop
 - c) directed loop (or) a circuit
 - d) A tree
 - e) A spanning tree.



Solution:

a) Path

1 - 2 - 5
1 - 3 - 4 - 5
1 - 3 - 5

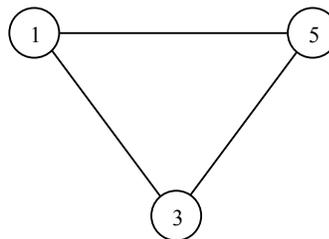
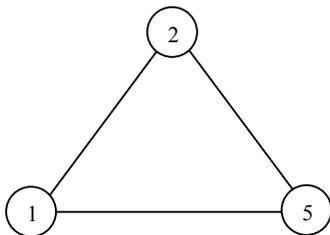
b) Loop

1 - 2 - 5 - 1
1 - 3 - 5 - 1
1 - 3 - 5 - 1

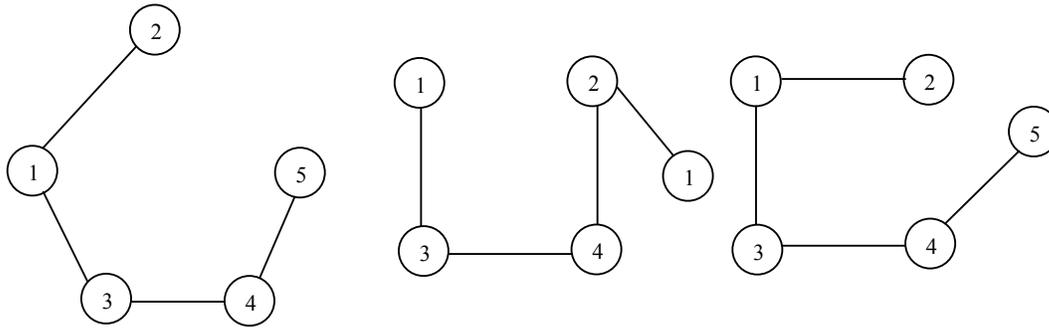
c) Directed Loop

1 - 2 - 5 - 1
1 - 3 - 4 - 5 - 1
1 - 3 - 5 - 1

d) Tree



e) Spanning Tree



CPM and PERT

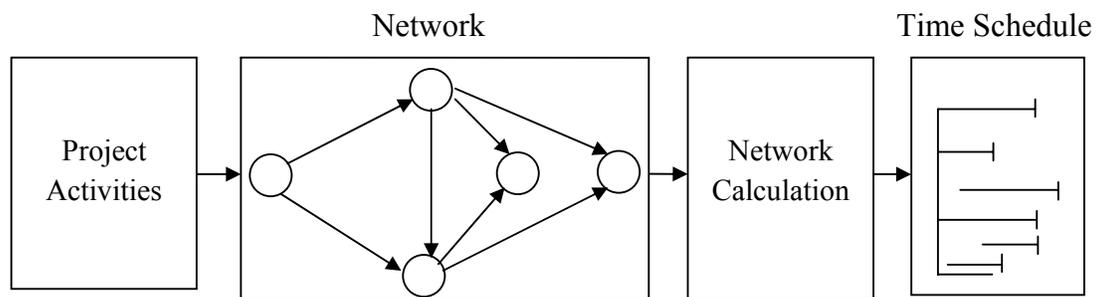
Critical Path Method (CPM) and Program Evaluation and Review Technique (PERT) are network based method designed to assist in the planning scheduling and control of project.

A project is defined as a collection of inter related activities with each activity consuming time and resources.

The objective of CPM and PERT is to provide analytic means for scheduling the activities. First we define the activities of the project the relationship and their time requirement.

Next, the project is translated into a network that shows the precedence relationship among the activities.

The third step involves carrying out specific network computation that facilitates the development of time scheduling for the project.



The two type techniques CPM and PERT which were developed independently differ in that CPM assumes deterministic activity duration. PERT assumes probabilistic duration.

Network Presentation

Each activity of the project is represented by a directional arc (more commonly known as arrow) point in the direction of progress in the project. The node of the network (also referred to as event) establish the precedence relationship among the different activities of the project.

Three rules are available for constructing the network.

Rule : 1

Each activity is represented by one and only one arrow in the network.

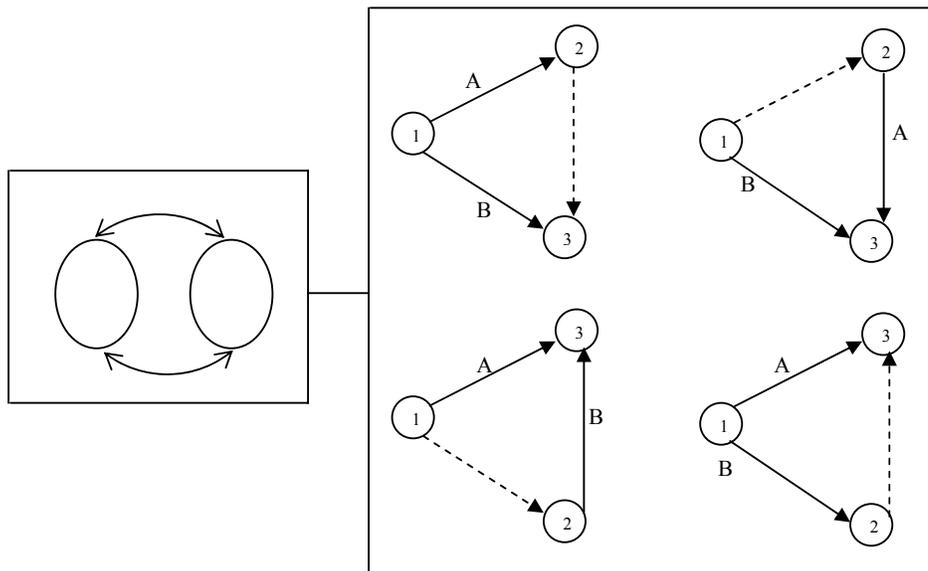
Rule: 2

Each activity must be identified by two distinct end node.

Rule : 3

To maintain the correct precedence relationship the following question must be answered as each activity is added to the network.

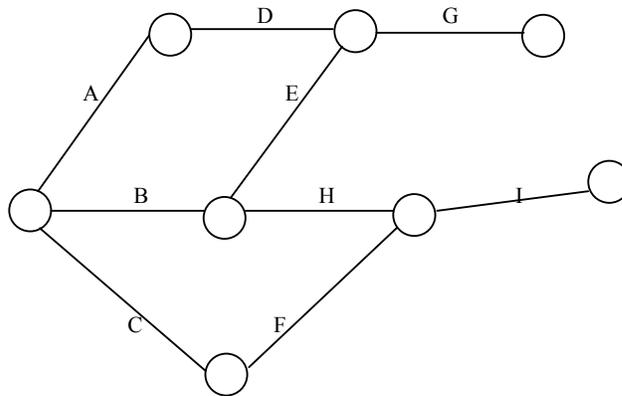
- a. What activities must immediately precede the current activity?
- b. What activities must follow the current activity?
- c. What activities must acquire concurrently with the current activity?



1. Construct the project network.

Activity	A	B	C	D	E	F	G	H	I
Predecessor	-	-	-	A	B	C	D,E	B	H,F

Solution.

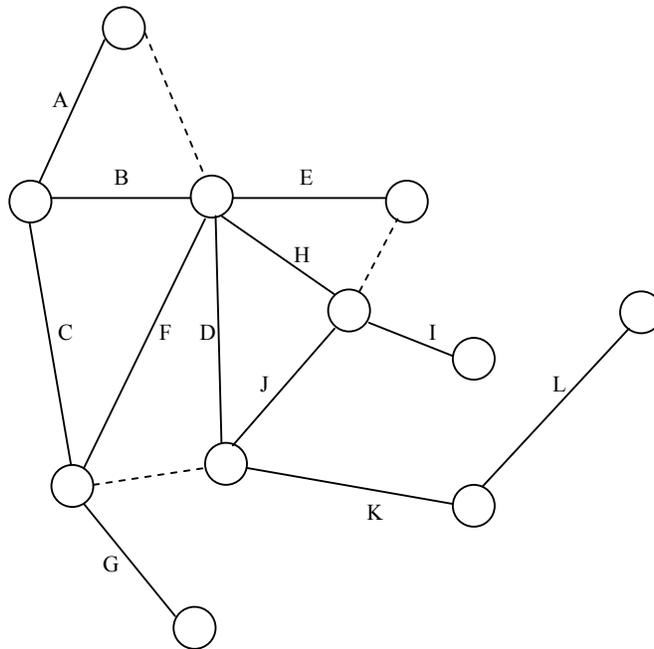


2. Construct the project network compress of activity A to L with the following relationship.

1. A, B and C be the first activity of the project can be executed
2. A and B proceed D
3. A and B proceed E, F and H
4. F and C proceed G
5. E and H proceed I and J
6. C, D, F and J proceed K
7. K proceed L
8. I, G and L are the terminal activities the project.

Solution.

Activity	A	—	—	D	D						
Predecessor	A	B	C	A,B							
Activity	A	B	C	D	E	F	G	H	I	J	K
	-	-	-	(A,B)	(A,B)	(A,B)	(F,C)	(A,B)	(E,H)	(E,H)	



Critical Path Computations

Critical

An activity is said to be critical when there is no Leeway in determine start and finish times.

Non critical

An non critical activity allows some scheduling slack, so that the start time of the activity may be advanced or delay within limits without affecting the completion date of the entire project.

Event

An event as a point in time at which activities are terminated and others are started.

Some definitions are as follows

\square_j = Earliest occurrence time of event j.

Δ_j = latest occurance time of event j

D_{ij} = Duration of activity (i, j)

The critical path calculation involve two passes the forward pass and backward pass. The forward pass determine the earlier occurrence time of the event and backward pass determine latest occurrence time of the event.

Forward Pass : (Earliest occurrence times of event j)

In this pass the competition starts at node 1 and advanced to end mode n.

Initial step

Set $\square_1 \equiv 0$ to indicate that the project start at time 0.

General step j

Given that nodes p, q and v are linked directly to node j by incoming activities P, D (q, i)..., (v, j) and that the earliest occurrence times of event j is computed as

$$\square_j = \max \{ \square_p + D_{pj}, \square_q + D_{qj}, \dots, \square_v + D_{vj} \}$$

The forward pass is complete when forward pass \square_n at node n is computed.

By defn \square_j represent the largest path i duration to node j.

Backward Pass : (Latest occurrence times (Δ))

Following the completion of the forward pass, the backward pass computation start at node n and regres to node 1.

Initial Step

Set $\Delta_n \equiv$ to indicate that the earliest and latest occurrence of the last event of the project are the same.

General Step j

Given that nodes P, q... and v are linked directly to node j by outcoming activities (j, p), D, q),... and (j, v) and that the latest occurrence time of event (node) p, , ... and v have already been computed, then the latest occurrence of time of event j is computed as

$$\Delta_j = \min \{ \Delta_p - D_{jp}, \Delta_q - D_{jq}, \dots, \Delta_v - D_{jv} \}$$

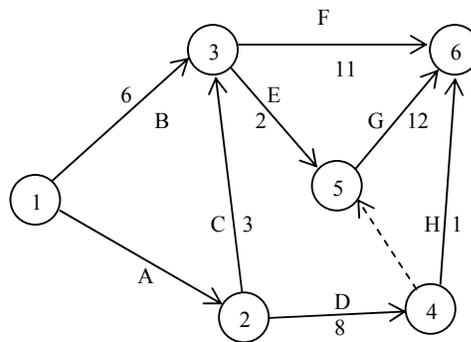
The backward pass is complete when Δ_1 at node 1 is computed.

Based on the preceding calculation an activity (i, j) will be critical if it satisfies 3 condition

- i. $\Delta i = \square i$
- ii. $\Delta j = \square j$
- iii. $\Delta j - \Delta i = \square j - \square i = D_{ij}$

The three condition state that the earliest and latest occurrence time of the end event i and j are equal and that the duration D_{ij} that time span exactly that does not satisfy all the conditions is thus not critical.

1. Determine the critical path for the project network all the duration are in days.



Forward pass

Step 1

$$\text{Set } \square 1 \equiv 0$$

Step 2

$$\square 2 = \square 1 + D_{12}$$

$$= 0 + 5$$

$$\square 2 = 5$$

Step 3

$$\square 3 = \max \{ \square 1 + D_{13}, \square 2 + D_{23} \}$$

$$= \max \{ 0 + 6, 5 + 3 \}$$

$$= \max \{ 6, 8 \}$$

$$= 8$$

Step 4

$$\begin{aligned} \square_4 &= \square_2 + D_{24} \\ &= 5 + 8 \\ &= 13 \end{aligned}$$

Step 5

$$\begin{aligned} \square_5 &= \max \{ \square_3 + D_{35} + \square_4 + D_{45} \} \\ &= \{ 8 + 2, 13 + 0 \} \\ &= 13 \end{aligned}$$

Step 6

$$\begin{aligned} \square_6 &= \max \{ \square_3 + D_{36} + \square_5 + D_{56}, \square_4 + D_{46} \} \\ &= \max \{ 8 + 11, 13 + 12, 13 + 1 \} \\ &= \max \{ 19, 25, 14 \} \\ &= 25 \end{aligned}$$

Backward pass The project can be completed in 25 days.

Backward pass:

$$\text{Node 6} \equiv \square_6 = 25$$

Node 5

$$\begin{aligned} \Delta_5 &= \Delta_6 - D_{56} \\ &= 25 - 12 \\ &= 13 \end{aligned}$$

Node 4

$$\begin{aligned} \Delta_4 &= \min \{ \Delta_6 - D_{46}, \Delta_5 - D_{45} \} \\ &= \min \{ 25 - 1, 13 - 0 \} \\ &= \min \{ 24, 13 \} \\ &= 13 \end{aligned}$$

Node 3

$$\begin{aligned}\Delta_3 &= \min \{ \Delta_6 - D_{36}, \Delta_5 - D_{35} \} \\ &= \min \{ 25 - 11, 13 - 2 \} \\ &= \min \{ 14, 11 \} \\ &= 11\end{aligned}$$

Node 2

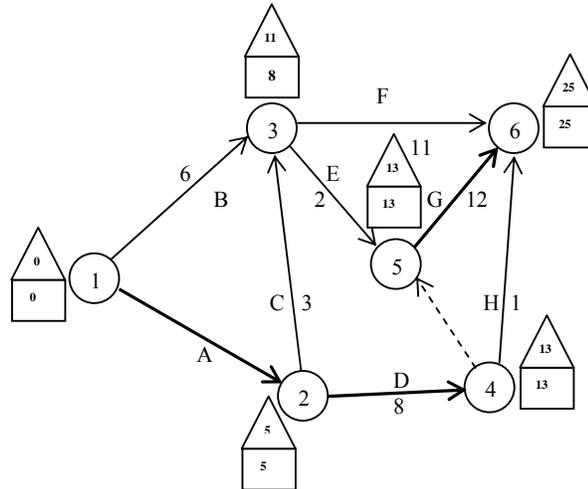
$$\begin{aligned}\Delta_2 &= \min \{ \Delta_3 - D_{23}, \Delta_4 - D_{24} \} \\ &= \min \{ 11 - 3, 13 - 8 \} \\ &= \min \{ 8, 5 \} \\ &= 5\end{aligned}$$

Node 1

$$\begin{aligned}\Delta_1 &= \min \{ \Delta_2 - D_{12}, \Delta_3 - D_{13} \} \\ &= \min \{ 5 - 5, 11 - 6 \} \\ &= \min \{ 0, 5 \} \\ &= 0\end{aligned}$$

Take $i=1, j=2$

- i. $\Delta_i = \square_i$
 $\Delta_i = \square_1 = 0$
- ii. $\Delta_j = \square_j$
 $\Delta_2 = \square_2 = 5$
- iii. $\Delta_i = \Delta_i = \square_j - \square_j$
 $\Delta_2 = \Delta_1 = \square_2 - \square_1 = 5$



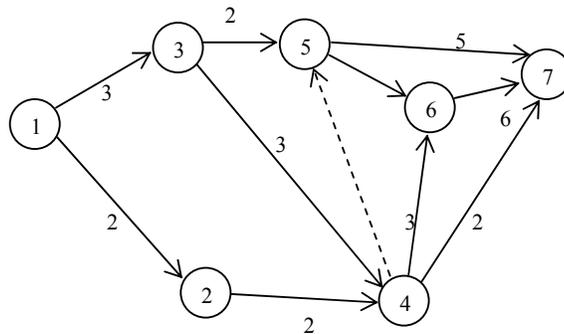
The critical path is 1→2→4→5→6

The sum of the duration of the critical activity

= duration of the project (25 days)

$$= 5 + 8 + 12 = 25$$

2. Determine the critical path for the project network of the duration are in days.



Solution:

Forward pass

Step 1:

$$\text{Set } \square 1 \equiv 0$$

Step 2:

$$\text{Set } \square 2 = \square 1 + D_{12}$$

$$= 0 + 2 = 2$$

Step 3

$$\begin{aligned}\square_3 &= \square_1 + D3 \\ &= 0 + 3 = 3\end{aligned}$$

Step 4

$$\begin{aligned}\square_4 &= \max\{\square_2 + D24 + \square_3 + D34\} \\ &= \max\{2 + 2, 3 + 3\} = 6\end{aligned}$$

Step 5

$$\begin{aligned}\square_5 &= \max\{\square_3 + D35 + \square_4 + D34\} \\ &= \max\{3 + 2, 6 + 0\} = 6\end{aligned}$$

Step 6

$$\begin{aligned}\square_6 &= \max\{\square_3 + D56 + \square_4 + D46\} \\ &= \max\{6 + 7, 6 + 3\} = 13\end{aligned}$$

Step 7

$$\begin{aligned}\square_7 &= \max\{\square_5 + D57 + \square_6 + D67, \square_4 + D47\} \\ &= \max\{6 + 5, 13 + 6, 6 + 2\} = 19\end{aligned}$$

The project can be completed in 19 days.

Backward pass

$$\text{Node 7} \equiv 19 - \Delta_7$$

Node 6

$$\begin{aligned}\Delta_6 &= \Delta_7 - D67 \\ &= 19 - 6 = 13\end{aligned}$$

Node 5

$$\begin{aligned}\Delta_5 &= \min\{\Delta_7-D_{51}, \Delta_6-D_{56}\}, \\ &= \min\{18-5, 13-7\} \\ &= \min\{14, 6\} = 6\end{aligned}$$

Node 4

$$\begin{aligned}\Delta_4 &= \min\{\Delta_7-D_{47}, \Delta_6-D_{46}\}, \\ &= \min\{19-2, 13-3, \Delta_6-D_{45}, 6-0\} \\ &= \min\{17, 10, 6\} = 6\end{aligned}$$

Node 3

$$\begin{aligned}\Delta_3 &= \min\{\Delta_5-D_{35}, \Delta_4-D_{34}\}, \\ &= \min\{6-2, 6-3\} = 3\end{aligned}$$

Node 2

$$\Delta_2 = \Delta_4 - D_{24} = 6 - 2 = 4$$

Node 1 $\Delta_1 = \min\{\Delta_2 - D_{12}, \Delta_3 - D_{13}\}$

$$\begin{aligned}&= \min\{4-2, 3-3\} \\ &= \{2, 0\} = 0\end{aligned}$$

Take $i=1, j=3$

i. $\Delta_i = \square_i = 0$

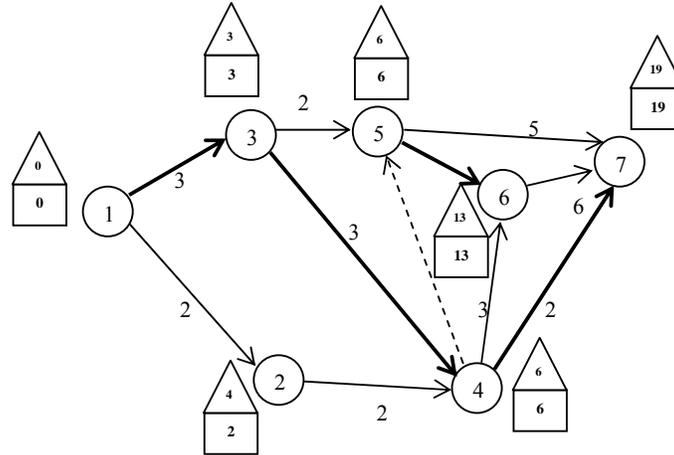
ii. $\Delta_i = \square_j$

$$\Delta_3 = \square_3 = 3$$

iii. $\Delta_j - \Delta_j = \square_j - \square_1$

$$\Delta_3 - \Delta_1 = \square_3 - \square_1$$

$$3 - 0 = 3 - 0 = 3$$



The critical path is (→3 →4 →5 →6 →7. The some of the duration of the critical activity

= duration of the project

= 3 + 3 + 7 + 6

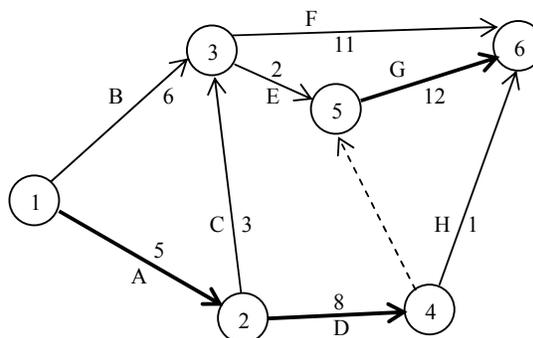
= 19 days.

Construction of the time schedule

The information obtained from the calculation can be used to develop the time schedule. We recognize that for an activity (i, j) forward pass \square_i represent the earliest start time and Δ_j represent the latest completion time. This means that (\square_i, Δ_j) delineates the (maximum) span during which the activity (i, j) may be scheduled.

Construction of Preliminary schedule

The method for constructing a preliminary schedule is illustrated by an example.

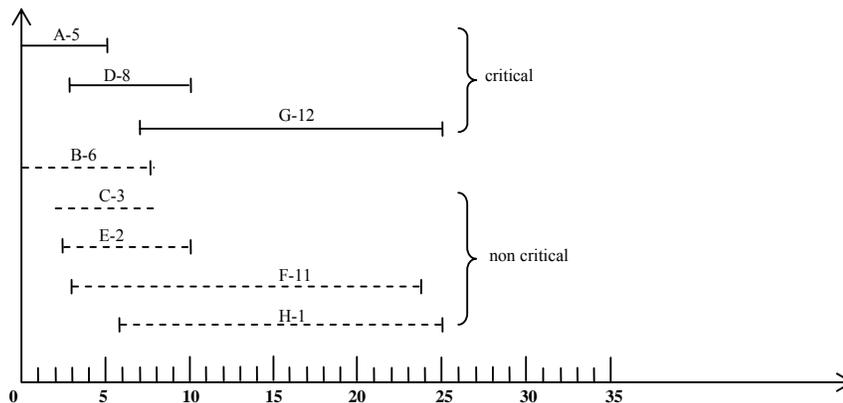


Determine the time schedule for the project Network

We can get a preliminary time schedule for the different activities of the project by delineating the respective time span. The project network provides this result.

To observation are in order

1. The critical activities (shown by solid lines) are scheduled on right after the other two ensured that the project is completed within its specified 25 days duration.
2. The non critical activities (shown by dashed line) compass span that are larger than the respective duration thus allowing slack in scheduling them within their allotted span.



Determination of the Floats

Floats are the slack times available within the allotted span of the non critical activity. The two most common floats are the total float and the free float.

The figure gives a convenient summary for computing the total float (TF_{ij}) and the free float (FF for an activity (i, j). The total float is the excess of the time span defined from the earliest occurrence of event to the latest assurance of event j over the duration of (i, j).

$$\text{ie, } TF_{ij} = \Delta_j - \square_i - D_{ij}$$

The free float is the excess of the time span defined from the earliest occurrence of event i to the earliest occurrence of even j over the duration of (i, j).

$$\text{ie, } FF_{ij} = \square_j - \square_i - D_{ij}$$

By defn

$$FF_{ij} \leq TF_{ij}$$

RED – FLAGGING RULE

For a non critical activity (i, j)

- a. If free float $FF_{ij} = TF_{ij}$ then the activity can be scheduled anywhere within in its $[\square_i, \Delta_j] \dots$ Without causing scheduled conflict.
- b. If $FF_{ij} < TF_{ij}$ then the start of the activity can be delayed by no more than FF_{ij} relative to its earliest start time (\square_i) without causing schedule conflict. Anyway larger than FF_{ij} (But not more than TF_{ij}) must be accompanying by an equal delay relative to (\square_j) in the start time of all the activities leaving node.

The implication of the rule is that a non critical activity (i, j) will be red flag if its $FF_{ij} < TF_{ij}$. This red flag is important only if we decide to delay the start of the activity passed its earliest start time \square_i in which case we must pay attention to the start time of the activities leaving node j to avoid schedule conflict.

INTEGER LINEAR PROGRAMMING

Integer linear programming (ILP) are linear programs in which same or all the variables are restricted to integer value.

Pure Linear Program

A pure integer program is one in which all the variables are integer

Mixed Integer program

A mixed Integer Program is one in which some of the variable are integer.

Capital budgeting Problem

There are m projects to be evaluated over a 'n' year planning horizon. The following table gives the expected returns to each project available funds for each year and the associated yearly.

Expenditures

Project	Expenditures per year						Returns
	1	2	3 j.....	n	
1	c_{11}	c_{12}	c_{1j}	c_{1n}	a_1
2	c_{21}	c_{22}	c_{2j}	c_{2n}	a_2
..
..
..
i	c_{i1}	c_{i2}	c_{ij}	c_{in}	a_i
..
..
..
m	c_{m1}	c_{m2}	c_{mj}	c_{mn}	A_m
Available funds	b_1	b_2	b_j	b_n	

The problem is to determine the project to be executed over the n-year horizon so that the total return is maximum. This is called the capital budgeting problem.

Hence the problem reduced to a “yes-no” for its project.

Defined x_i as follows $x_i = \begin{cases} 1 & \text{if project } i \text{ is selected} \\ 0 & \text{if project } i \text{ is not selected} \end{cases}$

Since the ILP model is $\max z = a_1x_1 + a_2x_2 + \dots + a_mx_m$

Subject to

$$C_{11}x_1 + c_{21}x_2 + \dots + c_{m1}x_m \leq b_1$$

$$C_{12}x_1 + c_{22}x_2 + \dots + c_{m2}x_m \leq b_2$$

..

..

$$C_{1n}x_1 + c_{2n}x_2 + \dots + c_{mn}x_m \leq b_n$$

$$x_1, x_2 \dots \dots x_m = (0, 1)$$

This is a pure ILP Model.

Five projects are being evaluated over a three year planning horizon. The following table gives the expected returns for each project and the associated yearly expenditure.

Project	Expenditure (million \$)			Returns (million \$)
	1	2	3	
1	5	1	8	20
2	4	7	10	40
3	3	9	2	20
4	7	4	1	15
5	8	6	10	30
Available funds (million \$)	25	25	25	

Solution:

Determine the project to be executed over the 3 year horizon.

The problem reduce to a “Yes-No” decision for each project.

Defined the binary variable x_j as

Defined x_j as follows $x_i = \begin{cases} 1 & \text{if project } i \text{ is selected} \\ 0 & \text{if project } i \text{ is not selected} \end{cases}$

Since the DIP model is gives as

$$\max z = 20x_1 + 40x_2 + \dots + 20x_3 + 15x_4 + 30x_5$$

Subject to

$$5x_1 + 4x_2 + 3x_3 + 7x_4 + 8x_5 \leq 25$$

$$x_1 + 7x_2 + 9x_3 + 4x_4 + 6x_5 \leq 25$$

$$8x_1 + 10x_2 + 2x_3 + x_4 + 10x_5 \leq 25$$

$$x_1, x_2, \dots, x_m = (0, 1)$$

This is a pure ILP model

FIXED CHARGE PROBLEM

There are three telephone companies in which a person can subscribe to their long distance service. The following table gives the flat monthly charge and charge per minutes for a long distance service fixed by the three companies.

Company	Flat monthly charge (Rs)	Charge per minute (Rs)
A	16	0.25
B	25	0.21
C	18	0.22

Mr.X usually makes an average of 200 minutes of long distance calls a month. He need not pay the flat monthly fees unless he make calls and he can calls among all three companies.

The problem is how should Mr. X use the three companies to minimize his monthly telephone bill. This problem is called fixed charge problem.

Formulation of ILP

Define x_1 = company A, long distance minutes per month.

x_2 = company B, long distance minutes per month.

x_3 = company C, long distance minutes per month.

$$y_1 = \begin{cases} 1 & \text{if } x_1 > 0 \\ 0 & \text{if } x_1 = 0 \end{cases}$$

$$y_2 = \begin{cases} 1 & \text{if } x_2 > 0 \\ 0 & \text{if } x_2 = 0 \end{cases}$$

$$y_3 = \begin{cases} 1 & \text{if } x_3 > 0 \\ 0 & \text{if } x_3 = 0 \end{cases}$$

Since Mr.X makes about 200 minutes per month $x_1+x_2+x_3=200$

The ILP model is minimize z

ie, $\min z=0.25x_1+0.21x_2+0.22x_3+16y_1+25y_2+18y_3$

subject to

$$x_1 + x_2 + x_3 < 200$$

$$x_1 \leq 200 y_1$$

$$x_2 \leq 200 y_2$$

$$x_3 \leq 200 y_3$$

$$x_1, x_2, x_3 \geq 0$$

$$y_1, y_2, y_3 \in (0, 1)$$

This is a mixed ILP model.

UNIT III

INTEGER PROGRAMMING ALGORITHM

Two general methods have been developed for generating the special constraints referred to

1. Branch-and-Bound (B&B) method
2. Cutting plane, method

1. Branch – and Bound method

Set an initial lower bound $z = -\alpha$ and the optimum objective value of ILP, set $i=0$.

Step 1

(Fathoming / Bounding)

Select LP_i the next sub problem to be examined. Solve LP_i and attempt to fathome it using one of there conditions.

- a. The optimal z value of LP_i cannot yield a better objective value then the current lower bound.
- b. LP_i yield a better feasible integer solution than the current lower bound.
- c. LP_i has no feasible solution.

Two cases will arise.

- a) If LP_i is fathomed update a lower bound if a better ILP solution is found. If all subproblems have been fathomed stop optimum.

if any otherwise, set $i=i+1$ and repeat step 1.

- b) If all LP_i is not fathomed goto step 2 to effect branching.

Step 2 (Branching)

Select one of the integer variable x_j , whose optimum values x_j^* in the LP_i solution is not integer eliminate the region $[x_j^*] < x_j < [x_j^*] + 1$

Where $[v]$ defined the largest integer $\leq V$ by creating two LP subproblems that correspond to $x_j \leq [x_j^*]$

and $x_j \geq [x_j^*] + 1$

set $I = i+1$ and goto step 1

The given steps apply to maximization problems. For minimization we replace the lower bound with an upper bound whose initial value is $z = +\alpha$

1. Consider the following ILP (B&B) method

$$\max z = 5x_1 + 4x_2$$

subject to

$$x_1 + x_2 \leq 5$$

$$10x_1 + 6x_2 \leq 45$$

$$x_1, x_2 \geq 0 \text{ and integer.}$$

Solution:

We consider to given LP as P_0 ,

$$\max z = 5x_1 + 4x_2$$

subject to

$$x_1 + x_2 \leq 5$$

$$10x_1 + 6x_2 \leq 45$$

$$x_1, x_2 \geq 0 \text{ and integer.}$$

Consider the first constraint as

$$x_1 + x_2 = 5$$

$$\text{put } x_1 = 0$$

$$x_2 = 5$$

point (0, 5)

$$\text{put } x_2 = 0$$

$$x_1 = 5$$

point (5, 0)

Consider the second constraint

$$10x_1 + 6x_2 = 45$$

Put $x_1=0$

$$x_2 = \frac{45}{6}$$

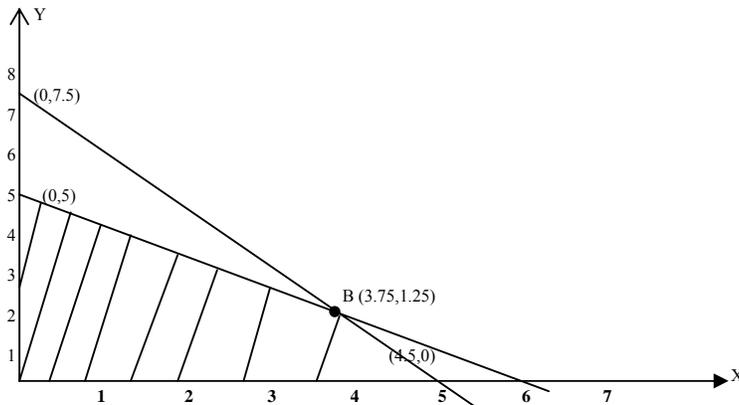
point $(0, \frac{45}{6})$ $(0, 7.5)$

put $x_2 = 0$

$$10x_1 = 45$$

$$x_1 = 4.5$$

point $(4.5, 0)$



Solve

$$x_1 + x_2 = 5 \quad (1)$$

$$10x_1 + 6x_2 = 45 \quad (2)$$

$$\underline{10x_1 + 10x_2 = 50}$$

$$4x_2 = 5$$

$$x_2 = \frac{5}{4} = 1.25$$

$$x_1 = 5 - 1.25$$

$$= 3.75$$

$$x_1 = 3.75, x_2 = 1.25$$

The point is (3.75, 1.25)

At (0, 0)

$$\max z = 5x_1 + 4x_2$$

$$z = 0$$

At (4.5, 0)

$$\max z = 5x_1 + 4x_2$$

$$z = 5.45 + 0$$

$$= 22.5$$

At (3.75, 1.25)

$$\text{Max } z = 5 \times 3.75 + 4 \times 1.25$$

$$= 18.75 + 5.00$$

$$= 23.75$$

At (0, 5)

$$\text{Max } z = 0 + 4 \times 5$$

$$Z = 20$$

$\therefore \max z = 23.75$ at B.

\therefore Optimal Solutions are $x_1=3.75$, $x_2=1.25$ and $x=23.75$.

Here x_1 and x_2 values are not integer

We choose x_1 as the branching variable.

LP₀ is subdivided into two branches

$$\text{LP}_1 = \text{LP}_0 + x_1 \leq 3$$

$$\text{LP}_2 = \text{LP}_0 + x_1 \geq 4$$

Solve : LP₁

Consider the LP1

$$\max z = 5x_1 + 4x_2$$

subject to

$$x_1 + x_2 \leq 5$$

$$10x_1 + 6x_2 \leq 45$$

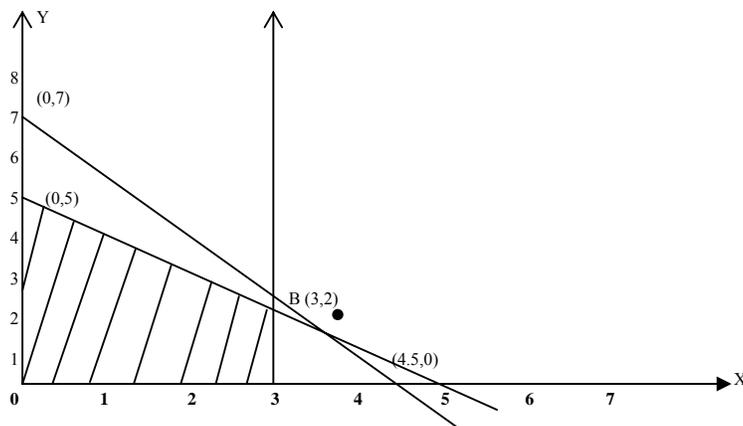
$$x_1 \leq 3$$

Consider the equation $x_1 + x_2 = 5$

The point is (0,5) and (5, 0)

Consider the eqn $10x_1 + 6x_2 = 45$

The point is the (0, 7.5) and (4.5, 0)



$$x_1 + x_2 = 5$$

$$x_1 = 3$$

$$x_2 = 5 - 3 = 2$$

The point is (3, 2)

At O (0,0)

$$x_1 = 0, x_2 = 0$$

$$z = 5x_1 + 4x_2 = 0$$

At A (3,0)

$$x_1 = 3, x_2 = 0$$

$$z = 5 \times 3 = 15$$

At B (3,2)

$$x_1 = 3, x_2 = 0$$

$$z = 5x_1 + 4x_2$$

$$= 5 \times 3 + 4 \times 0$$

$$= 15$$

At C (0,5)

$$z = 5 \times 0 + 4 \times 5$$

$$= 20$$

max $z = 15$

The optimal Solutions are $x_1=3, x_2=0$ and $z=15$

Here x_1 and x_2 are integer

Solve LP₂

Consider the max $z = 5x_1 + 4x_2$

Subject to

$$x_1 + x_2 \leq 5$$

$$10x_1 + 6x_2 \leq 45$$

$$x_1 \geq 4$$

Consider the equation $x_1 + x_2 = 5$

the point is (0, 5) and (5,0)

consider the equation $10x_1 + 6x_2 = 45$

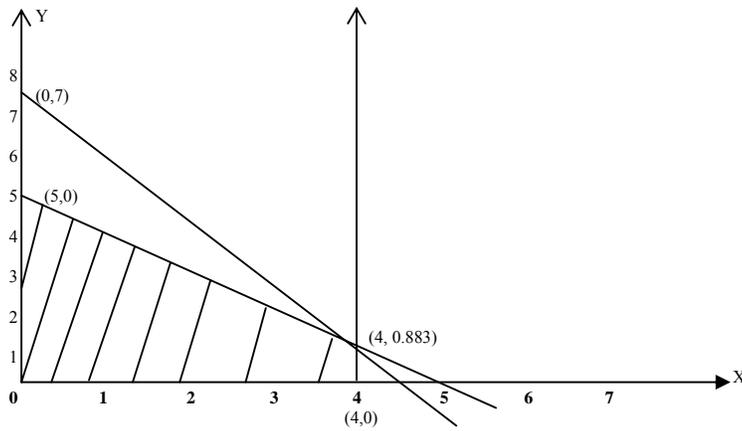
The point is (0, 7.5) (4.5, 0)

$$10x_1 + 6x_2 = 45$$

$$x_1 = 4$$

$$10 \times 4 + 6x_2 = 45$$

$$x_2 = 0.833$$



At 0 (0, 0)

$$z = 0$$

At A (4, 0)

$$z = 20$$

At B (4.50)

$$z = 22.5$$

At C (4, 0.833)

$$z = 5x_1 + 4x_2$$

$$\max z = 23.332$$

Here x_1 is integer and x_2 is not integer.

\therefore The Solution is not optimal

We choose x_2 as a branching variable. We subdivided LP_2 in subpbm in 2 sub pgn.

$$LP_3 = LP_2 + x_2 \leq 0$$

$$LP_4 = LP_2 + x_2 \geq 1$$

Solve LP_3

$$\max z = 5x_1 + 4x_2$$

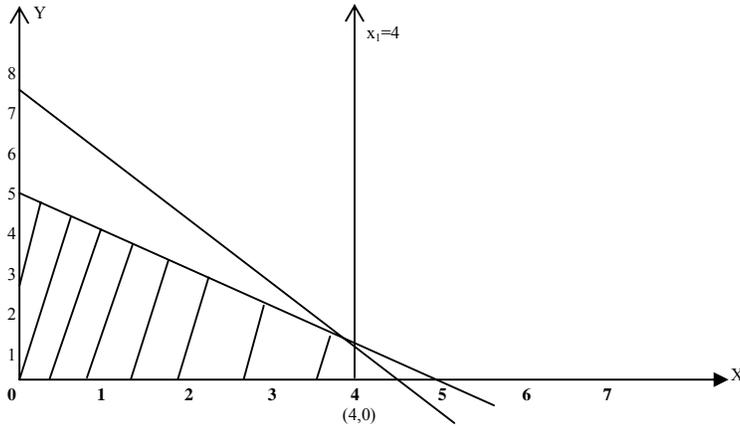
$$\text{S.to } x_1 + x_2 \leq 5$$

$$10x_1 + 6x_2 \leq 45$$

$$x_1 \geq 4$$

$$x_2 \leq 0$$

$$x_1, x_2 \geq 0$$



At O (0, 0)

$$z = 0$$

At A (4, 0)

$$z = 5x4 = 20$$

At B (4.5, 0)

$$z = 5x4.5 = 22.5$$

$$\max z = 22.5$$

The optimal Solution $x_1=4.5$, $x_2=0$ and $z=22.5$

x is not integer x_2 is integer. The Solution is not optimal

The LP₃ can be subdivided into two program.

$$LP_5 = LP_3 + x_1 \leq 4$$

$$LP_6 = LP_3 + x_1 \geq 5$$

Solve LP₅

$$\max z = 5x_1 + 4x_2$$

$$x_1 + x_2 \leq 5$$

$$100x_1 + 6x_2 \leq 45$$

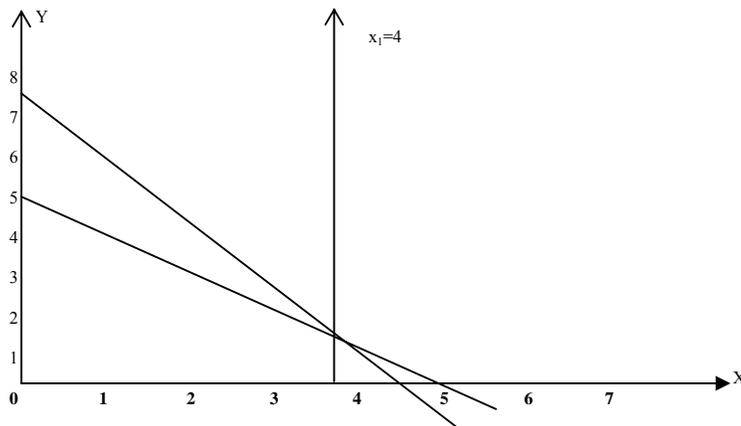
$$x_1 \geq 4$$

$$x_2 \leq 0$$

$$x_1 \leq 4$$

$$x_2 \geq 0$$

$$x_1, x_2 \geq 0$$



At O (0, 0)

$$z = 5(0) + 4(0) = 0$$

At A (4, 6)

$$z = 20$$

The optimal Solution $x_1=4, x_2=0, z=20$

The Solution is optimal

Solve LPs

$$\max z = 5x_1 + 4x_2$$

$$x_1 + x_2 \leq 5$$

$$10x_1 + 6x_2 \leq 5$$

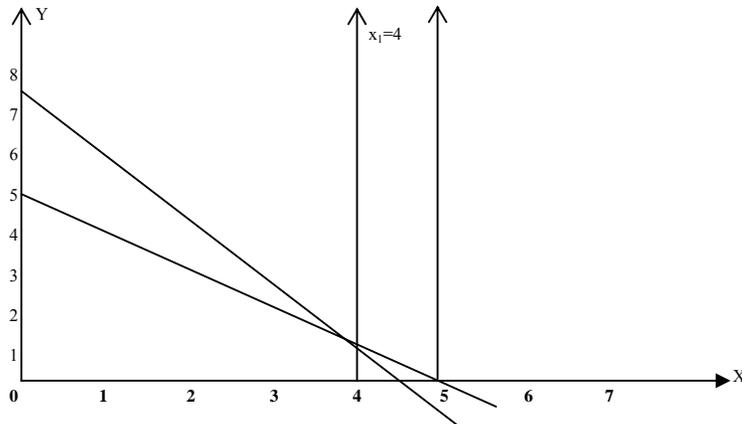
$$x_1 \geq 4$$

$$x_2 \leq 0$$

$$x_1 \geq 5$$

$$x_2 \geq 0$$

$$x_1, x_2 \geq 0$$



LP₆ is fathomed because it has no solution.

Solve LP₄

$$\text{Max } z = 5x_1 + 4x_2$$

$$\text{S.to } x_1 + x_2 \leq 5$$

$$0x_1 + 6x_2 \leq 45$$

$$x_1 \geq 4$$

$$x_2 \leq 1$$

LP has no Solution

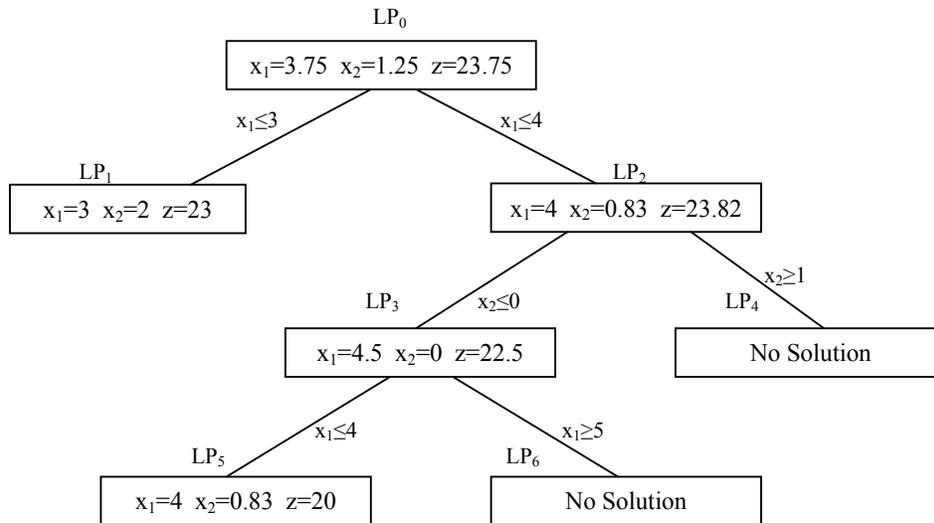
LP₄ is fathomed

In LP₅ the decision variable x_1, x_2 are integer.

$z = 20$ is a lower bound

The optimum value of z in LP₁ as 23

The lower bound is $z = 23$



Lower Bound

Additive Algorithm

The objective further is of the minimization time with all non negative coefficient.

All the constraint must be of the type (\leq) with negative R.H.S is necessary. These constraints are then converted to equation by using slack variable to the L.H.S. of the constraint.

1. Convert the following 0-1 problem to satisfy the starting requirement of the additive algorithm

$$\max z = 3x_1 - 5x_2$$

$$\text{s.to.} \quad x_1 + x_2 = 5$$

$$4x_1 + 6x_2 \geq 4$$

$$x_1, x_2 = (0,9)$$

Solution:

First convert the problem to minimization with all (\leq) constraints as follows

- a) Multiply by z by (-1) we get

$$\min w = -3x_1 + 5x_2$$

- b) Convert the constraint equation is to two constrain of the type (\leq) to obtain.

$$x_1 + x_2 \leq 5, \quad x_1 + x_2 \geq 5, \quad -x_1 - x_2 \leq 5,$$

c) Multiply the second constraint by (-1) we get $-4x_1-6x_2 \leq -4$

using the slack s_1, s_2, s_3 for the three constraint.

The problem is written as

$$\min w = -x_1+5x_2$$

subject to

$$x_1+x_2 + s_1 = 5$$

$$-x_1-x_2 + s_2 = -5$$

$$-4x_1-6x_2 + s_3 = -4$$

$$(x_1, x_2) = (0, 1)$$

$$s_1, s_2, s_3 \geq 0$$

Substitute $x_1=1-y_1$ and $x_2=y_2$ in the above LPP we get

$$\min w = -3(1-y_1)+5y_2$$

subject to

$$(1-y_1) + y_2 + s_1 = 5$$

$$-(1-y_1) - y_2 + s_2 = -5$$

$$-4(1-y_1) -6y_2 + s_3 = -4$$

$$(x_1, x_2) = (0, 1)$$

$$s_1, s_2, s_3, y_1, y_2 \geq 0$$

$$\min w = -3 + 3y_1 + 5y_2$$

$$\min w = 3y_1 + 5y_2 -3$$

subject to

$$-y_1 + y_2 + s_1 = 4$$

$$y_1 - y_2 + s_2 = -4$$

$$4y_1-6y_2 + s_3 = -4$$

$$s_1, s_2, s_3, y_1, y_2 \geq 0$$

$$(x_1, x_2) = (0, 1)$$

2. Solve the following (0-1) problem

$$\text{Maximize } w = 3y_1 + 2y_2 - 5y_3 - 2y_4 + 3y_5$$

Subject to

$$y_1 + y_2 + y_3 + 2y_4 + y_5 \leq 4$$

$$7y_1 + 3y_3 + 4y_4 + 3y_5 \leq 8$$

$$11y_1 - 6y_2 + 3y_4 - 3y_5 \geq 3$$

$$y_1, y_2, y_3, y_4, y_5 = (0, 1)$$

Solution

The problem is written as

$$\text{Minimize } z = -3y_1 - 2y_2 + 5y_3 + 2y_4 - 3y_5$$

Subject to

$$y_1 + y_2 + y_3 + 2y_4 + y_5 \leq 4$$

$$7y_1 + 3y_3 + 4y_4 + y_5 \leq 8$$

$$-11y_1 + 6y_2 - 3y_4 + 3y_5 \leq -3$$

$$y_1, y_2, y_3, y_4, y_5 = (0, 1)$$

substitute $y_1 = (1-x_1)$, $y_2 = (1-x_2)$, $y_3 = x_3$, $y_4 = x_4$, $y_5 = (1-x_5)$,

we get

$$\min z = -3(1-x_1) - 2(1-x_2) + 5x_3 + 2x_4 - 3(1-x_5)$$

subject to

$$(1-x_1) + (1-x_2) + x_3 + 2x_4 + (1-x_5) + s_1 = 4$$

$$7(1-x_1) - 13(x_3 - 4x_4 + 3(1-x_5)) + 5s_2 = 8$$

$$-1(1-x_1) - 6(1-x_2) - 3x_4 + 3(1-x_5) + x_3 = -3$$

$$x_1, x_2, x_3, x_4, x_5 = (0, 1)$$

$$s_1, s_2, s_3 \geq 0$$

$$\text{minimize } z = -3 + 3x_1 - 2 + 2x_2 + 5x_3 + 2x_4 - 3x_2 + 3x_5$$

subject to

$$1 - x_1 + 1 - x_2 + x_3 + 2x_4 - 1 - x_5 + s_1 = 4$$

$$7 - 7x_1 + 3x_3 - 4x_4 + 3 - 3x_5 + s_2 = 8$$

$$-11x_1 + 11x_1 + 6 - 6x_2 - 3x_4 + 3 - 3x_5 + s_3 = -3$$

$$x_1, x_2, x_3, x_4, x_5 = (0, 1)$$

$$s_1, s_2, s_3 \geq 0$$

$$\therefore \min z = 3x_1 + 2x_2 + 5x_3 + 2x_4 + 3x_5 - 8$$

subject to

$$-x_1 + x_2 + x_3 + 2x_4 - x_5 + s_1 = 1$$

$$-7x_1 + 3x_2 - 4x_4 - 3x_5 + s_2 = -2$$

$$11x_1 - 6x_2 - 3x_4 - 3x_5 + s_3 = -1$$

$$x_1, x_2, x_3, x_4, x_5 = (0, 1)$$

$$s_1, s_2, s_3 \geq 0$$

Starting Table

Basic	x_1	x_2	x_3	x_4	x_5	s_1	s_2	s_3	Solution
s_1	-1	-1	1	2	-1	1	-1	0	1
s_2	-7	0	3	-4	-3	0	1	0	-2
s_3	1	-6	0	-3	-3	0	0	1	-1
Objective coefficient	3	2	5	2	3	0	0	0	0

The slack solution is $(s_1, s_2, s_3) = (1, -2, -1)$

$$z = 0$$

S_3 cannot be selected as a branching variable because its constraint coefficient are non negative we conclude x_3 and consider x_2, x_3, x_4, x_5 as the only positive candidates for the branching variable.

The selection of the branching variable from among the candidate x_1, x_3, x_4 and x_5 is based on the uses of measure of slack is feasibility. This measurement it is based on the assumption zero variables x_j will be evaluated to level one is defined as $I_j = \sum_{\text{all } i} \min\{0, s_i - a_{ij}\}$ where s_i is the current value of slack I and a_{ij} is the constraint coefficient of the variable x_j in constraint 'i'.

Actually I_j is nothing but the sum of the negative slack resulting from evaluating x_j to level 1.

The complete formula can be simplified

$$I_j = \sum_{\text{all } i} \{\text{negative } s_i \text{ value given } x_j=1\}$$

For example:

We set $x_1=1$ $[I_j = \sum_{\text{all } i} \{0, s_i, a_{ij}\}]$

We get $s_1 = 1 - (-1) = 2$

$$s_2 = 2 - (-7) = 5$$

$$s_3 = -1 - 11 = -12$$

$$I_1 = \min\{0, 2, 5, -12\} = -12$$

Set $x_2 = 1$

We get $s_1 = 1 - (-1) = 2$

$$s_2 = -2 - 0 = -2$$

$$s_3 = -1 - (-6) = 5$$

$$I_2 = \min\{0, 2, -2, 5\}$$

$$I_2 = -2$$

Set $x_3 = 1$

We get $s_1 = 1 - 1 = 0$

$$s_2 = -2 - 3 = -5$$

$$s_3 = -1$$

$$I_3 = \min\{0, -1, -5, 0\}$$

$$I_3 = -5$$

Set $x_4 = 1$

We get $s_1 = 1 - 2 = -1$

$$s_2 = -2 + 4 = 2$$

$$s_3 = -1 + 3 = 2$$

$$I_4 = \min\{0, 2, 2, -1\}$$

$$I_4 = -1$$

Set $x_5 = 1$

We get $s_1 = 1 - (-1) = +2$

$$s_2 = -2 + 3 = 1$$

$$s_3 = -1 + 3 = 2$$

$$I_5 = \min\{0, 2, 2, 1, 2\}$$

$$I_5 = 0$$

$$I_5 = \max\{I_1, I_2, I_3, I_4, I_5\}$$

$$\max\{-12, -2, -5, -1, 0\}$$

$$I = 0 [x_5]$$

x_5 is selected as the branching variable. The 1000 branches in x_5 are

$x_5 = 0$ and $x_5 = 1$ and created of node 1 and 2 $x_5 = 1$.

Node 1 yield the feasible slack valued $(s_1, s_2, s_3) = (2, 1, 2)$ in I_5 .

$$z = 3x_1 + 2x_2 + 5x_3 + 2x_4 + 3x_5 + 0s_1 + 0s_2 + 0s_3$$

$$= 3x_0 + \dots + 3x_1 + 0x \dots$$

$$= 3$$

$x_5 = 0$

Node 2 yield the feasible slack value (S_1, S_2, S_3)

$$= (1, -2, -1), x_5 = 0$$

$$Z = 0$$

Node 1 is fathomed and $\bar{z} = 3$ defined the current upper bound on the optimal objective value.

The variable s_1, x_2, x_3 and x_4 are the candidates for the branching variable. The variables x_1 and x_3 are not possible to branch because their objectives coefficient (3 and 5) will yield a worse objective value than the current upper bound $\bar{z} = 3$. The remaining variable x_2 and x_4 .

$$\sum_{\text{all } i} \min(0, s_i - a_{ij})$$

$$x_2 = 1 \Rightarrow I_2 = -2$$

$$x_4 = 1 \Rightarrow I_4 = -1$$

We choose the maximum of I_2 and $I_4 - 1$ is the maximum of I_4 .

x_4 is the branching variable at node 2.

x_4 is branching into $x_4 = 1$ and $x_4 = 0$ leading to nodes 3 and 4

At node 3

$$x_4 = 1$$

We get $z = 3x_1 + 2x_2 + 5x_3 + 2x_4 + 3x_5 + 0s_1 + 0s_2 + 0s_3$

$$= 2 \times 1$$

$$= 2$$

$$x_4 = 0$$

We get $z = 0$

$$= 2 \times 1$$

$$= 2$$

The candidates of branching x_1, x_2, x_3 all the candidates variable executed and node 3 is fathomed. Next at a remaining defined by $x_4 = x_5 = 0$

$$x_2 = 1 \Rightarrow I_2 = -2$$

$$x_2 = 0 \Rightarrow I_2 = 1$$

$$(s_1, s_2, s_3) = (2, -2, 5)$$

$$z = 3x_1 + 2x_2 + 5x_3 + 2x_4 + 3x_5 + 0s_1 + 0s_2 + 0s_3$$

$$= 3(0) + 2(1)$$

$$z = 2$$

$$x_2 = 0$$

$$(s_1, s_2, s_3) = (1, -2, -1)$$

$$z = 3x_1 + 2x_2 + 5x_3 + 2x_4 + 3x_5 + 0s_1 + 0s_2 + 0s_3$$

$$z = 0$$

The variable x_1 and x_3 are excluded by the upper bound test. The remaining variable x_2 cannot be excluded by the upper bound infeasibility. Hence x_2 as a branching variable branches in 4 are node x_5 and 5.

The variable x_1 is excluded by the upper bound test and x_3 is excluded by both slack infeasibility.

This means that node is fathomed. Because neither x_1 (or) x_3 can produce a better feasible Solution. The optimal Solution is associated with node 1 yield $x_5=1$, $\bar{z}=3$ and all the other variables are 0.

In terms of the original variable

$$y_1 = 1 - x_1, y_2 = 1 - x_2, y_3 = x_3, y_4 = x_4, y_5 = 1 - x_5$$

$$y_1 = 1 - 0, y_2 = 1 - 0, y_3 = 0, y_4 = 0, y_5 = 0$$

$$w = 3y_1 + 2y_2 - 5y_3 - 2y_4 + 3y_5$$

$$= 3(1) + 2(1) - 5(0) - 2(0) + 3(0)$$

$$w = 5$$

2. Solve by the additive algorithm

$$\max z = 3x_1 + x_2 + 3x_3$$

subject to

$$-x_1 + 2x_2 + x_3 \leq 2$$

$$4x_1 - 3x_3 \leq 2$$

$$x_1 - 3x_2 - 2x_3 \leq 3$$

$$(x_1, x_2, x_3) = (0, 1)$$

Solution

$$\min w = -3x_1 - x_2 - 3x_3$$

subject to

$$-x_1 + 2x_2 + x_3 + s_1 = 2$$

$$4x_2 - 3x_3 + s_2 = 2$$

$$x_1 - 3x_2 - 2x_3 + s_3 = 3$$

$$(x_1, x_2, x_3) = (0, 1)$$

$$x_1 = 1 - y_1, x_2 = 1 - y_2, x_3 = 1 - y_3$$

$$\text{Min } w = -3 + 3y_1 - 1 + y_2 - 3 + 3y_3$$

$$= 3y_1 + y_2 + y_3 - 7$$

Subject to

$$1 + y_1 + 2 - 2y_2 + 1 - y_3 + s_1 = 2$$

$$\Rightarrow y_1 = 2y_2 - y_3 + s_1 = 0 \quad (1)$$

$$4 - 4y_2 - 3 + 3y_3 + s_2 = 2$$

$$\Rightarrow -4y_2 + 3y_3 + s_2 = 1 \quad (2)$$

$$1 - y_1 + 3 + 3y_2 + 2 - 2y_3 + s_3 = 3$$

$$-y_1 + 3y_2 - 2y_3 + s_3 = 0 \quad (3)$$

$$(y_1, y_2, y_3, s_1, s_2, s_3) = (0, 1)$$

Starting table

Basic	y_1	y_2	y_3	s_1	s_2	s_3	Solution
s_1	1	-2	-1	1	0	0	0
s_2	0	-4	3	-	1	0	1
s_3	-1	3	-2	0	0	1	3

Objective 3 1 3

coeff

The each Solution is $(s_1, s_2, s_3) = (0, 1, 3)$

$$I_j = \sum_{\text{all } i} \min \{0, s_i - a_{ij}\}$$

The complete formula can be simplified into

$$I_j = \sum_{\text{all } i} (-\text{ive } s_i \text{ value given } x_j=1)$$

We set $y_1=1$, we get

$$S_1=0-1= -1$$

$$S_2=1-0=1$$

$$S_3=1-(-1)=4$$

$$I_1=\min \{-1, 1, 4\}=-1$$

We get $y_2=1$

$$S_1=0-(-2)=2$$

$$S_2=1-(-4)=5$$

$$S_3=3-3 = 0$$

$$I_2=\min \{0, 2, 5, 0\} = 0$$

We get $y_3=1$

$$S_1=0-(-1)=1$$

$$S_2=1-3=-2$$

$$S_3=3-(-2) = 5$$

$$I_3=\min \{0, 1, -2, 5\} = -2$$

$$I = \max \{I_1, I_2, I_3\}$$

$$= \max \{-1, 0, -2\}$$

$$I = 0 = I_2$$

It yield the smallest of infeasibility.

y_1 is selected as a branching variable.

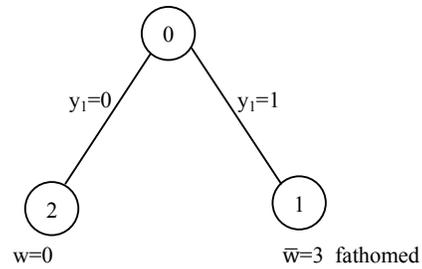
Two branches in y_1 are $y_1=1, y_1=0$

$$y_1=1, (s_1, s_2, s_3) = (-1, 1, 4)$$

$$w = 3y_1 + y_2 + 3y_3 + 0 + 0s_1 + 0s_2 + 0s_3$$

$$= 3(1) \dots$$

$$w = 3$$



Then y_2 as a branching variable

$$y_2=1 \text{ and } y_2=0$$

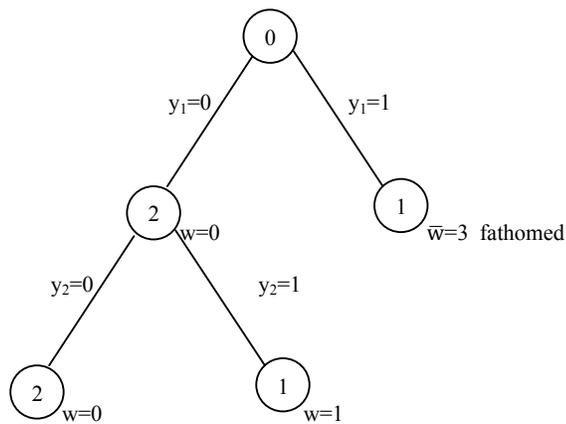
$$y_2=1$$

$$\Rightarrow w = 3(0) + 1 + 3(0) + 0(2) + 0(5) + 0(0)$$

$$w = 1$$

$$y_2 = 0$$

$$w = 0$$



Original Solution is $x_1=1-y_2$,

$$x_2 = 1 - y_2$$

$$x_3 = 1 - y_1$$

$$\text{put } y_1 = 1$$

\therefore original Solution

$$x_1 = 1 - 1 \Rightarrow x_1 = 0$$

$$x_2 = 1 - 0 \Rightarrow x_2 = 1$$

$$x_3 = 1 - 0 \Rightarrow x_3 = 1$$

$$z = 3(0) + 1 + 3(1)$$

$$z = 4$$

Cutting plane Algorithm

Algebra Development of cuts

The cutting plane algorithm start by solving the continuous Lp problem. In the optimum LP table be select one of the rows called the source row, for which the basic variable is non integer. The desire cut is constructed from the fractional for compomence of the coefficient of the source row. For this reason it is reputed to as to fractional cut.

Problem

Solve Problem using fractional cut

$$\text{Max } z = 7x_1 + 10x_2$$

Subject to

$$-x_1 + 3x_2 \leq 6$$

$$7x_1 + x_2 \leq 35$$

$$x_1, x_2 \geq 0 \text{ and integer}$$

Solution:

$$\text{max } z = 7x_1 + 10x_2$$

subject to

$$-x_1 + x_2 + s_2 = 35$$

$$x_1, x_2, s_1, s_2 \geq 0 \text{ and integer}$$

Basic	z_1	x_1	x_2	s_1	s_2	Solution	Ratop
z	1	-7	-10	0	0	0	-
s_1	0	-1	3	1	0	6	2
s_2	0	7	1	0	1	35	35

x_2 is the entering variable

s_1 is the leaving variable

3 is the pivot element.

Table 1

Basic	z_1	x_1	x_2	s_1	s_2	Solution	Ratop
Z	1	$-\frac{31}{3}$	0	$\frac{10}{3}$	0	20	-
x_2	0	$-\frac{1}{3}$	1	$-\frac{1}{3}$	0	2	-
s_2	0	$-\frac{22}{3}$	0	$-\frac{1}{3}$	1	33	$\frac{9}{2}$

New z row = old z row – (pivot element) New pivot row

Table 2

Basic	z_1	x_1	x_2	s_1	s_2	Solution	Ratop
Z	1	0	0	$\frac{63}{22}$	$\frac{3}{22}$	$66\frac{1}{2}$	$-\frac{1}{2}$
x_2	0	0	1	$\frac{7}{22}$	$\frac{1}{22}$	2	$\frac{3}{2}$
x_1	0	1	0	$-\frac{1}{22}$	$\frac{3}{2}$	$\frac{9}{2}$	$\frac{9}{2}$

New z row = $(1 \ -\frac{31}{3} \ 0 \ \frac{10}{3} \ 0 \ 20)$

The optimal continuous Solution $x_1=4\frac{1}{2}$, $x_2=3\frac{1}{2}$, $s_1=0$, $s_2=0$, $z=66\frac{1}{2}$

The current Solution is not optimal

The information of the optimum table can be written in the equation form.

$$z \text{ eqn} = z + 0x_1 + 0x_2 + \frac{63}{22} s_1 + \frac{31}{22} s_2 = 66\frac{1}{2}$$

$$x_2 \text{ eqn} = 0z + 0x_1 + 1x_2 + \frac{7}{22} s_1 + \frac{1}{22} s_2 = 3\frac{1}{2}$$

$$x_1 \text{ eqn} = 0z + 1x_1 + 0x_2 - \frac{1}{22} s_1 + \frac{3}{22} s_2 = 4\frac{1}{2}$$

In the optimum table any of the three equations can be used as a source row for generating the cut.

Let us arbitrarily select z equation for the purpose.

$$\text{Eqn : } z + \frac{63}{22} s_1 + \frac{31}{22} s_2 = 66\frac{1}{2} \text{ (z source row)}$$

To construct the fractional cut which of the non integer coefficient are factor into integer and fractional component provides in strictly, positive.

$$\text{[for example } \frac{5}{2} = 2 + \frac{1}{2}, -\frac{5}{2} = -3 + \frac{1}{2}$$

$$-\frac{7}{3} = \frac{-9+2}{3} = \frac{-9}{3} - \frac{2}{3} = -3 + \frac{2}{3}$$

$$\text{The factoring of z row yield } z + \frac{63}{22} s_1 + \frac{31}{22} s_2 = 66\frac{1}{2}$$

$$z + \left(2 + \frac{19}{22}\right) s_1 + \left(1 + \frac{9}{22}\right) s_2 = 66 + \frac{1}{2}$$

$$z + 2s_1 + s_2 - 66 = -\frac{19}{22} s_1 - \frac{9}{22} s_2 + \frac{1}{2}$$

Because s_1 and s_2 are non negative in the R.H.S. must be satisfied

$$-\frac{19}{22} s_1 - \frac{9}{22} s_2 + \frac{1}{2} \leq \frac{1}{2}$$

$$\frac{9}{22} s_1 - \frac{9}{22} s_2 + \frac{1}{2} \leq 0$$

This is the desired fractional cut. We can develop but also from x_1 source row and x_2 source row in the same manner.

Consider the x_1 row we have

$$x_1 - \frac{1}{22} s_1 + \frac{3}{22} s_2 = 4\frac{1}{2}$$

The factoring operation yield

$$x_1 + \left(-1 + \frac{21}{22}\right) + \frac{3}{22} s_2 = 4 + \frac{1}{2}$$

The associated cut is

$$\frac{-21}{22} s_1 - \frac{3}{22} s_2 \neq \frac{1}{2} \leq 0$$

$$x_2 + \frac{7}{22} s_1 + \frac{1}{22} s_2 = 3\frac{1}{2}$$

The Cub is given as

$$\frac{-7}{22} s_1 - \frac{1}{22} s_2 + \frac{1}{2} \leq 0$$

The optimum solution is $x_1 = 4$, $x_3 = 3$ and $z = 58$

UNIT IV

INVENTORY MODELS

DETERMINISTIC INVENTORY MODELS

Inventory control deals with ordering and stocking policies for items used in the manufacture of industrial products.

Items bought from vendors have the following costs associated with the purchase.

1. Cost of the product
2. Ordering cost per order
3. Carrying cost for holding the items
4. Shortage cost (back order costs)

Cost of the product C

This is represented as C Rs/Unit. Since the annual demand for the item is known and has to be met, this cost does not play a part in determining the order quantity.

Order cost C_o .

This is represented as Rs/order. The essential costs that contribute to this cost are:

1. Cost of people
2. Cost of stationary
3. Cost of communication – fax.
4. Cost of follow up – travel, fax
5. Cost of transportation
6. Cost of inspection and counting
7. Cost of rework, rejects.

Carrying cost / holding cost C_c .

This is represented as Rs/unit/year. The costs that contribute to carrying the items are

1. Cost of capital
2. Cost of people

3. Cost of space.
4. Cost of power
5. Cost of special facilities – air conditioners chillers, dust free environment.
6. Cost of pilferage
7. Cost of obsolescence

Sometimes when the cost of capital i , dominates the rest of the costs, carrying cost is expressed as $C_c = ic$ where c is the unit price in Rs. And i , the interest rate (cost of capital) in % / year.

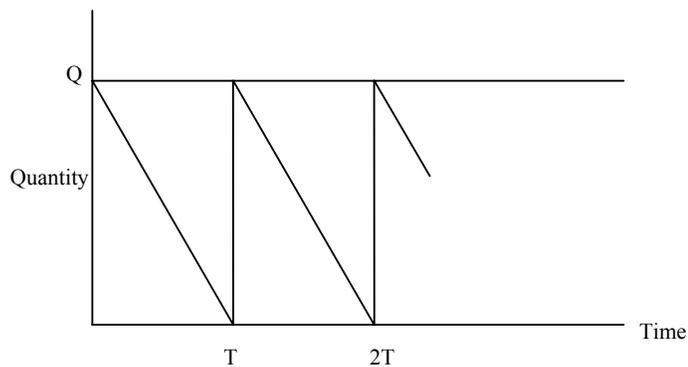
Shortage cost C_s .

This is represented as Rs/unit/year. The necessary costs that contribute to shortage cost are:

1. Loss of customer goodwill
2. Cost of losing a customer
3. Loss of profit associated with not delivering the product.

Continuous demand instantaneous Replenishment model

The inventory-time behavior is given by



The coefficients are:

Annual demand = D /year.

Order Cost = C_o .

Carrying cost = C_c .

Ordering Quantity = Q

$$\text{Number of orders / year} = D/Q$$

$$\text{Annual order cost} = \frac{DC_o}{Q}$$

$$\text{Annual inventory carrying cost} = \frac{Qc_c}{2}$$

$$\text{Total cost (Tc)} = \frac{DC_o}{Q} + \frac{Qc_c}{2}$$

The value of Q that minimizes the total cost is obtained by setting the first derivative to zero.

We get

$$\begin{aligned} \frac{\partial}{\partial Q} (Tc) &= DC_o \left(\frac{-1}{Q^2} \right) + \frac{c_c}{2} \\ &= -\frac{DC_o}{Q^2} + \frac{c_c}{2} \end{aligned}$$

The first derivative is zero.

$$\begin{aligned} &= -\frac{DC_o}{Q^2} + \frac{c_c}{2} \\ \frac{DC_o}{Q^2} &= \frac{c_c}{2} \end{aligned}$$

$$Q^2 = \frac{2DC_o}{c_c}$$

$$Q = \sqrt{\frac{2DC_o}{c_c}}$$

Substitute the value of Q in eqn (1) we get

$$\begin{aligned} T_c &= \frac{2DC_o}{\sqrt{\frac{2DC_o}{c_c}}} + \frac{Qc_c}{2} \\ &= \frac{Dc_o}{\sqrt{2DC_o}} \sqrt{C_c} + \frac{Qc_c}{2} \\ &= \frac{Dco \sqrt{C_c}}{\sqrt{2} \sqrt{DC_o}} + \frac{Qc_c}{2} \\ &= \frac{\sqrt{DC_o C_c}}{\sqrt{2}} + \frac{Qc_c}{2} \end{aligned}$$

$$\begin{aligned}
&= \frac{\sqrt{2}}{\sqrt{2}} \frac{\sqrt{Dc_0C_c}}{\sqrt{2}} + \frac{Qc_c}{2} \\
&= \frac{\sqrt{2Dc_0C_c}}{\sqrt{2}} + \frac{Qc_c}{2} \\
&= \frac{\sqrt{2Dc_0C_c}}{\sqrt{2}} + \frac{\sqrt{2Dc_0}}{C_c} \cdot \frac{C_c}{2} \\
T_c &= \frac{\sqrt{2Dc_0C_c}}{2} + \frac{\sqrt{2Dc_0}}{C_c} \cdot \frac{C_c}{2} \\
&= \frac{\sqrt{2Dc_0C_c}}{2} + \frac{\sqrt{2Dc_0}}{2} \\
&= 2\sqrt{Dc_0C_c}
\end{aligned}$$

It is denoted by $T_c^* = \sqrt{2Dc_0C_c}$ and $Q^* = \frac{\sqrt{2Dc_0}}{C_c}$

Q^* is called the economic order quantity.

Example : 1

$D = 1000/\text{year}$, $C_o = \text{Rs.}300/\text{order}$ and $C_c = \text{Rs.}4/\text{unit}/\text{year}$. Find the total cost, number of orders and increasing order of 15%.

$$\begin{aligned}
Q^* &= \frac{\sqrt{2Dc_0}}{2} \\
&= \frac{\sqrt{2 \times 10000 \times 300}}{4} \\
&= 1224.74
\end{aligned}$$

$$\begin{aligned}
T_c^* &= \sqrt{2Dc_0C_c} \\
&= \sqrt{2 \times 10000 \times 300 \times 4} = 4898.98
\end{aligned}$$

Number of orders / year (N) = $\frac{D}{Q}$

$$\begin{aligned}
&= \frac{10000}{1224.74} \\
&= 8.1649 \text{ order/year} \\
&= 8.165 \text{ order/year}
\end{aligned}$$

If we increase the order quantity by 15% and round it off to 1500

$$(D=10000 \quad Q = 10000 \times \frac{15}{100})$$

$$T_c = \frac{Dc_o}{Q} + \frac{Qc_c}{2}$$

$$T_c = \frac{10000 \times 300}{1500} + \frac{750 \times 4}{2}$$

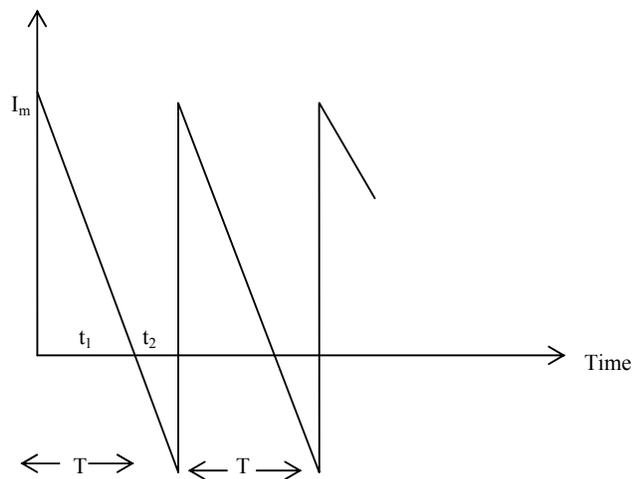
$$= 2000 + 3000$$

$$T_c = 5000$$

Which is approximately a 2% increase from the optimum cost.

Considering backordering

The maximum inventory held in $I_m = Q - S$. There is an inventory period of T_1 per cycle and a back period of T_2 per cycle.



The coefficient are

Annual demand = D/year .

Order cost = C_o

Carrying cost = C_c

Shortage (back order) cost = C_s

Order quantity = Q

Back order quantity = S

Maximum inventory in a cycle = I_m

Number of order / year = $\frac{D}{Q}$

Annual order cost = $\frac{DC_o}{Q}$

Average inventory in the system = $\frac{I_m}{2}$

Annual inventory carrying cost = $\frac{I_m C_c}{2}$

Average shortage in the system = $S/2$

Annual cost $T_c = \frac{DC_o}{Q} + \frac{I_m C_c}{2} \times \frac{T_1}{(T_1+T_2)} + \frac{C_3}{2} \times \frac{T_2}{T_1+T_2}$ (1)

From similar triangle, we get

$$\frac{T_1}{T_1+T_2} = \frac{Q-S}{Q}$$

and

$$\frac{T_2}{T_1+T_2} = \frac{S}{Q}$$

$$(1) \Rightarrow T_c = \frac{DC_o}{Q} + \frac{I_m C_c}{2} \times \frac{Q-S}{Q} + \frac{C_s}{2} \times \frac{S}{Q}$$

$$= \frac{DC_o}{Q} + \frac{(Q-S)C_c}{2} \times \frac{(Q-S)}{Q} + \frac{C_s}{2Q}$$

$$T_c = \frac{DC_o}{Q} + \frac{(Q-S)^2 C_c}{2Q} + \frac{C_s}{2Q} \quad (2)$$

(1) \Rightarrow Diff w.r.to 's'

$$\frac{\partial}{\partial s} (T_c) = 0 + \frac{2(Q-S)(-1)C_c}{2Q} + \frac{2S C_s}{2Q}$$

$$\frac{\partial}{\partial s} (T_c) = \frac{(S-Q)C_c}{Q} + \frac{SC_s}{Q}$$

Setting to zero

$$\frac{(S-Q)C_c}{Q} + \frac{SC_s}{Q} = 0$$

$$\frac{(S-Q)C_c}{Q} = - \frac{SC_s}{Q}$$

$$S C_c - Q C_c = -S C_s$$

$$S C_c + S C_s = Q C_c$$

$$S(C_c + C_s) = Q C_c$$

$$S = \frac{Q C_c}{C_c + C_s}$$

To find first differentiation T_c , partially diff w.r.to a and setting to 0

Deff w.r.to Q we get

$$(1) \Rightarrow T_c = \frac{DC_o}{Q} + \frac{(Q-S)^2 C_c}{2Q} + \frac{S^2 C_s}{2Q}$$

$$\frac{\partial}{\partial Q} (T_c) = DC_o \left(\frac{-1}{Q^2} \right) + \frac{2Q \cdot 2(Q-S)C_c - (Q-S)^2 C_c}{4Q^2} + \frac{S^2 C_s}{2} \left(\frac{-1}{Q^2} \right)$$

$$= -\frac{DC_o}{Q} + \frac{4Q(Q-S)C_c - 2(Q-S)^2 C_c}{4Q^2} - \frac{S^2 C_s}{2}$$

$$= -\frac{DC_o}{Q} + \frac{2Q C_c(Q-S) - (Q-S)^2 C_c}{2Q^2} - \frac{S^2 C_s}{2Q^2}$$

$$= -\frac{DC_o}{Q} + \frac{2 Q C_c \left(Q - \frac{Q C_c}{C_c + C_s} \right) - \left(Q - \frac{Q C_c}{C_c + C_s} \right)^2 C_c}{Q^2}$$

$$= \frac{C_s}{2Q^2} \cdot \left(\frac{Q C_c}{C_c + C_s} \right)^2$$

$$\frac{\partial}{\partial Q} (T_c) = -\frac{DC_o}{Q^2} + 2Q C_c \left[\frac{Q(C_c + C_s) - Q C_c}{C_c + C_s} \right] - \left[\frac{Q(C_c + C_s) - Q C_c}{C_c + C_s} \right]^2 C_c - \frac{Q^2 C_c^2}{(C_c + C_s)^2} \frac{C_s}{2Q^2}$$

$$= -\frac{DC_o}{Q^2} + \frac{2 Q C_c \times \frac{Q C_s}{C_c + C_s} - \frac{(Q^2 C_s^2) C_c}{(C_s + C_c)^2} - \frac{C^2 C_s}{2(C_s + C_s)^2}}{2Q^2}$$

$$= -\frac{DC_o}{Q^2} + \frac{2Q^2 C_c \cdot C_s}{2Q^2 (C_c + C_s)} - \frac{Q^2 C_s^2 C_c}{(C_s + C_s)^2 2Q^2} - \frac{C^2 C_s}{2(C_s + C_s)^2}$$

$$= \frac{-2DC_o(C_s + C_s)^2 + 2Q^2 C_c + C_s (C_c + C_s) - Q^2 C_s^2 C_c - Q^2 C_s^2 C_s}{2Q^2 (C_c + C_s)^2}$$

$$= \frac{-2DC_o(C_c^2 + C_s^2 + 2C_c \cdot C_s) + 2Q^2 C_c^2 C_s + 2Q^2 C_c C_s^2 - Q^2 C_s^2 C_c - Q^2 C_c^2 C_s}{2Q^2 (C_c + C_s)^2}$$

$$\frac{\partial}{\partial Q} (TC) = \frac{-2DC_o (C_c^2 + C_s^2 + 2C_c \cdot C_s) + Q^2 C_s^2 C_c + Q^2 C_c C_s^2}{2Q^2 (C_c + C_s)^2}$$

Setting to zero

$$0 = \frac{-2DC_o(C_c^2 + C_s^2 + 2C_c \cdot C_s) + Q^2 C_s \cdot C_c(C_c + C_s)}{2Q^2(C_c + C_s)^2}$$

$$-2DC_o(C_c + C_s)^2 + Q^2 C_s C_c(C_c + C_s) = 0$$

$$Q^2 C_s C_c(C_c + C_s) = 2DC_o(C_c + C_s)^2$$

$$Q^2 = \frac{2DC_o(C_c + C_s)}{C_s \cdot C_c}$$

$$Q = \sqrt{\frac{2DC_o(C_c + C_s)}{C_s C_c}} \Rightarrow Q^* = \sqrt{\frac{2DC_o(C_c + C_s)}{C_s C_c}}$$

Example : 2

D=10000/year, C_o= Rs.300/order and C_c=Rs.4/unit /year C_s=Rs.25 per unit/year. To find order Quantity, Back order quantity, Maximum inventory, Number of orders, and total cost.

Solution

$$Q^* = \sqrt{\frac{2DC_o(C_c + C_s)}{C_s C_c}}$$

$$= \sqrt{\frac{2 \times 10000 \times 300 (4 + 25)}{4 \times 25}}$$

$$= \sqrt{\frac{6000000 (29)}{100}} = \sqrt{1740000}$$

$$= Q^* = 1319.09$$

$$S = \frac{Q C_c}{C_c + C_s}$$

$$= \frac{1319.09 \times 4}{4 + 25} = \frac{5276.36}{29} = 181.943$$

We order 1319.09 but also build a backorder of 181.943 units. The maximum inventory held in a cycle is now

$$Im = Q^* - S^*$$

$$= 1319.09 - 181.943$$

$$Im = 1137.147$$

$$\text{Number orders/year } N = \frac{D}{Q}$$

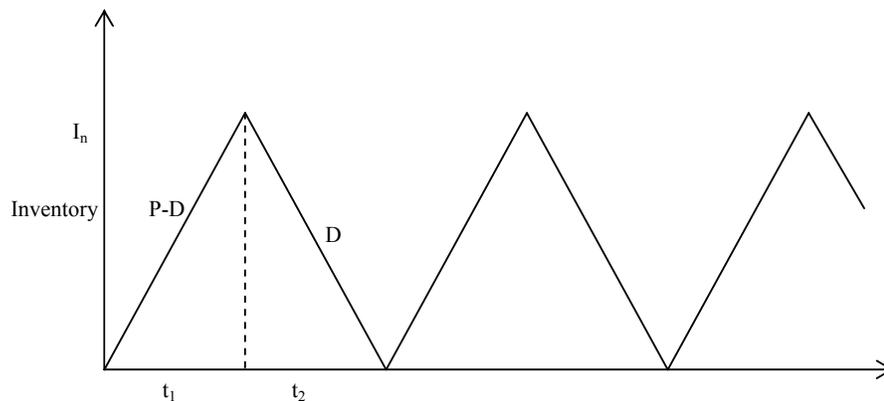
Each cycle is for 0.1319 years out of which the inventory cycle is 0.1137 years and the shortage cycle is 0.181 years. Then

$$\begin{aligned} T_c &= \frac{DC_o}{Q} + \frac{(Q-S)^2 C_c}{2Q} + \frac{S^2 C_s}{2Q} \\ &= \frac{10000 \times 300}{1319.09} + \frac{(1319.09 - 181.943)^2 4}{2 \times 1319.09} + \frac{(181.943)^2 \times 25}{2 \times 1319.09} \\ &= 2274.295 + 1960.599 + 313.694 \end{aligned}$$

$$T_c = 4548.59$$

Production consumption model

We produce and consume the item when it is produced. The, production rate is P per year ($P > D$). The production is for a period t_1 and the item is consumed for a period t_2 . The cycle time is $t_1 + t_2$.



The coefficients are :

Annual demand = D/year

Production rate = P/year

Order Cost = C_o

Carrying cost = C_c

Order Quantity = Q

Maximum inventory in a cycle = I_m

$$\text{Number of orders / year} = \frac{D}{Q}$$

$$\text{Annual order cost} = \frac{D C_o}{Q}$$

$$\text{Average inventory in the system} = \frac{I_m}{2}$$

$$\text{Annual inventory carrying cost} = \frac{I_m C_c}{2}$$

$$\text{Total cost } T_c = \frac{D C_o}{Q} + \frac{I_m C_c}{2} \quad (1)$$

We have $Q = pt_1$ and $I_m = (P-D)t_1$

From which $I_m = pt_1 - Dt_1$

$$= Q - Dt_1$$

$$= Q \left(1 - \frac{Dt_1}{Q} \right)$$

$$= Q \left(1 - \frac{Dt_1}{Pt_1} \right)$$

$$= Q \left(1 - \frac{D}{P} \right)$$

$$T_c = \frac{D C_o}{Q} + \frac{I_m C_c}{2}$$

$$T_c = \frac{D C_o}{Q} + \frac{Q \left(1 - \frac{D}{P} \right) C_c}{2}$$

The value of Q that minimizes the total cost are obtained by setting the first derivative with respect to a and S to zero.

Differentiate w.r.to 'Q'

$$\frac{\partial}{\partial Q} (T_c) = D C_o \left(\frac{-1}{Q^2} \right) + \frac{\left(1 - \frac{D}{P} \right) C_c}{2}$$

$$= \frac{D C_o}{Q^2} + \frac{\left(1 - \frac{D}{P} \right) C_c}{2}$$

Setting to zero. We get

$$\frac{-D C_o}{Q^2} + \frac{\left(1 - \frac{D}{P} \right) C_c}{2} = 0$$

$$\frac{DC_0}{Q^2} = \frac{\left(1 - \frac{D}{P}\right) C_c}{2}$$

$$Q^2 = \frac{2DC_0}{\left(1 - \frac{D}{P}\right) C_c}$$

$$Q = \sqrt{\frac{2DC_0}{\left(1 - \frac{D}{P}\right) C_c}}$$

Example : 3

D=10000/year, C_o = Rs/order and C_c =Rs.4/unit/year, P=20000, units/year

Soln:

$$\begin{aligned} \text{We have } Q^* &= \sqrt{\frac{2DC_0}{C_c \left(1 - \frac{D}{P}\right)}} \\ &= \sqrt{\frac{2 \times 10000 \times 300}{4 \left(1 - \frac{10000}{20000}\right)}} \\ &= \sqrt{\frac{6000000}{24 \times \frac{1}{4}}} = \sqrt{\frac{6000000}{2}} \\ &= \sqrt{3000000} \end{aligned}$$

$$Q^* = 1732.05$$

$$\begin{aligned} \text{Im} &= Q \left(1 - \frac{D}{P}\right) \\ &= 1732.05 \left(1 - \frac{10000}{20000}\right) \\ &= 1732.05 \left(\frac{1}{2}\right) \end{aligned}$$

$$\text{Im} = 866.025$$

$$Q = Pt_1$$

$$\begin{aligned} t_1 &= \frac{Q}{P} \\ &= \frac{1732.05}{20000} \end{aligned}$$

$$t_1 = 0.0866 \text{ years}$$

$$\text{No. of orders / years } N = \frac{Q}{D}$$

$$N = \frac{1732.05}{10000}$$

$$N = 0.173205$$

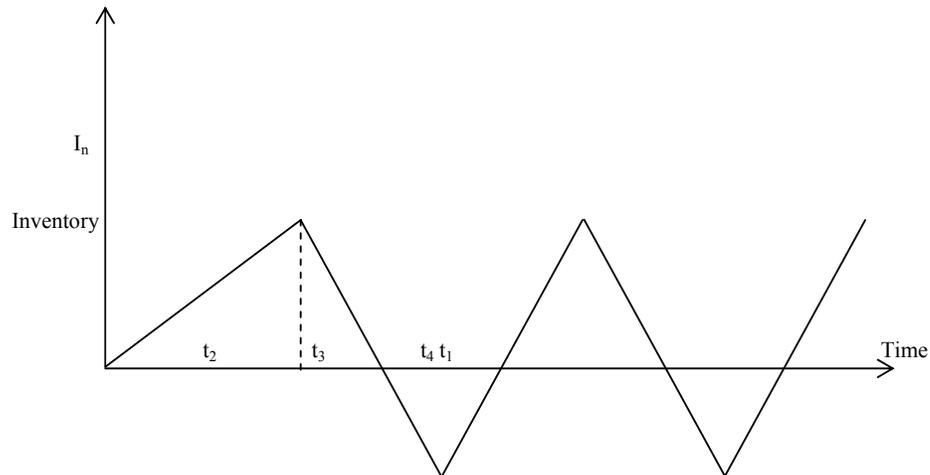
$$\begin{aligned} TC &= \frac{DC_0}{Q} + \frac{Q \left(1 - \frac{D}{P}\right) C_c}{2} \\ &= \frac{10000 \times 300}{1732.05} + \frac{1732.05 \left(1 - \frac{10000}{20000}\right)}{2} \\ &= \frac{3000000}{1732.05} + 1732.05 \end{aligned}$$

$$TC = 17320.05 + 1732.05$$

$$TC = 3464.10$$

Production Consumption model 3 with backordering

Here we allow a backordering and build a quantity of S , which is delivered as the items are produced in the next cycle.



The coefficients are:

$$\text{Annual demand} = D/\text{year}$$

$$\text{Production rate} = P/\text{year}$$

$$\text{Order cost} = C_0$$

$$\text{Carrying cost} = C_c$$

$$\text{Shortage (backorder) Cost} = C_s$$

$$\text{Order Quantity} = Q$$

$$\text{Backorder allowed} = S$$

$$\text{Maximum inventory in a cycle} = I_m$$

$$\text{Number of orders / year} = D/Q.$$

$$\text{Annual order cost} = \frac{DC_0}{Q}$$

$$\text{Average inventory in the system} = \frac{I_m}{2}$$

This is held for (t_2+t_3) time out of a cycle of T .

Annual inventory carrying cost

$$= \frac{I_m C_c}{2} \times \frac{t_2+t_3}{T}$$

$$\text{Average quantity back ordered} = \frac{S}{2}$$

This is held for a period (t_1+t_4) out of a cycle of T .

$$\text{Annual backorder cost} = \frac{S C_s}{2} \times \frac{t_1+t_4}{T}$$

$$\text{Then TC} = \frac{DC_0}{Q} + \frac{I_m C_c}{2} \times \frac{t_2+t_3}{T} + \frac{S C_s}{2} \times \frac{t_1+t_4}{T} \quad (1)$$

We also have

$$Q = p(t_1+t_2)$$

$$I_m = (P-D)t_2 = Dt_3 \quad (2)$$

$$S = (P-D)t_1 = Dt_4$$

$$T = t_1 + t_2 + t_3 + t_4$$

$$Q = DT$$

$$\frac{I_m}{Q} = \frac{(P-D)t_2}{P \times (t_1+t_2)}$$

$$= \frac{(P-D)}{P} \times \frac{t_2}{(t_1+t_2)}$$

$$\frac{t_1+t_2}{t_2} = \frac{P-D}{P} \times \frac{Q}{I_m}$$

$$\frac{t_1}{t_2} + \frac{t_2}{t_2} = \frac{(P-D)Q}{PI_m}$$

$$1 + \frac{t_1}{t_2} = \frac{PQ(1-\frac{D}{P})}{PI_m}$$

$$1 + \frac{t_1}{t_2} = \frac{Q(1-\frac{D}{P})}{I_m} \quad (3)$$

From similar triangles

$$\frac{t_1}{t_2} = \frac{S}{I_m} \quad (4)$$

$$(1) \Rightarrow 1 + \frac{t_1}{t_2} = \frac{Q(1-\frac{D}{P})}{I_m}$$

Substitute the value (4) in eqn (3) we get

$$1 + \frac{S}{I_m} = \frac{Q(1-\frac{D}{P})}{I_m}$$

$$I_m(1 + \frac{S}{I_m}) = Q(1 - \frac{D}{P})$$

$$I_m(\frac{I_m+S}{I_m}) = Q(1 - \frac{D}{P})$$

$$I_m + S = Q(1 - \frac{D}{P})$$

$$I_m = Q(1 - \frac{D}{P}) - S \quad (5)$$

Eqn (2) \Rightarrow

$$(P-D)t_2 = Dt_3$$

$$Pt_2 - Dt_2 = Dt_3$$

$$Pt_2 = Dt_2 + Dt_3$$

$$Pt_2 = D(t_2 + t_3)$$

$$\frac{Pt_2}{T} = \frac{D(t_2+t_3)}{T}$$

$$\text{ie } \frac{t_2+t_3}{T} = \frac{Pt_2}{T}$$

Again (2) $\Rightarrow I_m = (P-D)t_2$

$$PI_m = (P-D) Pt_2$$

$$\frac{PI_m}{P-D} = Pt_2$$

$$\frac{PI_m}{DT(P-D)} = \frac{Pt_2}{DT}$$

$$\frac{PI_m}{Q(P-D)} = \frac{Pt_2}{DT} \quad [\because DT=Q]$$

$$\therefore \frac{t_2+t_3}{T} = \frac{Pt_2}{DT} = \frac{PI_m}{Q(P-D)} \quad (6)$$

$$\text{Also } (P-D) t_1 = D t_4$$

$$P t_1 - D t_1 = D t_4$$

$$P t_1 = D = D(t_1 + t_4)$$

$$\frac{Pt_1}{DT} = \frac{t_1+t_4}{T} \quad (7)$$

and

$$S = (P-D) t_1$$

$$\frac{S}{P-D} = t_1$$

$$\frac{PS}{DT(P-D)} = \frac{Pt_1}{DT}$$

$$\frac{PS}{Q(P-D)} = \frac{Pt_1}{DT}$$

$$\frac{PS}{QP(1-\frac{D}{P})} = \frac{Pt_1}{DT}$$

$$\frac{S}{Q(1-\frac{D}{P})} = \frac{Pt_1}{DT} \quad (8)$$

(7) and (8)

$$\frac{t_1+t_4}{T} = \frac{Pt_1}{DT} - \frac{S}{Q(1-\frac{D}{P})}$$

Substitute all value in TC we get

$$(1) \Rightarrow TC = \frac{DC_o}{Q} + \frac{I_m C_c}{2} \left(\frac{P I_m}{Q(P-D)} + \frac{SC_3}{2} \left(\frac{S}{Q(1-\frac{D}{P})} \right) \right)$$

By eqn (6) and (8)

$$\begin{aligned}
&= \frac{DC_0}{Q} + \frac{I_m^2 C_c P}{2Q(P-D)} + \frac{S^2 C_s}{2Q(1-\frac{D}{P})} \\
&= \frac{DC_0}{Q} + [Q(1 - \frac{D}{P})-S]^2 \frac{C_c}{2} \frac{P}{Q(P-D)} + \frac{S^2 C_s}{2Q(1-\frac{D}{P})} \\
&= \frac{DC_0}{Q} + [Q(1 - \frac{D}{P})-S]^2 \frac{C_c}{2Q(P-D)} + \frac{S^2 C_s}{2Q(1-\frac{D}{P})}
\end{aligned}$$

The first derivative of T_c is zero

Diff w.r.to 's'

$$\frac{\partial}{\partial s} (T_c) + 0 + 2[Q(1 - \frac{D}{P})-s](-1) \frac{C_c}{2Q(1-\frac{D}{P})} + \frac{2SC_s}{2Q(1-\frac{D}{P})}$$

$$\frac{\partial}{\partial s} (T_c) + -[Q(1 - \frac{D}{P})-S] \frac{C_c}{Q(1-\frac{D}{P})} + \frac{SC_s}{Q(1-\frac{D}{P})}$$

$$= \frac{Q(1-\frac{D}{P})C_c}{Q(1-\frac{D}{P})} + \frac{SC_c}{Q(1-\frac{D}{P})} + \frac{SC_s}{Q(1-\frac{D}{P})}$$

$$= C_c + \frac{S(C_c + C_s)}{Q(1-\frac{D}{P})}$$

$$\frac{\partial}{\partial s} (T_c) = 0$$

$$C_c = \frac{S(C_c + C_s)}{Q(1-\frac{D}{P})}$$

$$S = \frac{Q(1-\frac{D}{P})C_c}{C_c + C_s}$$

$$T_c = \frac{DC_0}{Q} + [Q(1 - \frac{D}{P})-S]^2 \frac{C_c}{2Q(1-\frac{D}{P})} + \frac{S^2 C_s}{2Q(1-\frac{D}{P})}$$

$$\frac{\partial T_c}{\partial Q} = \frac{-DC_0}{Q^2} + 2Q(1 - \frac{D}{P}) \cdot 2 [Q(1 - \frac{D}{P})-s] (1 - \frac{D}{P}) C_c \frac{-[Q(1-\frac{D}{P})-s]^2 \cdot 2(1-\frac{D}{P})}{4Q^2(1-\frac{D}{P})} - \frac{S^2 C_s}{2Q^2(1-\frac{D}{P})}$$

$$= \frac{\frac{-DC_0}{Q^2} + 4Q(1-\frac{D}{P})^2 [Q(1-\frac{D}{P})-s] C_c - 2(1-\frac{D}{P}) C_c [Q(1-\frac{D}{P})-s]^2}{4Q^2(1-\frac{D}{P})^2} - \frac{S^2 C_s}{2Q^2(1-\frac{D}{P})}$$

$$\begin{aligned}
& \frac{-\frac{DC_0}{Q^2} + 2\{2Q(1-\frac{D}{P})^2 [Q(1-\frac{D}{P})-s] C_c - (1-\frac{D}{P}) C_c Q(1-\frac{D}{P})-s\}^2}{4Q^2(1-\frac{D}{P})^2} - \frac{S^2 C_s}{2Q(1-\frac{D}{P})} \\
&= \frac{-\frac{DC_0}{Q^2} + (1-\frac{D}{P}) C_c \{2Q(1-\frac{D}{P})Q(1-\frac{D}{P})-s\} - [Q(1-\frac{D}{P})-s]^2}{2Q^2(1-\frac{D}{P})^2} - \frac{S^2 C_s}{2Q^2(1-\frac{D}{P})} \\
&= \frac{-DC_0 (2(1-\frac{D}{P})) + C_c \{2Q(1-\frac{D}{P})Q(1-\frac{D}{P})-s\} - [Q(1-\frac{D}{P})-s]^2 S^2 C_s}{2Q^2(1-\frac{D}{P})} \\
&= \frac{-2DC_0 (1-\frac{D}{P}) + [Q(1-\frac{D}{P})-s] C_c (1-\frac{D}{P}) - [Q(1-\frac{D}{P})-s] - S^2 C_s}{2Q^2(1-\frac{D}{P})} \\
&= \frac{-2DC_0 (1-\frac{D}{P}) + C_c (1-\frac{D}{P})-s \cdot \{2Q-2Q\frac{D}{P}-Q+\frac{QD}{P}-s\} - S^2 C_s}{2Q^2(1-\frac{D}{P})} \\
&= \frac{-2DC_0 (1-\frac{D}{P}) + C_c [\{ [Q(1-\frac{D}{P})-s] \{ Q-\frac{QD}{P}-s \}] - S^2 C_s}{2Q^2(1-\frac{D}{P})} \\
\frac{\partial T_c}{\partial Q} &= \frac{-2DC_0 (1-\frac{D}{P}) + C_c [Q(1-\frac{D}{P})-s] [Q-\frac{D}{P})-s]}{2Q^2(1-\frac{D}{P})} \\
&= \frac{-2DC_0 (1-\frac{D}{P}) + C_c [Q(1-\frac{D}{P})-s]^2 - S^2 C_s}{2Q^2(1-\frac{D}{P})} \\
&= \frac{-2DC_0 (1-\frac{D}{P}) + C_c \left[Q(1-\frac{D}{P}) - \frac{Q(1-\frac{D}{P}) C_c}{C_c + C_s} \right]^2 - \left[\frac{Q(1-\frac{D}{P}) C_c}{C_c + C_s} \right]^2 C_s}{2Q^2(1-\frac{D}{P})} \\
&= \frac{-2DC_0 (1-\frac{D}{P}) + C_c [Q(1-\frac{D}{P})] \left[1 - \frac{C_c}{C_c + C_s} \right]^2 - \frac{Q^2(1-\frac{D}{P})^2 C_c^2}{(C_c + C_s)^2} C_s}{2Q^2(1-\frac{D}{P})} \\
&= \frac{-2DC_0 (1-\frac{D}{P}) + \frac{Q^2(1-\frac{D}{P})^2 C_c C_s}{(C_c + C_s)}}{2Q^2(1-\frac{D}{P})}
\end{aligned}$$

$$\frac{\partial}{\partial Q}(T_c) = 0$$

$$-2 DC_0 (1 - \frac{D}{P}) + Q^2 (1 - \frac{D}{P})^2 \frac{C_c C_s}{C_c + C_s} = 0$$

$$-2 DC_0 (1 - \frac{D}{P}) = Q^2 (1 - \frac{D}{P})^2 \frac{C_c C_s}{C_c + C_s}$$

$$Q = \sqrt{\frac{2DC_0(C_c + C_s)}{C_c(1 - \frac{D}{P})C_s}}$$

Example 4

$$D=10000, C_0=300, C_s=25, C_c=4, P=20000$$

Find Q^* , S , I_m , T_c .

$$\begin{aligned} \text{Soln } Q^* &= \sqrt{\frac{2DC_0(C_c + C_s)}{C_c(1 - \frac{D}{P})C_s}} \\ &= \sqrt{\frac{2 \times 10000(4 + 25)}{4 \times 25(1 - \frac{10000}{20000})}} \\ &= \sqrt{\frac{10000 \times 29 \times 6}{\frac{1}{2}}} \\ &= \sqrt{10000 \times 29 \times 6 \times 2} \\ &= \sqrt{2480000} \end{aligned}$$

$$Q^* = 1865.48$$

$$\begin{aligned} S &= \frac{QC_c(1 - \frac{D}{P})}{C_c + C_s} \\ &= \frac{1865.48 \times 4(1 - \frac{10000}{20000})}{4 + 25} \\ &= \frac{1865.48 \times 4(1 - \frac{1}{2})}{29} \\ &= \frac{1865.48 \times 4 \times \frac{1}{2}}{29} \end{aligned}$$

$$S = 128.65$$

$$\begin{aligned} I_m &= Q\left(1 - \frac{D}{P}\right) - S \\ &= 1865.48 \left(1 - \frac{10000}{20000}\right) - 128.65 \\ &= 1865.48 \left(\frac{1}{2}\right) - 128.65 \\ &= 932.74 - 128.65 \end{aligned}$$

$$I_m = 804.09$$

$$\begin{aligned}
T_c &= \frac{Dc_0}{Q} + [Q(1 - \frac{D}{P}) - S]^2 \frac{C_c}{2} \frac{1}{(1 - \frac{D}{P})} + \frac{S^2 C_s}{2Q(1 - \frac{D}{P})} \\
&= \frac{10000 \times 300}{1865.48} + [1865.48(1 - \frac{10000}{20000}) - 128.65]^2 \\
&\quad \frac{4}{2} \frac{1}{1865.48(1 - \frac{10000}{20000})} + \frac{(128.65)^2}{2 \times 1865.48(1 - \frac{10000}{20000})} \\
&= 1608.165 + [1865.48 \times \frac{1}{2} - 128.65]^2 \frac{2}{1865.48} \times \frac{1}{2} + \frac{16550.8225 \times 25}{3730.96 \times \frac{1}{2}} \\
&= 1608.165 + [932.74 - 128.65]^2 \frac{4}{1865.48} + \frac{827541.1256550.8225 \times 25}{3730.96} \\
&= 1608.165 + 1386.37 + 221.8 \\
T_c &= 3216.34
\end{aligned}$$

Inventory model with discount

Let us consider inventory models with discount

D=10000/year, C_o=Rs.300/order and C_c=Rs.4/unit/ year.

Annual demand D=10000

Order Cost C_o=Rs.300

Unit Price C = Rs.20

Interest = 20% per annum

Here the economic order quantity is

$$\begin{aligned}
Q^* &= \sqrt{\frac{2DC_0}{C_c}} \\
&= \sqrt{\frac{2 \times 10000 \times 300}{4}} = \sqrt{1500000}
\end{aligned}$$

$$Q^* = 1224.74$$

$$\begin{aligned}
T_c &= \sqrt{2DC_0 C_c} \\
&= \sqrt{2 \times 10000 \times 300 \times 4} \\
&= \sqrt{24000000} \quad T_c = 4898.98
\end{aligned}$$

Let us assume that the vender willing to give a 2% discount on unit price if the order quantity were 2000 (or) more and is willing to give a 4% discount if the order quantity were 5000 or more.

The total cost including the item cost Q ;EOQ is

$$\begin{aligned} \text{EOQ} &= T_c + D \times C \\ &= 4898.98 + 10000 \times 20 \\ \text{EOQ} &= 204898.98 \end{aligned}$$

If the order quantity is 2000 then

$$\begin{aligned} T_c &= \frac{DC_0}{Q} + Q \times C \times i \times \frac{0.98}{2} + DC \times 0.98 \\ &= \frac{10000 \times 300}{2000} + 2000 \times 20 \times \frac{20}{100} \times \frac{0.98}{2} + 10000 \times 20 \times 0.98 \\ &= 1500 + 4000 \times 0.98 + 196000 \\ &= 201420 \end{aligned}$$

If the order quantity is 5000 then

$$\begin{aligned} T_c &= \frac{DC_0}{Q} + \frac{Q \times c \times i \times (1-d)}{2} + D_c (1-d) \\ &= \frac{10000}{5000} \times 300 + 5000 \times \frac{20}{100} \times 20 \times \frac{0.96}{2} + 10000 \times 20 \times 0.96 \\ &= 600 + 9600 + 192000 \\ &= 202200 \end{aligned}$$

The total cost (including the item cost) is minimum when we avail a 2% discount at an order quantity of 2000 units.

Multiple items inventory

Constraints on total number of orders

Example 5

Let us consider two items with the data shown in the given table:

	Item 1	Item 2
Annual demand (D)	10000	20000
Order Cost (Co)	300	300
Unit Price c	20	25
Interest rate	20%	20%

Soln

Considering the 1st item. We have

$$\begin{aligned}Q_1^* &= \sqrt{\frac{2D_1C_0}{C_c}} \\&= \sqrt{\frac{2D_1C_0}{iC_1}} \quad \because C_c = ic \\&= \sqrt{\frac{2 \times 10000 \times 300}{\frac{20}{100} \times 20}} \\&= \sqrt{\frac{3000000}{2}} = \sqrt{1500000}\end{aligned}$$

$$Q_1^* = 1224.74$$

$$\begin{aligned}Tc_1 &= \sqrt{2D_1C_0C_c} \\&= \sqrt{2 \times 10000 \times 300 \times 4}\end{aligned}$$

$$Tc_1 = 4898.98$$

Consider the second item, we have

$$\begin{aligned}Q_2^* &= \sqrt{\frac{2D_2C_0}{iC_2}} \\&= \sqrt{\frac{2 \times 20000 \times 300}{\frac{20}{100} \times 25}} \\&= \sqrt{24000000}\end{aligned}$$

$$Q_2^* = 1549.19$$

$$\begin{aligned} Tc_2 &= \sqrt{2D_2C_0C_c} \\ &= \sqrt{2 \times 20000 \times 300 \times 5} \\ &= \sqrt{60000000} \end{aligned}$$

$$Tc_2 = 7745.97$$

$$\begin{aligned} \text{Number of orders / year } N_1 &= \frac{D_1}{Q_1} \\ &= \frac{10000}{1224.74} \end{aligned}$$

$$N_1 = 8.16$$

$$\begin{aligned} \text{Number of orders / year } N_2 &= \frac{D_2}{Q_2} \\ &= \frac{20000}{1549.19} \end{aligned}$$

$$N_2 = 12.91$$

We have, $N_1 + N_2 = 8.16 + 12.91 = 21.07$ orders/year.

We have more than thousand items in a factory and each has a certain inventory resulting in a very large amount of money stocked in goods.

Let us assume that we would restrict to 15 orders.

The problem formulation would to be

$$\text{minimize } \sum_{j=1}^n \frac{D_j C_0}{Q_j} + i \frac{Q_j C_j}{2}$$

$$\text{Subject to } \sum_{j=1}^n \frac{D_j}{Q_j} = N$$

$$Q_j \geq 0$$

When the constraint is violated by the individual EOQ's.

The constraint becomes

$$\sum_{j=1}^n \frac{D_j}{Q_j} = N$$

We are Lagrange on multiplier teaching introduce λ and write the Lagrangean function as

$$L = \sum_{j=1}^n \frac{D_j C_0}{Q_j} + \sum_{j=1}^n \frac{Q_j C_j}{2} + \sum_{j=1}^n \lambda \left[\frac{D_j}{Q_j} - N \right]$$

The optimum values of Q_j and λ can be obtained by equating the partial derivatives $\frac{\partial L}{\partial Q_j} = 0$ and $\frac{\partial L}{\partial \lambda}$ to zero.

$$\frac{\partial L}{\partial \lambda} = 0$$

$$\Rightarrow C_0 D_j \left(\frac{-1}{Q_j^2} \right) + \frac{i C_j}{2} + \lambda D_j \left(\frac{-1}{Q_j^2} \right) = 0$$

$$\Rightarrow - \left(\frac{C_0 D_j}{Q_j^2} \right) + \frac{i C_j}{2} - \left(\frac{\lambda D_j}{Q_j^2} \right) = 0$$

From which

$$\frac{-2 C_0 D_j + i C_j Q_j^2 - 2 \lambda D_j}{2 Q_j^2} = 0$$

$$\frac{i C_j Q_j^2 - 2 D_j (C_0 + \lambda)}{2 Q_j^2} = 0$$

$$\frac{i C_j}{2} - \frac{D_j (C_0 + \lambda)}{Q_j^2} = 0$$

$$\frac{i C_j}{2} = \frac{D_j (C_0 + \lambda)}{Q_j^2}$$

$$Q_j^2 = \frac{2 D_j (C_0 + \lambda)}{i C_j}$$

$$\frac{\partial L}{\partial \lambda} = 0 \text{ gives } \frac{\sum D_j}{Q_j} = N$$

To get the optimum value of λ , we substitute for Q_j^* in $\sum \frac{D_j}{Q_j} = N$

$$\Rightarrow \sum_{j=1}^n \frac{D_j}{\sqrt{\frac{2 D_j (C_0 + \lambda)}{i C_j}}} = N$$

$$\Rightarrow \sum_{j=1}^n \frac{D_j \sqrt{i C_j}}{\sqrt{2 D_j \cdot (C_0 + \lambda)}} = N$$

$$\Rightarrow \sum_{j=1}^n \frac{D_j \sqrt{ic_j}}{\sqrt{D_j \cdot (C_0 + \lambda)}} = \sqrt{2} N$$

Squaring,

$$\Rightarrow \sum_{j=1}^n \frac{D_j^2 ic_j}{D_j \cdot (C_0 + \lambda)} = 2N^2$$

$$\Rightarrow i \left[\sum_{j=1}^n \frac{\sqrt{D_j ic_j}}{2N^2} \right]^2 = C_0 + \lambda$$

$$\lambda = \frac{i \left[\sum_{j=1}^n \sqrt{D_j ic_j} \right]^2}{2N^2} - C_0$$

For the two item example with N=15 we have

$$\lambda = \frac{20}{100} \frac{(\sqrt{D_1 C_1 + D_2 C_2})^2}{2 \times 15^2} - 300$$

$$= \frac{1}{5} \frac{(\sqrt{1000 \times 20 + 20000 \times 25})^2}{2 \times 225} - 300$$

$$= \frac{1}{5} \frac{(\sqrt{1000 \times 20 + 20000 \times 25})}{450} - 300$$

$$\lambda = 292.20$$

$$Q_1 = \sqrt{\frac{2D_1(C_0 + \lambda)}{ic_1}}$$

$$= \sqrt{\frac{2 \times 10000 (300 + 292.20)}{\frac{20}{100} \times 20}}$$

$$= \sqrt{5000 \times 592.20}$$

$$Q_1 = \sqrt{1720.76}$$

$$Q_2 = \sqrt{\frac{2D_2(C_0 + \lambda)}{ic_2}}$$

$$Q_2 = \sqrt{\frac{2 \times 20000 (300 + 292.20)}{\frac{20}{100} \times 25}}$$

$$= \sqrt{8000 (592.20)}$$

$$Q_2 = 2176.607$$

$$\begin{aligned}
N_1+N_2 &= \frac{D_1}{Q_1} + \frac{D_2}{Q_2} \\
&= \frac{10000}{1720.76} + \frac{20000}{2176.607} \\
&= 15
\end{aligned}$$

$$\begin{aligned}
T_c &= \sum C_o \frac{D_j}{Q_j} + \sum Q_j \frac{C_j}{2} \\
&= C_o \frac{D_1}{Q_1} + C_o \frac{D_2}{Q_2} + i (Q_1 \frac{C_1}{2} + Q_2 \frac{C_2}{2}) \\
&= \frac{10000}{1720.76} \times 300 + \frac{1720.76 \times 4}{2} + \frac{20000}{2176.607} \times 300 + \frac{2176.607}{2} \times 5 \\
&= 13383.04
\end{aligned}$$

$$\frac{Q_1}{Q_1^*} = \frac{1720.76}{1224.75} = 1.405$$

$$\frac{Q_2}{Q_2^*} = \frac{2176.607}{1549.19} = 1.405$$

$$\frac{N_1+N_2}{N} = \frac{21.07}{15} = 1.405$$

If we had directly used the equation to find λ for $N=25$, we will get a negative value of λ indicating that the constraint is satisfied by the EOQ's.

GAME THEORY

Game theory deals with decision situations in which two intelligent opponents have conflicting objectives.

In a game conflict, two opponents, known as **players**, will each have a (finite or infinite) number of alternatives or **strategies**.

Associated with each pair of strategies is a **payoff** that one player pays to the other. Such games are known as two-person zero-sum games because the gain by one player equals the loss to the other.

Optional solution of two-person zero sum Games

The optimal solution selects one or more strategies for each player such that any change in the chosen strategies does not improve the pay off to either player. These solutions can be in the form of a single pure strategy or several strategies that are mixed according to predetermined probabilities.

Example

Two companies A and B sell two branches of the flu medicine company A advertises in radio (A_1), Television (A_2), and Newspaper (A_3) company B, in addition to using radio (B_1) television (B_2), and Newspapers (B_3), also mails brochures (B_4). Depending on the cleverness and intensity of the advertisement campaign each company can capture a portion of the market from the other. The following matrix summaries the percentage of the market captured or lost by company A.

The optional solution of the game calls for selecting strategies A_2 and B_2 that is both companies should use television advertisements. The payoff will be in favour of company, a because its market share will increase by 5%. In this case, we say that the value of the game is 5% and that A and B are using a saddle-point solution.

Solution of mixed strategy Games

Games with mixed strategies can be solved either graphically or by linear programming. The graphical solution is suitable for games in which atleast one player has exactly two pure strategies.

Graphical solution of games

We start with the case of $(2 \times n)$ game in which player A has two strategies.

The game assumes that player A mixes strategies A_1 and A_2 with the respective probabilities x_1 and $1-x_1$, $0 \leq x_1 \leq 1$.

Player B mixes strategies B_1 to B_n with the probabilities y_1, y_2, \dots , and y_n , where $y_j \geq 0$ for $j=1, 2, \dots, n$ and $y_1+y_2+\dots+y_n=1$.

Example

Consider the following 2×4 game. The payoff is for player A.

The game has no pure strategy solution and hence the strategies must be mixed.

A's expected payoffs corresponding to B's pure strategies are given as.

B's Pure strategy	A's expected payoff
1	$2x_1+4$
2	$-x_1+3$
3	x_1+2
4	$-7x_1+6$

The maximum of the lower envelope corresponds to the maximin solution point $x_1^* = .5$. This point is the player B's optimal mix is determined by the two strategies that define the lower envelope of the graphs. This means that B can mix strategies B_3 and B_4 in which case $y_1=y_2=0$ and $y_4=1-y_3$.

A's Pure strategy	B's expected payoff
1	$4y_3-1$
2	$-4y_3+6$

This process is equivalent to solving the equation.

$$4y_3-1 = -4y_3 + 6$$

The solution gives $y_3 = 7/8$

The value of the game as $V = 4 \times 7/8 - 1$

$$= 5/2$$

UNIT V

QUEUING MODELS

Elements of a Queuing Model

The principal actors in a queuing situation are the **customer** and the **server**. Customers are generated from a **source**. On arrival at the facility they can start service immediately or wait in a **queue**.

The arrival of customers to represented by the **inter arrival** time and the service is described by the service time per customer.

Queue size plays a role in the analysis of queues, and it may have a finite size as in the buffer area between two successive machines.

The **queue discipline** which represents the order in which customers are selected from a queue, is an important factor in the analysis of queuing models.

The most common discipline is first come, first served (FCFS). Other disciplines include last come, first served other disciplines include last come, first served (LCFS) and service in random order (SIRD).

Customers may also be selected from the queue based on some order of priority.

Role of exponential Distribution

Random inter arrival and service times are described quantitatively for the purpose of queuing modeling by the exponential distribution, which is defined as

$$f(t) = \lambda e^{-\lambda t} \quad t > 0$$

$$\text{where } E\{t\} = \frac{1}{\lambda}$$

Forgetfulness property

Given t is exponentially distributed as defined by $f(t)$, if S is the interval since the occurrence of the last event, then the forgetfulness property is described by a probability statement as:

$$P\{t > T+S/t > s\} = P\{t > T\}$$

To prove this result, we note that

$$P\{t > Y\} = 1 - P\{t < Y\} = e^{-\lambda t}$$

Thus,

$$\begin{aligned} P\{t > T+S | t > S\} &= \frac{P\{t > T+S, t > S\}}{P\{t > S\}} \\ &= \frac{P\{t > T+S\}}{P\{t > S\}} \\ &= \frac{e^{-\lambda(T+S)}}{e^{-\lambda S}} \\ &= e^{-\lambda T} \\ &= P\{(t > T)\} \end{aligned}$$

Derivation of the exponential Distribution

Theorem

The exponential distribution is based on three axioms.

Axiom : 1

Given $N(t)$, the number of events during the interval $(0, t)$, the probability process describing $N(t)$ has stationary independent increments in the sense that the probability of an event occurring in the interval $(T, T+S)$ depends only on the length of S .

Axiom 2

The Probability of an event occurring in a sufficiently small time interval $h > 0$ is positive but less than 1.

Axiom 3

In a sufficiently small interval $h > 0$, at most one event can occur – that is $P\{N(h) > 1\} = 0$.

Proof

Define $P_n(t)$ as the probability of n events occurring during t .

By axiom 1

The probability of no event occurring during $t+h$, $h>0$ and sufficiently small, is

$$P_o(t+h) = P_o(t) P_o(h)$$

Based on the remaining two axioms,

$$P_o(t) = e^{-\lambda t}, \quad t > 0$$

Where λ is a positive constant. Define $f(t)$ as the probability density function of the interval t between successive events, $t > 0$.

We have

$$P\{\text{inter event time} > T\} = P\{\text{no event during } T\}.$$

This statement translates to

$$\int_T^\infty f(t)dt = P_o(T), \quad T > 0$$

Substituting and rearranging terms we get

$$\int_0^T f(t)dt = 1 - e^{-\lambda T}, \quad t > 0$$

Taking the derivative of both sides with respect to T , we get the exponential distribution.

$$f(t) = \lambda e^{-\lambda t}, \quad t > 0$$

The mean value of the exponential is $\frac{1}{\lambda}$ time units, where λ is the rate at which the events are generated.

Pure Birth Model

Given that arrivals occur at the rate λ customers per unit time,

Then for small time interval $h > 0$.

$$\begin{aligned} P_o(h) &= e^{-\lambda h} = 1 - \lambda h + \frac{(\lambda h)^2}{2!} - \dots \\ &= 1 - \lambda h + o(h^2) \end{aligned}$$

During $h>0$, atmost one event (arrival) can occur. Thus, as $h \rightarrow 0$.

$$P_1(h) = 1 - P_0(h) \simeq \lambda h.$$

This result shows that the probability of an arrival occurring during h is directly proportional to h , with the arrival rate λ being the constant of proportionality.

Define $P_n(t)$ as the probability of n arrivals during t .

Thus, for $h > 0$ and sufficiently small,

$$P_n(t+h) \approx P_n(t) (1 - \lambda h) + P_{n-1}(t) \lambda h, \quad n > 0$$

$$P_0(t+h) \approx P_0(t) (1 - \lambda h), \quad n = 0$$

In the first equation, n arrivals will be realized during the $t+h$ if there are n arrivals during t and no arrivals during h or $n-1$ arrivals during t and one arrival during h .

Rearranging the terms and taking the limits as $h \rightarrow 0$, we get

$$\begin{aligned} P_n'(t) &= \lim_{h \rightarrow 0} \frac{P_n(t+h) - P_n(t)}{h} \\ &= -\lambda P_n(t) + \lambda P_{n-1}(t), \quad n > 0 \end{aligned}$$

$$\begin{aligned} P_0'(t) &= \lim_{h \rightarrow 0} \frac{P_0(t+h) - P_0(t)}{h} \\ &= -\lambda P_0(t) \end{aligned}$$

Where $P_n'(t)$ is the first derivative of $P_n(t)$ with respect to t .

The solution of the preceding difference differential equations yields.

$$P_n(t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

Which is the Poisson distribution with mean $E\{n/t\} = \lambda t$ arrivals during t .

The variance of the poisson distribution also equals λt .

Example

Babies are born in a sparsely populated state at the rate of one birth every 12 minutes. The time between births follows an exponential distribution. Find the following.

- The average number of births per year.
- The probability that no births will occur in any one day.
- The probability of issuing 50 birth certificates by the end of the next 3 hours given that 40 certificates were issued during the last 2 hours.

Soln:

The birth rate per day is computed as

$$\lambda = \frac{24 \times 60}{12} = 120 \text{ births/day}$$

The number at births per year in the state is

$$\begin{aligned} \lambda t &= 120 \times 365 \\ &= 43,800 \text{ births/year.} \end{aligned}$$

The probability of no births in any one day is computed from the poisson distribution as

$$P_0(1) = \frac{(120 \times 1)^0 e^{-120 \times 1}}{0!} = 0$$

Given $\lambda = \frac{60}{20} = 5$ births per hour.

$$P_{10}(1) = \frac{(5 \times 1)^{10} e^{-5 \times 1}}{10!} = 0.01813$$

Steady – State Measures of performance

The most commonly used measures of performance in a queuing situation are

L_s = Expected number of customers in system.

L_q = Expected number of customers in queue.

W_s = Expected waiting time in system

W_q = Expected waiting time in queue

\bar{C} = Expected number of busy service

Single Server models

(M/M/1) : (GD/ ∞ / ∞)

Using the notation of the generalized model, we have

$\lambda_n = \lambda$ and $\mu_s = \mu$ for all $n=0, 1, 2, \dots$,

Also, effective arrival $\lambda_{\text{eff}} = \lambda$ and

$$\lambda_{\text{lost}} = 0$$

Define $\rho = \frac{\lambda}{\mu}$

The expression for P_n

$$P_n = \rho^n P_0 \quad n=0, 1, 2..$$

We use the identify,

$$P_0 = (1 + \rho + \rho^2 + \dots) = 1$$

Assuming $\rho < 1$, the geometric series will have the finite sum $(\frac{p}{1-\rho})$ which yield $y_0 = 1 - \rho$.

$$P_n = (1 - \rho) \rho^n, \quad n=1, 2, \dots, (P < 1)$$

The measure of performance L_s can be derived in the following manner

$$\begin{aligned} L_s &= \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n (1 - \rho) \rho^n \\ &= (1 - \rho) \rho \frac{d}{d\rho} \sum_{n=0}^{\infty} \rho^n \\ &= (1 - \rho) \rho \frac{d}{d\rho} \frac{1}{1 - \rho} \\ &= \frac{\rho}{1 - \rho} \end{aligned}$$

$$W_s = \frac{L_s}{\lambda} = \frac{1}{\mu - (1 - \rho)}$$

$$W_q = W_s - \frac{1}{\mu} = \frac{\rho}{\mu - (1 - \rho)}$$

$$L_q = \lambda W_q = \frac{\rho^2}{1 - \rho}$$

$$\bar{C} = L_s - L_q = \rho$$

(M/M/1) (GD/N/∞)

This model differs from (M/M/1): (GD/∞/∞) in that there is a limit N or the number in the system (maximum queue length = N-1)

Once the number of customers in the system reaches N, no more arrivals are allowed.

$$\lambda_n = \begin{cases} \lambda & n=0, 1, \dots, N-1 \\ 0 & n=N, N+1, \dots \end{cases}$$

$$\mu_n = \mu, n=0, 1, \dots$$

Using $\rho = \frac{\lambda}{\mu}$ we get

$$P_n = \begin{cases} \rho^n P_0 & n \leq N \\ 0 & n > N \end{cases}$$

The value of P_0 is determined from the equation

$$\sum_{n=0}^{\infty} P_n = 1$$

$$P_0 (1 + \rho + \rho^2 + \dots + \rho^N) = 1$$

$$P_0 = \frac{1 - \rho}{1 - \rho^{N+1}}, \rho \neq 1$$

$$\frac{1}{N+1}, \rho = 1$$

Thus,

$$P_n = \frac{(1 - \rho)\rho^n}{1 - \rho^{N+1}}, \rho \neq 1 \quad n=0, 1, \dots, N$$

$$\frac{1}{N+1}, \rho = 1$$

$$\lambda_{\text{lost}} = \lambda P_N$$

$$\lambda_{\text{eff}} = \lambda - \lambda_{\text{lost}} = \lambda (1 - P_N)$$

The expected numbers of customers in the system is computed as

$$\begin{aligned} L_s &= \sum_{n=0}^N n P_n \\ &= \frac{1 - \rho}{1 - \rho^{N+1}} \sum_{n=0}^N n \rho^n \\ &= \frac{(1 - \rho)\rho}{1 - \rho^{N+1}} \frac{d}{d\rho} \left(\frac{1 - \rho^{N+1}}{1 - \rho} \right) \\ &= \frac{\rho \{1 - (N+1)\rho^N + N\rho^{N+1}\}}{(1 - \rho)(1 - \rho^{N+1})}, \rho \neq 1 \end{aligned}$$

$$\text{When } \rho=1, L_s = \frac{N}{2}$$