M.Sc. MATHEMATICS - I YEAR

DKM12 : REAL ANALYSIS

SYLLABUS

Unit I :

Basic topology – Metric spaces – compact sets – perfect sets – connected sets - convergent sequences – subsequences – upper and lower limits – some special sequences. [Chapter 2-2.1 to 2.45, chapter 3-3.1 to 3.20]

Unit II :

Series – Series of non-negative terms – The number e – The root and ratio tests – Power series – summation by parts – Absolute convergence – Addition and multiplication of series. [Chapter 3-3.21 to 3.50]

Unit III :

Continuity and Differentiation - Limit of functions – Continuous functions – Continuity and compactness – Continuity and connectedness – Monotonic functions – Infinite limits and limits at infinity – Differentiation – Mean value theorems – Continuity of Derivatives – L'Hospital rule – Taylor's theorem. [Chapter 4-4.1 to 4.34 & Chapter 5.1 to 5.15]

Unit IV :

The Riemann–Steiltjes integral and Sequences and series of functions – Existence of the integral – Properties of the integral – Integration and Differentiation – Integratin of vector–valued functions - Uniform convergence – Uniform convergence and continuity – Uniform convergence and intergration. [Chapter 6-6.1 to 6.25 & Chapter 7-7.1 to 7.16]

Unit V:

Uniform Convergence and differentiation – Equicontinuity – Equicontinuous families of functions – Stone Weierstrass' theorem – some special functions. [Chapter 7-7.17 to 7.26 & Chapter 8.1 to 8.6]

Text :

Rudin - Principles of Mathematical Analysis (Tata McGrows Hill) Third Edition, Chapters 2 to 8.

1. UNIT I

Basic Topology

Definition 1.1 ***Metric space:*** *A set $X(\neq \emptyset)$ whose elements we shall called points is said to be a metric space if with any two points p, q of X there is associated a real number $d(p,q)$, called the distance from p to q, such that*

1. $d(p,q) > 0$ *if* $p \neq q$,
2. $d(p,q) = d(q,p) \quad \forall p, q \in X$,
3. $d(p,q) \leq d(p,r) + d(r,p) \quad \forall p, q, r \in X$ *(Triangle inequality),*
4. $d(p,q) = 0$ *if* $p = q$.

Note 1.2 *Any function with these three properties is called a distance function (or) metric.*

Example 1.3 *1.* $\mathbb{R}^1$ *with usual metric* $d(x,y) = |x-y|$ *is a metric space.*
2. The euclidean space $\mathbb{R}^k = \{(x_1, x_2, ..., x_k) = \bar{x} | x_i \in \mathbb{R}^1\}$ *with usual metric*

$$d(\bar{x}, \bar{y}) = |\bar{x} - \bar{y}| = \sqrt{\sum_{i=1}^{k} (x_i - y_i)^2}, \bar{x}, \bar{y} \in \mathbb{R}^k$$

Note 1.4 *Usually a non-empty set X with a metric d denoted by (X, d) is called as metric space.*

Remark 1.5 *Every subset Y of a metric space X is a metric space (with the same metric of) in its own right. For if conditions 1, to 4, of the Definition 1.1 hold for $p, q, r \in X$, then they also hold if you restrict p, q, r to lie in Y.*

Definition 1.6
1. $(a,b) = \{x | a < x < b\}-$ *segment.*
2. $[a,b] = \{x | a \leq x \leq b\}$- *interval.*
3. $(a,b] = \{x | a < x \leq b\}$- *Half open interval.*
4. $[a,b) = \{x | a \leq x < b\}$- *Half open interval.*

Definition 1.7 ***k-cell:*** *If $a_i < b_i$ $i = 1, 2, ..., k$ then $\{\bar{x} = (x_1, ..., x_2) | a \leq x_i \leq b_i, i = 1, 2, ..., k\}$ is called a k-cell.*

Note 1.8 *One-cell is a interval. Two cell is a rectangle. Three cell is cuboid.*

Definition 1.9 ***Convex Set:*** *A set E subset of $\mathbb{R}^k$ is convex if $\lambda\bar{x} + (1 - \lambda)\bar{y} \in E$ whenever $\bar{x}, \bar{y} \in E$ and $0 < \lambda < 1$.*

Definition 1.10 ***Open ball:*** *If $\bar{x} \in \mathbb{R}^k, r > 0$, the open ball or (closed ball) B with center at $\bar{x}$ and radius r is defined to be the set $\{\bar{y} \in \mathbb{R}^k | \, |\bar{x} - \bar{y}| < r\}$ or $\{\bar{y} \in \mathbb{R}^k | \, |\bar{x} - \bar{y}| \leq r\}$.*

$$\text{i.e., open ball } B(\bar{x}, r) = \{\bar{y} \in \mathbb{R}^k | \, |\bar{x} - \bar{y}| < r\}$$
$$\text{closed ball } B[\bar{x}, r] = \{\bar{y} \in \mathbb{R}^k | \, |\bar{x} - \bar{y}| \leq r\}$$

Lemma 1.11 *Balls are convex.*
Proof: Let $B(\bar{x}, r)$ be a open ball and let $\bar{y}, \bar{z}$ lie in a open ball B.
$\Rightarrow |\bar{y} - \bar{x}| < r$ and $|\bar{z} - \bar{x}| < r$

$$\begin{aligned}
0 \leq \lambda \leq 1 \Rightarrow 0 \leq 1 - \lambda \Rightarrow & |\lambda\bar{y} + (1 - \lambda)\bar{z} - \bar{x}| \\
&= |\lambda\bar{y} + (1 - \lambda)\bar{z} - (\lambda\bar{x} + (1 - \lambda)\bar{x})| \\
&= |\lambda(\bar{y} - \bar{x}) + (1 - \lambda)(\bar{z} - \bar{x})| \\
&\leq \lambda \, |\bar{y} - \bar{x}| + (1 - \lambda) \, |\bar{z} - \bar{x}| \\
&< \lambda r + (1 - \lambda) r = r \\
&\Rightarrow \lambda \, |\bar{y} + (1 - x)\bar{z} - \bar{x}| < r \\
&\Rightarrow \lambda\bar{y} + (1 - \lambda)\bar{z} \text{ lies in the open ball } B.
\end{aligned}$$

$\Rightarrow$ Every open ball is convex. Similarly every closed ball is convex.

Note 1.12 *Every k-cell is convex.*

Definition 1.13 ***Neighbourhood of a point:*** *Let X be a metric space. The neighbourhood a point p is $=\{q \in X | d(p, q) < r\}$ and is denoted by $N_r(p)$.*

Note 1.14 $N_r(p) = (p - r, p + r)$ *in* $\mathbb{R}$.

Definition 1.15 ***Limit point:*** *Let $p \in X$ and $E \subset X$. The point p is said to be the limit point of E, if every neighbourhood of p contains a point q of E other than p.*

Note 1.16 *p is a limit point of E.* $\Rightarrow N_r(p) \cap E - \{p\} \neq \emptyset \; \forall r > 0$.

Example 1.17 $A = \{0, 1, 1/2, ...\}$; $N_r(0) = (-r, r) \; \forall r > 0$. *By Archimedian principle $\forall r > 0$ there exists an +ve integer n such that $n \cdot r > 1$*

$$\begin{aligned}
&\Rightarrow r > 1/n \\
&\Rightarrow r > 1/n \\
&\Rightarrow 0 < 1/n < r \\
&\Rightarrow 1/n \in (-r, r) \\
&\Rightarrow (A - \{0\}) \cap (-r, r) \neq \emptyset \\
&\Rightarrow (A - \{0\}) \cap N_r(0) \neq \emptyset \; \forall r > 0 \\
&\Rightarrow 0 \textit{ is a limit point of } A.
\end{aligned}$$

Clam: 1 *is not a limit point. Consider* $N_{1/4}(1) = (1 - 1/4, 1 + 1/4) = (3/4, 5/4)$. $\therefore (3/4, 5/4) \cap (A - \{0\}) = \emptyset$ *(i.e.),* $N_{1/4}(1) \cap (A - \{1\}) = \emptyset \Rightarrow 1$ *is not a limit point of* A. *Similarly we can prove that* $1/n$ *is not a limit point* $\forall n \in N$. *Hence* 0 *is the only limit point of* A.

Definition 1.18 ***Isolated point:*** *Let* X *be a metric space and* E *subset of* X. *If a point* $p \in E$ *is not a limit point of* E. *Then we say that* p *is an isolated point of* E. *In the above example* $1, 1/2, 1/3, ...$ *are the isolated point of* A.

Definition 1.19 ***Closed set:*** *Let* X *be a metric space and* $E \subset X, E$ *is said to be closed in* X, *if every limit point of* E *is a point of* E. *In the previous example* A *is closed in* R *since* $\{0\} \subset A$.

Definition 1.20 ***Interior point:*** *Let* X *be a metric space and* $E \subset C$. *A point* p *is an interior point of* E. *If there exists neighbourhood* $N(p)$ *such that* N *is contained in* E $(N \subset E)$.

Definition 1.21 ***Open set:*** *Let* X *be a metric space and* $E \subset X$. E *is said to be open in* X *if every point of* E *is an interior point of* E.

Note 1.22 *Let* E' *denote the set of all limit points of* E. *Let* E° *denote the set of all interior points of* $E.E^\circ \subseteq E$ *always.* E *is closed if* $E' \subset E$ *and* E *is open if* $E = E^\circ$.

Definition 1.23 ***Perfect set:*** *Let* X *be a metric space and* $E \subset X.E$ *is said to be perfect in* X *if* E *is closed and if every point of* E *is a limit point of* E.

Note 1.24 E *is perfect if* $E = E'$.

Definition 1.25 ***Complement of a set:*** *Complement of a set is defined as* $E^c = \{p \in X | p \notin E\}$.

Definition 1.26 ***Bounded Set:*** *Let* X *be a metric space and* $E \subset X$. E *is said to be bounded in* X *if there exists a real number* M *and a point* $q \in X$ *such that* $d(p, q) < M \ \forall p \in E$.

Definition 1.27 ***Dense Set:*** E *is dense in* X *if every point of* X *is a limit point of* E *or a point of* E *or both. If* E *is dense in* X, *then* $X = \bar{E} = E \cup E'$.

Example 1.28 Q *is dense in* R.

Theorem 1.29 *Every neighbourhood is an open set.*
Proof: Consider a neighbourhood $N_r(p)$ (neighbourhood of p with radius $r > 0$). To prove: $N_r(p)$ open. Let $q \in N_r(p)$. Enough to prove: q is an interior point of N_r. Now $q \in N_r(p) \Rightarrow d(p,q) < r$. Let $S = r - d(p,q)$. Claim: $N_S(q) \subset N_r(p)$

$$\begin{aligned} & r \in N_S(q) \\ & \Rightarrow d(r,q) < S = r - d(p,q) \\ & \Rightarrow d(p,q) + d(r,q) < r \\ & \Rightarrow d(p,r) < r \\ & \Rightarrow r \in N_r(p) \\ & \therefore N_S \subset N_r(p) \end{aligned}$$

Hence the claim. That is an interior pt of $N_r(p)$. Since q is an arbitrary. Every point of $N_r(p)$ is an interior point. $\Rightarrow N_r(p)$ is open. $\therefore$ Every neighbourhood is open.

Theorem 1.30 *If p is a limit point of E. Then every neighbourhood of p contains infinitely many points of E.*
Proof: Suppose there exists a neighbourhood N of p contains only finitely many points of E.
Let $q_1, q_2, ..., q_n$ be those points of E in N differ from p. $\{q_1, q_2, ..., q_n \in (N \cap E - \{p\})$. Let $r = min\{d(p,q_i) | i = 1...n\}$. Clearly, $r > 0$. Now the neighbourhood $N_r(p)$ contains no point q of E. such that $q \neq p$. Then p is not a limit point of E which is a contradiction to p is a limit point of E. $\therefore$ Every neighbourhood of p contains infinitely many points of E.

Corollary 1.31 *Any finite set has no limit point.*
Proof: Let X be a metric space and $E \subset X$ be a finite set. To prove: E has no limit points. If p is limit point of E. Then every neighbourhood of p contains infinitely many points of E.(by above theorem) This is a contradiction to E is a finite set. Hence a finite set has no limit point.

Theorem 1.32 *Let $\{E_\alpha\}$ be a (finite or infinite) collection of set E_α. Then $(\bigcup E_\alpha)^c = \bigcap E_\alpha^c$.*
Proof: Let $x \in (\bigcup E_\alpha)^c$.

$$\begin{aligned} & \Leftrightarrow x \notin \bigcup E_\alpha \\ & \Leftrightarrow x \notin E_\alpha \ \forall \alpha \\ & \Leftrightarrow x \in E_\alpha^c \ \forall \alpha \\ & \Leftrightarrow x \in \bigcap E_\alpha^c \\ \therefore (\bigcup E_\alpha)^c & = \bigcap E_c^\alpha \end{aligned}$$

Theorem 1.33 *A set E is an open iff its complement is closed.*
Proof: Let E be an open set. To prove: E^c is closed. Let q be a limit point of $E^c \Rightarrow$ Every neighbourhood of q contains atleast one point p of E^c such that $p \neq q$. $\Rightarrow q$ is not an interior point of E. ($\because E$ is open) ($\because N_r(q) \cap E^c - \{q\} \neq \emptyset \ \forall r > 0$ (i.e.), $N_r(q) \nsubseteq E \ \forall r > 0) \Rightarrow q \notin E \Rightarrow q \in E^c$. Since q is arbitrary. E^c contains all its limit point. $\therefore E^c$ is closed. Conversely, let E^c be closed. To prove: E is open. Let $q \in E$. To prove: q is an interior point of E. Since $q \in E \Rightarrow q \notin E^c \Rightarrow q$ is not a limit point of E^c. Which implies, there exists neighbourhood of N of q such that $N \cap (E^c - \{q\}) = \emptyset$ (i.e.) $N \cap E^c = \emptyset (\because q \notin E^c) \Rightarrow N \subset E \Rightarrow q$ is an interior point of E. Since q is arbitrary. Every point of E is an interior point of E. $\Rightarrow E$ is open.

Corollary 1.34 *A set F is closed iff its complement is open.*
Proof: $F = (F^c)^c$ is closed. $\Leftrightarrow F^c$ is open. (by previous theorem)

Theorem 1.35 *(a) For any collection $\{G_\alpha\}$ of open sets $\bigcup_\alpha G_\alpha$ is open (or) Arbitrary union of open sets is open.*
(b) For any collection $\{F_\alpha\}$ of closed sets $\bigcap_\alpha F_\alpha$ is closed (or) Arbitrary intersection of closed sets is closed.
(c) For any finite collection $\{G_1, G_2, ..., G_n\}$ of open sets $\bigcap_{i=1}^n$ is open (or) Finite intersection of open sets is open.
(d) For any finite collection $\{F_1, F_2, ..., F_n\}$ of closed sets $\bigcup_{i=1}^n F_i$ is closed (or) Finite union of closed sets is closed.
Proof: (a) To prove: $\bigcup_\alpha G_\alpha$ is open where each G_α is open. Let $p \in \bigcup_\alpha G_\alpha \Rightarrow p \in G_\alpha$ for some $\alpha \Rightarrow$ there exists a neighbourhood N of p such that $N \subset G_\alpha$ ($\because G_\alpha$ is open) $\Rightarrow N \subset G_\alpha \subset \bigcup_\alpha G_\alpha \Rightarrow N \subset \bigcup_\alpha G_\alpha \Rightarrow p$ is an interior point of $\bigcup_\alpha G_\alpha$. Since p is arbitrary, every point of $\bigcup_\alpha G_\alpha$ is an interior point. $\Rightarrow \bigcup_\alpha G_\alpha$ is open.
(b) To prove: $\bigcap_\alpha F_\alpha$ is closed where each F_α is closed $\forall \alpha$. (i.e.) To prove $(\bigcap_\alpha F_\alpha)^c$ is open. $(\bigcap_\alpha F_\alpha)^c = \bigcup_\alpha F_\alpha^c$. F_α is closed $\Rightarrow F_\alpha^c$ is open. By (a) $\bigcup_\alpha F_\alpha^c$ is open. $\Rightarrow (\bigcap_\alpha F_\alpha)^c$ is open. $\Rightarrow \bigcap_\alpha F_\alpha$ is closed.
(c) To prove: $\bigcap_{i=1}^n G_i$ is open when G_i is open $\forall i = 1, ..., n$. Let $x \in \bigcap_{i=1}^n G_i \Rightarrow x \in G_i \ \forall i = 1$ to n. For each i, there exists a neighbourhood $N_{r_i}(x)$ such that $N_{r_i}(x) \subset G_i \ \forall i = 1, 2, ..., n (\because G_i$ is open). Let $r = min\{r_1, r_2, ..., r_n\} \Rightarrow N_r(x) \subset N_{ri}(x) \ \forall i \Rightarrow N_r(x) \subset G_i \ \forall i \Rightarrow N_r(x) \subset \bigcap_{i=1}^n G_i \Rightarrow x$ is an interior point of $\bigcap_{i=1}^n G_i$. Since x is arbitrary, every point of $\bigcap_{i=1}^n G_i$ is an interior point. $\therefore \bigcap_{i=1}^n G_i$ is open.
(d) To prove: $\bigcup_{i=1}^n F_i$ is closed when F_i is closed $\forall i$. (i.e.) To prove $(\bigcup_{i=1}^n F_i)^c$ is open. $(\bigcup_{i=1}^n F_i)^c = \bigcup_{i=1}^n F_i^c$. Now, $\forall i F_i$ is closed $\Rightarrow F_i^c$ is open. By (c), $\bigcap_{i=1}^n F_i^c$ is open. $\Rightarrow (\bigcup_{i=1}^n F_i)^c$ is open. $\Rightarrow \bigcup_{i=1}^n F_i$ is closed.

Note 1.36 *Arbitrary intersection of open sets need not be open.*

Example 1.37 *Consider $G_n = (-1/n, 1/n)$ in R with usual metric. $\Rightarrow G_n$ is open $\forall n$. Now, $\bigcap_{n=1}^\infty G_n = \bigcap_{n=1}^\infty (-1/n, 1/n) = \{0\}$ is not open.*

Result 1.38 *Arbitrary Union of closed sets need not be closed.*
Proof: Consider $F_n = (-\alpha, -1/n) \cup (1/n, \alpha)\ \forall n$. (i.e.) $F_n^c = (-1/n, 1/n)\ \forall n$ $\Rightarrow$ F_n^c is open $\Rightarrow$ F_n is closed $\forall n$. Now, $(\bigcup_{n=1}^{\infty} F_n)^c = \bigcap_{n=1}^{\infty} F_n^c = \bigcap_{n=1}^{\infty}(-1/n, 1/n) = \{0\}$ is not open in R. $\Rightarrow (\bigcup F_n)^c$ is not open in R. $\Rightarrow$ $\bigcup F_n$ is not closed in R.

Definition 1.39 *If X is a metric space and $E \subset X$ and if E' denotes the set of all limit points of E in X. Then the closure of E is the set $\bar{E} = E \cup E'$.*

Theorem 1.40 *If X is a metric space and $E \subset X$. Then*

1. *$\bar{E}$ is closed.*
2. *$E = \bar{E}$ iff E is closed.*
3. *$\bar{E} \subset F_\alpha \forall$ closed set $F_\alpha \subset X$ such that $E \subset F_\alpha$.*

Proof: **(1)** To prove:$\bar{E}$ is closed. (i.e.) To prove $\bar{E}^c$ is open. Let $p \in \bar{E}^c$
$\Rightarrow p \in E^c \cap E'^c \Rightarrow p \in E^c$ and $p \in E'^c$ $(\because \bar{E} = E \cup E' \bar{E}^c = E^c \cap (E')^c)$
$\Rightarrow p \notin E$ and $p \notin E' \Rightarrow p \notin E$ and p is not a limit point of E
$\Rightarrow$ there exists a neighbourhood N of p such that $N \cap (E - \{p\}) = \emptyset$ and $p \notin E$
$\Rightarrow N \cap E = \emptyset$ ......(1)
$\Rightarrow$ every point of N is not a limit point of E ($\because N$ is open) $\Rightarrow N \subset E'^c$.
From (1), $N \subset E^c \Rightarrow N \subset \bar{E}^c \cap E^c = (E \cup E')^c = \bar{E}^c \Rightarrow N \subset \bar{E}^c$
$\Rightarrow p$ is an interior point of $\bar{E}^c \Rightarrow$ Since p is an arbitrary. $\therefore$ Every point of $\bar{E}^c$ is an interior point. $\Rightarrow \bar{E}^c$ is open. $\Rightarrow \bar{E}$ is closed.
(2) E is closed. $\Rightarrow E' \subset E \Rightarrow E \cup E' \subset E \Rightarrow \bar{E} \subset E$. But $E \subset \bar{E}$ always. $\therefore E = \bar{E}$. Conversely, $E = \bar{E} = E \cup E' \Rightarrow E' \subset E \Rightarrow E$ is closed.
(3) Let $p \in \bar{E} \Rightarrow p \in E \cup E' \Rightarrow p \in E$ or $p \in E'$. If $p \in E$ then $p \in F[\because E \subset F]$ Let $p \in E' \Rightarrow p$ is a limit point of $E \Rightarrow$ Every neighbourhood of p contains atleast one point $q \in E$ such that $q \neq p \Rightarrow$ Every neighbourhood of p contains atleast one point $q \in F$ such that $q \neq p[\because E \subset F] \Rightarrow p$ is a limit point of $F \Rightarrow p \in F(\because F$ is closed$) \Rightarrow \bar{E} \subset F$.

Theorem 1.41 *Let E be a non-empty set of real numbers, which is bounded above. Let $y = \sup E$ then $y \in \bar{E}$. Hence $y \in E$ if E is closed.*
Proof: Let $y = \sup E$. By the definition of sup$\forall$ real $h > 0$ there exists $X \in E$ such that $y - h < x < y \Rightarrow y - h < x < y + h\ \forall\ h > 0$ and $x \in E \Rightarrow N_h(y) \cap E - \{y\} \neq \emptyset\ \forall h > 0 \Rightarrow y$ is a limit point of $E \Rightarrow y \in E' \subset \bar{E} \Rightarrow y \in \bar{E}$. If E is closed then $E = \bar{E}$. Hence $y \in E$ if E is closed.

Note 1.42 *Let X be a metric space and $Y \subset X$. Then Y itself is a metric space under the same metric in X.*

Definition 1.43 ***Open relative:*** *Suppose $E \subset Y \subset X$ and E is open relative to Y if $\forall p \in E$ there exists $r_p > 0$ such that $d(p, q) < r_p, q \in Y \Rightarrow q \in E$.*

Note 1.44 $N_{r_p}(p) \cap Y \subset E$.

Example 1.45 $(a, b) \subset R \subset R \times R$. *Here segment* (a, b) *is open in* R *but not open in* $R \times R$.

Theorem 1.46 *Suppose* $Y \subset X$, *a subset* E *of* Y *is open relative to* Y *iff* $E = Y \cap G$ *for some open subset* G *of* X.
Proof: Suppose E is open relative to Y. Then $\forall p \in E$ there exists $r_p > 0$ such that $d(p, q) < r_p, q \in Y \Rightarrow q \in E$....... (1)
Let $V_p = \{q \in X | d(p, q) < r_p\} \Rightarrow V_p$ is neighbourhood in $X \Rightarrow V_p$ is open in X. Let $G = \bigcup_{p \in E} V_p \Rightarrow G$ is open in X {Arbitrarty $\bigcup$ of open set is open}.
Claim: $E = Y \cap G$. Let $p \in E \Rightarrow p \in V_p$ ($\because V_p$ is neighbourhood of p) and $p \in V$ ($\because E \subset Y$) $\Rightarrow p \in V_p \subset \bigcup V_p = G$ and
$p \in Y \Rightarrow p \in G \cap Y \Rightarrow E \subset G \cap Y$....... (2)
Let $q \in Y \cap G \Rightarrow q \in G$ and $q \in Y \Rightarrow q \in \bigcup_{p \in E} V_p$ and $q \in Y \Rightarrow q \in V_p$ for some $p \in E$ and $q \in Y \Rightarrow d(p, q) < r_p$ and $q \in Y$ for some $p \Rightarrow q \in E$ (by (1)) $\Rightarrow Y \cap G \subset E$......(3)
By (2) and (3), $E = y \cap G$. Conversely, suppose $E = G \cap Y$ for some open set G in X. To prove: $E \subset Y$ is open relative to Y. Let $p \in \bar{E} \Rightarrow p \in G \cap Y$ for some open set G in X. $\Rightarrow p \in Y$ and $p \in G \Rightarrow p \in Y$ and $V_p \subset G$ where V_p is a neighbourhood of p in $X \Rightarrow Y \cap V_p \subset Y \cap G = E \Rightarrow E$ is open relative to Y.

Compact Set:

Definition 1.47 *Let* X *be a metric space. By an open cover of a set* E *in* X *we mean a collection* $\{G_\alpha\}$ *of open sets in* X *such that*

$$E \subset \bigcup_{\alpha} G_\alpha.$$

Example 1.48 *Consider the collection,* $I = \{(-n, n) | n \in N\}$ *is a family of open sets in* R *clearly* I *is an open cover for* R.

Definition 1.49 *A subset* K *of metric space* X *is said to be compact, if every open cover of* K *contains a finite subcover (or) A set* K *is compact in* X *and*

$$K \subset \bigcup_{\alpha} G_\alpha \cdot G_\alpha$$

is open in X, *which implies, there exists* $\alpha_1, \alpha_2, ..., \alpha_n$ *such that*

$$K \subset \bigcup_{i=1}^{n} G_{\alpha_i}.$$

Result 1.50 *Let* X *be a metric space. Let* $A = \{X_1, X_2, ..., X_n\}$ *be a finite set in* X. *Clearly* A *is compact.*

Theorem 1.51 *Suppose $K \subset Y \subset X$. Then, K is compact relative to X iff K is compact relative to Y.*
Proof: Suppose K is compact relative to X. To prove: K is compact relative to Y. Let $\{V_\alpha\}$ be collection of open set in Y and $K \subset \bigcup_\alpha V_\alpha$. Now V_α is open in $Y \Rightarrow$ there exists an open set G_α in X such that $V_\alpha = G_\alpha \cap Y \ \forall\alpha$. Now $K \subset \bigcup_\alpha V_\alpha \Rightarrow K \subset \bigcup_\alpha (G_\alpha \cap Y) \Rightarrow K \subset (\bigcup_\alpha G_\alpha) \cap Y \Rightarrow K \subset \bigcup_\alpha G_\alpha$. G_α is open in X. Since K is compact relation to X, there exists $\alpha_1, \alpha_2, ..., \alpha_n$ such that $K \subset \bigcup_{i=1}^n G_{\alpha_i}$. Now $K \cap Y \subset (\bigcup_{i=1}^n G_{\alpha_i}) \cap Y \Rightarrow K \subset \bigcup_{i=1}^n (G_{\alpha_i} \cap Y) \Rightarrow K \subset \bigcup_{i=1}^n V_{\alpha_i} \Rightarrow K$ is compact relative to Y. Conversely, suppose K is compact relative to Y. To prove: K is compact relative to X. Let $\{G_\alpha\}$ be collection of open set in X. Now, $K \subset \bigcup_\alpha G_\alpha \Rightarrow K \cap Y \subset (\bigcup_\alpha G_\alpha) \cap Y \Rightarrow K \subset \bigcup_\alpha (G_\alpha \cap Y)$ where $V_\alpha = G_\alpha \cap Y \Rightarrow K \subset \bigcup_\alpha V_\alpha$ [V_α is open in Y]. Since K is compact relative to Y, there exists $\alpha_1, \alpha_2, ..., \alpha_n$ such that $K \subset \bigcup_{i=1}^n V_{\alpha_i} = \bigcup_{i=1}^n (G_{\alpha_i} \cap Y)$ (i.e.) $K \subset \bigcup_{i=1}^n G_{\alpha_i} \cap Y \Rightarrow K \subset \bigcup_{i=1}^n G_{\alpha_i} \Rightarrow K$ is compact relative to X.

Theorem 1.52 *Compact subsets of a metric are closed.*
Proof: Let K be a compact subset of a metric X. To prove: K is closed, it is enough to prove that K^c is open. If $q \in K$. Let V_q and W_q be neighbourhood of p and q respectively of radius less than $d(p,q)/2 \Rightarrow V_q \cap W_q = \emptyset \ \forall q \in K$. $\{W_q | q \in K\}$ is an open cover for K. Since K is compact there exist $q_1, q_2, ..., q_n \in K$ such that $K \subset \bigcup_{i=1}^n W_{q_i}$. Let $W = \bigcup_{i=1}^n W_{q_i}$ and $V = V_{q_1} \cup V_{q_2} ... \cup V_{q_n}$. Clearly, V is a neighbourhood of p. Also $V \cap W = \emptyset \Rightarrow V \subset W^c \subset K^c \Rightarrow V \subset K^c \Rightarrow p$ is an interior point of $K^c \Rightarrow K^c$ is open $\{\because p$ is arbitrary$\} \Rightarrow K$ is closed.

Theorem 1.53 *Closed subset of a compact sets are compact.*
Proof: Suppose $F \subset K \subset X$, where F is closed with respect to X and K is compact. To prove: F is compact. Let $\{V_\alpha\}$ be an open cover for F. Now F is closed $\Rightarrow F^c$ is open. Let $\Omega = \{V_\alpha\} \cup \{F^c\}$. Now, Ω is an open cover for K. As K is compact, there exists an finite subcover ϕ of Ω such that ϕ covers $K \Rightarrow \phi$ covers F ($\because F \subset K$). If $F^c \in \phi$ then $\phi - \{F^c\}$ covers F. $\therefore F$ is compact.

Corollary 1.54 *F is closed and K is compact. Then $F \cap K$ is compact.*
Proof: Since K is compact subset of a metric space $\Rightarrow K$ is closed. [by Theorem 1.52] $\Rightarrow K \cap F$ is closed. [$\because F$ is closed] Now $F \cap K \subset K \Rightarrow F \cap K$ is compact, by Theorem 1.53

Theorem 1.55 *If $\{K_\alpha\}$ is a collection of compact subset of a metric set X, such that the intersection of every finite subcollection of K_α is non-empty , then $\bigcap K_\alpha$ is non-empty.*
Proof: Fix a member K_1 of $\{K_\alpha\}$ and put $G_\alpha = K_\alpha^c$. Assume that no point of K_1 belongs to every K_α (i.e.) $K_1 \cap (\bigcap_\alpha K_\alpha) = \emptyset \Rightarrow K_1 \subset (\bigcap K_\alpha)^c = \bigcup_\alpha K_\alpha^c = \bigcup_\alpha G_\alpha \Rightarrow K_1 \subset \bigcup_\alpha G_\alpha$. Since $\{G_\alpha\}$ is an open cover for K_1 and K_1

is compact, there exists $\alpha_1, ..., \alpha_n$ such that $K_1 \subset \bigcup_{i=1}^{n} G_{\alpha_i} = (\bigcup_{i=1}^{n} K_{\alpha_i}^c) = (\bigcap_{i=1}^{n} K_{\alpha_i})^c \Rightarrow K_1 \cap (\bigcap_{i=1}^{n} K_{\alpha_i}) = \emptyset$. This is a contradiction to the above hypothesis. $\therefore$ Our assumption is wrong. $\therefore$ We have $\bigcap_{\alpha} K_{\alpha} \neq \emptyset$.

Corollary 1.56 *$\{K_n\}$ is a sequences of non-empty compact set such that $K_n \supset K_{n+1} (n = 1, 2, ...)$ then $\bigcap_{n=1}^{\infty} K_n$ is non-empty.*
Proof: Since $K_n \supset K_{n+1}$ $\forall n$. We have every finite intersection of K_n is non-empty. $\therefore$ by above theorem $\bigcap_{n=1}^{\infty} K_n$ is non-empty.

Theorem 1.57 Bolzono weistras theorem: *If E is a finite subset of a compact set k. Then E has a limit point in K.*
Proof: Suppose no point of k is a limit point of E. Then for each $q \in k$ there exists a neighbourhood V_q of q such that V_q contains atmost one point of E namely, q if $q \in E$. Let $\{V_q | q \in k\}$ be an open cover for k. Clearly, no finite subcollection of $\{V_q\}$ covers E and same is true for k. [Since $E \subset k$] This is a contradiction to the fact that k is compact. $\therefore$ Our assumption is wrong. $\therefore$ E has a limit point in k.

Theorem 1.58 *If $\{I_n\}$ is a sequence of intervals in R such that $I_n \supset I_{n+1}$ $n = 1, 2, ...$ Then $\bigcap_{n=1}^{\infty} I_n$ is non-empty.*
Proof: Let $I_n = [a_n, b_n]$ $n = 1, 2, ...$ Let $E = \{a_n / n \in N\} \Rightarrow E$ is bounded above by b_1 Let x be the least upper bound of E. (i.e.) $x = \sup\ E$. If m and n are positive integers, then $a_n \leq a_{m+n} \leq x \leq b_{m+n} \leq b_m \forall m \Rightarrow x \leq b_m$ $\forall m$ and $a_m \leq x\ \ \leq m \Rightarrow a_m \leq x \leq b_m$ $\forall m \Rightarrow x \in [a_m, b_m]$ $\forall m \Rightarrow x \in I_m$ $\forall m \Rightarrow x \in \bigcap_{n=1}^{\infty} I_n \therefore x \in \bigcap_{n=1}^{\infty}$ is non-empty.

Theorem 1.59 *Let k be a the integer $\{I_n\}$ is a sequence of k cells such that $I_n \supset I_{n+1} \supset I_{n+2} ...$ Then $x \in \bigcap_{n=1}^{\infty} I_n \neq \phi$.*
Proof: Given $I_n = \{\bar{x} = (x_1, x_2, ..., x_n) \in \mathbb{R}^k | a_{n,j} \leq x_j \leq b_{n,j}, j = 1, 2, ..., k$ and $n = 1, 2, ...\}$. Given $I_n \supset I_{n+1} \supset I_{n+2} ...$ Let $I_{n,j} = [a_{n,j}, b_{n,j}]$ $1 \leq j \leq k$ and $n = 1, 2, ...$ For each $j, \{I_{n,j}\}$ is a sequence of intervals such that $I_{n,j} \supset I_{n+1,j}$ $n = 1, 2, 3, 4... \Rightarrow \bigcap_{n=1}^{\infty} I_{n,j} \neq \emptyset$ for each j (By Theorem 1.58). Let $x_j \in \bigcap_{n=1}^{\infty} I_{n,j}$ for each $j = 1$ to $k \Rightarrow$ for each $j,\ x_j \in I_{n,j}$ $\forall n = 1, , 2, ...$ Let $\bar{x} = \{x_1, x_2, ..., x_k\} \in I_n$ $\forall n = 1, 2, ... \Rightarrow \bar{x} \in \bigcap_{n=1}^{\infty} I_n \Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem 1.60 *Every k-cell is compact.*
Proof: $I = \{\bar{x} = \{x_1, x_2, ..., x_k \in \mathbb{R}^k | a_i \leq x_i \leq b_i\}$, put $S = [\sum_{i=1}^{k} (b_i - a_i)^2]^{\frac{1}{2}}$. Now, for each $\bar{x}, \bar{y} \in I$, $|\bar{x} - \bar{y}| \leq S$. To prove: I is compact. Suppose I is not compact. $\Rightarrow$ There exists an open cover $\{G_\alpha\}$ of I such that it has no finite subcover for I. Put $c_j = \frac{a_j + b_j}{2}$. The intervals $[a_j, b_j]$ and $[c_j, b_j]$. Then determine 2^k ,k-cells Q_i such that $I = \bigcup_{i=1}^{2^k} Q_i$. Then atleast one of these cells Q_i, say I_1 cannot be covered by any finite subcollection of G_α. Proceeding like this we have
(a) $I \supset I_1 \supset I_2 \supset ...$

(b) Each I_n is not covered by any finite subcollection of $\{G_\alpha\}$ and
(c) $\bar{x}, \bar{y} \in I_n, |\bar{x} - \bar{y}| \leq \frac{\delta}{2^n}$
by (a)$\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1} \supset I_{n+2}...$, $n = 1, 2, ... \Rightarrow \bigcap_{n=1}^{\infty} I_n \neq \emptyset$ for each j (By Theorem 1.58) $\Rightarrow \bar{x} \in \bigcap_{n=1}^{\infty} I_n \Rightarrow \bar{x} \in I_n \; \forall n = 1, 2, ... \Rightarrow \bar{x} \in G_\alpha$ for some α $[\because I_n \subset I \subset \bigcup_\alpha G_\alpha] \Rightarrow$ There exists a neighbourhood $N_r(\bar{x})$ such that $N_r(\bar{x}) \subset G_\alpha [\because G_\alpha$ is open$] \Rightarrow \{\bar{y} | \, |\bar{x} - \bar{y}| < r\} \subset G_\alpha$..... (1)
Since $r > 0, \delta > 0$. There exists a positive integer n such that $n \cdot r > \delta$ (by Archimedian principle) $\Rightarrow 2^n \cdot r > n \cdot r > \delta \Rightarrow 2^n \cdot r > \delta \Rightarrow r > S \cdot 2^{-n} \Rightarrow r > \frac{\delta}{2^n}$.... (2)
Let $\bar{y} \in I_n \Rightarrow |\bar{x} - \bar{y}| < \frac{\delta}{2^n} [\because \bar{x} \in I_n \; \forall n] \Rightarrow |\bar{x} - \bar{y}| < r \Rightarrow \bar{y} \in N_r(\bar{x}) \Rightarrow I_n \subset N_r(\bar{x}) \subset G_\alpha \Rightarrow\Leftarrow$ (b). $\therefore$ Our assumption is wrong. $\therefore$ Every k-cell is compact.

Theorem 1.61 *A set in $\mathbb{R}^k$ has one of the following three properties then it has the other two.*
(a) E is closed and bounded.
(b) E is compact.
(c) Every infinite subset of E has a limit point in E.
Proof: $(a) \Rightarrow (b)$ Assume that E is closed and bounded. To prove: E is compact. Since E is bounded, $E \subset I$ for some k-cell I. By the above theorem I is compact. $\therefore E$ is a closed subset of compact set I. $\Rightarrow E$ is compact.
$(b) \Rightarrow (c)$ The proof is obvious from, Theorem 1.57.
$(c) \Rightarrow (a)$ Suppose every infinite subset of E has a limit point in E. To prove E is closed and bounded. Suppose E is not bounded. $\Rightarrow$ There exists $\bar{x}_n \in E$ such that $|\bar{x}_n| > n$ $(n = 1, 2, ...)$. Let $S = \{\bar{x}_n | \, |\bar{x}_n| > n, \; n = 1, 2, ...\}$......(*)
Clearly, S is a infinite subset of E and S has no limit points in $\mathbb{R}^k$. Which implies, S has no limit points in E $[\because E \subset \mathbb{R}^k]$ (Suppose $\bar{x}$ is a limit point of S. Then $N_r(\bar{x})$ contains infinitely many points of S $\forall \bar{y} \in S$. Now, $||\bar{y}| - |\bar{x}|| < |\bar{y} - \bar{x}| < r \Rightarrow |\bar{y}| < |\bar{x}| + r < m$ for some integer $m \Rightarrow |\bar{y}| < m$ for integer $\bar{y}$ in S. There exists $n > m$ such that $\bar{y} = \bar{x}_n \in S$ and $|\bar{x}_n| < m \Rightarrow |\bar{x}_n| < m < n \Rightarrow |\bar{x}_n| < n, \bar{x}_n \in S \Rightarrow\Leftarrow$ to (*)) $\therefore E$ is bounded. Suppose E is not closed. There exists a point $\bar{x}_0$ in $\mathbb{R}^k$ such that $\bar{x}_0$ a limit point of E, but $\bar{x}_0 \notin E \Rightarrow$ Every neighbourhood of $\bar{x}_0$ contains a point $\bar{y}$ of E such that $\bar{y} \neq \bar{x}_0$ (i.e.) For $n = 1, 2, ..., N_{\frac{1}{n}}(\bar{x}_0)$ contains a point $\bar{x}_n$ of E, $\bar{x}_n \neq \bar{x}_0$. Let $S = \{\bar{x}_n | |\bar{x}_n - \bar{x}_0| < \frac{1}{n} \; n = 1, 2, ...\}$. $\therefore S$ is infinite. [otherwise $|\bar{x}_n - \bar{x}_0|$ would have a constant positive value for infinitely many n] Also $\bar{x}_0$ is the only limit point of S. Suppose there is a point $\bar{y} \in \mathbb{R}^k$ such that $\bar{y} \neq \bar{x}_0$ and

$\bar{y}$ is a limit point of S. Consider

$$\begin{aligned} |\bar{y}-\bar{x}_0| &= |\bar{y}-\bar{x}_n+\bar{x}_n-\bar{x}_0| \\ &\leq |\bar{y}-\bar{x}_n|+|\bar{x}_n-\bar{x}_0| \\ -|\bar{y}-\bar{x}_0| &\geq -|\bar{y}-\bar{x}_n|-|\bar{x}_n-\bar{x}_0| \\ \Rightarrow |\bar{x}_n-\bar{y}| &\geq |\bar{y}-\bar{x}_0|-|x_n-x_0| \\ &> |\bar{y}-\bar{x}_0|-\frac{1}{n}.......(1) \end{aligned}$$

Now as $|\bar{x}_0-\bar{y}|>0$ and $2\in\mathbb{Z}^+$ such that there exists an positive integer m such that $m|\bar{x}_0-\bar{y}|>2$ [By Archimedian principle]

$$\begin{aligned} &\Rightarrow n|\bar{x}_0-\bar{y}|>2\ \forall n\geq m \\ &\Rightarrow \frac{1}{2}|\bar{x}_0-\bar{y}|>\frac{1}{n}\ \forall n\geq m \\ &\Rightarrow -\frac{1}{2}|\bar{x}_0-\bar{y}|<-\frac{1}{n} \\ \text{By (1)} \Rightarrow |\bar{x}_n-\bar{y}| &\geq |\bar{x}_0-\bar{y}|-\frac{1}{n} \\ &\geq |\bar{x}_0-\bar{y}|-\frac{1}{2}|\bar{x}_0-\bar{y}| \\ &= \frac{1}{2}|\bar{x}_0-\bar{y}| = r \text{ (say) } \forall n\geq m \\ \therefore |\bar{x}_n-\bar{y}| &\geq r\ \forall n\geq m. \end{aligned}$$

(i.e.) There exists a neighbourhood $\bar{y}$ such the neighbourhood contains only finite number of points of S, it is a contradiction to the assumption that $\bar{y}$ is a limit point of S. $\therefore$ Our assumption is wrong. Hence $\bar{y}$ is not a limit point of S. $\therefore S$ has only one limit point $\bar{x}_0$ in $\mathbb{R}^k$ and x_0 is not in $E \Rightarrow S$ has no limit points in E. (i.e.) S is an infinite subset of E and it has no limit point in E. $\Rightarrow\Leftarrow$ hypothesis (c). $\therefore E$ is closed.

Theorem 1.62 ***Heine-Borel theorem:*** *Any subset $E of \mathbb{R}^k$ is closed and bounded iff E is compact.*

Remark 1.63 *The Heine-Borel theorem need not be true for any general metric space.*

Example 1.64 *Let X be an infinite set. Define a discrete metric d on X,*

$$d(p,q)=\begin{cases}0 & \text{if } p=q \\ 1 & \text{if } p\neq q\end{cases}$$

Let A be any infinite subset of X. To prove: A is closed and bounded. Clearly, A is bounded in $X[\because d(p,q)\leq 1\ \ \forall p,q\in A]$. Let $\{x\}$ be a subset *of X. Claim: $\{x\}$ is open in X. Choose $r = 1$. Then, $N_r(x) = \{y \in X | d(x,y) < r\} = \{y \in X | d(x,y) < 1\} = \{x\}$. But every neighbourhood is open. $\therefore$ $\{x\}$ is open. $\therefore$ Every singleton set in the discrete metric set is open. Now, $A = \bigcup_{x \in A}\{x\}$. $\therefore$ A is open in X. $\therefore$ Every subset of X is open in $X \Rightarrow A^c$ subset of X is open in $X \Rightarrow A$ is closed in X $\therefore$ Every subset of a discrete metric space X is both open and closed. $A = \bigcup_{x \in A}\{x\} \Rightarrow \{\{x\} | x \in A\}$ is a open cover for A but it has no finite subcover. $\therefore$ A is not compact. $\therefore$ Heine-Borel theorem need not be true for any general metric space.*

Theorem 1.65 ***Weistras theorem:*** *Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.*
Proof: Let E be an infinite subset of $\mathbb{R}^k \Rightarrow E \subset I$ for some k-cell $I \subset \mathbb{R}^k$. But I is compact. By Bolzona Weistras property, E has a limit point in $I \subset \mathbb{R}^k \Rightarrow E$ has a limit point in $\mathbb{R}^k$.

Perfect Set:

Theorem 1.66 *Let P be a non-empty perfect set in $\mathbb{R}^k$. Then P is uncountable.*
Proof: Given P is a perfect set in $\mathbb{R}^k \Rightarrow P$ is closed and all the points of P are the limit point of $P \Rightarrow P$ is infinite $\Rightarrow P$ is either countable or uncountable. If P is countable then $P = \{\bar{x}_1, \bar{x}_2, ..., \bar{x}_n ....\}$. We construct the sequence of neighbourhood $\{V_n\}$ by the method of induction on n. Let $V_1 = \{\bar{y} \in \mathbb{R}^k | \, |\bar{y} - \bar{x}_1| < r\}$; $\bar{V_1} = \{\bar{y} \in \mathbb{R}^k | \, |\bar{y} - \bar{x}_1| \leq r\}$. Obviously, $V_1 \cap P \neq \emptyset$. $\therefore$ Induction true for $n = 1$. Since every point of P are the limit points, there exists a neighbourhood $V_2(\bar{x}_2)$ such that (i) $\bar{V_2} \subset V_1$, (ii) $\bar{x}_1 \notin V_2$ and (iii) $V_2 \cap P \neq \emptyset$. Suppose V_n has been constructed so that (i) $\bar{V_n} \subset V_{n-1}$, (ii) $\bar{x}_{n-1} \notin \bar{V_n}$ and (iii) $V_n \cap P \neq \emptyset$. Suppose every point of P are the limit points there exists a neighbourhood $V_{n+1}(\bar{x}_{n+1})$ such that (i) $\bar{V}_{n+1} \subset V_n$, (ii) $\bar{x}_n \notin \bar{V}_{n+1}$ and (iii) $V_{n+1} \cap P \neq \emptyset$. $\therefore$ by proceeding we have the $\{V_n\}$ of neighbourhood. Put $K_n = \bar{V_n} \cap P \; \forall n$.......*
$\bar{x}_n \notin \bar{V}_{n+1} \; \forall n \Rightarrow \bar{x}_n \notin K_{n+1}$ $[K_{n+1} = \bar{V}_{n+1} \cap P] \Rightarrow$ no points of P lies in $\bigcap_{n=1}^{\infty} K_n$...... (1)
Now, $K_n = \bar{V_n} \cap P \Rightarrow K_n \subset P \; \forall n \Rightarrow \bigcap K_n \subset K_n \subset P$...... (2)
From (1) and (2), $\bigcap K_n = \emptyset$...... (3)
As $\bar{V_n}$ is a subset of $\mathbb{R}^k$ and $\bar{V_n}$ is closed and bounded $\Rightarrow \bar{V_n}$ is compact. Now, P is closed $\Rightarrow \bar{V_n} \cap P$ is closed and $\bar{V_n} \cap P \subset \bar{V_n}$. (i.e.) $\bar{V_n} \cap \mathbb{R}^k$ is compact[$*$] and also $\bar{V}_{n+1} \subset V_n \subset \bar{V_n} \Rightarrow \bar{V}_{n+1} \cap P \subset \bar{V_n} \cap P \Rightarrow K_{n+1} \subset K_n \; \forall n$. $\therefore$ We have a $\{K_n\}$ of compact such that $K_n \supset K_{n+1}$. $\therefore$ by Theorem 1.55, $\bigcap K_n \neq \emptyset \Rightarrow\Leftarrow$ to (3). $\therefore$ Our assumption is wring. $\therefore$ P is uncountable.

Corollary 1.67 *Every $[a,b](a < b)$ is uncountable. In particular, the set of all real numbers is uncountable.*
Proof: We know that, Every closed interval is perfect set in $\mathbb{R}^1 \Rightarrow [a,b]$ is uncountable $\Rightarrow \mathbb{R}^1$ is uncountable.

Definition 1.68 ***The Cantor Set:*** *Define the cantor set P and show that*

1. *P in non-empty.*
2. *P is closed and bounded.*
3. *P is compact.*
4. *P is perfect or dense in itself.*
5. *P contains no segment.*

The construction of cantor set: The construction of cantor set shows that there exists a perfect sets in $\mathbb{R}^1$ which contains no segment. Let $E_0 = [0, 1]$. Remove the segment $(\frac{1}{3}, \frac{2}{3})$ from $[0, 1]$ and Let $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Remove the middle 3^{rd} of these intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Let $E_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ and each interval is of length $= \frac{1}{9}$, continuing in this way, we obtain a sequence of compact sets
(a) $E_0 \supset E_1 \supset E_2 ...$
(b) E_n is the union of 2^n intervals.
(i.e.) $E = [0, \frac{1}{3^n}] \cup [\frac{2}{3^n}, \frac{3}{3^n}] \cup ... \cup [\frac{3^n-3}{3^n}, \frac{3^n-2}{3^n}] \cup [\frac{3^n-1}{3^n}, 1]$ and each of length 3^{-n}. Let $P = \bigcap_{n=1}^{\infty} E_n$. The set P is called the cantor set.
Step 1: To prove: $P \neq \emptyset$. Since each E_n is closed and bounded and also $E_n \subset \mathbb{R}^1$ for each n. By Heine-Borel theorem each E_n is compact. $\therefore$ We have $\{E_n\}$ of compact sets such that $E_n \supset E_{n+1}$ $\forall n$. By Theorem 1.55, $\bigcap_{n=1}^{\infty} E_n \neq \emptyset \Rightarrow P \neq \emptyset$.
Step 2: To prove: P is closed and bounded. Since each E_n is closed and bounded. $\Rightarrow \bigcap_{n=1}^{\infty} E_n$ is closed and bounded. $\Rightarrow P$ is closed and bounded.
Step 3: To prove: P is compact. Now, $P \subset \mathbb{R}^1$ and P is closed and bounded. $\therefore$ By Heine-borel theorem, P is compact.
Step 4: To prove: P is perfect. (i.e.) To prove P is closed and ever point of P are the limit points of P. By step 2, P is closed. Take $x \in P \Rightarrow x \in \bigcap_{n=1}^{\infty} E_n \Rightarrow X \in E_n$ $\forall n$. Let I_n be an interval of E_n which contains x. [$\because E_n$ is the union of 2^n closed intervals] Let S be any segment containing x. Choose n large enough so that $I_n \subset S$. Let x_n be an end point of I_n such that $x_n \neq x \Rightarrow x_n P$. Since end point of I_n should be the points of $P \Rightarrow x$ is a limit point of P. [$\because S \cap (P - \{x\}) \neq \emptyset$] Since x is arbitrary, every point P are the limit points. $\therefore P$ is perfect.
Step 5: P is perfect $\Rightarrow P$ is uncountable.
Step 6: P contains no segment from the construction of the cantor set. Obviously P does not contain segment of the from $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$........ (1) where $k, m \in Z^+$. Let (α, β) be any segment and if (α, β) contains a segment (1) only if $3^{-m} < \frac{\beta-\alpha}{6}$. But P does not contains the segments (1). $\therefore P$ does not contains the segments (α, β). Since (α, β) is arbitrary. $\therefore P$ contains no segment.

Connected Sets:

Definition 1.69 ***Separated Sets:*** *Any two subsets A and B of a metric space X are said to be separated if $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$.*

Example 1.70 $A = (2, 3)$, $B = (3, 4)$ *and* $C = (3, 4)$. *Then A and B are separated.* $\bar{A} = [2, 3]$; $\bar{B} = [3, 4]$; $\bar{C} = [3, 4]$. *Now,* $\bar{A} \cap B = [2, 3] \cap (3, 4) = \emptyset$; $A \cap \bar{B} = [2, 3] \cap [3, 4] = \emptyset$. $\therefore$ *A and B are separated.* $\bar{A} \cap C = [2, 3] \cap [3, 4] = \{3\} \neq \emptyset \Rightarrow$ *A and C are not separated.*

Remark 1.71

1. *Separated Sets are disjoint.*
2. *Disjoint Sets need not be separated.*

Definition 1.72 ***Connected Sets:*** *A set $E \subset X$ is said to be connected if E is not a union of two non-empty separated sets.*

Theorem 1.73 *A subset E of the real line $\mathbb{R}^1$ is connected iff it has the following property. If $x \in E, y \in E$ and $x < z < y$ then $z \in E$ (or) Find all the connected subsets of the real line.*

Proof: Suppose E is connected. To prove: If $x, y \in E, x < z < y$ then $x \in E[E$ is an interval] Suppose there exists $x, y \in E$ and some $z \in (x, y)$ such that $z \notin E$. Then $E = A_z \cup B_z$ where $A_z = E \cap (-\alpha, z)$; $B_z = E \cap (z, \alpha)$; $A_z \neq \emptyset$; $B_z \neq \emptyset$ [$\because x \in A_z$ and $x \in B_z$]. Now, $\bar{A}_z \cap B_z = \emptyset$; $A_z \cap \bar{B}_z = \emptyset$. $\therefore$ $A_z$and B_z are non-empty separated sets. $A_z \cup B_z = (E \cap (-\alpha, z)) \cup (E \cap (z, \alpha)) = E \cap [(-\alpha, z) \cup (z, \alpha)] = E \cap \{R - \{z\}\} = E$ [$z \notin E$ and $E \subset R - \{z\}$]. $\therefore$ E can be expressed as the union of two-non-empty separated sets. $\therefore$ E is not connected. This is a contradiction. Hence, if $\forall x \in E, y \in E$ and $x < z < y$ then $z \in E$. Conversely, Suppose if $\forall x \in E, y \in E$ and $x < z < y$. Then $z \in E$....... (1)

To prove:E is connected. Suppose E is not connected. $\Rightarrow$ E can be expressed as union of two non-empty separated sets. $\therefore E = A \cup B$ where A and B are two non-empty separated sets. Choose $x \in A, y \in B$ such that $x < y$. Now, $A \cap [x, y]$ is a set of real numbers and it is bounded above by y and also has a $\sup z$. (i.e.) $z = \sup(A \cap [x, y]) \Rightarrow z \in \overline{A \cap [x, y]} \subset \bar{A}$ [by Theorem 7] $\Rightarrow z \in \bar{A} \Rightarrow z \notin B$ [$\because A \cap [x, y] \subset A$] $\because z = \sup(A \cap [x, y]) \Rightarrow z \geq \alpha$ $\forall \alpha \in A \cap [x, y]$. In particular $x \leq z, z \leq y$. But $z \notin B$ $\therefore z < y$ $\therefore x \leq z < y$....... (2)

$x \in A, x < y$ there exists $z \notin B$ $x < z < y$. Now, $z \in \bar{A} \Rightarrow z \in A \cup A' \Rightarrow z \in A$ or $z \in A'$

Case (i): If $z \in A \Rightarrow z \notin \bar{B}$ [$\because A \cap \bar{B} = \emptyset$] $\Rightarrow$ There exists a point z such that $z < z_1 < y$ and $z_1 \notin B$. Also $z_1 \notin A$[$\because z_1 \notin A$, $x < z_1 < y$ and $z_1 \in (x, y) \subset [x, y] \Rightarrow z_1 \in A \cap [x, y]$ $\therefore z = \sup(A \cap [x, y])$ and $z_1 > z \Rightarrow\Leftarrow$] $\therefore z_1 \notin A \cup B \Rightarrow z_1 \notin E \Rightarrow\Leftarrow$ to (1)

Case (ii): If z is not in A and $z \in A'$ $\therefore$ z is a limit point of A. Also

$x < z < y$ and $x, y \in E$. Since z is a limit point of $A, z \in \bar{A} \Rightarrow z \notin B[\because \bar{A} \cap B = \emptyset] \therefore z \notin A$ and $z \notin B \Rightarrow z \notin A \cup B = E$. $\therefore z \notin E \Rightarrow\Leftarrow$ to (1) $\therefore$ From case (i) and (ii) the contradiction shows that E is connected.

Problem 1.74 *Let E' be the set of all limit points of the set E. Prove that E' is closed and also prove that E and $\bar{E}$ have the same limit points, Do E and E' always have the same limit point?*

Proof: To prove: E' is closed. Let E'' denoted the set of all limit points of E'. It $E'' = \emptyset$ then E' is closed. Suppose $E'' \neq \emptyset$. Let $x \in E'' \Rightarrow x$ is a limit point of E'. There exists $r > 0$ such that $N_r(x)$ contains a point Y of E' such that $Y \neq E' \Rightarrow Y \in E' \Rightarrow Y$ is a limit point of$E$. $\Rightarrow$ Every neighbourhood of Y contains infinitely many points of E. $\Rightarrow$ Every neighbourhood of x contains infinitely many points if E. $\Rightarrow x$ is a limit point of E. $\Rightarrow x \in E' \therefore E'' \subset E' \therefore E'$ contains all its limit points. E' in closed. To prove: E and E' have same limit points. (i.e.) To prove $E' = \bar{E}'$. Let $x \in E' \Rightarrow x$ is a limit points of E. There exists $r > 0, N_r(x)$ contains points Y of E such that $y \neq x \Rightarrow \forall r > 0, N_r(x)$ contains Y of $\bar{E}$ such that $y \neq x \Rightarrow x$ is a limit point of $\bar{E}$. $\Rightarrow x \in \bar{E}' \therefore E' \subseteq \bar{E}'$.......(1)

Let $x \in \bar{E}' \Rightarrow x$ is a limit point of $\bar{E}$. $\Rightarrow x \in \bar{E}$ [$\because \bar{E}$ is closed] $\Rightarrow x$ is a limit point of $E \cup E' \Rightarrow \therefore x$ is a limit point of E (or) x is a limit point of $E' \Rightarrow x \in E'$ or $x \in E'' \subset E'[\because E'$ is closed$] \Rightarrow x \in E' \therefore \bar{E}' \subset E'$...... (2)

From (1) and (2), $E' = \bar{E}'$. To prove E and E' need not have the same limit point. Let $E = \{0, 1, \frac{1}{2}, ...\}$; $E' = \{0\}$. Then E has limit point $\{0\}$ only and E' have the no limit point. $\therefore E$ and E' need not have the same limit point.

Problem 1.75 *Let $K \subset \mathbb{R}^1$ consists of numbers $0, \frac{1}{n}$, $(n = 1, 2, ...)$. Prove that K is compact without using Heine-Borel theorem.*

Proof: Let $\{G_\alpha\}$ be an open cover for K. $\Rightarrow$ Now $0 \in K \Rightarrow 0 \in G_{\alpha_1}$ for some α_1. Since G_{α_1} is open there exists a neighbourhood $N_\epsilon(0) \subset G_{\alpha_1}$, $(-\epsilon, \epsilon) \subset G_{\alpha_1}$. By Archimedian Principle, there exists $m \in \mathbb{Z}^+$ such that $m \cdot \epsilon > 1 \Rightarrow n \cdot \epsilon \geq m \cdot \epsilon > 1 \ \forall n \geq m \Rightarrow \frac{1}{n} < \epsilon \ \forall n \geq m \Rightarrow \frac{1}{n} \in (-\epsilon, \epsilon) \ \forall n \geq m \Rightarrow 0$ and $\frac{1}{n} \in G_{\alpha_1} \ \forall n \geq m$. There exists $\alpha_2, ..., \alpha_m$ such that $\frac{1}{i-1} \in G_{\alpha_i}, i = 1, 2, ..., m \Rightarrow K \subset \bigcup_{i=1}^{n} G_{\alpha_i}$. $\therefore K$ is compact.

Problem 1.76 *Given an example of an open cover of the segment $(0, 1)$ which has no finite subcover (or) prove that $(0, 1)$ are not compact.*

Proof: Consider the family of open intervals $\mathcal{F} = \{(\frac{1}{1+n}, n) | n = 1, 2, ...\}$. Clearly $\mathcal{F}$ is an open cover for $(0, 1)$. (i.e.) $(0, 1) \subset \cup_{n=1}^{\infty}(1/1 + n, n)$. Also we cannot find any subcollection from $\mathcal{F}$ covering $(0, 1) \therefore$ The open cover $\mathcal{F}$ has no finite subcover for $(0, 1) \Rightarrow (0, 1)$ is not compact.

Note 1.77 *In general $(a, b) \subseteq \mathbb{R}^1$ is not compact. Since $\{(a + \frac{1}{n+1}, b) | n \in Y\}$ it is an open cover for (a, b) and it has no finite subcover covering (a, b). $\therefore (a, b)$ is not compact.*

Example 1.78 *Prove that: Set of all irrational is uncountable.*
Proof: $\mathbb{R}$ is uncountable (by Corollary 1.67) and also $\mathbb{Q}$ is countable. If $\{\text{irrational}\}$ is countable. $= \mathbb{Q} \cup \{\text{irrational}\} =$ countable $\Rightarrow\Leftarrow$ to (1) $\therefore$ irrational is uncountable.

Example 1.79 *Construct a bounded set of real numbers with exactly 3 limit points.*
Proof: $E = \{1 + \frac{1}{n}, 2 + \frac{1}{n}, 3 + \frac{1}{n} | n \in N\} \subseteq \mathbb{R}$. It has exactly 3 limit points namely $1, 2, 3$. Since $X < 5$ for all $x \in E \Rightarrow E$ is bounded.

Note 1.80 $E = \{\frac{1}{n}\} \cup \{\frac{1}{n} + \frac{1}{m}\} | m, n \in \mathbb{Z}^+\} \cup \{0\} \subseteq \mathbb{R}$. *It is closed and bounded subset of* $\mathbb{R}^1$. $\therefore E$ *is compact.*

Example 1.81 *Let* E° *denote the set of all interior points of a set* E.
(a) Prove that E° *is always open.*
(b) Prove that E *is open iff* $E = E^\circ$.
(c) If $G \subset E$ *and* G *is open prove that* $G \subset E^\circ$.
(d) Prove that the complement of E° *is the closure of the complement of* E^c. *(i.e.)* $E^{\circ^c} = \bar{E}^c$. *Do* E *and* $\bar{E}$ *always have the same interiors? Do* E *and* E° *always have same closure?*
Proof: **(a)** Prove that E° is open. Let $x \in E^\circ \Rightarrow x$ is an interior point of E. $\Rightarrow$ There exists $r > 0$ such that $N_r(x) \subset E$. Claim:$N_r(x) \subset E^\circ$. Let $y \in N_r(x) \Rightarrow$ There exists $S > 0$ such that $N_S(y) \subset N_r(x) \subset E$.[$\because N_r(x)$ is open] $\Rightarrow y \in N_S(y) \subset E \Rightarrow y$ is an interior point of E. $\Rightarrow y \in E^\circ \Rightarrow N_r(x) \subset E^\circ \therefore x$ is an interior point of E°. Since x is arbitrary. Every point of E° in an interior point. $\therefore E^\circ$ is open.
(b) Suppose E is open. To prove $E = E^\circ \Rightarrow E$ is open. Clearly, $E^\circ \subset E$ $\because$ E is open, $E \subset E^\circ$. $\therefore E = E^\circ$. Conversely: $E = E^\circ \Rightarrow$ Every point of E is an interior point of E. $\Rightarrow E$ is open.

Convergent Sets
Numerical sequence and series:

Definition 1.82 *Let* X *be a metric space. Let* $F : N \to X$ *be a function defined by* $f(n) = p_n$. *Then* $p_1, p_2, ..., p_n$ *is called sequence in* X. *Determined by the function* F *and it is denoted by* $\{p_n\}$.

Definition 1.83 $\{p_n\}$ *is said to converge to a point* p *in* X *if given* $\epsilon > 0$ *there exists a positive integer* N *such that* $d(p_n, p) < \epsilon$ $\forall n \geq N$ *and we write* $p_n \to p$ *as* $n \to \infty$ *or*

$$\lim_{n\to\infty} p_n = p$$

If $\{p_n\}$ *does not converge then* $\{p_n\}$ *diverges.*

Definition 1.84 *The set of all points* $\{p_1, p_2, ..., p_n\}$ *is called the range of the sequence* $\{p_n\}$. *The range set is either finite or infinite.*

Definition 1.85 *A sequence is said to be bounded. If its range is bounded.*

Example 1.86 .

1. $S_n = \{\frac{1}{n}\}$ $n = 1, 2, ...$ *Clearly,* $S_n \to 0$. $\therefore \{S_n\}$ *is a bounded sequences and the range* S_n *is infinite.*

2. $\{n\}$ *is not a convergent sequences. It is a divergent sequence.* $\therefore$ *It is a unbounded sequences.* $\therefore$ *range is infinite.*

3. $S_n = i^n$, $n = 1, 2, ...$. *This is not a convergent sequence.* $\therefore$ *It is a divergent sequence. The range of* S_n *is finite.* $\therefore$ *Sequence* $\{S_n\}$ *is bounded, range of* $S_n = \{1, -1, i, -i\}$.

Theorem 1.87 *Let* $\{p_n\}$ *be a sequence in a metric space* X. *Then,*
(a) $\{p_n\}$ *converges to* $p \in S$. p *iff every neighbourhood of* p *contains all but finitely many of the terms of sequence* $\{p_n\}$.
(b) It $p \in X, p' \in X$ *and* $\{p_n\}$ *converges to* p *and* p' *then* $p = p'$
(c) If $\{p_n\}$ *converges then* $\{p_n\}$ *is bounded.*
(d) $E \subset X$ *and if* p *is limit points of* E. *Then there is a sequence* $\{p_n\}$ *in* E *such that*

$$p = \lim_{n\to\infty} p_n.$$

Proof: **(a)**Suppose $\{p_n\}$ converges to a point p. Let V be a neighbourhood of p. Since V is open, there exists $\epsilon > 0$, such that $N_\epsilon(p) \subset V$. Since $\{p_n\}$ converges to p. Given $\epsilon > 0$ there exists a positive integer N such that $d(p_n, p) < \epsilon \quad \forall n \geq N$. $\therefore p_n \in N_\epsilon(p) \quad \forall n \geq N \Rightarrow p_n \in N_\epsilon(p) \subset V$ $\forall n \geq N \Rightarrow p_n \in V \quad \forall n \geq N \Rightarrow V$ contains all but finitely many terms of the sequence $\{p_n\}$. Conversely, every neighbourhood of p contains all but finitely many points of sequences $\{p_n\}$. Fix $\epsilon > 0, V = \{q \in X | d(p, q) < \epsilon\}$. Then V is a neighbourhood of p. By assumption, there exists N such that $p_n \in V \ \forall n \geq N \Rightarrow d(p, p_n) < \epsilon \ \forall n \geq N \Rightarrow p_n \to p$ as $n \to \infty$.
(b) The limit of a convergent sequence is unique. Let $\epsilon > 0$ be given let $p \neq p'$ and $p_n \to p$ and $p_n \to p'$. $\because p_n \to p$, there exists a positive integer N_1 such that $d(p_n, p) < \epsilon/2 \ \forall n \geq N_1$. As $p_n \to p'$ there exists a positive integer N_2 such that $d(p_n, p') < \epsilon/2 \ \forall n \geq N_2$; $N = ma \times \{N_1, N_2\}$. Now, $\forall n \geq N, d(p, p') \leq d(p, p_n) + d(p_n, p') < \epsilon/2 + \epsilon/2 = \epsilon$. Since ϵ is arbitrary, $d(p, p') = 0 \Rightarrow p = p'$.
(c) Every convergent sequences is bounded sequences. Suppose sequence $\{p_n\}$ converges to a point p. Then there exists a positive integer N such that $d(p_n, p) < 1 \ \forall n \geq N$. Let $r = max\{d(p_1, p), ..., d(p_N, p), 1\} \Rightarrow d(p_n, p) < r$ $\forall n \Rightarrow$ The range of sequence $\{p_n\}$ is bounded. $\Rightarrow \{p_n\}$ is bounded.
(d) Given that p is a limit point of the set E. $\Rightarrow$ For each there exists a neighbourhood $N_{1/n}(p)$ contains a point p_n of E such that $p_n \neq p$ $\therefore$ $d(p_n, p) < 1/n \ \forall n$. Given $\epsilon > 0$ choose N such that $N{\cdot}\epsilon > 1$. (i.e.) $N > 1/\epsilon$. It $n > N, d(p_n, p) < 1/n < 1/N < \epsilon \therefore d(p_n, p) < \epsilon \ \forall n > N \Rightarrow p_n \to p$ as $n \to \infty$.

Theorem 1.88 *Suppose* $\{S_n\}$ *and* $\{t_n\}$ *are complex sequences and*

$$\lim_{n\to\infty} s_n = s, \lim_{n\to\infty} t_n = t.$$

Then,

1.
$$\lim_{n\to\infty}(s_n + t_n) = s + t.$$

2.
$$\lim_{n\to\infty}(cs_n) = cs, \lim_{n\to\infty}(c + s_n) = c + s \text{ for any number } c.$$

3.
$$\lim_{n\to\infty} s_n t_n = st.$$

4.
$$\lim_{n\to\infty}(\frac{1}{s_n}) = \frac{1}{s}(s_n \neq 0 \ \forall n, s \neq 0).$$

Proof: **(1)** Given $\{s_n\}$ converges to s. Given $\epsilon > 0$ there exists a positive integer n_1 such that $|s_n - s| < \epsilon/2 \quad \forall n \geq n_1$. As $\{t_n\}$ converges to t. Given ϵ there exists a positive integer n_2 such that $|t_n - t| < \epsilon/2$ $\forall n \geq n_2$. Let $N = max\{n_1, n_2\} \Rightarrow |s_n + t_n - (s + t)| = |s_n - s + t_n - t| \leq |s_n - s| + |t_n - t| < \epsilon/2 + \epsilon/2 = \epsilon \quad n \geq N \therefore s_n + t_n \to s + t$ as $n \to \infty$.
(2) Given $\{s_n\}$ converges to s. Let $\epsilon > 0$ be given. Then there exists a positive integer N such that $|s_n - s| < \epsilon \, \forall n \geq N$. $|c + s_n - (s + c)| = |s_n - s| < \epsilon \ \forall n \geq N$. $\therefore c + s_n \to c + s$ as $n \to \infty$. Now to prove $cs_n \to cs$ as $n \to \infty$.
Case (i): $c \neq 0$. Given $s_n \to s$. Let $\epsilon > 0$ be given. Then there exists a positive integer N such that $|s_n - s| < \frac{\epsilon}{|c|} \ \forall n \geq N$, $|cs - n - cs| = |c| \, |s_n - s| < |c| \frac{\epsilon}{|c|} = \epsilon \ \forall n \geq N$. $\therefore cs_n \to cs$ as $n \to \infty$.
Case (ii): If $c = 0$ then clearly $cs_n \to cs$.
(3) To prove: $s_n t_n \to st$. Let $\epsilon > 0$ be given. Given $s_n \to s \Rightarrow$ there exists positive integer n_1 such that $|s_n - s| < \sqrt{\epsilon} \quad \forall n \geq n_1$. As $t_n \to t \Rightarrow$ there exists positive integer n_2 such that $|t_n - t| < \sqrt{\epsilon} \quad \forall n \geq n_2, N = max\{n_1, n_2\}$. $\therefore |(s_n - s)(t_n - t)| = |s_n - s| \, |t_n - t| < \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon \quad \forall n \geq N$. $\therefore (s_n - s)(t_n - t) \to 0$ as $n \to \infty$. Now,

$$\begin{aligned} s_n t_n - st &= (s_n - s)(t_n - t) + s(t_n - t) + t(s_n - s) \\ \lim_{n\to\infty} s_n t_n - st &= \lim_{n\to\infty}(s_n - s)(t_n - t) + \lim_{n\to\infty} s(t_n - t) + \lim_{n\to\infty} t(s_n - s) \\ &= 0 \ [\because s_n - s \to 0, \ t_n - t \to 0, \ (s_n - s)(t_n - t) \to 0] \\ \therefore \lim_{n\to\infty} s_n t_n &= st. \end{aligned}$$

(4) Given that $\{s_n\}$ converges to s. Let $\epsilon > 0$ be given. There exists a positive integer N_1 such that

$$|s_n - s| < \frac{|s|}{2}\ \forall\ n \geq N_1$$
$$\text{Always}\ |s_n - s| \geq |s| - |s_n|$$
$$\Rightarrow \frac{|s|}{2} > |s_n - s| \geq |s| - |s_n|$$
$$\Rightarrow \frac{|s|}{2} > |s| - |s_n|$$
$$\Rightarrow |s| - |s_n| < \frac{|s|}{2}$$
$$\Rightarrow |s| - \frac{|s|}{2} < |s_n|$$
$$\Rightarrow \frac{|s|}{2} < |s_n|\ \ \forall n \geq N_1$$

Now $s_n \to s \Rightarrow$ There exists a positive integer N_2 such that $|s_n - s| < \epsilon\frac{|s|^2}{2}$ $\forall n \geq N_2$. Let $N = max\{N_1, N_2\}$

$$\left|\frac{1}{s_n} - \frac{1}{s}\right| = \frac{|s_n - s|}{|s_n|\,|s|}$$
$$< \epsilon\frac{|s|^2}{2} \cdot \frac{2}{|s|\,|s|}\ \ [\because \frac{|s|}{2} < |s_n|]$$
$$= \epsilon\ \forall n \geq N$$
$$\Rightarrow \frac{1}{s_n} \to \frac{1}{s}\ \text{as}\ n \to \infty.$$

Theorem 1.89 *1. Suppose $\bar{x}^n \in \mathbb{R}^k$, $(n = 1, 2, ...)$ and $\bar{x}_n = \{\alpha_{1,n}, \alpha_{2,n}, ..., a_{k,n}\}$. Then $\{\bar{x}_n\}$ converges to $\bar{x} = (\alpha_1, \alpha_2, ..., \alpha_k) \Leftrightarrow$*

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j,\ 1 \leq j \leq k.$$

2. Suppose $\{\bar{x}_n\}, \{\bar{y}_n\}$ are sequences in $\mathbb{R}^k, \{\beta_n\}$ is a sequence of real numbers and $\bar{x}_n \to \bar{x}, \bar{y}_n \to \bar{y}, \beta_n \to \beta$. Then,

$$\lim_{n\to\infty}(\bar{x}_n + \bar{y}_n) = \bar{x} + \bar{y}\ \text{and}\ \lim_{n\to\infty}\beta_n\bar{x}_n = \beta\bar{x}.$$

Proof: **(1)** Suppose $\bar{x}_n \to \bar{x}$. Given $\epsilon > 0$ there exists a positive integer

N such that $|\bar{x}_n - \bar{x}| < \epsilon \; \forall n \geq N$

$$\Rightarrow \sqrt{\sum_{j=1}^{k} (\alpha_{j,n} - \alpha_j)^2} < \epsilon \; \forall \; n \geq N$$
$$\Rightarrow \sum_{j=1}^{k} (\alpha_{j,n} - \alpha_j)^2 < \epsilon^2 \; \forall \; n \geq N$$
$$\Rightarrow (\alpha_{j,n} - \alpha_j)^2 < \sum_{j=1}^{k} (\alpha_{j,n} - \alpha_j)^2 < \epsilon^2 \; \forall \; n \geq N$$
$$\Rightarrow |\alpha_{j,n} - \alpha_j| < \epsilon \; \forall \; n \geq N, \; 1 \leq j \leq k$$
$$\therefore \lim_{n \to \infty} \alpha_{j,n} = \alpha_j \; 1 \leq j \leq k$$

Conversely, Suppose

$$\lim_{n \to \infty} \alpha_{j,n} = \alpha_j, \; (1 \leq j \leq k)$$

Let $\epsilon > 0$ be given, there exists a positive integer N_j such that $|\alpha_{j,n} - \alpha_j| < \epsilon/\sqrt{k} \;\; \forall n \geq N_j$. Let $N = max\{N_1, N_2, ..., N_k\}$.

$$\Rightarrow |x_n - \bar{x}| = \sqrt{\sum_{j=1}^{k} (\alpha_{j,n} - \alpha_j)^2}$$
$$< \sqrt{\sum_{j=1}^{k} (\epsilon/\sqrt{k})^2} \; \forall \; n \geq N$$
$$< \sqrt{k\epsilon^2/k} = \sqrt{\epsilon^2}$$
$$= \epsilon \; \forall \; n \geq N$$
$$\therefore |x_n - \bar{x}| < \epsilon \; \forall \; n \geq N$$
$$\therefore (\bar{x}^n) \to \bar{x} \text{ as } n \to \infty.$$

(2) Given $\bar{x}_n \to \bar{x}$ and $\bar{y}_n \to \bar{y}$ as $n \to \infty \Rightarrow \alpha_{j,n} \to \alpha_j; \; \gamma_{j,n} \to \gamma_j$ as $n \to \infty, \; 1 \leq j \leq k$ where $\bar{x}_n = (\alpha_{1,n}, \alpha_{2,n}, ..., \alpha_{k,n}); \; \bar{y}_n = (\gamma_{1,n}, \gamma_{2,n}, ..., \gamma_{k,n}); \; \bar{x} = (\alpha_1, \alpha_2, ..., \alpha_k)$ and $\bar{y} = (\gamma_1, \gamma_2, ..., \gamma_k)$. Now $\alpha_{j,n} + \gamma_{j,n} \to \alpha_j + \gamma_j$ as $n \to \infty, \; j = 1$ to $k \Rightarrow \bar{x}_n + \bar{y}_n \to \bar{x} + \bar{y}$ as $n \to \infty$ (by (1)). Given $\beta_n \to \beta, \bar{x}_n \to \bar{x}$ as $n \to \infty \Rightarrow \beta_n \to \beta, \alpha_{j,n} \to \alpha_j$ as $n \to \infty \; \forall j \Rightarrow \beta_n \alpha_{j,n} \to \beta \alpha_j$ as $n \to \infty \; \forall j \Rightarrow \beta_n \bar{x}_n \to \beta \bar{x}$ as $n \to \infty$. (by using (1))

Definition 1.90 ***Subsequences:*** *Given a sequence $\{p_n\}$ consider a $\{n_k\}$ of positive integers such that $n_1 < n_2 < n_3 \cdots$. Then the sequence $\{p_{n_i}\}$ is called a subsequence of $\{p_n\}$*

Note 1.91 *If $\{p_{n_i}\}$ converges, its limit is called subsequencial limit of $\{p_n\}$.*

Theorem 1.92 .

1. *If $\{p_n\}$ is a sequence in a compact metric space X. Then some subsequence of $\{p_n\}$ converges to a point of X.*

2. *Every bounded sequence in $\mathbb{R}^k$ contains converges subsequence.*

Proof: (1) Let E=Range of $\{p_n\}$.
Case (i): Suppose E is finite. Then there is a point p in E and a sequence $\{n_i\}$ with $n_1 < n_2 < n_3 \cdots$ such that $p_{n_1} = p_{n_2} = \cdots = p$. The subsequence $\{p_n\}$ so obtained converges to p.
Case (ii): Suppose E is infinite. $\Rightarrow E$ is an infinite subset of a compact metric space X. $\Rightarrow E$ has a limit point p in X. [Theorem 1.57] Choose $n_1, d(p, p_{n_1}) < 1$. Choose $n_2 < n_1$, such that $d(p, p_{n_2}) < 1/2$. Having chosen $n_1, n_2, ..., n_{i-1}$, there exists an integer $n_i > n_{i-1}$ such that $d(p, p_{n_i} < 1/i)(\because$ every neighbourhood of p contains infinite many point of E). Choose $\epsilon > 0$ such that there exists a positive integer N such that $\epsilon N > 1$ (Archimedean principle) (i.e.) $N > 1/\epsilon$. Then for every $i > N$, $d(p, p_{n_i}) < 1/i < 1/N < \epsilon \ \forall i > N \Rightarrow \{p_{n_i}\} \to p$.
(b) Let $\{p_n\}$ be a bounded sequence in $\mathbb{R}^k$. $\Rightarrow$ Range of $\{p_n\}$ is bounded. Range of $\{p_n\}$ is a subset of some K-cell I. As I is compact, by (a) since I compact, $\{p_n\}$ contains a convergent subsequence in $I \subset \mathbb{R}^k$. $\Rightarrow$ Every bounded sequence in $\mathbb{R}^k$ has a convergence subsequence.

Definition 1.93 ***Cauchy Sequence:*** *A sequence $\{p_n\}$ in a metric space X is said it to be a Cauchy sequences, if for every $\epsilon > 0$ there is an integer N such that $d(p_n, p_m) < \epsilon \ \forall n, m \geq N$.*

Definition 1.94 ***Diameter:*** *If $E \subset X$ and $S = \{d(a, b)|a, b \in E\}$ then the diameter of $E = \sup\ S$ (i.e.) $dia(E) = \sup\{d(a, b)|a, b \in E\}$.*

Note 1.95 *If $\{p_n\}$ is a sequence in X,and $E_N = \{p_N, p_{N+1}, ...\}$ and p_n is a Cauchy sequence in X iff*

$$\lim_{N\to\infty} dia(E_N) = 0 \ or \ dia(E_N) \to 0 \ as \ N \to \infty.$$

Theorem 1.96

1. *If $\bar{E}$ is the closure of the set E in a metric space X, then $dia(\bar{E}) = dia(E)$.*

2. *If $\{k_n\}$ is a sequence of compact sets such that $k_n \supset k_{n+1}$, $(n = 1, 2, ...)$ and if*

$$\lim_{n\to\infty} dia(k_n) = 0, \quad then \bigcap_{n=1}^{\infty} k_n$$

contains exactly one point.

Proof: **(1)** Since $E \subset \bar{E}$, diameter $E \leq$ diameter $\bar{E}$. Fix $\epsilon > 0, p, q \in \bar{E}$ by the definition of $\bar{E}$, these are points $p', q' \in E$ such that $d(p, p') < \epsilon$ and $d(q, q') < \epsilon$. Now,

$$\begin{aligned} d(p,q) &\leq d(p,p') + d(p',q') + d(q',p) \\ &\leq d(p',q') + \epsilon + \epsilon \\ &= d(p',q') + 2\epsilon \end{aligned}$$

Since ϵ is arbitrary, $d(p,q) < d(p',q') \Rightarrow d(p,q) < d(p',q') < \sup\ d(p',q') = dia(E) \Rightarrow d(p,q) < dia(E)\, \forall\, p,q \in \bar{E}$. Taking sup, we get $dia\bar{E} < dia(E)$. $\therefore$ $dia(E) = dia(\bar{E})$.
(2) Let $K = \bigcap_{n=1}^{\infty} K_n \Rightarrow K$ is non-empty. (by Theorem 1.58). To prove: K contains exactly one point. Suppose K contains more than one point, then $dia(K) > 0$. Also $K \subset K_n\ \forall n \Rightarrow 0 < dia(K) < dia(K_n)\ \forall n \Rightarrow 0 < dia(K_n) = 0 \Rightarrow\Leftarrow$

$$\lim_{n\to\infty} dia(K_n) = 0$$

$\therefore$ K contains exactly one point.

Theorem 1.97 *A subsequential limit of $\{p_n\}$ in a metric space X form a closed subset of X.*
proof: Let E^* be the set of all subsequential limits of $\{p_n\}$ and let q be a limit point of E^*. To prove: $q \in E^*$ Choose n_1 so $p_{n_1} \neq q$. (If no such n_1 exists, E^* has only one point and there is nothing to prove) Put $S = d(p_{n_1}, q)$. Choose $n_2 > n_1$ such that $d(p_{n_2}, q) < \frac{S}{2}$ and $p_{n_2} \neq q (\because q$ is a limit point). Suppose $n_1, n_2, ..., n_{i-1}$ are chosen. Since q is a limit point, there exists $x \in E^*$ such that $d(x,q) < \frac{S}{2^i}$. Since $x \in E^*$ there exists an $n_i > n_{i-1}$ with

$$\begin{aligned} d(p_{n_i}, x) &< \frac{S}{2^i} \\ d(p_{n_i}, q) &< d(p_{n_i}, x) + d(x,q) \\ &< \frac{S}{2^i} + \frac{S}{2^i} = \frac{S}{2^{i-1}} \\ (i.e.)\ d(p_{n_i}, q) &< \frac{S}{2^{i-1}} \end{aligned}$$

$\Rightarrow$ (i.e.) we get a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ such that p_{n_i} converges to $q \Rightarrow q$ is a subsequential limit of $\{p_n\} \Rightarrow q \in E^*$. Since q is arbitrary, E^* contains all its limit points. $\therefore E^*$ is closed.

Theorem 1.98 *(a) In any metric space X, every convergent sequences is a Cauchy sequence.*
(b) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X,

then $\{p_n\}$ converges to some point of X.
(c) In $\mathbb{R}^k$, every Cauchy sequence converges.
Proof: (a) Let $\{p_n\}$ be a sequence in X such that $\{p_n\}$ converges to p. Given $\epsilon < 0$ there exists a positive integer N such that $(d_{p_n}, p) < \epsilon/2\ \forall n \geq N$. Now, $\forall n, m \geq N$, $d(p_n, p_m) \leq d(p_n, p) + d(p, p_m) < \epsilon/2 + \epsilon/2 = \epsilon\ \forall n, m \geq N$. $\therefore \{p_n\}$ is Cauchy sequence in X.
(b) Let $\{p_n\}$ be a Cauchy sequence in a compact metric space X. For each $N = 1, 2, 3...$, $E_N = \{p_N, p_{N+1}, ...\}$. Also $\{p_n\}$ is Cauchy sequence $\Rightarrow$ diam $E_N \to 0$ as $N \to \infty \Rightarrow$ diam $\bar{E}_N \to 0$ as $N \to \infty$[$\because$ diam E_N=diam $\bar{E}_N$ by Theorem 1.96]. Now $\bar{E}_N$ is a closed subset of a compact metric space $X \Rightarrow \bar{E}_N$ is compact and also $\bar{E}_{N+1} \subset \bar{E}_N$ for each N. By Theorem 1.96, $\bigcap_{n=1}^{\infty} \bar{E}_n$ contains exactly one point, p (say) in X. $p \in \bar{E}_N$ for each N. Since diam $\bar{E}_N \to 0$ as $N \to \infty$. Given $\epsilon > 0$ there exists an integer N_0 such that diam $\bar{E}_N < \epsilon\ \forall N \geq N_0 \Rightarrow d(p, q) < \epsilon\ \forall q \in \bar{E}_N\ \forall N \geq N_0$. In particular, $d(p, q) < \epsilon\ \forall q \in \bar{E}_{N_0} \Rightarrow d(p, p_n) < \epsilon\ \forall n \geq N_0$. $\therefore \{p_n\}$ converges to a point in X.
(c) Let $\{p_n\}$ be Cauchy sequence in $\mathbb{R}^k$. Let $E_N = \{p_N, p_{N+1}, ...\}$. Since $\{p_n\}$ is a Cauchy sequence $\Rightarrow$ diam $E_N \to 0$ as $N \to \infty \Rightarrow$ diam $E_N < 1$ for some N. Let E be the range of the sequence $\{p_n\} \Rightarrow E = \{p_1, p_2 ... p_{N_1}\} \cup E_N$. As E_N is bounded and $\{p_1, p_2, ..., p_{N-1}\}$ is a finite set. $\therefore E$ is bounded set in $\mathbb{R}^k$. $\Rightarrow \{p_n\}$ is bounded in $\mathbb{R}^k$. By Heine-Borel theorem E has a compact closure in $\mathbb{R}^k$. (i.e.) $\bar{E}$ is compact in $\mathbb{R}^k$. $\Rightarrow \{p_n\}$ is a Cauchy sequence in $\bar{E}$ and $\bar{E}$ is compact. By (b), $\{p_n\}$ converges to a point in $\bar{E} \subset \mathbb{R}^k \Rightarrow$ Every Cauchy sequence in $\mathbb{R}^k$ converges.

Definition 1.99 ***Complete metric space:*** *A metric space X is said to be complete metric space if every Cauchy sequence in X converges to a point in X.*

Example 1.100 *(i) $\mathbb{R}^k$ is complete.*
(ii) Every compact metric space is complete.

Theorem 1.101 *Every closed subset E of a complete metric space x is complete.*
Proof: Given that E is closed subset of a complete metric space x. To prove: E is complete. Let $\{x_n\}$ be a Cauchy Sequence in $E \Rightarrow \{x_n\}$ is a Cauchy Sequence in x. Given that x is complete. $\Rightarrow \{x_n\}$ converges to a point x in x. $\Rightarrow$ Every neighbourhood of x contains all but finitely many terms of $\{x_n\}$. $\Rightarrow$ Every neighbourhood of x contains a point of $\{x_n\}$ other than x. [$\because x_n \neq x$] $\Rightarrow N_r(x) \cap E - \{x\} \neq \emptyset\ \forall r > 0 \Rightarrow x$ is a limit point of E. $\Rightarrow x \in E$ [$\because E$ is closed] $\Rightarrow \{x_n\}$ converges to x and $x \in E$. $\therefore E$ is complete.

Definition 1.102 *A sequence $\{s_n\}$ of real numbers is said it to be monotonic increasing if $s_n \leq s_{n+1}$ ($\forall n = 1, 2, ...$) and monotonic decreasing if $s_n \geq s_{n+1}$ ($\forall n = 1, 2, ...$).*

Note 1.103 *A $\{s_n\}$ is said it to be monotonic if it is monotonic increasing or monotonic decreasing.*

Theorem 1.104 *Suppose $\{s_n\}$ is monotonic then the $\{s_n\}$ converges iff it is bounded.*
Proof: Suppose $\{s_n\}$ converges $\Rightarrow$ $\{s_n\}$ is bounded.(by Theorem 1.87) Conversely, suppose $\{s_n\}$ is bounded. Let E be the range of the sequence $\{s_n\}$ and Let s is least upper bound of E. For every $\epsilon > 0$, there exists an integer N such that $s - \epsilon < s_N \leq s \Rightarrow s - \epsilon < s_n \leq s$ $(\forall n \geq N)(\because s_n$ is monotonic) (If not $s - \epsilon$ would be an upper bound) $\Rightarrow s - \epsilon < s_n \leq s < s + \epsilon$ $\forall n \geq N \Rightarrow s - \epsilon < s_n \leq s + \epsilon \Rightarrow |s_n - s| < \epsilon$ $\forall n \geq N \Rightarrow s_n \to s$ as $n \to \infty$

Upper and Lower bounds

Definition 1.105 *Let $\{s_n\}$ be a sequence of real numbers with the following properties*

1. *For ever real number M, there is an integer N such that $s_n \geq M$ $\forall n \geq N$ then we write $s_n \to \infty$.*
2. *$\forall M$, there is an integer N such that $s_n \leq M, \forall n \geq N$, then we write $s_n \to -\infty$.*

Definition 1.106 *Let s_n be a sequence of real numbers, E be the set of numbers x (in extended real number system such that $s_{n_k} \to x$ for all sub sequences $\{s_{n_k}\}$. The set E contains all subsequential limits defined above, plus possible, the number α to $-\alpha$. Let $s^* = \sup\ E$ and $s_* = \inf\ E$.*

Theorem 1.107 *Let $\{s_n\}$ be a sequence of real numbers. E and s^* as defined above. Then s^* has the following properties.*
(a) $s^* \in E$
(b) If $x > s^*$ then there is an integer N such that $n > N \Rightarrow s_n < x$
Moreover s^* is the only number with the properties $(a) + (b)$. This result is true for s_* also.
Proof:(a) Case (i): Suppose $s^* = \infty$. Since $\sup E = \infty, E$ is not bounded above. Then $\{s_n\}$ is not bounded above and there is a subsequence $\{s_{N_k}\}$ which converges to ∞. $\therefore$ ∞ is a subsequential limit. Hence $\infty \in E$. (i.e.) $s^* \in E$.
Case (ii): Suppose s^* is real. Then E is bounded above. $\therefore$ atleast one subsequential limit exists say $\lambda \in E$. $\Rightarrow$ E is non-empty. $\therefore$ E is a non-empty set of real numbers and bounded above also $s^* = \sup\ E \Rightarrow s^* \in \bar{E}$ [by Theorem 1.41] $\Rightarrow s^* \in E$ [since by Theorem 1.40 E is closed $\Leftrightarrow E = \bar{E}$]
Case (iii): Suppose $s^* = -\infty \Rightarrow E$ contains only one element namely $(-\infty)$ and there is no subsequential limits. $\Rightarrow$ For any real numbers $s_n > m$ for atmost finite number of values of n. ((i.e.) $s_n \leq N$ $\forall n \geq N$ for some integer N) so that $s_n \to -\infty$. $\therefore s^* = -\infty \in E$ $\therefore$ From all the three cases

$s^* \in E$.
(b) Suppose there is a number $x > s^*$ such that $s_n \geq x$ for infinitely many values of n. $\Rightarrow$ There exists a number $y \in E$ such that $y \geq x > s^* \Rightarrow\Leftarrow$ to s^* is the supremum of $E \Rightarrow s_n < x$ for all $n \geq N_1$ for some integer N. **Uniqueness:** Suppose there are two numbers p and q satisfy both (a) and (b) such that $p \neq q$. Without loss of generality $p < q$. Choose x such that $p < x < q$. If $x > p$, then by (b) there exists a integer N such that $s_n < x < q\ \forall n \geq N \Rightarrow q$ is not in $E \Rightarrow q$ cannot satisfy the property (a). $\therefore s^*$ is unique.

Theorem 1.108 *If $s_n \leq t_n \forall n \geq N, N$ is fixed, then*

$$\lim_{n\to\infty} \inf\ s_n \leq \lim_{n\to\infty} \inf\ t_n \text{ and } \lim_{n\to\infty} \sup\ s_n \leq \lim_{n\to\infty} \sup\ t_n.$$

Proof: Given $s_n \leq t_n \quad \forall n \geq N \Rightarrow \inf\ s_n \leq t_n \quad \forall n \geq N$. Therefore $\inf\ s_n \leq t_n \quad \forall n \geq N \Rightarrow$

$$\lim_{n\to\infty} \inf\ s_n \leq \lim_{n\to\infty} \inf\ t_n$$

Similarly, $s_n \leq t_n \quad \forall n \geq N \Rightarrow s_n \leq \sup\ t_n \quad \forall n \geq N \Rightarrow \sup\ s_n \leq \sup\ t_n \Rightarrow$

$$\lim_{n\to\infty} \sup\ s_n \leq \lim_{n\to\infty} \sup\ t_n.$$

Remark 1.109 ***Sandwitch number:*** *For $0 \leq x_n \leq s_n \quad \forall n \geq N$ and if $s_n \to 0$ then $x_n \to 0$.*

Theorem 1.110 ***Some Special Sequences:***
(a) If $p > 0$ then

$$\lim_{n\to\infty} \frac{1}{n^p} = 0.$$

(b) If $p > 0$ then

$$\lim_{n\to\infty} \sqrt[n]{p} = 1.$$

(c)

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

(d) If $p > 0, \alpha$ is real then

$$\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0.$$

(e) If $|x| < 1$ then

$$\lim_{n\to\infty} x^n = 0.$$

Proof: **(a)** Given $p > 0$ there exists an integer N such that $N > \frac{1}{\epsilon^{1/p}}$. Now, $\left|\frac{1}{n^p} - 0\right| = \left|\frac{1}{n^p}\right| \leq \frac{1}{N^p} < \epsilon [\because p < 0]$.

(b) Case (i): Suppose $p > 1$. Let $x_n = \sqrt[n]{p} - 1 \geq 0[\because p > 1]$. $\therefore \sqrt[n]{p} = 1 + x_n \Rightarrow p = (1 + x_n)^n = 1 + nx_n + n_{c_2}x_n^2 + .. + x_n^n \Rightarrow p \geq 1 + nx_n[\because x_n \geq 0] \Rightarrow p - 1 \geq nx_n \Rightarrow 0 \leq x_n \leq \frac{p-1}{n}$. Since $\frac{p-1}{n} \to 0$ as $n \to \infty \Rightarrow x_n \to 0$ (by the above remark) $\Rightarrow$

$$\begin{aligned} &\lim_{n\to\infty} x_n = 0 \\ \Rightarrow &\lim_{n\to\infty} \sqrt[n]{p} = 0 \\ \Rightarrow &\lim_{n\to\infty} \sqrt[n]{p} = 1 \\ \Rightarrow &(\sqrt[n]{p}) \to 1 \text{ as } n \to \infty. \end{aligned}$$

Case (ii): Suppose $p = 1$. Then $\sqrt[n]{p} = 1 \Rightarrow (\sqrt[n]{p}) = 1 \to 1$ as $n \to \infty$.
Case (iii): Suppose $0 < p < 1$. Now, $p < 1 \Rightarrow 1/p > 1$. By Case (i) $\sqrt[n]{p} \to 1$ as $n \to \infty$. $\Rightarrow \frac{1}{\sqrt[n]{p}} \to 1$ as $n \to \infty$. $\Rightarrow \sqrt[n]{p} \to 1$ as $n \to \infty$.
(c)

$$\lim_{n\to\infty} \sqrt[n]{n} =$$

Let $x_n = \sqrt[n]{n} - 1 \geq 0(\because n \geq 1) \Rightarrow \sqrt[n]{n} = 1 + x_n \Rightarrow n = (1 + x_n)^n = 1 + nx_n + n_{c_2}x_n^2 + ... + x_n^n, n \geq n_{c_2}x_n^2 \Rightarrow n \geq \frac{n(n-1)}{2}x_n^2 \Rightarrow x_n^2 \leq \frac{2}{n-1}$ $\forall n \geq 2 \Rightarrow 0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$ $\forall n \geq 2$. Now, $\sqrt{\frac{2}{n-1}}$ as $n \to \infty$. By the above remark $x_n \to 0$ as $n \to \infty$. $\therefore \sqrt[n]{n} \to 1$ as $n \to \infty$.
(d) Let k be any positive integer such that $k > \alpha$. Let $n > 2k$,

$$\begin{aligned} (1+p)^n &= 1 + np + \frac{n(n-1)}{2}p^2 + ... + n_{c_{k-1}}p^{k-1} + ... + p^n \\ &\geq n_{c_k}p^k \\ &= \frac{n(n-1)\cdots(n-(k-1))}{1\cdot 2\cdots k}p^k \\ &> \frac{\frac{n}{2}\frac{n}{2}\cdots\frac{n}{2}}{k!}p^k \\ &= \frac{n^k}{2^k k!}p^k \\ &> \frac{n^k}{2^k}\frac{p^k}{k!} \\ \frac{1}{(1+p)^n} &< \frac{2^k}{n^k}\frac{k!}{p^k} \\ \frac{n^\alpha}{(1+p)^n} &< \frac{2^k k!}{p^k}\frac{1}{n^{k-\alpha}} \\ \Rightarrow 0 \leq \frac{n^\alpha}{(1+p)^n} &< \frac{2^k k!}{p^k}\frac{1}{n^{k-\alpha}} \end{aligned}$$

Also $\frac{1}{n^{k-\alpha}} \to 0$ as $n \to \infty(\because k - \alpha > 0$ by (a))
By the above remark,

$$\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$$

(e) $|x| < 1 \Rightarrow \frac{1}{|x|} > 1 \Rightarrow \frac{1}{|x|} = 1 + p, p > 0$, put $\alpha = 0$ in (d). We have $\frac{1}{(1+p)^n} \to 0$ as $n \to \infty \Rightarrow |x|^n \to 0$ as $n \to \infty \Rightarrow x^n \to 0$ as $n \to \infty$.

2. UNIT II

Series:
Let

$$\sum_{n=1}^{\infty} a_n$$

be a series and let

$$s_n = a_1 + a_2 + .. + a_n = \sum_{n=1}^{\infty} a_k$$

the *nth* partial sum of the series $\sum a_n$. we can form a sequence $\{s_n\}$ and this $\{s_n\}$ is called sequence of partial sum of the series.

Definition 2.1 *If $\{s_n\} \to s$ as $n \to \infty$ then we write*

$$\sum_{n=1}^{\infty} a_n = s$$

and the series $\sum a_n$ converges to s. s is called sum of the series.

Note 2.2
1. *If $\{s_n\}$ diverges then the series is said to diverge.*
2. *For convergence we shall consider the series of the form*

$$\sum_{n=0}^{\infty} \alpha_n.$$

Theorem 2.3 *A series of non-negative term converges iff its partial sum forms a bounded sequence.*
Proof: Suppose $\sum a_n$ converges. $\Rightarrow \{s_n\}$ converges. $\Rightarrow \{s_n\}$ is bounded. (Theorem 1.85(c)). Conversely: Suppose $\{s_n\}$ is bounded. Then $\{s_n\}$ is monotonic increasing $\Rightarrow \{s_n\}$ converges. (Theorem 1.102) $\Rightarrow \sum a_n$ converges.

Theorem 2.4 ***Cauchy's Criterian:*** *$\sum a_n$ converges iff $\forall \epsilon > 0$, there exists an integer N such that*

$$\left|\sum_{k=n}^{m} a_k\right| < \epsilon \quad \text{if } m \geq n \geq N.$$

Proof: Let $\sum a_n$ converges. Let $s_n = a_1 + a_2 + ... + a_n \Rightarrow \{s_n\}$ converges. $\Rightarrow \{s_n\}$ is Cauchy sequence. Given $\epsilon > 0$ there exists an integer N such that $|s_m - s_n| < \epsilon \;\; \forall m \geq n \geq N \Rightarrow$

$$\left|\sum_{k=n}^{m} a_k\right| < \epsilon \; \forall m \geq n \geq N.$$

Conversely, suppose

$$\left|\sum_{k=n}^{m} a_k\right| < \epsilon \ \forall m \geq n \geq N.....(1)$$

for all $\epsilon > 0$ and for some integer N. To prove, $\sum a_n$ converges. (1)$\Rightarrow$ $|s_m - s_n| < \epsilon \ \ \forall m \geq n \geq N$. Every Cauchy sequence converges. $\Rightarrow \{s_n\}$ converges. $\Rightarrow \sum a_n$ converges.

Theorem 2.5 *If $\sum a_n$ converges, then*

$$\lim_{n\to\infty} a_n = 0.$$

Proof: Given $\sum a_n$ converges. By Cauchy's criterian there exists N such that

$$\left|\sum_{k=n}^{m} a_k\right| < \epsilon \ \forall m \geq n \geq N. \text{ Taking } m = n,$$
$$|a_n| < \epsilon \ \forall n \geq N$$
$$\Rightarrow a_n \to 0 \text{ as } n \to \infty.$$

Note 2.6 *Converse of the above theorem and need not be true. Consider $\{1/n\}$, $1/n \to 0$ as $n \to \infty$. But $\sum 1/n$ diverges.*

Theorem 2.7 ***Comparison test:***
(a) If $|a_n| < C_n$ for $n \geq N_0$ where N_0 is some fixed integer and if $\sum C_n$ converges then $\sum a_n$ converges.
(b) If $a_n \geq d_n \geq 0 \ \ \forall n \geq N_0$ and if $\sum d_n$ diverges then $\sum a_n$ also diverges.
Proof: (a) Given $\sum C_n$ converges. By Cauchy's criterion. Given $\epsilon > 0$ there exists +ve integer $N \geq N_0$ such that

$$\left|\sum_{k=n}^{m} a_k\right| < \epsilon \ \forall m \geq n \geq N.$$
$$\text{Now } \left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| \leq \sum_{k=n}^{m} C_k < \epsilon \ \forall m \geq n \geq N$$
$$\therefore \left|\sum_{k=n}^{m} a_k\right| < \epsilon \ \forall m \geq n \geq N.$$

$\therefore \sum a_n$ converges.
(b) Given $0 \leq d_n \leq a_n \ \ n \geq N_0$. Suppose $\sum a_n$ converges. $\sum d_n$ converges by (a) $\Rightarrow\Leftarrow$. $\therefore \sum a_n$ diverges.

Series of non negative terms:

Theorem 2.8 *If* $0 \leq x < 1$ *then*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \ x \geq 1$$

then the series diverges.

Proof: Let $\{s_n\}$ be a sequence of partial sum of the series $\sum x^n$. Suppose $0 \leq x \leq 1$
$s_n = 1 + x + x^2 + ... + x^n = \frac{1-x^n}{1-x}$. Since $x^{n+1} \to 0$ as $n \to \infty$ if $0 \leq x < 1$ (by Theorem 1.108(e)) $\Rightarrow s_n \to \frac{1}{1-x}$ as $n \to \infty$ if $0 \leq x < 1 \Rightarrow \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. suppose $x = 1, \ s_n = n + 1 \Rightarrow \{s_n\}$ diverges. $\Rightarrow \{s_n\}$ unbounded diverges. $\therefore \sum x^n$ diverges. Suppose $x > 1 \Rightarrow x^n > 1 \Rightarrow \sum x^n > \sum 1 \ \ (0 \leq 1 < x)$. $\therefore$ $\sum 1$ is diverges. $\therefore$ By comparison test. $\sum x^n$ diverges.

Theorem 2.9 ***Cauchy's condensation test:*** *Suppose* $a_1 \geq a_2 \geq ... \geq 0$ *then the series*

$$\sum_{n=1}^{\infty} a^n$$

converges iff

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + ...$$

converges.

Proof: Let $s_n = a_1 + a_2 + ... + a_n; \ t_k = a_1 + 2a_2 + ... + 2^k a_2^k$.
Case (i): $n < 2^k$

$$\begin{aligned} s_n &\leq a_1 + (a_2 + a_3) + ... + (a_{2^k} + a_{2^k+1} + ... + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + ... + 2^k a_{2^k} \\ &= t_k \\ s_n &\leq t_k ......(1) \end{aligned}$$

Case (ii): $n < 2^k$

$$\begin{aligned} s_n &\geq a_1 + a_2 + (a_3 + a_4) + .. + (a_{2^{k-1}+1} + ... + a_{2^k}) \\ &\geq \frac{a_1}{2} + a_2 + 2a_4 + ... + 2^{k-1} a_{2^k} \\ 2s_n &\geq a_1 + 2a_2 + 2^2 a_4 + .. + 2^k a_{2^k} = t_k \\ 2s_n &\geq t^k ......(2) \end{aligned}$$

From (1) and (2), $\{s_n\}$ and $\{t_n\}$ are either both bounded or both unbounded. (i.e.) $\{s_n\}$ is bounded $\Leftrightarrow \{t_k\}$ is bounded. $\Rightarrow \sum a_n$ converges. $\Leftrightarrow \sum 2^k a_{2^k}$ converges. (by Theorem 2.3)

Theorem 2.10 *$\sum \frac{1}{n^p}$ converges if $p > 1$ and $\sum \frac{1}{n^p}$ converges if $p \leq 1$.*
Proof: $\{\frac{1}{n}\}$ is a decreasing sequence. $\Rightarrow \frac{1}{n} \geq \frac{1}{n+1} \Rightarrow \frac{1}{n^p} \geq \frac{1}{(n+1)^p}$ $\forall p > 0$
Case (i): Suppose $p > 0$. Consider the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k} = \sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{k-kp} = \sum_{k=0}^{\infty} 2^{k(1-p)}$$

By Theorem reft16, $\sum x^k$ converges if $0 \leq x < 1$, diverges if $x \geq 1$. Now,

$$\sum_{k=0}^{\infty} 2^{k(1-p)} = \sum_{k=0}^{\infty} (2^{1-p})^k \text{converges if } p > 1.$$

$$\sum_{k=0}^{\infty} (2^{1-p})^k \text{ diverges if } p \leq 1.$$

Case (ii): If $p \leq 0$ then $\{\frac{1}{n^p}\}$ is an unbounded sequence $\Rightarrow \{\frac{1}{n^p}\}$ diverges. $\therefore \sum 1/n^p$ diverges if $p \leq 0$. $\therefore \sum \frac{1}{n^p}$ converges $p > 1$. $\sum \frac{1}{n^p}$ diverges $p \leq 1$.

Theorem 2.11 *If $p > 1$,*

$$\sum_{k=0}^{\infty} \frac{1}{n(\log n)^p}$$

converges and if $p \leq 1$ this series diverges.
Proof: $\{\log n\}$ is an increasing sequence. $\Rightarrow$ $\frac{1}{n(\log n)^p}$ is a decreasing sequence. Consider

$$\sum_{k=1}^{\infty} 2^k \frac{1}{2^k (\log 2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{(k \log 2)^p} = \frac{1}{(\log 2)^p} \sum_{k=1}^{\infty} \frac{1}{k^p}.$$

converges if $p > 1$, diverges of $p \leq 1$. [By Theorem 2.10] By Cauchy's condensation test,

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$$

converges if $p > 1$, diverges of $p \leq 1$.

Problem 2.12 *Test the converges of the series*

$$\sum_{n=3}^{\infty} \frac{1}{n(\log n) \cdot \log(\log n)}.$$

Proof: $\{n \log n \ \log(\log n)\}$ is an increasing sequence. $\Rightarrow \{\frac{1}{n \log n \ \log(\log n)}\}$ is a decreasing sequence. Consider,

$$\begin{aligned}
\sum_{k=2}^{\infty} 2^k a_{2^k} &= \sum_{k=2}^{\infty} 2^k \frac{1}{2^k \log 2^k \log(\log 2^k)} \\
&= \sum_{k=2}^{\infty} \frac{1}{k \log 2 \log(k \log 2)} \\
&= \frac{1}{\log 2} \sum_{k=2}^{\infty} \frac{1}{k \log(k \log 2)}
\end{aligned}$$

Now

$$\begin{aligned}
\log 2 &< 1 \\
\Rightarrow k \log 2 &< k \ k > 0 \\
\Rightarrow \log(k \log 2) &< \log k \\
\Rightarrow k \log(k \log 2) &< k(\log k) \\
\Rightarrow \frac{1}{k \log(k \log 2)} &> \frac{1}{k \log k} \\
\Rightarrow \sum_{k=2}^{\infty} \frac{1}{k \log(k \log 2)} &> \sum_{k=2}^{\infty} \frac{1}{k \log k}
\end{aligned}$$

By previous problem put $p = 1 \sum \frac{1}{k \log k}$ diverges. By comparison test $\sum \frac{1}{k \log(k \log 2)}$ diverges $\Rightarrow \frac{1}{\log 2} \sum \frac{1}{k \log(k \log 2)}$. $\therefore$ By condensation test, the given sequence diverges.

Definition 2.13 $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... = \sum \frac{1}{n!}$.

Note 2.14 *The above definition is well defined.*
Proof: Now $e = \sum 1/n!$. Let

$$\begin{aligned}
s_n &= \sum_{k=0}^{n} \frac{1}{k!} = 1 + \frac{1}{1!} + ... + \frac{1}{n!} \\
&< 1 + \frac{1}{1^2} + \frac{1}{2^1} + \frac{1}{2^2} + ... + \frac{1}{2^{n-1}} \\
&< 1 + \frac{1}{1^2} + \frac{1}{2^1} + \frac{1}{2^2} + ... + \frac{1}{2^n} + ... \\
&= 1 + \frac{1}{1 - \frac{1}{2}} \\
&= 1 + \frac{1}{\frac{1}{2}} = 1 + 2 \\
&= 3 \\
\therefore s_n &< 3 \ \forall n
\end{aligned}$$

$\therefore$ $\{s_n\}$ is a bounded sequence. Since $\{s_n\}$ is monotonic increasing and bounded, $\{s_n\}$ is converges. $\Rightarrow \sum \frac{1}{n!}$ converges. $\therefore e$ is well defined.

Theorem 2.15

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e.\ Let\ s_n = 1+\frac{1}{1!}+\frac{1}{2!}+...+\frac{1}{n!}.$$

Proof: Let

$$\begin{aligned} t_n &= (1+\frac{1}{n})^n \\ &= 1+n\cdot\frac{1}{n}+\frac{n(n-1)}{2}\frac{1}{n^2}+\frac{n(n-1)(n-2)}{1\cdot2\cdot3}\frac{1}{n^3}+... \\ &\quad +\frac{n(n-1)\cdots2\cdot1}{1,2\cdots n}\frac{1}{n^n} \\ &= 1+1+\frac{1(1-\frac{1}{n})}{2}+\frac{1(1-\frac{1}{n})(1-\frac{2}{n})}{1\cdot2\cdot3}+... \\ &\quad +(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{(n-2)}{n})(1-\frac{\overline{n-1}}{n})\frac{1}{n!}.....(a) \\ &< 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+...+\frac{1}{n!} \\ &= s_n \end{aligned}$$

$$\therefore t_n < s_n\ \forall n$$

$$\Rightarrow \lim_{n\to\infty}\sup\ t_n < \lim_{n\to\infty}\sup\ S_n = e.....(1)[\because \lim_{n\to\infty}s_n = e]$$

Consider $m \le n$, Using (a)

$$t_n \ge 1+1+(1-\frac{1}{n})\frac{1}{2!}+...+(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{m-1}{n})\frac{1}{m!}$$

keeping m, fixed and letting $n\to\infty$ we have

$$\lim_{n\to\infty}\ \inf t_n \ge 1+\frac{1}{1!}+\frac{1}{2!}+...+\frac{1}{m!} = s_m$$

$$\lim_{n\to\infty}\ \inf t_n \ge s_m\ \forall m$$

$$\text{Letting } m\to\infty \Rightarrow \lim_{n\to\infty}\ \inf t_n \ge e......(2)$$

From (1) and (2),

$$\lim_{n\to\infty}\ \inf t_n \ge e \ge \lim_{n\to\infty}\ \sup t_n.....(B)$$

$$\lim_{n\to\infty}\ \inf t_n \ge \lim_{n\to\infty}\ \sup t_n$$

$$\text{Always } \lim_{n\to\infty}\ \inf t_n \le \lim_{n\to\infty}\ \sup t_n$$

$$\Rightarrow \lim_{n\to\infty}\ \inf t_n = \lim_{n\to\infty}\ \sup t_n$$

$$\Rightarrow \lim_{n\to\infty} t_n \text{ exists and } \lim_{n\to\infty} t_n = e$$

$$\therefore \lim_{n\to\infty}(1+\frac{1}{n})^n = e$$

Lemma 2.16 *Prove that* $0 < e - s_n < \frac{1}{n!n}$.
Proof: Clearly, $e - s_n > 0 \ \forall n$

$$\begin{aligned}
e - s_n &= \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + ... \\
&= \frac{1}{(n+1)!}[1 + \frac{1}{n+2} + \frac{1}{(n+2)(n+3)} + ...] \\
&< \frac{1}{(n+1)!}(1 + \frac{1}{n+2} + \frac{1}{(n+1)^2} + ...) \\
&= \frac{1}{(n+1)!}(\frac{1}{1 - \frac{1}{n+1}}) \\
&= \frac{1}{(n+1)!}(\frac{n+1}{n+1-1}) \\
&= \frac{1}{n!}\frac{1}{n} \\
\therefore 0 < e - s_n &< \frac{1}{n!n}
\end{aligned}$$

Lemma 2.17 *Prove that* e *is irrational.*
Proof: Suppose e is rational. $e = \frac{p}{q}$, $q \neq 0$; $\gcd(p,q) = 1$; p, q are integer.
By the above lemma $0 < e - S_q < \frac{1}{q!q} \Rightarrow 0 < (e - s_q)q! < \frac{1}{q}$........ (1)
Now, $q!e$ is an integer. $[\because q!e = q!\frac{p}{q} = (q-1)!p =$ an integer$]$

$$\begin{aligned}
q!s_q &= q![1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{q!}] \\
&= q! + q! + 3 \cdot 4 \cdots q + ... + q + 1 \\
&= \text{an integer} \\
q \geq 1 &\Rightarrow \frac{1}{q} \leq 1 \\
\therefore \ (1) \ &\Rightarrow 0 < q!(e - s_q) < \frac{1}{q} \leq 1 \\
&0 < (e - s_q)q! < 1
\end{aligned}$$

This means that $q!(e - s_q)$ is an integer lying between 0 and 1. $\therefore e$ must be irrational.

Root and Ratio test

Theorem 2.18 ***Root test:*** *Given* $\sum a_n$ *and*

$$\alpha = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|}$$

(a) if $\alpha < 1$, $\sum a_n$ *converges.*
(b) if $\alpha > 1$, $\sum a_n$ *diverges.*

(c) if $\alpha = 1$ then the test gives no information.
Proof: (a) If $\alpha < 1$ then there exists β with $\alpha < \beta < 1$, and an integer N such that $\sqrt[n]{|a_n|} < \beta \ \forall n \geq N$ (By Theorem 1.105(b)), $|a_n| < \beta^n \quad \forall n \geq N$. But $\sum \beta^n$ converges ($\because \beta < 1$) $\therefore$ By comparison test, $\sum a_n$ converges.
(b) If $\alpha > 1$, by Theorem 1.105(a); there is a sequence $\{n_k\}$ such that $\sqrt[n_k]{|a_{n_k}|} \to \alpha$ as $k \to \infty[\because \alpha$ is a subsequence limit$] \Rightarrow |a_n| > 1$ for infinitely many values of n. $\{a_n\}$ does not convergers to 0. $\therefore \sum a_n$ diverges [By Theorem 2.5]
(c) Suppose $\alpha = 1$. Consider the series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$. Take $a_n = \frac{1}{n}$. Then

$$\begin{aligned} a_n^{\frac{1}{n}} &= (\frac{1}{n})^{\frac{1}{n}} \\ &= \frac{1}{n^{\frac{1}{n}}} \\ \lim_{n\to\infty} \sup a_n^{\frac{1}{n}} &= \lim_{n\to\infty} \sup \frac{1}{n^{\frac{1}{n}}} = 1 \ [\because \lim_{n\to\infty} n^{\frac{1}{n}} = 1] \end{aligned}$$

Then $\sum 1/n$ diverges. $a_n = 1/n^2$

$$\lim_{n\to\infty} \sup a_n^{\frac{1}{n}} = \lim_{n\to\infty} \sup(\frac{1}{n^{\frac{1}{n}}})^2 = 1$$

But $\sum \frac{1}{n^2}$ converges. $\therefore$ The root test fails.

Theorem 2.19 ***Ratio test:*** *Consider the series $\sum a_n$*
(a) It converges if

$$\lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right| < 1$$

(b) It diverges if $\left|\frac{a_{n+1}}{a_n}\right| \geq 1 \ \forall n \geq N$.
Proof: (a) Let

$$\alpha = \lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right| < 1 \ \text{ and } \alpha < 1.$$

Then there exists β with $\alpha < \beta < 1$ and an integer N such that

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta \ \forall n \geq N.$$
$$|a_{n+1}| < \beta\,|a_n| \ \ \forall n \geq N.$$
$$|a_N + 1| < \beta\,|a_N|$$
$$|a_N + 2| < \beta\,|a_{N+1}| < \beta \cdot \beta \cdot |a_N| = \beta^2\,|a_N|$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$|a_N + p| < \beta^p\,|a_N| \ \ \forall p \geq 0.$$

Take $n = N + p \ \forall p \geq 0$

$$|a_n| < \beta^{n-N}\,|a_N| \ \ \forall n \geq N.$$
$$= \beta^{-N}\,|a_N|\,\beta^n$$
$$(i.e.)\,|a_n| < (\beta^{-N}\,|a_N|)\beta^n$$

Now $\sum \beta^n$ converges ($\because \beta < 1$) $\therefore \sum \alpha_n$ converges, by comparison test.
(b)

$$\left|\frac{a_{n+1}}{a_n}\right| \geq 1 \ \forall n \geq n_0$$
$$\Rightarrow |a_{n+1}| \geq |a_N| \ \ \forall n \geq n_0$$
$$\Rightarrow (a_n) \nrightarrow 0 \ \text{ as } n \to \infty [\because |a_n| \text{ is an increasing sequence.}$$
$$(i.e.) 0 \leq |a_1| \leq |a_1| \leq ...]$$
$$\Rightarrow \sum a_n \ \text{ diverges.}$$

Note 2.20

$$\lim_{n\to\infty} \sup \left|\frac{a_n + 1}{a_n}\right| = 1 \ \ \textit{gives no information.}$$

Proof: Consider

$$\lim_{n\to\infty} \sup \left|\frac{a_n + 1}{a_n}\right| = 1$$

$$\text{Consider the series } \ \sum \frac{1}{n}$$

$$\text{Now } a_n = \frac{1}{n} \text{ and } a_{n+1} = \frac{1}{n+1}$$
$$\frac{a_{n+1}}{a_n} = \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}}$$
$$\lim_{n\to\infty} \sup \left|\frac{a_n + 1}{a_n}\right| = \lim_{n\to\infty} \frac{1}{1 + \frac{1}{n}} = 1$$

Observe, $\sum \frac{1}{n}$ diverges. Consider $\sum \frac{1}{n^2}$

$$a_n = \frac{1}{n^2};\ a_{n+1} = \frac{1}{(n+1)^2}$$
$$\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} = \frac{1}{(1+1/n)^2}$$
$$\lim_{n\to\infty} \sup \left|\frac{a_n + 1}{a_n}\right| = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^2} = 1$$

Note that $\sum \frac{1}{n^2}$ converges. $\therefore \lim_{n\to\infty} \sup \left|\frac{a_n+1}{a_n}\right| = 1$ gives no information.

Problem 2.21 *Consider the series* $1/2 + 1/3 + 1/2^2 + 1/3^2 + ...$
Let

$$a_n = \begin{cases} \frac{1}{2^{\frac{n+1}{2}}} & \text{if } n \text{ is odd} \\ \frac{1}{3^{\frac{n}{2}}} & \text{if } n \text{ is even} \end{cases}$$
$$a_n^{1/n} = \begin{cases} \frac{1}{2^{\frac{n+1}{2n}}} & \text{if } n \text{ is odd} \\ \frac{1}{3^{\frac{n}{2n}}} & \text{if } n \text{ is even} \end{cases}$$
$$= \begin{cases} \frac{1}{2^{\frac{1}{2}+\frac{1}{2n}}} & \text{if } n \text{ is odd} \\ \frac{1}{3^{\frac{1}{2}}} & \text{if } n \text{ is even} \end{cases}$$
$$\lim_{n\to\infty} \inf \sqrt[n]{|a_n|} = \frac{1}{\sqrt{3}};\ \lim_{n\to\infty} \sup \sqrt[n]{|a_n|} = \frac{1}{\sqrt{2}} < 1$$

$\therefore \sum a_n$ *converges*

Note 2.22

$$\lim_{n\to\infty} \sup |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} (\frac{3}{2})^{\frac{n}{2}} \frac{1}{2} = \infty$$
$$\lim_{n\to\infty} \inf |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} (\frac{2}{3})^{\frac{n}{2}} \sqrt{2} = 0$$

Here we observe that whenn is odd. $|\frac{a_{n+1}}{a_n}| = \frac{2^{\frac{n+1}{2}}}{3^{\frac{n}{2}}} = (\frac{2}{3})^{\frac{n}{2}}\sqrt{2} \leq 1\ \forall$ *odd* $n \geq n_0$. $\therefore$ *We need not apply ratio test.*

Problem 2.23 *Test the converges series* $\frac{1}{2}+1+\frac{1}{8}+\frac{1}{4}+\frac{1}{32}+\frac{1}{16}+\frac{1}{128}+\frac{2}{64}+...$
(i.e.)$\frac{1}{2} + 1 + \frac{1}{2^3} + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^4} + \frac{1}{2^7} + \frac{1}{2^6} + ...$
Solution:

$$a_n = \begin{cases} \frac{1}{2^n} & \text{if } n \text{ is odd} \\ \frac{1}{2^{n-2}} & \text{if } n \text{ is even} \end{cases}$$
$$a_n^{\frac{1}{n}} = \begin{cases} \frac{1}{2} & \text{if } n \text{ is odd} \\ \frac{1}{2^{1-\frac{2}{n}}} & \text{if } n \text{ is even} \end{cases}$$

$$\lim_{n\to\infty} \sup a_n^{\frac{1}{n}} = \frac{1}{2} < 1$$

$\therefore \sum a_n$ converges.

Note 2.24 *Let n is even*

$$\begin{aligned}\frac{a_{n+1}}{a^n} &= \frac{2^{n-2}}{2^{n+1}} \;(\because a_n = \frac{1}{2^{n-2}})\\ &= \frac{2^n 2^{-2}}{2^n 2^1} = \frac{1}{2^3}\\ &= 1/8\end{aligned}$$

When, n is odd

$$\begin{aligned}\frac{a_{n+1}}{a^n} &= \frac{1}{2^{n-1}} \cdot 2^n \;(\because a_n = \frac{1}{2^n}\\ &= \frac{1}{2^{-1}} = 2\\ \therefore |\frac{a_{n+1}}{a^n}| &= \frac{1}{8} < 1 \;\forall n \geq n_0\end{aligned}$$

There is no need to apply ratio test.

Remark 2.25

$$\lim_{n\to\infty} \sup |\frac{a_{n+1}}{a^n}| = 2;\; \lim_{n\to\infty} \inf |\frac{a_{n+1}}{a^n}| = \frac{1}{8}.$$

Theorem 2.26 *For any sequence $\{c_n\}$ of +ve numbers,*
(a)

$$\lim_{n\to\infty} \sup \sqrt[n]{c_n} \leq \lim_{n\to\infty} \sup \frac{c_{n+1}}{c_n}$$

(b)

$$\lim_{n\to\infty} \inf \frac{c_{n+1}}{c_n} \leq \lim_{n\to\infty} \inf \sqrt[n]{c_n}$$

Proof: Let

$$\alpha = \lim_{n\to\infty} \sup \frac{c_{n+1}}{c_n}$$

Suppose $\alpha = \infty$ then there is nothing to prove. If α is a real number, then there exists $\beta > \alpha$ under integer N such that $\frac{c_{n+1}}{c_n} < \beta \;\forall n \geq N$ [by Theorem

1.105(b)]

$$\frac{c_{N+1}}{c_N} < \beta$$
$$\frac{c_{N+2}}{c_{N+1}} < \beta$$
$$\frac{c_{N+3}}{c_{N+2}} < \beta$$
$$\vdots$$
$$\frac{c_{N+p}}{c_{N+p-1}} < \beta$$

multiplying all these inequalities

$$\frac{c_{N+p}}{c_N} < \beta^p \ \forall p \geq 0$$
$$\Rightarrow c_{N+p} < \beta^p c_N \ \forall p \geq 0$$

put $n = N + p$

$$c_n < \beta^{n-N} c_N = (c_N \beta^{-N})\beta^n$$
$$\Rightarrow c_n^{\frac{1}{n}} < (c_N \beta^{-N})^{\frac{1}{n}} \beta$$
$$\lim_{n\to\infty} \sup c_n^{\frac{1}{n}} < \beta [\because \lim_{n\to\infty} (c_N \beta^{-N})^{\frac{1}{n}} = 1]$$

This is true for every $\beta > \alpha$

$$\therefore \lim_{n\to\infty} \sup c_n^{\frac{1}{n}} \leq \alpha = \lim_{n\to\infty} \sup \frac{c_{n+1}}{c_n}$$
$$\therefore \lim_{n\to\infty} \sup \sqrt[n]{c_n} \leq \lim_{n\to\infty} \sup \frac{c_{n+1}}{c_n}$$

(b) Let

$$\alpha = \lim_{n\to\infty} \inf \frac{c_{n+1}}{c_n}.$$

If $\alpha = -\infty$ there is nothing to prove. If α is finite then thee exists a +ve real number $\beta < \alpha$, and an integer N such that

$$\frac{c_{n+1}}{c_n} > \beta \ \forall n \geq N \text{ (by Theorem 1.105(b) for} \inf x < s* \Rightarrow s_n \geq x)$$
$$\Rightarrow \frac{c_{N+1}}{c_N} > \beta$$
$$\Rightarrow \frac{c_{N+2}}{c_{N+1}} > \beta$$
$$\vdots$$
$$\frac{c_{N+p}}{c_{N+p-1}} > \beta$$

multiplying all these inequalities, $\frac{c_{N+p}}{c_N} < \beta^p \ \forall p \geq 0$. put $n = N + p$

$$\frac{c_n}{c_N} > \beta^{n-N}$$
$$\Rightarrow c_n > c_N \beta^{n-N}$$
$$\Rightarrow \sqrt[n]{c_n} > \sqrt[n]{c_N \beta^{-N}} \beta$$
$$\lim_{n\to\infty} \inf \sqrt[n]{c_n} > \beta \ (\because \lim_{n\to\infty} \sqrt[n]{c_N \beta^{-N}} = 1)$$

This is true for every $\beta < \alpha$

$$\therefore \lim_{n\to\infty} \inf \sqrt[n]{c_n} \geq \alpha$$
$$= \lim_{n\to\infty} \inf \frac{c_{n+1}}{c_n}$$
$$\therefore \lim_{n\to\infty} \inf \frac{c_{n+1}}{c_n} \leq \lim_{n\to\infty} \inf \sqrt[n]{c_n}.$$

Power Series

Definition 2.27 *Given a $\{c_n\}$ of complex numbers, the series $\sum_{n=0}^{\infty} c_n x_n$ is called a power series. The numbers c_n are called coefficient of the series and z is a complex number.*

Note 2.28

1. *The series will converge or diverge depending upon the choice of z.*
2. *Every power series there is associated a circle of convergence such that the given power series converge if z is the interior of the circle and diverges if z is exterior of the circle.*

Theorem 2.29 *Given the power series*

$$\sum_{n=0}^{\infty} C_n z^n \text{ and } \alpha = \lim_{n\to\infty} \sup \sqrt[n]{|C_n|}$$

and $R = \frac{1}{\alpha}$ *then* $\sum C_n z^n$ *converges if* $|z| < R$ *and diverges if* $|z| > R$. *(*R *is called the radius of convergence of* $\sum C_n z^n$*)*
Proof: Let

$$\begin{aligned} a_n &= C_n z^n \\ |a_n| &= |C_n||z|^n \\ \lim_{n\to\infty} \sup \sqrt[n]{|a_n|} &= \lim_{n\to\infty} \sup \sqrt[n]{|C_n|}|z| \\ &= \alpha|z| \\ &= \frac{|z|}{R}(\because \alpha = \frac{1}{R}) \end{aligned}$$

By root test $\sum C_n z^n$ converges if $\frac{|z|}{R} < 1$ (i.e.)if $|z| < R$ and $\sum C_n z^n$ diverges if $\frac{|z|}{R} > 1$ (i.e.) if $|z| > R$.

Problem 2.30 *Find the radius of convergence of* $\sum n^n z^n$.
Solution: Let

$$\begin{aligned} c_n &= \sum n^n z^n \\ 1/R &= \lim_{n\to\infty} \sup \sqrt[n]{|c_n|} \\ &= \lim_{n\to\infty} \sup \sqrt[n]{|n_n|} \\ &= \lim_{n\to\infty} n \\ 1/R &= \infty \\ R &= 0 \end{aligned}$$

$\therefore \sum n^n z^n$ is digit on the whole plane.

Note 2.31

$$\begin{aligned} \lim_{n\to\infty} \inf \frac{c_{n+1}}{c_n} &\leq \lim_{n\to\infty} \inf \sqrt[n]{n} \\ &\leq \lim_{n\to\infty} \sup \sqrt[n]{c_n} \\ &\leq \lim_{n\to\infty} \sup \frac{c_{n+1}}{c_n} \\ \textit{If } \lim_{n\to\infty} \frac{c_{n+1}}{c_n} \quad \textit{exists.} \Rightarrow \lim_{n\to\infty} \inf \frac{c_{n+1}}{c_n} &= \lim_{n\to\infty} \sup \frac{c_{n+1}}{c_n} \\ \Rightarrow \lim_{n\to\infty} \inf \sqrt[n]{c_n} &= \lim_{n\to\infty} \sup \sqrt[n]{c_n} \\ \textit{and } \Rightarrow \lim_{n\to\infty} \sqrt[n]{c_n} &= \lim_{n\to\infty} \frac{c_{n+1}}{c_n} \\ \textit{Hence } \frac{1}{R} &= \lim_{n\to\infty} \sup \sqrt[n]{c_n} \\ &= \lim_{n\to\infty} \sqrt[n]{c_n} \\ \frac{1}{R} &= \lim_{n\to\infty} \frac{c_{n+1}}{c_n}. \end{aligned}$$

Problem 2.32 *Find the radius of convergence of* $\sum \frac{z^n}{n!}$
Solution: Here, $c_n = \frac{1}{n!}$; $c_{n+1} = \frac{1}{(n+1)!}$. Now,

$$\frac{c_{n+1}}{c_n} = \frac{1}{n+1}$$
$$\frac{1}{R} = \lim_{n\to\infty} \frac{c_{n+1}}{c_n}$$
$$= \lim_{n\to\infty} \frac{1}{n+1} = \frac{1}{\infty} = 0$$
$$R = \infty$$

$\therefore \sum \frac{z^n}{n!}$ converges $\forall z$.

Problem 2.33 *Find the radius of convergence of* $\sum z^n$
Solution: Here, $c_n = 1$; $c_{n+1} = 1$. Now, $\frac{1}{R} = \lim_{n\to\infty} \frac{c_{n+1}}{c_n} = 1 \Rightarrow R = 1$. $\therefore \sum z^n$ converges if $|z| < 1$ and $\sum z^n$ diverges if $|z| > 1$.

Problem 2.34 $\sum \frac{z^n}{n^2}$ *has radius of converges and prove that the power series converges for all* z *within* $|z| \leq 1$.
Solution: Here, $c_n = \frac{1}{n^2}$; $c_{n+1} = \frac{1}{(n+1)^2}$. Now,

$$\frac{1}{R} = \lim_{n\to\infty} \frac{c_{n+1}}{c_n}$$
$$= \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$
$$= \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^2}$$
$$\frac{1}{R} = 1$$
$$R = 1$$

$\therefore \sum \frac{z^n}{n^2}$ converges if $|z| < 1$. When $|z| = 1$, consider $|\frac{z^n}{N^2}| = \frac{|z^n|}{|n^2|} = \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ converges , By comparison test. $\sum \frac{z^n}{n^2}$ converges if $|z| < 1$ and $\sum \frac{z^n}{n^2}$ converges within and on the circle $|z| = 1$. $\therefore \sum \frac{z^n}{n^2}$ converges $\forall z$ with $|z| \leq 1$.

Summation by Parts Given two sequences $\{a_n\}$ and $\{b_n\}$. Put

$$A_n = \sum_{k=0}^{n} a_k \quad \text{if } n \geq 0.$$

Put $A_{-1} = 0$. Then for $0 \leq p \leq q$

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

Proof:

$$\begin{aligned}
A_n &= a_0 + a_1 + ... + a_{n-1} + a_n = A_{n-1} + a_n \\
A_n - A_{n-1} &= a_n \\
\sum_{n=p}^{q} A_n b_n &= \sum_{n=p}^{q-1} (A_n - A_{n-1}) b_n \\
&= \sum_{n=p}^{q} a_n b_n - \sum_{n=p}^{q} A_{n-1} b_n \\
&= \sum_{n=p}^{q} A_n b_n - [A_{p-1} b_p + A_p b_{p+1} + ... + A_{q-1} b_q] \\
&= \sum_{n=p}^{q} A_n b_n - \sum_{n=p-1}^{q-1} A_n b_{n+1} \\
&= \sum_{n=p}^{q-1} A_n b_n + A_q b_q - [\sum_{n=p}^{q-1} A_n b_{n+1} + A_{p-1} b_p] \\
&= \sum_{n=p}^{q-1} A_n (b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.
\end{aligned}$$

Note 2.35 *The above formula is called partial summation formula. It is used to investigate the series of the form* $\sum a_n b_n$.

Theorem 2.36 ***Dirichlet Test:***
(a) Suppose the partial summation A_n *of* $\sum a_n$ *form a bounded sequence.*
(b) $b_0 \geq b_1 \geq b_2 \geq ...$
(c) If

$$\lim_{n\to\infty} b_n = 0.$$

Then $\sum a_n b_n$ *converges.*
Proof: Given that $\{A_n\}$ is a sequence of partial sum of the series $\sum a_n$. Also given that $\{A_n\}$ is bounded by (a) $\Rightarrow$ There exists a real number M such that $|A_n| \leq M \ \ \forall M$. Also by (c) $\lim_{n\to\infty} b_n = 0 \Rightarrow$ Given $\epsilon = 0$ there exists a +ve integer N such that $|b_n - 0| < \epsilon/2M \ \ \forall n \geq N$ (i.e.) $|b_n| < \epsilon/2M$ $\forall n \geq N$....(1)

For $N \leq p \leq q$,

$$\begin{aligned}
|\sum_{n=p}^{q} a_n b_n| &= \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p \\
&\leq M|\sum_{n=p}^{q-1}(b_n - b_{n+1}) + b_q + b_p| \\
&= M|(b_p - b_{p+1}) + (b_{p+1} - b_{p+2}) + ... + (b_{q-1} - b_q) + b_q + b_p| \\
&= M|(b_p - b_q) + b_q + b_p| \\
&= 2M|b_p|
\end{aligned}$$

$$|\sum_{n=p}^{q} a_n b_n| \leq 2M|b_p| < 2M \cdot \frac{\epsilon}{2M} = \epsilon \ [\because p \geq N \quad \text{using (1)}]$$

$$\therefore |\sum_{n=p}^{q} a_n b_n| < \epsilon \ \forall q \geq p \geq N$$

By cauchy's criterian,

$$\sum_{n=1}^{\infty} a_n b_n$$

converges

Theorem 2.37 *(Leibnitz Test)*
(a) Suppose $|c_1| \geq |c_2| \geq |c_3| \geq ...$
(b) $c_{2m-1} \geq 0, c_{2m} \leq 0 (m = 1, 2, 3, ..)$
(c)

$$\lim_{n\to\infty} c_n = 0.$$

Then $\sum c_n$ *converges.*
Proof: By (b) $c_n = (-1)^{n+1}|c_n|$. Take $a_n = (-1)^{n+1}, b_n = |c_n|$. Let $\{A_n\}$ be a sequence of partial summation of the series $\sum a_n = \sum(-1)^{n+1} \Rightarrow \{A_n\}$ is a bounded sequence. Also by (a) $|c_1| \geq |c_2| \geq |c_3| \geq ....$ Also using (c)

$$\lim_{n\to\infty} |c_n| = 0$$

$\therefore$ By the Dirichlet's Test, $\sum(-1)^{n+1}|c_n| = \sum c_n$ converges.

Note 2.38 *The series for which condition (b) holds are called alternating series.*

Theorem 2.39 *Suppose the radius of convergence of* $\sum c_n z^n$ *is* 1. *and suppose* $c_0 \geq c_1 \geq c_2 ....$ *and* $\lim_{n\to\infty} c_n = 0$. *Then* $\sum c_n z^n$ *converges, at every point of the circle* $|z| = 1$ *except possibly at* $z = 1$.

Proof: Consider the series $\sum c_n z^n$. Let $\{A_n\}$ be the sequence of partial sums of the series $\sum z^n$

$$\begin{aligned}\therefore |A_n| &= |1 + z + z^2 + ... + z^n| \\ &= \left|\frac{1 - z^{n+1}}{1 - z}\right| = \frac{|1 - z^{n+1}|}{|1 - z|} \\ &\leq \frac{1 - |z|^{n+1}}{|1 - z|} \\ &= \frac{2}{|1 - z|} \text{ if } |z| = 1, z \neq 1 \\ |A_n| &\leq \frac{2}{|1 - z|}\end{aligned}$$

$\Rightarrow \{A_n\}$ is bounded.
Also $c_0 \geq c_1 \geq ...$ and

$$\lim_{n\to\infty} c_n = 0$$

$\therefore$ By Dirichels test, $\sum c_n z^n$ converges if $|z| = 1$ and $z \neq 1$. Also given that the radius convergence of $\sum c_n z^n$ is 1. $\therefore$ The series $\sum c_n z^n$ converges at every point in and on the circle $|z| = 1$ except at $z = 1$.

Definition 2.40 ***Absolute convergence:*** *The series $\sum a_n$ is said to be converge absolutely if $\sum |a_n|$ converges.*

Theorem 2.41 *If $\sum a_n$ converges absolutely then $\sum |a_n|$ converges.*
Proof: Suppose $\sum a_n$ converges absolutely $\Rightarrow \sum a_n$ converges. Given $\epsilon > 0$ there exists an integer N such that

$$\sum_{k=m}^{n} |a_k| < \epsilon \; \forall n \geq m \geq N.....(1)$$

Also

$$\begin{aligned} |\sum_{k=m}^{n} a_k| &\leq \sum_{k=m}^{n} |a_k| < \epsilon \; \forall n \geq m \geq N \quad \text{by(1)} \\ \Rightarrow |\sum_{k=m}^{n} a_k| &< \epsilon \; \forall n \geq m \geq N\end{aligned}$$

$\Rightarrow \sum a_n$ converges. The converse of the above theorem is not true.

Example 2.42 *Consider the series $\sum_{n=1}^{\infty}(-1)^{n-1}$ converges but it is not absolutely convergent.*
Proof: For $c_n = (-1)^{n-1}$; $c_{2m-1} = (-1)^{2m-1-1} = 1 \geq 0$; $c_{2m} = (-1)^{2m-1} =$

$-1 < 0$; $|c_n| = 1 \forall n$; $|c_1| \geq |c_2| \geq ....$ Now, $\{\frac{1}{n}\}$ is a monotonic decreasing sequence and

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

By Leibnitz test $\sum(-1)^{n-1}\frac{1}{n}$ converges.

$$\sum_{n=1}^{\infty} \left|(-1)^{n-1}\frac{1}{n}\right| = \sum \frac{1}{n} \text{ diverges.}$$

But it is not absolutely convergence. $\therefore$ convergence $\not\Rightarrow$ absolutely convergence.

Note 2.43 *For series of +ve terms convergence and absolutely convergence are the same.*

Theorem 2.44 ***Addition and Multiplication of series:***
$\sum a_n = A$; $\sum b_n = B$. *Then* $\sum(a_n + b_n) = A + B$; $\sum ca_n = cA$ *for any fixed* c.
Proof: Let $\{A_n\}$ be a sequence of partial sums of the series $\sum a_n$ and $\{B_n\}$ be a sequence of partial sum of the series $\sum b_n$. Now $\sum a_n = A$; $\sum b_n = B \Rightarrow A_n \to A$ and $B_n \to B$ as $n \to \infty \Rightarrow A_n + B_n \to A + B$ as $n \to \infty$

$$\begin{aligned} (i.e.) \lim_{n\to\infty}(A_n + B_n) &= A + B \\ \Rightarrow \lim_{n\to\infty}(\sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k) &= A + B \\ \Rightarrow \lim_{n\to\infty}\sum_{k=1}^{n}(a_k + b_k) &= A + B \\ \sum_{k=1}^{\infty}(a_k + b_k) &= A + B \end{aligned}$$

clearly $cA_n \to cA$ as $n \to \infty$

$$\begin{aligned} (i.e.) \lim_{n\to\infty} c\sum_{k=1}^{n}(a_k &= cA) \\ \lim_{n\to\infty}\sum_{k=1}^{n}(ca_k) &= cA \\ \sum_{k=1}^{\infty} ca_k &= cA \end{aligned}$$

Cauchy's Product:
Given $\sum a_n,\ \sum b_n$ we put

$$\begin{aligned}
c_n &= b_n a_0 + b_{n-1}a_1 + ... + b_0 a_n \\
&= \sum_{k=0}^{n} a_k b_n - k \\
(\sum a_n)(\sum b_n) &= a_0 b_0 + (a_0 b_1 + a_1 b_0) + ... + (a_0 b_n + a_1 b_{n-1} + ... + a_n b_0) \\
&= c_0 + c_1 + c_2 + ... + c_{n-1} + ... \\
&= \sum c_n
\end{aligned}$$

Example 2.45 *Cauchy's product of two convergent series need not be convergent.*
Proof: *Consider the series*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}.$$

Here $\left\{\frac{1}{\sqrt{n+1}}\right\}$ *to a decreasing sequence and* $\frac{1}{\sqrt{n+1}} \to 0$ *as* $n \to \infty$. $\therefore$ *By Leibnitz test,*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}} \text{ converges.}$$

Consider the product of two series

$$\begin{aligned}
\sum a_n = \sum \frac{(-1)^n}{\sqrt{n+1}} &= \sum b_n \\
\text{Now } c_n &= \sum_{k=0}^{n} a_k b_{n-k} \\
&= \sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \frac{(-1)^{n-k}}{\sqrt{n-k+1}} \\
&= (-1) \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}\sqrt{n-k+1}} \\
\text{Now } (k+1)(n+1-k) &= nk + k - k^2 + n + 1 - k \\
&= nk - k^2 + n + 1 \\
&= (n+1) - (k^2 - nk) \\
&= (\frac{n^2}{4} + n + 1) - (k^2 + \frac{n^2}{4} - nk) \\
&= (\frac{n}{2} + 1)^2 - (k - \frac{n}{2})^2 \\
&\leq (\frac{n}{2} + 1)^2
\end{aligned}$$

$$\therefore (k+1)(n+1-k) \leq (\frac{n}{2}+1)^2$$
$$\Rightarrow \sqrt{(k+1)(n+1-k)} \leq (n/2+1)$$
$$\Rightarrow \frac{1}{\sqrt{(k+1)(n+1-k)}} \geq \frac{1}{\frac{n}{2}+1}$$

$$\begin{aligned}
|c_n| &= \left|(-1)^n \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n+1-k)}}\right| \\
&= \left|\sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n+1-k)}}\right| \\
&= \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n+1-k)}} \geq \sum_{k=0}^{n} \frac{1}{\frac{n}{2}+1} \\
&= \frac{1}{\frac{n}{2}+1}\sum_{k=0}^{n} 1 = \frac{n+1}{\frac{n}{2}+1} = \frac{2(n+1)}{(n+2)} \\
&= \frac{2(1+\frac{1}{n})}{1+\frac{2}{n}} \\
|c_n| &\geq \frac{2(1+\frac{1}{n})}{1+\frac{2}{n}}
\end{aligned}$$

$\Rightarrow c_n$ *does not converges to* 0 *as* $n \to \infty \Rightarrow \sum c_n$ *diverges.*

Note 2.46 *The product of two convergent series converges if atleast one of the two series converges absolutely.*

Theorem 2.47 ***Merten's Theorem:***
(a) Suppose $\sum a_n$ *converges absolutely.*
(b) Suppose $\sum a_n = A$
(c) Suppose $\sum a_n = B$
(d) $c_n = \sum_{k=0}^{n} a_k b_{n-k} (n = 0, 1, 2...)$.
Then

$$\sum_{n=0}^{\infty} c_n = AB.$$

Proof:

$$A_n = \sum_{k=0}^{n} a_k;\ B_n = \sum_{k=0}^{n} b_k;\ c_n = \sum_{k=0}^{n} c_k.$$

Let

$$\begin{aligned}
\beta_n &= B_n - B\ \forall n \\
&= c_0 + c_1 + ... + c_n \\
&= a_0b_0 + (a_0b_1 + a_1b_0) + ... + (a_0b_{n-1} + ... + a_nb_0) \\
&= a_0((b_0 + b_1 + ... + b_n) + a_1(b_0 + b_1 + .. + b_{n-1}) + a_nb_0) \\
&= a_0B_n + a_1B_{n-1} + ... + a_nB_0
\end{aligned}$$

$$= a_0(B + \beta_n) + a_1(B + \beta_{n-1}) + ... + a_n(B + \beta_0) \ (\because \beta_n = B_n - B)$$
$$= B(a_0 + a_1 + ... + a_n) + (a_0\beta_n + a_1\beta_{n-1} + .. + a_n\beta_0)$$
$$= BA_n + \gamma_n \text{ where } \gamma_n = a_0\beta_n + a_1\beta_{n-1} + ... + a_n\beta_0$$

Claim $c_n \to AB$ as $n \to \infty$; $A_n \to A$ as $n \to \infty \Rightarrow BA_n \to AB$ as $n \to \infty$. If enough to prove $\gamma_n \to 0$ as $n \to \infty$. Given $\sum a_n$ converges absolutely. $\Rightarrow \sum |a_n|$ converges.

$$(i.e.) \sum_0^\infty |a_n| = \alpha$$
$$\text{Now } \lim_{n\to\infty} \beta_n = \lim_{n\to\infty}(B_n - B)$$
$$= B - B$$
$$= 0$$

Given $\epsilon > 0$ there exists an integer N such that

$$|\beta_n - 0| < \epsilon \ \forall \ n \geq N$$
$$\Rightarrow |\beta_n| < \epsilon \ \forall \ n \geq N......(1)$$
$$|\gamma_n| = |a_0\beta_n + a_1\beta_{n-1} + ... + a_n\beta_0|$$
$$= |\beta_n a_0 + \beta_{n-1}a_1 + ... + \beta_N a_{n-N} + \beta_{N-1}a_{n-N+1} + ... + \beta_0 a_n|$$
$$\leq |\beta_n a_0 + \beta_{n-1}a_1 + ... + \beta_N a_{n-N}| + |\beta_{N-1}a_{n-N+1} + ... + \beta_0 a_n|$$
$$< \epsilon(|a_0| + |a_1| + ... + |a_{n-N}|) + |\beta_{N-1}a_{n-N+1} + ... + \beta_0 a_n| \text{ By (1)}$$
$$< \beta_{N-1}a_{n-N+1} + ... + \beta_0 a_n| + \epsilon(|a_0| + |a_1| + ... + |a_n|)$$
$$= \beta_{N-1}a_{n-N+1} + ... + \beta_0 a_n| + \epsilon\alpha$$
$$\therefore |\gamma_n| < |\beta_{N-1}a_{n-N+1} + ... + \beta_0 a_n| + \epsilon\alpha$$

keeping N fixed and letting $n \to \infty$ we have

$$\lim_{n\to\infty} \sup |\gamma_n| \leq \epsilon\alpha$$

Since ϵ is arbitrary, we have,

$$\lim_{n\to\infty} |\gamma_n| = 0$$
$$\Rightarrow c_n \to AB \text{ as } n \to \infty$$
$$\Rightarrow \sum_{n=0}^\infty c_n = AB.$$

3. UNIT III

Continuity and Differentiation

Let X, Y be the metric spaces. Suppose $E \subset X, f$ maps E into Y and p is a limit point of E we write $f(x) \to q$ as $x \to p$ or

$$\lim_{x \to p} f(x) = q.$$

If there is a point $q \in Y$ with the following property, for every $\epsilon > 0$ there exists $S > 0$ such that $d_y(f(x), q) < \epsilon \forall x \in E$ for which $0 < d_X(x, p) < S$. (i.e.)

$$\lim_{x \to p} f(x) = q.$$

if given $\epsilon > 0$ there exists $S > 0$ such that $0 < d_X(x, p) < S \Rightarrow d_Y(f(x), q) < \epsilon$.

Definition 3.1 *Let X and Y be any two metric spaces and $E \subset X$. Let f and g be any complex functions defined on E then we define $f+g$ as follows. $(f+g)(x) = f(x) + g(x)$*

Theorem 3.2 *Let X and Y be any two metric spaces and $E \subset X$. p is a limit point of E. Then*

$$\lim_{x \to p} f(x) = q \text{ iff } \lim_{n \to \infty} f(p_n) = q$$

for every sequence $\{p_n\}$ in E such that $p_n \neq p$ and

$$\lim_{n \to \infty} p_n = p.$$

Proof: Suppose

$$\lim_{x \to p} f(x) = q$$

$\Rightarrow$ Given $\epsilon > 0$, there exists $S > 0$ such that $0 < d_X(x, p) < S \Rightarrow d_Y(f(x), q) < \epsilon \ \forall x \in E$.....(1)

$\{p_n\}$ is a sequence of points in E such that $\{p_n\} \to p$ as $n \to \infty (p_n \neq p)$ (This is possible $\because p$ is a limit point of E) $\Rightarrow$ there exists N depending on S such that $d_X(p_n, p) < S \ \forall n \geq N$. Now By (1) we have, $d_Y(f(p_n), q) < \epsilon \ \forall n \geq N$ (i.e.)

$$\lim_{n \to \infty} f(p_n) = q.$$

Conversely, Suppose

$$\lim_{n \to \infty} f(p_n) = q$$

for every $\{p_n\}$ in E such that $p_n \neq p$ and

$$\lim_{n \to \infty} p_n = p$$

To Prove

$$\lim_{x \to p} f(x) = q$$

Suppose this result is false, for some $\epsilon > 0$ and for every $S > 0$ such that $d_X(x, p) < S \Rightarrow d_Y(f(x), q) \geq \epsilon$. Let $S_n = \frac{1}{n}$, $n = 1, 2, 3...$ For $S > 0$ without loss of generality choose a point $p \in E$ such that $d_X(p_1, p) < S_1(= 1) \Rightarrow d_Y(f(p_1), q) \geq \epsilon$. Similarly, for $S_2 > 0$ choose a point $p_2 \in E$ such that $d_X(p_2, p) < S_1 = (1/2) \Rightarrow d_Y(f(p_2), q) \geq \epsilon$. Proceeding for $S_n > 0$, choose a point $p_n \in E$ such that $d_X(p_n, p) < S_1(= 1/n) \Rightarrow d_Y(f(p_n), q) \geq \epsilon$. $\therefore$ we have a sequence $\{p_n\}$ in E such that $d_X(p_n, p) < \frac{1}{n} \Rightarrow d_Y(f(p_n), q) \geq \epsilon$. Now $\{p_n\} \to p$ as $n \to \infty$ [$\because$ $1/n \to 0$ as $n \to \infty$]. But $f(p_n)$ does not converge to q $\therefore$ our assumption is wrong. Hence for every $\epsilon > 0$ there exists $S > 0$ such that $d_X(x, p) < S \Rightarrow d_Y(f(x), q) < \epsilon \ \ \forall x \in E$.

$$\therefore \lim_{x \to p} f(x) = q.$$

Corollary 3.3 *If f has a limit at p then this limit is unique.*
Proof: Suppose q is a limit of f at p. (i.e.)

$$\lim_{x \to p} f(x) = q.$$

$\therefore$ By the previous theorem, we have

$$\lim_{n \to \infty} f(p_n) = q$$

for every $\{p_n\}$ in E such that $p_n \neq p$ and $p_n \to p$. But we know that, Every convergence sequence converges to a unique limit. $\therefore$ f has a unique limit at p.

Definition 3.4 *Suppose we have two complex f and g then $f \pm g, fg, \lambda f$, $\frac{f}{g}(g \neq 0)$ are defined on a set E as follows.*

1. $(f + g)(x) = f(x) + g(x)$.
2. $(f \cdot g)(x) = f(x) \cdot g(x)$
3. $(\lambda f)(x) = \lambda f(x)$
4. $(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$.

Similarly we define $\bar{f}, \bar{g}$ map E into $\mathbb{R}^k$. Then we can define $\bar{f} \pm \bar{g}, \bar{f}\bar{g}, \lambda\bar{f}$, $\frac{\bar{f}}{\bar{g}}, (\bar{g} \neq 0)$.

Definition 3.5 ***Continuous at a point:*** *Suppose X, Y are metric spaces and $E \subset X, p \in E$ and f maps E into Y. Then f is said to be continuous at p if for every $\epsilon > 0$, there exists a $S > 0 \Rightarrow 0 < d_X(x, p) < S \Rightarrow d_Y(f(x), f(p)) < \epsilon \ \forall x \in E$.*

Remark 3.6 *Suppose f is continuous at $p \Rightarrow$ for every $\epsilon > 0$ there exists $S > 0$ such that $0 < d_X(x,p) < S \Rightarrow d_Y(f(x), f(p)) < \epsilon \ \forall x \in E \Rightarrow x \in N_S(p) \Rightarrow f(x) \in N_\epsilon(f(p)) \ \forall x \in E \Rightarrow f(N_S(p)) \subset N_\epsilon(f(p))$.*

Theorem 3.7 *Let X, Y be metric space and $E \subset X$. p is a limit point of E and $f : E \to Y$. Then f is continuous at p iff*

$$\lim_{x \to p} f(x) = f(p)$$

Proof: Suppose f is continuous at p. $\Leftrightarrow$ for every $\epsilon > 0$ there exists $S > 0$ such that $0 < d_X(x,p) < S \Rightarrow d_Y(f(x), f(p)) < \epsilon \quad \forall x \in E \Leftrightarrow$

$$\lim_{x \to p} f(x) = f(p)$$

Theorem 3.8 *Suppose X, Y, Z are metric space and $E \subset E$. f maps E into Y, g maps the range of f into Z and h is a mapping of E into Z defined by $h(x) = g(f(x))$. If f is continuous at $p \in E$ and if g is continuous at $f(p)$ then h is continuous at p. (The function h is called composite of f and g and we write as $h = g \circ f$)*
Proof: Let $\epsilon > 0$ be given and g is continuous at $f(p)$. $\therefore \eta > 0$ such that $d_Y(y, f(p)) < \eta \Rightarrow d_Z(g(y), g(f(p))) < \epsilon, \ y \in f(E)$...... (1)
Since f is continuous at p for this $\eta > 0$, there exists $S > 0$ such that $d_X(x,p) < S \Rightarrow d_Y(f(x), f(p)) < \eta \quad \forall x, y \in E$

$$(i.e.) d_Y(f(x), f(p)) < \eta, f(X) \in f(E)$$
$$\Rightarrow d_Z(g(f(x)), (g(f(p)) < \epsilon \text{ by } (1)$$
$$\Rightarrow d_Z(g \circ f(x), (g \circ f)(p)) < \epsilon$$
$$\Rightarrow d_Z(h(x), h(p)) < \epsilon \ (h = g \circ f).$$

$\therefore$ we have, $d_X(x,p) < S \Rightarrow d_Z(h(x), h(p)) < \epsilon \ \forall x \in E \Rightarrow h$ is continuous at p.

Theorem 3.9 *A mapping f of a metric space X into a metric space Y is continuous on X iff $f^{-1}(E)$ is open in X for every open get E in Y.*
Proof: Suppose f is continuous on X. Let V be a open get in Y. To Prove: $f^{-1}(V)$ is open in X. Let $p \in f^{-1}(V)$; $p \in f^{-1}(V) \Rightarrow f(p) \subset V$. Since V is open, there exists $\epsilon > 0$ such that $N_\epsilon(f(p)) \subset V$....... (1)
Since f is continuous at p, for $\epsilon > 0$ there exists $S > 0$ such that $f(N_S(p)) \subset N_\epsilon(f(p))$...... (2)
From (1) and (2), $\Rightarrow f(N_S(p)) \subset V \Rightarrow N_S(p) \subset f^{-1}V \Rightarrow p$ is an interior point of $f^{-1}(V)$. Since p is arbitrary, $f^{-1}(V)$ is open in X. Conversely: Suppose $f^{-1}(V)$ is open in X for every open set V in Y. To Prove: f is continuous at $p, p \in X$. Let $\in > 0$ be given. Consider an open set $N_\epsilon(f(p))$ in Y, $f^{-1}(N_\epsilon(f(p)))$ is open in X. Now, $\Rightarrow p \in f^{-1}(N_\epsilon(f(p))) \Rightarrow p$ is an interior point of $f^{-1}(N_\epsilon(f(p))) \Rightarrow$ there exists $S > 0$ such that $N_S(p) \subset f^{-1}(N_\epsilon(f(p))) \Rightarrow f(N_S(p)) \subset N_\epsilon(f(p)) \Rightarrow f$ is continuous at p.

Corollary 3.10 *A mapping f of a metric space X into a metric space Y is continuous iff $f^{-1}(C)$ is closed in X for every closed set C in Y.*
Proof: Let C be a closed set in $Y.C^c$ is open in $Y \Rightarrow f^{-1}(C^c)$ is open in X. (by Theorem 3.9) $\Rightarrow [f^{-1}(C)]^c$ is open in $X \Rightarrow f^{-1}(C)$ is closed in X. Conversely: Suppose $f^{-1}(C)$ is closed in X for every closed set C in Y. To Prove: f is continuous on X. Let A be an open set in $Y \Rightarrow A^c$ is closed in $Y \Rightarrow f^{-1}(A^c)$ is closed in X. (by our assumption) $\Rightarrow [f^{-1}(A)]^c$ is closed in $X \Rightarrow f^{-1}(A)$ is open in X. $\Rightarrow f$ is continuous on X. (by the previous theorem)

Theorem 3.11 *Let f and g be complex continuous function in a metric space X, then $f+g, f\cdot g, \frac{f}{g}(g \neq 0)$ are continuous on X.*
Proof: At isolated point of X there is nothing prove. Fix a point $p \in X$ and suppose p is a limit point of X. Since f and g are continuous at p.

$$\lim_{x\to p} f(x) = f(p);\ \lim_{x\to p} g(x) = g(p)$$

Now,

$$\lim_{x\to p}(f+g)(x) = \lim_{n\to\infty}(f+g)p_n$$

where $p_n \to p$ as $n \to \infty$ and $p_n \neq p$

$$\begin{aligned}\lim_{x\to p}(f+g)(x) &= \lim_{n\to\infty}(f(p_n)+g(p_n))\\ &= \lim_{n\to\infty} f(p_n) + \lim_{n\to\infty} g(p_n)\\ &= f(p)+g(p)\end{aligned}$$

similarly the other results follow.

Theorem 3.12 *Let $f_1, f_2, ..., f_k$ be real functions in a metric space X. Let $\bar{f}$ be the mapping X into $\mathbb{R}^k$. defined by $\bar{f}(x) = (f_1(x), f_2(x), ..., f_k(x))x \in X$. Then*
(a) $\bar{f}$ is continuous iff each of the functions $f_1, f_2, ..., f_k$ is continuous.
(b) $\bar{f}$ and $\bar{g}$ are continuous mapping of X into $\mathbb{R}^k$ then $\bar{f}+\bar{g}, \bar{f}\cdot\bar{g}$ are continuous on $X(f_1, f_2, ..., f_k$ are called components of $\bar{f})$.
Proof: Suppose $\bar{f}$ is continuous at every $p \in X$. Then given $\epsilon > 0$ there exists $S > 0$ such that

$$\begin{aligned}|\bar{f}(x)-\bar{f}(p)| &< \epsilon \text{ if } 0 < d_X(x,p) < S\\ \Rightarrow \left(\sum_{i=1}^{k}(f_i(x)-f_i(p))^2\right)^{1/2} &< \epsilon \text{ if } 0 < d_X(x,p) < S\\ \Rightarrow |f_i(x)-f_i(p)| < \left(\sum_{i=1}^{k}(f_i(x)-f_i(p))^2\right)^{1/2} &< \epsilon\ \forall i = 1,2,...,k\\ \Rightarrow |f_i(x)-f_i(p)| < \epsilon\ \forall i = 1,2,...,k &\text{ if } 0 < d_X(x,p) < S\end{aligned}$$

$\Rightarrow$ each f_i is continuous at p, $(1 \le i \le k,\ p \in X)$ $\Rightarrow$ each f_i is continuous on X, $(1 \le i \le k)$. Conversely, Suppose f_i is continuous on X for each $i = 1, ..., k \Rightarrow f_i$ is continuous at every $p \in X \Rightarrow$ Given $\epsilon > 0$ there exists $S_i > 0$ such that $0 < d_X(x,p) < S_i \Rightarrow |f_i(x) - f_i(p)| < \frac{\epsilon}{\sqrt{k}}$ $\forall i = 1, 2, ..., k$. Let $S = min(S_1, S_2, ..., S_k)$. Now,

$$\begin{aligned} 0 < d_X(x,p) < S_i \Rightarrow |f_i(x) - f_i(p)| &< \frac{\epsilon}{\sqrt{k}} \ \forall i = 1,2,...,k \\ \Rightarrow |f_i(x) - f_i(p)|^2 &< \frac{\epsilon^2}{(\sqrt{k})^2} \\ \Rightarrow \sum_{i=1}^{k} |f_i(x) - f_i(p)|^2 &< \frac{\epsilon^2}{k} \cdot k \\ &= \epsilon^2 \\ \Rightarrow \sqrt{\sum_{i=1}^{k} |f_i(x) - f_i(p)|^2} &< \epsilon \\ \Rightarrow |\bar{f}(x) - \bar{f}(p)| &< \epsilon \\ (i.e.) 0 < d_X(x,p) < S \Rightarrow |\bar{f}(x) - \bar{f}(p)| &< \epsilon \end{aligned}$$

$\Rightarrow \bar{f}$ is continuous at every $p \in X \Rightarrow \bar{f}$ is continuous on X
(b) Let $\bar{f} = (f_1, f_2, ..., f_k)$ and $\bar{g} = (g_1, g_2, ..., g_k)$. Now, $\bar{f} + \bar{g} = (f_1 + g_1, f_2 + g_2, ..., f_k + g_k)$; $\bar{f} \cdot \bar{g} = (f_1 \cdot g_1, f_2 \cdot g_2, ..., f_k \cdot g_k)$. Given $\bar{f}$ and $\bar{g}$ are continuous. by (a), each f_i, g_i are continuous $(i \le i \le k)$ (by Theorem 3.11) $\Rightarrow f_i + g_i, f_i \cdot g_i$ are continuous. (by (a))

Theorem 3.13 *Let $\bar{x} = (x_1, x_2, ..., x_k) \in \mathbb{R}^k$ define $\phi_i : \mathbb{R}^k \to \mathbb{R}$ by $\phi_i(\bar{x}) = x_i$, $(i = 1, 2, ..., k)$. ϕ_i is called the coordinate function, then ϕ_i is continuous.*
Proof: Let $\bar{x}, \bar{y} \in \mathbb{R}^k$. Given $\epsilon > 0$ choose $S = \epsilon$ such that

$$\begin{aligned} |\bar{x} - \bar{y}| &< S \\ \Rightarrow |\phi_i(\bar{x}) - \phi_i(\bar{y})| &= |x_i - y_i| \\ &< \left(\sum_{i=1}^{k} |x_i - y_i|^2 \right)^{1/2} \\ &= |\bar{x} - \bar{y}| \\ &< \epsilon \end{aligned}$$

$\Rightarrow \phi_i$ is continuous on $\mathbb{R}^k$

Theorem 3.14 *Every polynomial in $\mathbb{R}^k$ is continuous.*
Proof: By the above theorem $\phi_i : \mathbb{R}^k \to \mathbb{R}$ is continuous for every i. Now, $\phi_i^2(\bar{x}) = \phi_i(\bar{x}) \cdot \phi_i(\bar{x}) = x_i \cdot x_i = x_i^2$ $\forall i$. In general $\phi_i^{n_i}(\bar{x}) = x_i^{n_i}$ $\forall i$. By

Theorem 3.11, $\phi_i^{n_i}$ is continuous. Now,

$$\begin{aligned}(\phi_1^{n_1} \cdot \phi_2^{n_2} \cdots \phi_k^{n_k})\bar{x} \\ &= \phi_1^{n_1}(\bar{x}) \cdot \phi_2^{n_2}(\bar{x}) \cdots \phi_k^{n_k}(\bar{x}) \\ &= x_1^{n_1} \cdot x_2^{n_2} \cdots x_k^{n_k}\end{aligned}$$

Now $\phi_1^{n_1} \cdot \phi_2^{n_2} \cdots \phi_k^{n_k}$ is a monomial function, where $n_1, n_2, ..., n_k$ are positive integers. Every monomial function is continuous $C_{n_1,n_2,...,n_k}$ is a complex constant $\Rightarrow C_{n_1,n_2,...,n_k} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdots \cdot x_k^{n_k}$ is continuous on $\mathbb{R}^k$. $\Rightarrow \sum C_{n_1,n_2,...,n_k} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdots \cdot x_k^{n_k}$ is continuous on $\mathbb{R}^k$. $\Rightarrow$ Every polynomial is continuous on $\mathbb{R}^k$.

Continuity and Compact: A mapping $\bar{f}$ on a set E into X is said to be bounded, if there is a real number m such that $|\bar{f}(x)| < m \ \forall x \in X$.

Theorem 3.15 *Suppose f is continuous function on a compact metric space X into a metric space Y. Then $f(X)$ is compact. (i.e., continuous image of a compact metric space is compact)*

Proof: Given that X is compact. To Prove: $f(X)$ is compact. Let $\{V_\alpha\}$ be an open cover for $f(X) \Rightarrow$ each V_α is open in Y. Now, Given f is continuous $\Rightarrow f^{-1}(V_\alpha)$ is open in X for each $\alpha \Rightarrow \{f^{-1}(V_\alpha)\}$ is open cover for X. Since X is compact, there exists finitely may indices $\alpha_1, \alpha_2, ..., \alpha_n$ such that

$$\begin{aligned}X &\subset f^{-1}(V_{\alpha_1}) \cup f^{-1}(V_{\alpha_2}) \cup \cdots \cup f^{-1}(V_{\alpha_n}) \\ &= \bigcup_{i=1}^{n} f^{-1}(V_{\alpha_i}) \\ \Rightarrow f(X) &\subset \bigcup_{i=1}^{n} f f^{-1}(V_{\alpha_i}) \subset \bigcup_{i=1}^{n} V_{\alpha_i}\end{aligned}$$

$\Rightarrow \{V_\alpha\} \Rightarrow$ has a finite sub cover. $\therefore f(X)$ is compact.

Theorem 3.16 *If $\bar{f}$ is continuous mapping of a compact metric space X into $\mathbb{R}^k$. Then $\bar{f}(X)$ is closed and bounded. $\therefore \bar{f}$ is bounded.*

Proof: Given $\bar{f}$ is continuous and X is compact. $\Rightarrow \bar{f}(x)$ is a compact subset of $\mathbb{R}^k$. $\Rightarrow \bar{f}(x)$ is closed and bounded. (by Heine Borel theorem) Now, in particular $\Rightarrow \bar{f}(x)$ is bounded $\Rightarrow \bar{f}$ is bounded.

Theorem 3.17 *Suppose f is a continuous real function on a compact metric space X and $M = \sup_{p \in X} f(p)$ and let $m = \inf_{p \in X} f(p)$. Then, there exists a points $p, q \in X$ such that $f(p) = m_1, \ f(q) = m_2$ (i.e., f attains maximum M at p and minimum m at q)*

Proof: We know that, If E is bounded and $y = \sup E$ and $X = \inf E$ then $x, y \in \bar{E}$. Since f is continuous and X is compact $\Rightarrow f(X)$ is closed and bounded [By the above Theorem 3.16] and since $f(X)$ is bounded. $m, M \in \overline{f(X)} = f(X)$ ($\because f(X)$ is closed) $\Rightarrow m, M \in f(X) \Rightarrow$ there exists $p, q \in X$ such that $M = f(p), \ m = f(q)$.

Theorem 3.18 *Suppose f is continuous $1-1$ mapping of a compact metric space X into a metric space Y. Then the inverse mapping f^{-1} defined on Y by $f^{-1}(f(X)) = X$ is a continuous mapping of Y onto X.*
Proof: Suppose f is a continuous $1-1$ mapping of a compact metric space X into a metric space Y and also $f^{-1}(f(X)) = X$. To Prove: f^{-1} is continuous on Y, it is enough to prove that $(f^{-1})(V)$ is open in Y for every open set V in X. Let V be a open set in $X \Rightarrow V^c$ is closed in X. Since X is compact, V^c is compact in X. Since f is continuous, $f(V^c)$ is compact in $Y \Rightarrow f(V^c)$ is closed in $Y \Rightarrow (f(V^c))^c$ is closed in $Y \Rightarrow f(V)$ is open in Y. ($\because f$ is $1-1$ and onto) $\Rightarrow (f^{-1}(V))^{-1}$ is open in $Y \Rightarrow f^{-1}$ is continuous on Y.

Definition 3.19 ***(Uniformly Continuous)*** *Let X and Y be any two metric space then the $f : X \to Y$ is said it to be uniformly continuous on X if for every $\epsilon > 0$ there exists a $S > 0$ such that $d_X(p, q) < S \Rightarrow d_Y(f(p), f(q)) < \epsilon$ $\forall p, q \in X$.*

Theorem 3.20 *Let f be a continuous mapping of a compact metric space X into a metric space Y then f is uniformly continuous. (i.e.) Continuous function defined on a compact metric space is uniformly continuous.*
Proof: Let $\epsilon > 0$ be given let f is continuous on $X \Rightarrow f$ is continuous at every point $p \in X$. Now, f is continuous at $p \Rightarrow$ there exists a positive real $\phi(p)$ such that $d_X(p, q) < \phi(p) \Rightarrow d_Y(f(p), f(q)) < \epsilon \ \forall q \in X$....... (1)
Let $J(p) = N_{\frac{\phi(p)}{2}}\{p\} \Rightarrow J(p)$ is a closed in $X \Rightarrow J(p)$ is a open in X. $\therefore$ $\{J(p)|p \in X\}$ is an open cover for X. Since X is compact, there exists finitely may $p \in S$. $p_1, p_2, ..., p_n$ such that $X \subset \bigcup_{i=1}^{n} J(p_i)$. Let $S = min\{(\frac{\phi(p)}{2}, ..., \frac{\phi(p)}{2})\}$. Clearly, $S > 0$. Let p, q be points in X such that $d_X(p, q) < S$. Now,

$$\begin{aligned}
p \in X \subset \bigcup_{i=1}^{n} J(p_i) & \\
\Rightarrow p \in J(p_m) & \text{ for some } m, 1 \le m \le n \\
\Rightarrow d_X(p, p_m) &< \frac{\phi(p_m)}{2} < \phi(p_m) \\
\Rightarrow d_Y(f(p), f(p_m)) &< \epsilon/2.......(2) \ (by(1)) \\
\text{Now } d_X(q, p_m) &< d_X(q, p) + d(p, p_m) \\
&< S + \frac{\phi(p_m)}{2} \\
&< \frac{\phi(p_m)}{2} + \frac{\phi(p_m)}{2} \\
&= \phi(m) \\
(i.e.) \ d_X(q, p_m) &< \phi(p_m) \\
\Rightarrow d_Y(f(q), f(p_m)) &< \epsilon/2 \text{ by}(1)........(3)
\end{aligned}$$

$$\Rightarrow d_Y(f(p), f(q)) < d_Y(f(q), f(p_m)) + d_Y(f(p_m)f(q))$$
$$= \epsilon/2 + \epsilon/2 \text{ (by (2) and (3))}$$
$$\therefore d_X(p, q) < S \Rightarrow d_Y(f(p), f(q)) < \epsilon$$

$\Rightarrow f$ is uniformly continuous on X.

Theorem 3.21 *Let E be a non-compact set in $\mathbb{R}^1$. Then*
(a) there exists a continuous function on E which is not bounded,
(b) there exists continuous and bounded function on which has no maximum if in addition E is bounded,
(c) there exists a continuous function on E which is not uniformly continuous.
Proof: Case(i): Suppose E is bounded.
(a) To Prove: f is continuous but not bounded. Since E is bounded, there exists a limit point of x_0 of E such that $x_0 \notin E$. [$\because E$ is not closed]. Define a map $f : E \to \mathbb{R}^1$ by $f(x) = \frac{1}{x-x_0}$, $x \in E$. $\therefore f$ is continuous on E. To Prove: f is unbounded on E. Since x_0 is a limit point of E. $N_r(x_0) \cap E \neq \emptyset$ $\forall r > 0 \Rightarrow$ there exists x_1 such that $x_1 \in N_r(x_0) \cap E \Rightarrow x_1 \in N_r(x_0)$ and $x_1 \in E$

$$\Rightarrow |x_1 - x_0| < r \text{ and } x_1 \in E$$
$$\Rightarrow \frac{1}{|x_1 - x_0|} > \frac{1}{r} \text{ and } x_1 \in E$$
$$\Rightarrow |f(x_1)| > \frac{1}{r} \text{ and } x_1 \in E \ \forall r > 0$$

$\forall r > 0$ there exists $x \in E$ such that $|f(x)| > \frac{1}{r} \Rightarrow f$ is unbounded on E.
(b) Define $g : E \to R$ by $g(x) = \frac{1}{1+(x-x_0)^2}$, $x \in E$. Clearly, g is continuous. Now, $0 < g(x) < 1 \Rightarrow g(x)$ is a bounded function. Clearly, $\sup_{x \in E} g(x) = 1$. But $g(x) < 1 \ \ \forall x \in E$. $\therefore g$ has no maximum on E.
(c) Let $f : E \to R$ be defined by $f(x) = \frac{1}{x-x_0}$, $x \in E$, where x_0 is a limit point of E. Clearly, f is continuous on E. Let $\epsilon > 0$ be given. Let $S > 0$ be arbitrary choose a point $x \in E$ such that $|x - x_0| < S$ and taking t very close to x_0 so as to satisfy $|t - x| < S$. Then,

$$|f(t) - f(x)| = \left|\frac{1}{t - x_0} - \frac{1}{x - x_0}\right|$$
$$= \left|\frac{x - x_0 - t + x_0}{(t - x_0)(x - x_0)}\right|$$
$$= \frac{|x - t|}{|t - x_0||x - x_0|}$$
$$> \frac{1}{t - x_0} > \epsilon$$

(If we choose $x \in (x_0 - S, x_0), t \in (x_0, x_0 + S)$ and $|x - t| < S$ or $t \in (x_0 - S, x_0), x \in (x_0, x_0 + S)$ and $|x - t| < S \Rightarrow |t - x| > |x - x_0|$) So we have taken t very close to x_0 and we made the difference $|f(t) - f(x)| > \epsilon$ although $|t - x| < S$. Since this is true for every $S > 0 \Rightarrow f$ is not uniformly continuous.

Case(ii): Suppose E is not bounded.

(a) Define $f : E \to R$ by $f(x) = x$. Clearly, f is continuous on E and f is not bounded on E. $\therefore$ there exists function on E which is not bounded.

(b) Define $g : E \to R$ by $g(x) = \frac{x^2}{1+x^2} \Rightarrow g$ is continuous. Now, as $x^2 < 1 + x^2 \Rightarrow g(x) = \frac{x^2}{1+x^2} < 1$. $\therefore 0 < g(x) < 1 \quad \forall x \in E$. $\therefore g$ is a bounded. $\therefore g$ is a continuous and bounded function. $\sup_{x \in E} g(x) = 1$. But g has no maximum on E.

(c) If the boundedness is omitted then the result fails. Let E be the set of all integers. Then every function defined on E is uniformly continuous on $E \Rightarrow$ for every $\epsilon > 0$ choose $S < 1$ such that $|X - Y| < S \Rightarrow |f(x) - f(y)| = 0 < \epsilon$

Continuity and Connectedness:

Theorem 3.22 *If f is a continuous mapping on a metric space X into a metric space Y and E is a connected subset of X. Then $f(E)$ is connected. i.e., continuous image of a connected subset of a metric space is connected.*

Proof: Given E is connected subset of X. To Prove: $f(E)$ is a connected subset of Y. Suppose $f(E)$ is not connected. $\Rightarrow f(E) = A \cup B$ where A and B are non-empty separated sets. Put $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$

$$\begin{aligned} G \cup H &= (E \cap f^{-1}(A)) \cup (E \cap f^{-1}(B)) \\ &= E \cap (f^{-1}(A) \cup f^{-1}(B)) \\ &= E \cap (f^{-1}(A \cup B)) \\ &= E \cap E \\ G \cup H &= E \end{aligned}$$

Clearly $G \neq \emptyset \quad H \neq \emptyset \quad (\because A \neq \emptyset, B \neq \emptyset)$. Claim: G and H are separated sets. i.e., To Prove $\bar{G} \cap H = \emptyset, G \cap \bar{H} = \emptyset$. Now

$$\begin{aligned}
G &= E \cap f^{-1}(A) \\
&\Rightarrow G \subset f^{-1}(A) \subset f^{-1}(\bar{A}) \\
&\Rightarrow \bar{G} \subset \overline{f^{-1}(\bar{A})} = f^{-1}(\bar{A}) \; [\because \bar{A} \text{ is closed and } \\
&\qquad\qquad\qquad\qquad f \text{ is continuous } \Rightarrow f^{-1}(\bar{A})] \\
&\Rightarrow f(\bar{G}) \subset f f^{-1}(\bar{A}) \subset \bar{A} \\
&\Rightarrow f(\bar{G}) \subset \bar{A} \\
H &= E \cap f^{-1}(B) \\
&\Rightarrow H \subset f^{-1}(B) \Rightarrow f(H) \subset f f^{-1}(B) = B \\
&\Rightarrow f(H) \subset B \\
&\Rightarrow f(\bar{G}) \cap f(H) \subset \bar{A} \cap B = \emptyset \; (\because A \text{ and } B \text{ are separated sets}) \\
&\Rightarrow f(\bar{G}) \cap f(H) = \emptyset \\
&\Rightarrow f(\bar{G} \cap H) = \emptyset \\
&\Rightarrow \bar{G} \cap H = \emptyset \\
&\text{similarly, } G \cap \bar{H} = \emptyset
\end{aligned}$$

$\therefore$ G and H are separated sets. $\Rightarrow$ E can be expressed as a union of two non-empty separated sets. $\Rightarrow$ E is not connected. $\Rightarrow\Leftarrow$ to E is connected. $\therefore f(E)$ is connected.

Theorem 3.23 ***Intermediate Value Theorem:*** *Let f be a continuous real valued function on $[a, b]$. If $f(a) < f(b)$ and c is the number such that $f(a) < c < f(b)$ then there exists a point $x \in (a, b)$ such that $f(x) = c$.*
Proof: Every interval in $\mathbb{R}$ is connected and f is continuous. By the previous theorem, $f[a, b]$ is connected in $\mathbb{R}$. $\Rightarrow f[a, b]$ is interval in $\mathbb{R}$. Let $f(a), f(b) \in f[a, b] \Rightarrow [f(a), f(b)] \subset f[a, b]$. Now, $f(a) < c < f(b) \Rightarrow c \in f[a, b] \Rightarrow c = f(x)$ for some $x \in [a, b]$.

Remark 3.24 *Converse not true.*
Proof: If any two points x_1 and x_2 and for any member c between $f(x_1)$ and $f(x_2)$ there is a point x in $[x_1, x_2]$ such that $f(x) = c$ then f may be discontinuous. For example:

$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Choose $x_1 \in (-\frac{\pi}{2}, 0), x_2 \in (0, \frac{\pi}{2})$. Clearly $x_1 < x_2$; $f(x_1)$ =negative $f(x_2)$=positive. $\therefore f(0) = 0$. f is continuous all the points except at 0.

Differentiation:

Definition 3.25 *Let f be real value function defined on $[a,b]$, for any $x \in [a,b]$ form the quotient $\phi(t) = \frac{f(t)-f(x)}{t-x}, a < t < b, t \neq x$, and defined*

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

provided the limit exists.

Remark 3.26 *1. If f' is defined at a point, we say that f is differentiable at x.*
2. If f' is defined at every point of a set $E \subset [a,b]$, we say that f is differentiable on E.

Theorem 3.27 *Let f be defined on $[a,b]$. If f is differentiable at a point x in $[a,b]$, then f is continuous at x.*
Proof: Given f is differentiable at x. (i.e.)

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \text{ exists.}$$

To Prove: f is continuous at x (i.e.)To Prove

$$\lim_{t \to x} f(t) = f(x)$$

Now

$$\begin{aligned}
f(t) - f(x) &= \frac{f(t) - f(x)}{t - x}(t - x) \\
\lim_{t \to x}(f(t) - f(x)) &= \lim_{t \to x} \left[\frac{f(t) - f(x)}{t - x}(t - x) \right] \\
&= \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \cdot \lim_{t \to x}(t - x) \\
&= f'(x) \cdot 0 \\
&= 0 \\
\lim_{t \to x}(f(t) - f(x)) &= 0 \\
\text{(or) } \lim_{t \to x} f(t) &= f(x)
\end{aligned}$$

$\therefore f$ is continuous at x.

Remark 3.28 *Converse of above theorem is not true. For example $f(x) = |x|$ is continuous but not differentiable at origin.*

Theorem 3.29 *Suppose f and g are defined on $[a,b]$ and are differentiable at at point x in $[a,b]$ then $f + g, fg, \frac{f}{g}$ are differentiable at x.*
(a) $(f + g)'(x) = f'(x) + g'(x)$

(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
(c) $(\frac{f}{g})'(x) = \frac{g(x)f'(x)-g'(x)f(x)}{g^2(x)},\ g(x) \neq 0.$
Proof: Given f and g are differentiable at x.

$$(i.e.)f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \text{ and } g'(x) = \lim_{t\to x} \frac{g(t)-g(x)}{t-x} \text{ exists.}$$

(a)

$$\begin{aligned}
\phi(t) &= \frac{(f+g)(t)-(f+g)(x)}{t-x} \\
&= \frac{f(t)+g(t)-(f(x)+g(x))}{t-x} \\
\phi(t) &= \frac{f(t)-f(x)}{t-x} + \frac{g(t)-g(x)}{t-x}
\end{aligned}$$

Taking limits as $t \to x$

$$\begin{aligned}
\lim_{t\to x} \phi(t) &= \lim_{t\to x} \left\{ \frac{f(t)-f(x)}{t-x} + \frac{g(t)-g(x)}{t-x} \right\} \\
&= \lim_{t\to x} \frac{f(t)-f(x)}{t-x} + \lim_{t\to x} \frac{g(t)-g(x)}{t-x} \\
(i.e.)(f+g)'(x) &= f'(x) + g'(x)
\end{aligned}$$

(i.e.) $(f+g)$ is differentiable at x.
(b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$. Let $h = fg$. Now,

$$\begin{aligned}
(h(t)-h(x)) &= (fg)(t) - (fg)(x) \\
&= f(t)g(t) - f(x)g(x) \\
&= f(t)g(t) - f(t)g(x) + f(t)g(x) - f(x)g(x) \\
&= f(t)(g(t)-g(x)) + g(x)(f(t)-f(x)) \\
\frac{h(t)-h(x)}{t-x} &= f(t)\frac{(g(t)-g(x))}{t-x} + g(x)\frac{(f(t)-f(x))}{t-x} \\
\lim_{t\to x} \frac{h(t)-h(x)}{t-x} &= \lim_{t\to x} \left\{ f(t)\frac{g(t)-g(x)}{t-x} + g(x)\frac{f(t)-f(x)}{t-x} \right\} \\
&= \lim_{t\to x} f(t) \lim_{t\to x} \frac{g(t)-g(x)}{t-x} + \lim_{t\to x} g(x) \lim_{t\to x} \frac{f(t)-f(x)}{t-x} \\
h'(x) &= f(x)g'(x) + g(x)f'(x) \\
(fg)'(x) &= f(x)g'(x) + g(x)f'(x)
\end{aligned}$$

fg is differentiable at x.

(c) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x)-g'(x)f(x)}{g^2(x)}$. Let $h = \frac{f}{g}$.

$$\begin{aligned}
(h(t)-h(x)) &= \frac{f}{g}(t) - \frac{f}{g}(x) \\
&= \frac{f(t)}{g(t)} - \frac{f(x)}{g(x)} \\
&= \frac{f(t)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(t)}{g(t)g(x)} \\
&= \frac{g(x)(f(t)-f(x)) - f(x)(g(t)-g(x))}{g(t)g(x)} \\
\frac{h(t)-h(x)}{t-x} &= \frac{g(x)(f(t)-f(x)) - f(x)(g(t)-g(x))}{g(t)g(x)(t-x)} \\
\lim_{t\to x}\frac{h(t)-h(x)}{t-x} &= \lim_{t\to x}\frac{g(x)}{g(t)g(x)}\left(\frac{f(t)-f(x)}{t-x}\right) - \lim_{t\to x}\frac{f(x)}{g(t)g(x)}\left(\frac{g(t)-g(x)}{t-x}\right) \\
&= \frac{g(x)}{g^2(x)}\lim_{t\to x}\frac{f(t)-f(x)}{t-x} - \frac{f(x)}{g^2(x)}\lim_{t\to x}\frac{g(t)-g(x)}{t-x} \\
h'(x) &= \frac{g(x)f'(x)-g'(x)f(x)}{g^2(x)} \\
\left(\frac{f}{g}\right)'(x) &= \frac{g(x)f'(x)-g'(x)f(x)}{g^2(x)}
\end{aligned}$$

Since $f'(x), g'(x)$ exists and $g(x) \neq 0$, $\left(\frac{f}{g}\right)'(x)$ exists.

Example 3.30 *(1) The derivative of any constant is zero.*
(2) $f(x) = x \Rightarrow f'(x) = 1$
(3) $f(x) = n \Rightarrow f'(x) = nx^{n-1}$

Theorem 3.31 ***Chain Rule:*** *Suppose f is continuous on $[a,b]$, $f'(x)$ exists at some point x in $[a,b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If $h(t) = g(f(t)), a \leq t \leq b$ then h is differentiable at x, and $h'(x) = g'(f(x))f'(x)$.*
Proof: Given

$$f'(x) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x} \text{ exists, } t \in [a,b].$$

Let $h(t) = g(f(t))$. To Prove: $h'(x) = g'(f(x))f'(x)$. Since f is differentiable at $x \in [a,b]$

$$f'(x) = \lim_{t\to x}\frac{f(t)-f(x)}{t-x} \text{ exists, } t \in [a,b] \text{ exists.}$$

$$(i.e.)\ f'(x) + u(t) = \frac{f(t)-f(x)}{t-x},\ t \in [a,b] \text{ where } \lim_{t\to x} u(t) = 0$$

$$\Rightarrow (f'(x)+u(t))(t-x) = f(t)-f(x) ......(1)$$

Let $y = f(x)$. Now g is differentiable at $y(= f(x))$

$$\begin{aligned}
g'(y) &= \lim_{s\to y} \frac{g(s) - g(y)}{s - y}, s \in I \\
(i.e.)\ g'(y) + v(s) &= \frac{g(s) - g(y)}{s - y}, s \in I \text{ where } \lim_{s\to y} v(s) = 0 \\
(g'(y) + v(s))(s - y) &= g(s) - g(y) ....(2)
\end{aligned}$$

Let $s = f(t)$. Now,

$$\begin{aligned}
h(t) - h(x) &= g(f(t)) - g(f(x)) \\
&= (g'(f(x)) + v(s))(s - y)\ (by(2)) \\
h(t) - h(x) &= g'(f(x) + v(s))(f(t) - f(x)) \\
&= g'(f(x) + v(s))(f'(x) + u(t))(t - x)\ (by(1)) \\
\frac{h(t) - h(x)}{t - x} &= g'(f(x) + v(s))(f'(x) + u(t)) \\
\lim_{t\to x} \frac{h(t) - h(x)}{t - x} &= \lim_{t\to x} \{g'(f(x) + v(s))(f'(x) + u(t))\} \\
h'(x) &= \lim_{t\to x} g'(f(x) + v(s)) \lim_{t\to x} (f'(x) + u(t)) \\
&= \lim_{s\to y} (g'(f(x)) + v(s)) f'(x) \\
&= g'(f(x)) f'(x) \\
\therefore h'(x) &= g'(f(x)) f'(x)
\end{aligned}$$

Example 3.32 *Let*

$$f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Find $f'(x)(x \neq 0)$, and show that $f'(0)$ does not exist.
Solution:

$$\begin{aligned}
f(x) &= x \sin \frac{1}{x} \\
f'(x) &= x \cos \left(\frac{1}{x}\right) \left(\frac{-1}{x^2}\right) + \sin \left(\frac{1}{x}\right) \\
&= -\frac{1}{x} \cos \left(\frac{1}{x}\right) + \sin \left(\frac{1}{x}\right) \\
&= \sin \left(\frac{1}{x}\right) - \left(\frac{1}{x}\right) \cos \left(\frac{1}{x}\right), x \neq 0.
\end{aligned}$$

since $x \neq 0 f'(x)$ exists. To Prove: $f'(0)$ does not exists.

$$\begin{aligned} f'(0) &= \lim_{t\to 0} \frac{f(t)-f(0)}{t-0} \\ &= \lim_{t\to 0} \frac{t\sin\frac{1}{t}-0}{t-0} \\ &= \lim_{t\to 0} \sin\frac{1}{t} \text{ which does not exists.} \end{aligned}$$

$\therefore f'(0)$ does not exists.

Example 3.33 *Let*

$$f(x) = \begin{cases} x^2\sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Find $f'(x)(x \neq 0)$, show that $f'(0) = 0$
Solution: Let

$$\begin{aligned} f(x) &= x^2\sin\frac{1}{x} \\ f'(x) &= x^2(\cos\left(\frac{1}{x}\right))\left(\frac{-1}{x^2}\right) + 2x\cdot\sin\frac{1}{x} \\ &= 2x\cdot\sin\frac{1}{x} - \cos\frac{1}{x}, x \neq 0 \\ f'(0) &= \lim_{t\to 0}\frac{f(t)-f(0)}{t-0} \\ &= \lim_{t\to 0}\frac{x^2\sin\frac{1}{t}-0}{t-0} \\ &= \lim_{t\to 0} t\sin\frac{1}{t} \\ &= 0 \ (\because \left|t\sin\frac{1}{t}\right| \leq 1) \end{aligned}$$

$$\therefore f'(0) = 0$$

Mean Value Theorems:

Definition 3.34 ***Local Maximum, Local Minimum:*** *Let f be a real function defined on a metrics space X. We say that f has local maximum at a point p in X if there exists $\delta > 0$ such that $f(q) \leq f(p)$ $\forall q \in X$ with $d(p,q) < \delta$. f has a local minimum at p in X, if $f(p) \leq f(q)$ $\forall q \in X$ such that $d(p,q) < \delta$.*

Theorem 3.35 *Let f be defined on $[a,b]$; if f has a local maximum at a point $x \in (a,b)$ and if f' exists, then $f'(x)$=0. The analogous statement for local minimum is also true.*
Proof: Case(i) Assume that f has local maximum at x. To Prove: $f'(x) =$

0. Since f has local maximum at x, there exists $\delta > 0$ such that $(q, x) < \delta \Rightarrow f(q) \leq f(x)$

$$\begin{aligned}
\text{If } x - \delta < t < x \text{ then } \frac{f(t) - f(x)}{t - x} &\geq 0 \\
\Rightarrow \lim_{t \to x} \frac{h(t) - h(x)}{t - x} &\geq 0 \\
(i.e.)\ f'(x) &\geq 0 \ ....(1) \\
\text{If } t^x < x^t < x + \delta \text{ then } \frac{f(t) - f(x)}{t - x} &\leq 0 \\
\Rightarrow \lim_{t \to x} \frac{h(t) - h(x)}{t - x} &\leq 0 \\
\Rightarrow f'(x) &\leq 0 \ ....(2)
\end{aligned}$$

Since $f'(x)$ exists, (1),(2) $\Rightarrow f'(x) = 0$.
Case(ii) Assume that f has a local minimum at x. We show that $f'(x)$=0. Then there exists $\delta > 0$ such that $d(q, x) < \delta \Rightarrow f(q) \geq f(x)$

$$\begin{aligned}
\text{If } x - \delta < t < x \text{ then} \frac{f(t) - f(x)}{t - x} &\leq 0 \\
\Rightarrow \lim_{t \to x} \frac{f(t) - f(x)}{t - x} &\leq 0 \\
(i.e.)\ f'(x) &\leq 0 \ ....(3) \\
\text{If } x < t < x + \delta \text{ then } \frac{f(t) - f(x)}{t - x} &\geq 0 \\
\Rightarrow \lim_{t \to x} \frac{f(t) - f(x)}{t - x} &\geq 0 \\
\Rightarrow f'(x) &\geq 0 \ ....(4)
\end{aligned}$$

Since $f'(x)$ exists, and from (3) and (4) we get $f'(x)$=0.

Theorem 3.36 ***Generalised Mean Value Theorem:*** *If f and g are continuous real functions on $[a, b]$, which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which $[f(b) - f(a)]g'(x) = [g(b) - g(a)]f'(x)$.*
proof: Let $h(t) = [f(b) - f(a)]g(t) - [g(b) - g(a)]f(t),\ t \in [a, b]$. Since f and g are differentiable in (a, b), $h(t)$ is also differentiable in (a, b). Now,

$$\begin{aligned}
h(a) &= [f(b) - f(a)]g(a) - [g(b) - g(a)]f(a) \\
&= f(b)g(a) - f(a)g(a) - g(b)f(a) + g(a)f(a) \\
&= f(b)g(a) - g(b)f(a) \\
h(b) &= [f(b) - f(a)]g(b) - [g(b) - g(a)]f(b) \\
&= f(b)g(b) - f(a)g(b) - g(b)f(b) + g(a)f(b) \\
&= g(a)f(b) - f(a)g(b)
\end{aligned}$$

Claim: $h'(x) = 0$ for some $x \in (a,b)$. If $h(t)$ is a constant then $h'(x) = 0\ \forall x \in (a,b)$. If $h(t) < h(a), a < t < b$, then by Intermediate value theorem, there exists x in (a,b) at which h is minimum. $\therefore h'(x) = 0$ (by Theorem 3.35). If $h(t) > h(a)$ then h attains its maximum at some point $x \in (a,b)$. $\therefore$ $h'(x) = 0$ (by Theorem 3.35) (i.e.)

$$(f(b) - f(a))g'(x) - (g(b) - g(a))f'(x) = 0$$
$$(f(b) - f(a))g'(x) = (g(b) - g(a))f'(x)$$

Theorem 3.37 ***Mean Value Theorem:*** *If f is a real continuous function on $[a,b]$ which is differentiable at (a,b) then there is a point $x \in (a,b)$ at which $f(b) - f(a) = (b-a)f'(x)$.*
Proof: Put $g(x) = x$ in theorem 3.36. $\therefore g'(x) = 1 \Rightarrow (f(b) - f(a)) = (b-a)f'(x)$.

Theorem 3.38 *Suppose f is differentiable in (a,b).*
(a) If $f'(x) \geq 0\ \forall x \in (a,b)$, then f is monotonically increasing.
(b) If $f'(x) = 0\ \forall x \in (a,b)$, then f is a constant.
(c) If $f'(x) \leq 0\ \forall x \in (a,b)$, then f is monotonically decreasing.
Proof: (a) By theorem 3.37, If $x_1 < x_2$, then there exists $x_1 < x < x_2$ such that $f(x_2) - f(x_1) = (x_2 - x_1)f'(x)$...... (1)
If $f'(x) \geq 0$ then (1) $\Rightarrow f(x_2) - f(x_1) \geq 0$ ($\because (x_2 - x_1)f'(x) \geq 0$) $\Rightarrow f(x_1) \leq f(x_2)$ (i.e.) f is an increasing function
(b) If $f'(x)$=0 then (1) $\Rightarrow f(x_2) - f(x_1) = 0 \Rightarrow f(x_2) = f(x_1)$. $\therefore f$ is constant.
(c) If $f'(x) \leq 0$ then (1)$\Rightarrow f(x_2) - f(x_1) \leq 0 \Rightarrow f(x_1) \geq f(x_2)$. $\therefore f$ is an decreasing function.

The Continuity Of Derivatives

Theorem 3.39 *Suppose f is a real differentiable function on $[a,b]$ and suppose $f'(a) < \lambda < f'(b)$, then there is a point $x \in (a,b)$ such that $f'(x) = \lambda$. A similar result holds if $f'(a) > \lambda > f'(b)$.*
Proof: Let $g(t) = f(t) - \lambda t, t \in [a,b]$ then, $g'(t) = f'(t) - \lambda$; $g'(a) = f'(a) - \lambda < 0$. $\therefore$ there exists $a < t_1 < b$ such that $g(t_1) < g(a)$. Also, $g'(b) = f'(b) - \lambda > 0$. $\therefore$ there exists $a < t_2 < b$ such that $g(t_2) < g(b)$. $\therefore g$ attains minimum at $x \in (a,b)$. $\therefore g'(x)$=0 (by Theorem 3.35) (i.e.) $f'(x) - \lambda = 0 \Rightarrow f'(x) = \lambda$.

Corollary 3.40 *If f is differentiable on $[a,b]$, then f' is cannot have any simple discontinuity on $[a,b]$. But f' may have discontinuity of second kind.*
Proof: f' takes every value between $f(a)$ and $f(b)$. Let $a < x < b$. If f' is not continuous at x, then

1. $f'(x+), f'(x-)$ exists,

2. $f'(x+) \neq f'(x-)$,

3. $f'(x-) = f'(x+) \neq f'(x) \Rightarrow\Leftarrow$

$\therefore$ f' cannot have any simple discontinuity. In Example 3.33 f' has a discontinuity of second kind at $x \in [a, b]$.

Theorem 3.41 ***L'Hospital's Rule:*** *Suppose f and g are differentiable in (a, b) and $g'(x) \neq 0$ $\forall x \in (a, b)$ where $-\infty \leq a < b \leq \infty$. Suppose $\frac{f'(x)}{g'(x)} \to A$ as $x \to a$........ (1).*
If $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$........ (2) (or) if $g(x) \to \infty$ as $x \to a$....... (3), then $\frac{f(x)}{g(x)} \to A$ as $x \to a$....... (4). (The analogous statement is true if $x \to b$ (or) if $g(x) \to -\infty$ in (3)).
Proof: Case(i): Let $-\infty \leq A < \infty$. We choose r and q such that $A < r < q$. Given

$$\lim_{x \to a} \frac{f'(x)}{g'(x)} = A$$

Then there exists $c \in (a, b)$ such that $a < x < c \Rightarrow \frac{f'(x)}{g'(x)} < r$....... (i)
Now if $a < x < y < c$ then by generalised mean value theorem, there exists $t \in (a, b)$ such that $\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)} < r$........ (ii)
Suppose $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$. Then by taking limits as $x \to a$, then (ii) we get $\frac{f(y)}{g(y)} \leq r < q$........ (iii)
Suppose $g(x) \to \infty$ as $x \to a$, then by keeping y fixed in (ii) we can find $c_1 \in (a, y)$ such that $g(x) > g(y)$ and $g(x) > 0$ $\forall x \in (a, c_1)$. Multiply (ii) by $\frac{g(x)-g(y)}{g(x)}$, we get

$$\begin{aligned}\frac{f(x) - f(y)}{g(x)} &< r\left(\frac{g(x) - g(y)}{g(x)}\right)\\ \Rightarrow \frac{f(x)}{g(x)} - \frac{f(y)}{g(x)} &< r\left(1 - \frac{g(y)}{g(x)}\right)\\ \Rightarrow \frac{f(x)}{g(x)} &< r - r\frac{g(y)}{g(x)} + \frac{f(y)}{g(x)}\end{aligned}$$

Since $g(x) \to \infty$ as $x \to a$, there exists $c_2 \in (a, c_1)$ such that $\frac{f(x)}{g(x)} < r$ $\forall x \in (a, c_2)$ (or) $\frac{f(x)}{g(x)} < q$ $\forall x \in (a, c_2)$.......(iv)
suppose $-\infty < A \leq \infty$. By choosing $p < A$ as above, we can show that there exists $c_3 \in (a, b)$ such that $p < \frac{f(x)}{g(x)}$ $\forall a < x < c_3$.....(v)
Thus in all cases $\frac{f(x)}{g(x)} \to A$ as $x \to a$. Hence

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

.

Derivatives Of Higher Order

Definition 3.42 *If f has a derivative f' on an interval and if f' is differentiable, we see the second derivative f'' exists. Similarly if $f^{n-1}(x)$ is differentiable we say $f^{(n)}$ exists.*

Theorem 3.43 ***Taylor's Theorem:*** *Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists $\forall t \in (a, b)$. Let α, β be distinct points of $[a, b]$ and define*

$$p(t) = \sum_{n=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k,$$

then there exists a point $x \in (\alpha, \beta)$ such that $f(\beta) = p(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n$.
Proof: If n=1, then $f(\beta) = f(\alpha) + f'(x)(\beta-\alpha)$; $\frac{f(\beta)-f(\alpha)}{\beta-\alpha} = f'(x)$. This is just the mean value theorem. Suppose $n > 1$. Define a number M such that $f(\beta) = p(\beta) + M(\beta-\alpha)^n$........(1)
Let $g(t) = f(t) - p(t) - M(t-\alpha)^n$........ (2)
Now,

$$\begin{aligned}
g(\alpha) &= f(\alpha) - p(\alpha) - M(\alpha-\alpha)^n \\
&= f(\alpha) - p(\alpha) \\
g(\alpha) &= f(\alpha) - f(\alpha) \ (\because p(\alpha) = f(\alpha)) \\
&= 0 \\
g(\beta) &= f(\beta) - p(\beta) - M(\beta-\alpha)^n \\
&= 0 \ (by\ (1)).......(4) \\
\text{Also } g^{(n)}(t) &= f^{(n)}(t) - 0 - Mn!.......(5) \\
g^{(k)}(\alpha) &= f^{(k)}(\alpha) - p^{(k)}(\alpha) \\
&= f^{(k)}(\alpha) - f^{(k)}(\alpha) \\
&= 0......(6)
\end{aligned}$$

(i.e.) $g(\alpha) = g'(\alpha) = \cdots = g^{n-1}(\alpha) = 0$. Since $g(\alpha) = 0$ and $g(\beta) = 0$, there exists $x_1 \in (\alpha, \beta)$, by mean value theorem, such that $g'(x_1)$=0. Now since $g'(\alpha) = 0$; $g'(x_1) = 0$ again by mean value theorem there exists $x_2 \in (\alpha, x_1)$ such that $g"(x_2) = 0$. Proceeding this way we get $\alpha < x_n < x_{n-1}$, such that $g^{(n)}(x_n) = 0$ (i.e.) $f^{(n)}(x_n) - Mn! = 0$ (by (5)). $\therefore M = \frac{f^n(x_n)}{n!}$, sub M in (1) $\Rightarrow f(\beta) = p(\beta) + \frac{f^{(n)}(x_n)}{n!}(\beta-\alpha)^n$, $\forall x \in (\alpha, x_{n-1})$

4. UNIT IV

The Riemann-Steiltjes integral and Sequences and series of functions

Definition 4.1 *Let $[a, b]$ be an interval. By a partition P of $[a, b]$ we mean a finite set of points $x_0, x_1, ..., xn$, where $a = x_0 \le x_1 \le, ..., \le x_{i-1} \le x_i \le , ..., \le x_n = b$.*

Remark 4.2 *1.* $\Delta x_i = x_i - x_{i-1}\ \forall i = 1, 2, ..., n$.

2. *Let f be a bounded real function on $[a, b]$ then $m_i = \inf f(x), M_i = \sup f(x) \quad \forall x_{i-1} \le x \le x_i$.*

3. $$\begin{aligned} L(P, f) &= \sum_{i=1}^{n} m_i \Delta x_i \\ U(P, f) &= \sum_{i=1}^{n} m_i \Delta x_i \\ L(P, f) &\le \int_a^b f(x)dx \le U(P, f) \\ L(P, f) &\le U(P, f). \end{aligned}$$

4. $\int_{\underline{a}}^{b} f(x)dx = \sup L(P, f)$

5. $\int_a^{\bar{b}} f(x)dx = \inf U(P, f)$ *(The* $\inf$ *and* $\sup$ *are taken over all partition P of $[a, b]$).*

6. *If the upper and lower reimann interval over is same then f is said to be Reimann integrable over $[a, b].f \in \mathcal{R}(\mathcal{R}$ is the set of all Reimann integrable functions)*

7. $$\int_{\underline{a}}^{b} f(x)dx = \int_a^{\bar{b}} f(x)dx = \int_a^b f(x)dx$$

Result 4.3 *For every partition P of $[a, b]$ and every bounded function f there exists 2 real numbers m, M such that $m(b-a) \le L(P, f) \le U(P, f) \le M(b-a)$.*
Solution: Let $m = \inf f(x)$ and $M = \sup f(x), a \le x \le b$. Let $P =$

$\{x_0, x_1, ..., x_n\}$ be the given partition of $[a, b]$,

$$\begin{aligned} m &\leq m_i \leq M_i \leq M \\ m\Delta x_i &\leq m_i\Delta x_i \leq M_i\Delta x_i \leq M\Delta x_i \ (\Delta x_i \geq 0) \\ \sum_{i=1}^{n} m\Delta x_i &\leq \sum_{i=1}^{n} m_i\Delta x_i \leq \sum_{i=1}^{n} M_i\Delta x_i \leq \sum_{i=1}^{n} M\Delta x_i \\ m(\sum_{i=1}^{n} \Delta x_i) &\leq L(P, f) \leq U(P, f) \leq M \sum_{i=1}^{n} \Delta x_i........(1) \\ \text{Now, } \sum_{i=1}^{n} \Delta x_i &= \Delta x_1 + \Delta x_2 + ... + \Delta x_n \\ &= (x_1 - x_0) + (x_2 - x_1) + ... + (x_n - x_{n-1}) \\ &= x_n - x_0 \\ &= b - a........(2) \end{aligned}$$

sub (2) in (1) we get, $m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a)$.

Definition 4.4 *Let α be a monotonically increasing function on $[a, b]$. Corresponding to each partition P of $[a, b]$*
we define $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. Clearly, $\Delta\alpha_i \geq 0$

$$\begin{aligned} L(P, f, \alpha) &= \sum_{i=1}^{n} m_i\Delta\alpha_i \\ U(P, f, \alpha) &= \sum_{i=1}^{n} M_i\Delta\alpha_i \\ \sup L(P, f, \alpha) &= \int_{\underline{a}}^{b} f d\alpha \\ U(P, f, \alpha) &= \int_{a}^{\bar{b}} f d\alpha \end{aligned}$$

where infimum and suprimum are taken over all partitions. If

$$\int_{\underline{a}}^{b} f d\alpha = \int_{a}^{\bar{b}} f d\alpha,$$

then f is Reimann Stieljes integrable with respect to,

$$\int_{a}^{b} f d\alpha = \int_{\underline{a}}^{b} f d\alpha = \int_{a}^{\bar{b}} f d\alpha,$$

we also write $f \in \mathcal{R}(\alpha)$.

Note 4.5 *By taking $\alpha(x) = x$, we see that the Reimann integral is the special case of Riemann's Stieltjes integral.*

Definition 4.6 *The partition P^* of $[a,b]$ is called a refinement of P if $P \subset P^*$. Given two partition P_1 and P_2, we say that $P = P_1 \cup P_2$ is the common refinement of P_1 and P_2.*

Theorem 4.7 *If P^* is an refinement of P, then $L(P,f,\alpha) \leq L(P^*,f,\alpha)$ and $U(P^*,f,\alpha) \leq U(P,f,\alpha)$.*
Proof: Let $P = \{x_0, x_1, ..., x_{i-1}, x_i, ..., x_n\}$ be a partition of $[a,b]$ and let $P^* = \{x_0, x_1, x_2, ..., x_{i-1}, x^*, x_i, ..., x_n\}$ be an refinement of P. Let

$$\begin{aligned} m_i &= \inf f(x),\ x_{i-1} \leq x \leq x_i \\ w_1 &= \inf f(x),\ x_{i-1} \leq x \leq x^* \\ w_2 &= \inf f(x),\ x^* \leq x \leq x_i \end{aligned}$$

$\therefore w_1 \geq m_i$ and $w_2 \geq m_i$. Now,

$$\begin{aligned} L(P^*,f,\alpha) &= m_1\Delta\alpha_1 + m_2\Delta\alpha_2 + ... + m_{i-1}\Delta\alpha_{i-1} + w_1(\alpha(x^*) - \alpha(x_{i-1})) \\ &\quad + w_2(\alpha(x_i) - \alpha(x^*)) + m_{i+1}\Delta\alpha_{i+1} ... + m_n\Delta\alpha_n ......(1) \\ L(P,f,\alpha) &= m_1\Delta\alpha_1 + m_2\Delta\alpha_2 + ... + m_{i-1}\Delta\alpha_{i-1} + m_i\Delta\alpha_i \\ &\quad + m_{i+1}(\Delta\alpha_{i+1}) + ... + m_n\Delta\alpha_n ......(2) \end{aligned}$$

(1)-(2) $\Rightarrow$

$$\begin{aligned} L(P^*,f,\alpha) - L(P,f,\alpha) &= w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i) - \alpha(x^*)) - m_i\Delta\alpha_i \\ &= w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i) - \alpha(x^*)) \\ &\quad - m_i(\alpha(x_i) - \alpha(x_{i-1})) \\ &= w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i) - \alpha(x^*)) \\ &\quad - m_i(\alpha(x_i) - \alpha(x^*)) - m_i(\alpha(x^*) - \alpha(x_{i-1})) \\ &= (w_1 - m_i)(\alpha(x^*) - \alpha(x_{i-1})) \\ &\quad + (w_2 - m_i)(\alpha(x_i) - \alpha(x^*)) \\ &\geq 0 (\because w_1 \text{ and } w_2 \geq m_i) \\ L(P^*,f,\alpha) - L(P,f,\alpha) &\geq 0 \\ \Rightarrow L(P,f,\alpha) &\leq L(P^*,f,\alpha) \\ \therefore L(P,f,\alpha) &\leq L(P^*,f,\alpha) \end{aligned}$$

Let $P^* = \{x_0, x_1, ..., x_{i-1}, x^*, x_i, ..., x_n\}$ be refinement of P. Let

$$\begin{aligned} M_i &= \sup f(x), x_{i-1} \leq x \leq x_i \\ w_1 &= \sup f(x), x_{i-1} \leq x \leq x^* \\ w_2 &= \sup f(x), x^* \leq x \leq x_i \\ \therefore w_1 &\geq M_i \text{ and } w_2 \geq M_i \end{aligned}$$

Now

$$\begin{aligned}U(P^*, f, \alpha) &= M_1\Delta\alpha_1 + M_2\Delta\alpha_2 + ... + M_{i-1}\Delta\alpha_{i-1} + w_1(\alpha(x^*) - \alpha(x_{i-1}))\\ &\quad + w_2(\alpha(x_i) - \alpha(x^*)) + M_{i+1}\Delta\alpha_{i+1} + ... + M_n\Delta\alpha_n ....(1)\\ U(P, f, \alpha) &= M_1\Delta\alpha_1 + M_2\Delta\alpha_2 + ... + M_{i-1}\Delta\alpha_{i-1} + M_i\Delta\alpha_i\\ &\quad + M_{i+1}(\Delta\alpha_{i+1}) + ... + M_n\Delta\alpha_n ......(2)\end{aligned}$$

(1)-(2) $\Rightarrow$

$$\begin{aligned}U(P^*, f, \alpha) - U(P, f, \alpha) &= w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i)\\ &\quad - \alpha(x^*)) - M_i\Delta\alpha_i\\ &= w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i) - \alpha(x^*))\\ &\quad - M_i(\alpha(x_i) - \alpha(x_{i-1}))\\ &= w_1(\alpha(x^*) - \alpha(x_{i-1})) + w_2(\alpha(x_i) - \alpha(x^*))\\ &\quad - M_i(\alpha(x_i) - \alpha(x^*)) - M_i(\alpha(x^*) - \alpha(x_{i-1}))\\ &= (w_1 - M_i)(\alpha(x^*) - \alpha(x_{i-1}))\\ &\quad + (w_2 - M_i)(\alpha(x_i - \alpha(x^*)))\\ &\le 0 (\because w_1 \text{ and } w_2 \le M)\end{aligned}$$

$$\begin{aligned}(i.e.)\ U(P^*, f, \alpha) - U(P, f, \alpha) &\le 0\\ \Rightarrow U(P^*, f, \alpha) &\le U(P, f, \alpha)\\ \therefore U(P^*, f, \alpha) &\le U(P, f, \alpha)\end{aligned}$$

If P^* contains k-points more than P, we repeat this reasoning k-times and get the result.

Theorem 4.8

$$\int_{\underline{a}}^{b} f d\alpha \le \int_a^{\bar{b}} f d\alpha.$$

Proof: Let P_1 and P_2 be two partition of $[a, b]$ and let $P^* = P_1 U P_2$. (i.e.) P^* is a common refinement of P_1 and P_2. $L(P_1, f, \alpha) \le L(P^*, f, \alpha) \le U(P^*, f, \alpha) \le U(P_2, f, \alpha) \Rightarrow L(P_1, f, \alpha) \le U(P_2, f, \alpha)$. Keeping P_1 fixed and taking infimum over all partition P_2, we get

$$L(P, f, \alpha) \le \int_a^{\bar{b}} f d\alpha.$$

Now, by taking suprimum over all partition P_1 we get

$$\int_{\underline{a}}^{b} f d\alpha \le \int_a^{\bar{b}} f d\alpha.$$

Theorem 4.9 ***Criterion for Riemann Integrability:*** *Let* $f \in \mathcal{R}(\alpha)$ *iff* $\forall \in > 0$, *there exists a partition* P *such that* $U(P, f, \alpha) - L(P, f, \alpha) < \in$.

Proof: Let $\in > 0$, there exists a partition P such that $U(P,f,\alpha) - L(P,f,\alpha) < \in$
Claim: $f \in \mathcal{R}(\alpha)$. We know that

$$U(P,f,\alpha) \geq \bar{\int_a^{b}} f d\alpha .....(1)$$

$$L(P,f,\alpha) \leq \underline{\int_a^b} f d\alpha .....(2)$$

$$(2) \times -1 \Rightarrow -L(P,f,\alpha) \geq -\underline{\int_a^b} f d\alpha .....(3)$$

$$(1) + (3)\ U(P,f,\alpha) - L(P,f,\alpha) \geq \bar{\int_a^{b}} f d\alpha - \underline{\int_a^b} f d\alpha$$

$$(or)\ \bar{\int_a^{b}} f d\alpha - \underline{\int_a^b} f d\alpha \leq U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$$

Since ϵ is arbitrary,

$$\underline{\int_a^b} f d\alpha = \bar{\int_a^{b}} f d\alpha . (i.e.)\ f \in \mathcal{R}(\alpha).$$

Conversely: Assume $f \in \mathcal{R}(\alpha)$. To Prove: let $\epsilon > 0$, there exists a partition P such that $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$
let $\epsilon > 0$ be given
Then there exists two partition P_1 and P_2 such that
$U(P_1,f,\alpha) < \int_a^b f d\alpha + \frac{\epsilon}{2}$.....(4) and $\int_a^b f d\alpha - \frac{\epsilon}{2} < L(P_2,f,\alpha)$.......(5)
Let $P = P_1 U P_2$ (i.e.) P is the common refinement of P_1 and P_2
Now

$$\begin{aligned} U(P,f,\alpha) &\leq U(P_1,f,\alpha) \\ &\leq \int_a^b f d\alpha + \frac{\epsilon}{2} \text{ (by (4))} \\ &< L(P_2,f,\alpha) + \frac{\epsilon}{2} + \frac{\epsilon}{2} \text{ (by (5))} \\ &= L(P_2,f,\alpha) + \epsilon \\ &\leq L(P,f,\alpha) + \epsilon \\ \therefore U(P,f,\alpha) - L(P,f,\alpha) &< \epsilon \end{aligned}$$

Theorem 4.10 *Let P be a partition $\in$: $U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$...(1)*
(a) if (1) holds for some P and ϵ then (1) holds for every refinement of P.
(b) if (1) holds for $P = \{x_0, x_1, ..., x_n\}$ and s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$ then

$$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \Delta\alpha_i < \epsilon$$

(c) if $f \in \mathcal{R}(\alpha)$ and the hypothesis of (b) holds then

$$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| < \epsilon.$$

Proof: (a) Let P^* be a refinement of P. We know that

$$\begin{aligned} U(P^*, f, \alpha) &\leq U(P, f, \alpha)......(2) \\ L(P^*, f, \alpha) &\leq L(P, f, \alpha) \text{ (by Theorem 4.7)} \\ -L(P^*, f, \alpha) &\leq -L(P, f, \alpha)......(3) \end{aligned}$$

(2)+(3) gives

$$\begin{aligned} U(P^*, f, \alpha) - L(P^*, f, \alpha) &\leq U(P, f, \alpha) - L(P, f, \alpha) \\ &< \epsilon \text{ (by (1))} \\ (i.e.) U(P^*, f, \alpha) - L(P^*, f, \alpha) &< \epsilon \end{aligned}$$

(b) $s_i, t_i \in [x_{i-1}, x_i]$; $f(s_i), f(t_i) \in f[x_{i-1}, x_i]$; $m_i \leq f(s_i), f(t_i) \leq M_i$

$$\begin{aligned} \therefore |f(s_i) - f(t_i)| &\leq M_i - m_i \ (\because M_i - m_i \geq 0) \\ \Rightarrow |f(s_i) - f(t_i)|\Delta\alpha_i &\leq (M_i - m_i)\Delta\alpha_i \\ \Rightarrow \sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i &= \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i \\ &= \sum_{i=1}^{n} M_i\Delta\alpha_i - \sum_{i=1}^{n} m_i\Delta\alpha_i \\ &= U(P, f, \alpha) - L(P, f, \alpha) \text{ (by (1))} \\ \therefore \sum_{i=1}^{n} |f(s_i) - f(t_i)|\Delta\alpha_i &< \epsilon. \end{aligned}$$

(c) We have

$$\begin{aligned} m_i &\leq f(t_i) \leq M_i \\ \Rightarrow m_i\Delta\alpha_i &\leq f(t_i)\Delta\alpha_i \leq M_i\Delta\alpha_i \\ \Rightarrow \sum_{i=1}^{n} m_i\Delta\alpha_i &\leq \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \leq \sum_{i=1}^{n} M_i\Delta\alpha_i \\ \Rightarrow L(P, f, \alpha) &\leq \sum_{i=1}^{n} f(t_i)\Delta\alpha_i \leq U(P, f, \alpha)......(4) \\ L(P, f, \alpha) &\leq \int_a^b f d\alpha \leq U(P, f, \alpha)......(5) \end{aligned}$$

(4) and (5) $\Rightarrow$

$$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| \leq U(P,f,\alpha) - L(P,f,\alpha)$$
$$= \epsilon \text{ (by (1))}$$
$$\left|\sum_{i=1}^{n} f(t_i)\Delta\alpha_i - \int_a^b f d\alpha\right| < \epsilon.$$

Theorem 4.11 *If f is continuous on $[a,b]$ then $f \in \mathcal{R}(\alpha)$.*
Proof: Let $\epsilon > 0$ be given. Choose $\eta > 0$ such that $[\alpha(b) - \alpha(a)]\eta < \epsilon$...(1)
Since f is continuous on $[a,b]$ and $[a,b]$ is compact, f is uniformly continuous.
Then there exists $\delta > 0$ such that $|x - \epsilon| < \delta \Rightarrow |f(x) - f(\epsilon)| < \eta$..... (2)
Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a,b]$ such that $\Delta x_i < \delta \therefore$ (2) guarantees that $|M_i - m_i| < \eta$ (i.e.) $M_i - m_i < \eta$......(3)
Now,

$$\begin{aligned}
U(P,f,\alpha) - L(P,f,\alpha) &= \sum_{i=1}^{n} M_i\Delta\alpha_i - \sum_{i=1}^{n} m_i\Delta\alpha_i \\
&= \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i \\
&< \eta(\sum_{i=1}^{n}\Delta\alpha_i) \text{ (by (3))} \\
&= \eta[\Delta\alpha_1 + \Delta\alpha_2 + ... + \Delta\alpha_n] \\
&= \eta[(\alpha(x_1) - \alpha(x_0)) + (\alpha(x_2) - \alpha(x_1)) + ... + (\alpha(x_n) - \alpha(x_{n-1}))] \\
&= \eta(\alpha(x_n) - \alpha(x_0)) \\
&= \eta[\alpha(b) - \alpha(a)] \\
&< \epsilon
\end{aligned}$$

$\therefore U(P,f,\alpha) - L(P,f,\alpha) < \epsilon$ (by Theorem 4.9)

By Theorem 4.9, $f \in \mathcal{R}(\alpha)$.

Theorem 4.12 *If f is monotonic on $[a,b]$ and if α is continuous in $[a,b]$, then $f \in \mathcal{R}(\alpha)$.*
Proof: Let
epsilon > 0 be given. For every positive integer n, we choose a partition P such that $\Delta\alpha_i = \frac{\alpha(b)-\alpha(a)}{n}$. This is possible since α is continuous.
Case(i): f is monotonic increasing. $\therefore M_i = f(x_i);\ m_i = f(x_{i-1})\ \forall i =$

$1, 2, ..., n$. Now,

$$
\begin{aligned}
&U(P,f,\alpha) - L(P,f,\alpha)\\
&\quad = \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i\\
&\quad = \sum_{i=1}^{n} (M_i \Delta\alpha_i - m_i \Delta\alpha_i)\\
&\quad = \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i\\
&\quad = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(\frac{\alpha(b) - \alpha(a)}{n})\\
&\quad = \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})]\\
&\quad = \frac{\alpha(b) - \alpha(a)}{n} \{(f(x_1) - f(x_0)) + (f(x_2) - f(x_1)) + ...\\
&\qquad + (f(x_n) - f(x_{n-1}))\}\\
&\quad = \frac{\alpha(b) - \alpha(a)}{n} [f(x_n) - f(x_0)]\\
&\quad = \frac{\alpha(b) - \alpha(a)}{n} (f(b) - f(a))\\
&\quad < \epsilon \;\text{ as } n \to \infty.\\
&\therefore f \in \mathcal{R}(\alpha).
\end{aligned}
$$

Case(ii): f is monotonic decreasing. $\therefore M_i = f(x_i);\; m_i = f(x_{i-1})\; \forall i = 1, 2, ..., n$. Now,

$$
\begin{aligned}
&U(P,f,\alpha) - L(P,f,\alpha)\\
&\quad = \sum_{i=1}^{n} (M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i) \Delta\alpha_i\\
&\quad = \sum_{i=1}^{n} (M_i \Delta\alpha_i - m_i \Delta\alpha_i)\\
&\quad = \sum_{i=1}^{n} (M_i - m_i) \Delta\alpha_i\\
&\quad = \sum_{i=1}^{n} (f(x_{i-1}) - f(x_i))(\frac{\alpha(b) - \alpha(a)}{n})\\
&\quad = \frac{\alpha(b) - \alpha(a)}{n} \sum_{i=1}^{n} [f(x_{i-1}) - f(x_i)]
\end{aligned}
$$

$$= \frac{\alpha(b)-\alpha(a)}{n}\{(f(x_0)-f(x_1))+(f(x_1)-f(x_2))+... + (f(x_{n-1})-f(x_n))\}$$
$$= \frac{\alpha(b)-\alpha(a)}{n}[f(x_0)-f(x_n)]$$
$$= \frac{\alpha(b)-\alpha(a)}{n}(f(a)-f(b))$$
$$< \epsilon \text{ as } n \to \infty.$$
$$\therefore f \in \mathcal{R}(\alpha).$$

Hence the proof.

Theorem 4.13 *Suppose f is bounded on $[a,b], f$ has only finitely many point of discontinuity on $[a,b]$ and α is continuous at every point at which f is discontinuous, then $f \in \mathcal{R}(\alpha)$.*
Proof: Let $\epsilon > 0$ be given. Put $M = sup|f(x)|$. Let E be the set of points at which f is discontinuous. Since E is finite and α is continuous at every point of E, we can cover E by finitely many disjoint $[u_j, v_j] \subset [a,b]$ such that the sum of the corresponding differences

$$\sum_j [\alpha(v_j) - \alpha(u_j)] < \epsilon.$$

Also we place these intervals in such a way that every point of $E \cap (a,b)$ lies in the interval of some $[u_j, v_j]$. Remove the segments (u_j, v_j) from $[a,b]$. The remaining set K is compact. hence f is uniformly continuous on K. $\therefore$ there exists $\delta > 0$ such that $|s-t| < \delta \Rightarrow |f(s)-f(t)| < \epsilon \;\; \forall s,t \in K$. We form a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$ as follows. Each u_j occurs in P, each v_j occurs in P. No point of any segment (u_j, v_j) occurs in P. If x_{i-1} is not one of the u_j's then $\Delta x_i < \delta$. we observe that $M_i - m_i \leq 2\mu$, $\forall i$ and $M_i - m_i \leq \epsilon$ unless x_{i-1} is one of the u_j's. $\therefore U(P,f,\alpha) - L(P,f,\alpha) \leq [\alpha(b)-\alpha(a)]\epsilon + 2M\epsilon$. (By Theorem 4.11) Since ϵ is arbitrary, Theorem 4.9 guarantees that $f \in \mathcal{R}(\alpha)$.

Theorem 4.14 *Suppose $f \in \mathcal{R}(\alpha)$ on $[a,b], m \leq f \leq M, \phi$ is continuous on $[m,M]$ and $h(x) = \phi(f(x))$ on $[a,b]$, then $h \in \mathcal{R}(\alpha)$ on $[a,b]$.*
Proof: Let $\epsilon > 0$ be given. Since $\phi : [m,M] \to R$ is continuous and $[m,M]$ is compact, ϕ is uniformly continuous. $\therefore$ There exists $\delta > 0$ such that $\delta < \epsilon, |s-t| < \delta \Rightarrow |\phi(s) - \phi(t)| < \epsilon$ for $s,t \in [m,M]$...... (1)
Since $f \in \mathcal{R}(\alpha)$, there exists a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$ such that $U(P,f,\alpha) - L(P,f,\alpha) < \delta^2$...... (2)
To Prove: $h \in \mathcal{R}(\alpha)$. Let $M_i^* = \sup h(x), x_{i-1} \leq x \leq x_i$ and $m_i^* = \inf h(x), x_{i-1} \leq x \leq x_i$. Let $A = \{i | 1 \leq i \leq n, M_i - m_i < \delta\}$; $B =$

$\{i|1 \leq i \leq n, M_i - m_i \geq \delta\}$

$$\begin{aligned}
\text{for } i \in A, |M_i - m_i| < \delta &\Rightarrow |\phi(M_i) - \phi(m_i)| < \epsilon \text{ (by (1))} \\
\Rightarrow |M_i^* - m_i^*| &< \epsilon......(3) \\
\text{For } i \in B, |M_i^* - m_i^*| &\leq |M_i^*| + |m_i^*| \\
&\leq k + k \quad \text{where } k = sup|\phi(t)|, t \in [m, M] \\
|M_i^* - m_i^*| &\leq 2k.....(4) \\
\text{Also } \delta \sum_{i \in B} \Delta\alpha_i &\leq \sum_{i \in B} (M_i - m_i)\Delta\alpha_i \\
&\leq \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i \\
&= \sum_{i=1}^{n} M_i \Delta\alpha_i - \sum_{i=1}^{n} m_i \Delta\alpha_i \\
&= U(P, f, \alpha) - L(P, f, \alpha) \\
&< \delta^2 \text{ (by (2))} \\
(i.e.)\ \delta \sum_{i \in B} \Delta\alpha_i &< \delta^2 \\
\Rightarrow \sum_{i \in B} \Delta\alpha_i &< \delta.....(5)
\end{aligned}$$

$$\begin{aligned}
\text{Now } U(P, h, \alpha) - L(P, h, \alpha) &= \sum_{i=1}^{n} M_i^* \Delta\alpha_i - \sum_{i=1}^{n} m_i^* \Delta\alpha_i \\
&= \sum_{i=1}^{n} (M_i^* - m_i^*)\Delta\alpha_i \\
&= \sum_{i \in A} (M_i^* - m_i^*)\Delta\alpha_i + \sum_{i \in B} (M_i^* - m_i^*)\Delta\alpha_i \\
&< \epsilon \sum_{i \in A} \Delta\alpha_i + 2k \sum_{i \in B} \Delta\alpha_i \text{ (by (3) and (4))} \\
&< \epsilon \sum_{i=1}^{n} \Delta\alpha_i + 2k \sum_{i \in B} \Delta\alpha_i \\
&< \epsilon[\alpha(b) - \alpha(a)] + 2k\delta \\
&< \epsilon[\alpha(b) - \alpha(a)] + 2k\epsilon \ (\because \delta < \epsilon) \\
&= \epsilon[\alpha(b) - \alpha(a) + 2k]
\end{aligned}$$

(i.e.) $U(P, h, \alpha) - L(P, h, \alpha) < \epsilon[\alpha(b) - \alpha(a) + 2k]$
since ϵ is arbitrary, Theorem 4.9, implies that $h \in \mathcal{R}(\alpha)$.

Lemma 4.15 *If $f \in \mathcal{R}(\alpha)$ and $f \geq 0$ on $[a, b]$ then $\int_a^b f d\alpha \geq 0$.*

Proof: Since $f \geq 0,\ M_i \geq 0 \forall_i$.

$$\therefore \sum_{i=1}^{n} M_i \Delta\alpha_i \geq 0$$
$$\Rightarrow U(P,h,\alpha) \geq 0$$
$$\Rightarrow \inf U(P,h,\alpha) \geq 0$$
$$\Rightarrow \int_a^b f d\alpha \geq 0.$$

Properties of Integral

Theorem 4.16 *(a) If $f_1, f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$ then $f_1 + f_2 \in \mathcal{R}(\alpha), cf_1 \in \mathcal{R}(\alpha)$ for every constant c and $\int_a^b (f_1+f_2)d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha, \int_a^b cf_1 d\alpha = c\int_a^b f_1 d\alpha$.*
(b) If $f_1(x) \leq f_2(x)$ on $[a,b]$ then $\int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$.
(c) If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and $a < c < b$, then $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and on $[a,b]$ and $\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$
(d) If $f \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $|f(x)| \leq M$ then $|\int_a^b f d\alpha| \leq [\alpha(b) - \alpha(a)]$.
(e) If $f \in R(\alpha_1)$ and $f \in R(\alpha_2)$ then $f \in R(\alpha_1 + \alpha_2)$ and $\int_a^b f d(\alpha_1+\alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$. If $f \in \mathcal{R}(\alpha)$ and c is positive constant then $f \in \mathcal{R}(\alpha)$ and $\int_a^b f d\alpha = c\int_a^b f d\alpha$.

Proof: **(a)** Let $\epsilon > 0$ be given. Since $f_1 \in \mathcal{R}(\alpha)$ and $f_2 \in [a,b]$, there exists two partitions P_1 and P_2 of $[a,b]$ such that $U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha) < \epsilon$..... (1) and $U(P_2, f_2, \alpha) - L(P_2, f_2, \alpha) < \epsilon$.....(2)
Let $P = P_1 \cup P_2$ be the common refinement of $[a,b]$.

$$\therefore U(P_1, f_1, \alpha) \leq U(P_1, f_1, \alpha)$$
$$L(P_1, f_1, \alpha) \leq L(P_1, f_1, \alpha)$$
$$\Rightarrow U(P, f_1, \alpha) + L(P_1, f_1, \alpha) \leq U(P_1, f_1, \alpha) + L(P, f_1, \alpha)$$
$$\Rightarrow U(P, f_1, \alpha) - L(P_1, f_1, \alpha) \leq U(P_1, f_1, \alpha) - L(P_1, f_1, \alpha)$$
$$U(P, f_1, \alpha) - L(P, f_1, \alpha) < \epsilon \text{ (by (1))}.......(3)$$
$$\text{Similarly } U(P, f_2, \alpha) - L(P, f_2, \alpha) < \epsilon \text{ (by (2))}.......(4)$$

(3)+(4)$\Rightarrow$

$$U(P, f_1, \alpha) + U(P, f_2, \alpha) - (L(P, f_1, \alpha)) + L(P, f_2, \alpha) < 2\epsilon......(5)$$

$$\begin{aligned}\text{Now } L(P, f_1, \alpha) + L(P, f_2, \alpha) &\leq L(P, f_1 + f_2, \alpha)\\ &\leq U(P, f_1 + f_2, \alpha)\\ &\leq U(P, f_1, \alpha) + U(P, f_2, \alpha)......(6)\end{aligned}$$

(5),(6)$\Rightarrow U(P, f_1 + f_2, \alpha) - L(P, f_1 + f_2, \alpha) < 2\epsilon$. $\therefore f_1 + f_2 \in \mathcal{R}(\alpha)$ on $[a,b]$.
To prove:

$$\int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

Since $f_1,\ f_2 \in \mathcal{R}(\alpha)$, there exists partition P_1 and P_2 of $[a,b]$

$$U(P_1, f_1, \alpha) < \int_a^b f_1 d\alpha + \epsilon \text{ (by Theorem 4.9).......(1*)}$$
$$U(P_2, f_2, \alpha) < \int_a^b f_2 d\alpha + \epsilon \text{........(2*)}$$

(1)+(2)⇒

$$U(P_1, f_1, \alpha) + U(P_2, f_2, \alpha) < \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + 2\epsilon \text{......(3*)}$$

Let $P = P_1 \cup P_2$

$$U(P, f_1, \alpha) \leq U(P_1, f_1, \alpha) \text{.......(4*)}$$
$$U(P, f_2, \alpha) \leq U(P_2, f_2, \alpha) \text{.......(5*)}$$

(4*)+(5*)⇒

$$\begin{aligned} U(P, f_1, \alpha) + U(P, f_2, \alpha) &\leq U(P_1, f_1, \alpha) + \leq U(P_2, f_2, \alpha) \\ &< \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + 2\epsilon \text{......(6*) (by (3*))} \\ U(P, f_1 + f_2, \alpha) &\leq U(P, f_1, \alpha) + U(P, f_2, \alpha) \\ &< \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + 2\epsilon \text{ (by (6*))} \end{aligned}$$

Taking infimum over all partition P,

$$\int_a^b (f_1 + f_2) d\alpha < \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha + 2\epsilon$$

Since ϵ is arbitrary,

$$\int_a^b (f_1 + f_2) d\alpha \leq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \text{.....(7*)}$$

Replacing f_1 and f_2 in (7*) by $-f_1$ and $-f_2$ respectively we get,

$$\int_a^b (-f_1 - f_2) d\alpha \leq \int_a^b (-f_1) d\alpha + \int_a^b (-f_2) d\alpha$$
$$\Rightarrow \int_a^b (f_1 + f_2) d\alpha \geq \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \text{.....(8*)}$$

From (7*)and(8*) we get,

$$\int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

To Prove: $cf_1 \in \mathcal{R}(\alpha)$ where c is a constant.
For any partition P, of $[a, b]$

$$U(P, cf_1, \alpha) = \begin{cases} cU(P, f_1, \alpha) & c \geq 0 \\ cL(P, f_1, \alpha) & c \leq 0 \end{cases}$$

and

$$L(P, cf_1, \alpha) = \begin{cases} cL(P, f_1, \alpha) & c \geq 0 \\ cU(P, f_1, \alpha) & c \leq 0 \end{cases}$$

$$U(P, cf_1, \alpha) - L(P, cf_1, \alpha) = \begin{cases} c(U(P, f_1, \alpha) - L(P, f_1, \alpha)) & c \geq 0 \\ -c(U(P, f_1, \alpha) - L(P, f_1, \alpha)) & c \leq 0 \end{cases}$$

$$U(P, cf_1, \alpha) - L(P, cf_1, \alpha) = |c|(U(P, f_1, \alpha) - L(P, f_1, \alpha)).....(1A)$$

Since $f_1 \in \mathcal{R}(\alpha)$ there exists a partition P of $[a, b]$ such that

$$U(P, f_1, \alpha) - L(P, cf_1, \alpha) < \frac{\epsilon}{|c|}......(2A)$$

Sub (2A) in (1A), we get

$$U(P, cf_1, \alpha) - L(P, cf_1, \alpha) < |c|\frac{\epsilon}{|c|}$$
$$U(P, cf_1, \alpha) - L(P, cf_1, \alpha) < \epsilon$$
$$\therefore cf_1 \in \mathcal{R}(\alpha).$$

To Prove:

$$\int_a^b cf_1 d\alpha = \int_a^b cf_1 d\alpha$$

$$\begin{aligned}
\text{If } c \geq 0, \text{ then } U(P, cf_1, \alpha) &= cU(P, f_1, \alpha) \\
\Rightarrow \inf U(P, cf_1, \alpha) &= \inf(cU(P, f_1, \alpha)) \\
\Rightarrow \inf U(P, cf_1, \alpha) &= c \inf U(P, cf_1, \alpha) \\
\Rightarrow \int_a^b cf_1 d\alpha &= \int_a^b cf_1 d\alpha \\
\text{If } c \leq 0, \text{ then } L(P, cf_1, \alpha) &= cU(P, f_1, \alpha) \\
&= -|c|U(P, f_1, \alpha) \ (\because c \leq 0) \\
\Rightarrow \sup L(P, cf_1, \alpha) &= \sup(-|c|U(P, f_1, \alpha)) \\
&= |c| \sup(-U(P, f_1, \alpha)) \\
&= -|c| \inf(U(P, f_1, \alpha)) \\
\Rightarrow \int_a^b cf_1 d\alpha &= -|c| \int_a^b f_1 d\alpha \\
&= c \int_a^b f_1 d\alpha \\
\text{When } c = 0, \int_a^b cf_1 d\alpha &= \int_a^b f_1 d\alpha \ (= 0)
\end{aligned}$$

To Prove:
$$f_1 \leq f_2 \Rightarrow \int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$$

Proof of b: Given $f_1 \leq f_2 \Rightarrow f_2 - f_1 \geq 0$

$$\begin{aligned}
&\Rightarrow \int_a^b (f_2 - f_1) d\alpha \geq 0\\
&\Rightarrow \int_a^b f_2 + \int_a^b (-f_1) d\alpha \geq 0\\
&\Rightarrow \int_a^b f_2 d\alpha + \int_a^b (-f_1) d\alpha \geq 0 \text{ (by (a))}\\
&\Rightarrow \int_a^b f_2 d\alpha - \int_a^b f_1 d\alpha \geq 0\\
&\Rightarrow \int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha
\end{aligned}$$

Proof of (c): Given $f \in \mathcal{R}(\alpha)$ on $[a, b]$ and $a < c < b$ for $\epsilon < 0$, there exists a partition P of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \epsilon ......(1B)$$

Let $P^* = P \cup \{c\}$. Now P^* is a refinement of P and induces two partitions P_1 and P_2 of $[a, c]$ and $[c, b]$ respectively. Now,

$$\begin{aligned}
U(P, f, \alpha) &\geq U(P^*, f, \alpha)\\
&= U(P_1, f, \alpha) + U(P_2, f, \alpha).....(2B)\\
\Rightarrow U(P_1, f, \alpha) &\leq U(P, f, \alpha)......(3B)\\
\text{and } \ U(P_2, f, \alpha) &\leq U(P, f, \alpha).......(4B)\\
L(P, f, \alpha) &\leq L(P^*, f, \alpha)\\
&= L(P_1, f, \alpha) + L(P_2, f, \alpha).......(5B)\\
-L(P, f, \alpha) &\geq -L(P_1, f, \alpha) - L(P_2, f, \alpha)\\
-L(P_1, f, \alpha) &\leq -L(P, f, \alpha)......(6B)\\
\text{and } \ -L(P_2, f, \alpha) &\leq -L(P, f, \alpha).......(7B)\\
(3B) + (6B) \Rightarrow U(P_1, f, \alpha) - L(P_1, f, \alpha) &\leq U(P, f, \alpha) - L(P, f, \alpha) \text{ (by (1B))}\\
&< \epsilon\\
\therefore f &\in \mathcal{R}(\alpha) \text{ on } [a, c].\\
(4B) + (7B) \Rightarrow U(P_2, f, \alpha) - L(P_2, f, \alpha) &\leq U(P, f, \alpha) - L(P, f, \alpha) \text{ (by (1B))}\\
&< \epsilon\\
\therefore f &\in \mathcal{R}(\alpha) \text{ on } [c, b].
\end{aligned}$$

To Prove:
$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$$

$$\begin{aligned}
(2B) \Rightarrow U(P,f,\alpha) &\geq U(P_1,f,\alpha)+U(P_2,f,\alpha) \\
&\geq \int_a^c f d\alpha + \int_c^b f d\alpha \\
\Rightarrow \inf U(P,f,\alpha) &\geq \int_a^c f d\alpha + \int_c^b f d\alpha \\
\int_a^b f d\alpha &\geq \int_a^c f d\alpha + \int_c^b f d\alpha ......(8B) \\
(5B) \Rightarrow L(P,f,\alpha) &\leq L(P_1,f,\alpha)+L(P_2,f,\alpha) \\
&\leq \int_a^c f d\alpha + \int_c^b f d\alpha \\
\Rightarrow \sup U(P,f,\alpha) &\leq \int_a^c f d\alpha + \int_c^b f d\alpha \\
\int_a^b f d\alpha &\leq \int_a^c f d\alpha + \int_c^b f d\alpha ....(9B)
\end{aligned}$$

$\therefore$ (8B) and (9B), we get

$$\int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$$

Proof of (d): Given $f \in \mathcal{R}(\alpha)$ and $|f(x)| \leq M$
To Prove: $|\int_a^b f d\alpha| \leq [\alpha(b)-\alpha(a)]$
we have, for any partition P of $[a,b]$,

$$\begin{aligned}
\int_a^b f d\alpha &\leq U(P,f,\alpha) \\
\left|\int_a^b f d\alpha\right| &\leq |U(P,f,\alpha)| \\
&= \left|\sum_{i=1}^n M_i \Delta\alpha_i\right| \\
&< \sum_{i=1}^n |M_i \Delta\alpha_i| \\
&= \sum_{i=1}^n |M_i| \Delta\alpha_i \ (\because \Delta\alpha_i \geq 0) \\
&\leq \sum_{i=1}^n M \Delta\alpha_i \ (\because |f(x)| \leq M) \\
&= M \sum_{i=1}^n \Delta\alpha_i \\
\left|\int_a^b f d\alpha\right| &\leq M[\alpha(b)-\alpha(a)]
\end{aligned}$$

Proof of (e): Given $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$. To Prove: $f \in \mathcal{R}(\alpha_1+\alpha_2)$.

Let $\alpha = \alpha_1 + \alpha_2$. For any partition p of $[a, b]$,

$$\begin{aligned}
U(P, f, \alpha) &= \sum_{i=1}^{n} M_i \Delta\alpha_i \\
&= \sum_{i=1}^{n} M_i(\alpha(x_i) - \alpha(x_{i-1})) \\
&= \sum_{i=1}^{n} M_i[(\alpha_1 + \alpha_2)(x_i) - (\alpha_1 + \alpha_2)(x_{i-1})] \\
&= \sum_{i=1}^{n} M_i[\alpha_1(x_i) + \alpha_2(x_i)] - [\alpha_1(x_{i-1}) + \alpha_2(x_{i-1})] \\
&= \sum_{i=1}^{n} M_i[\alpha_1(x_i) - \alpha_1(x_{i-1})] + \sum_{i=1}^{n} M_i[\alpha_2(x_i) - \alpha_2(x_{i-1})] \\
U(P, f, \alpha) &= U(P, f, \alpha_1) + U(P, f, \alpha_2).......(1C)
\end{aligned}$$

Similarly $L(P, f, \alpha) = L(P, f, \alpha_1) + L(P, f, \alpha_2).......(2C)$

since $f \in \mathcal{R}(\alpha_1)$ and $f \in \mathcal{R}(\alpha_2)$, there exists partitions P_1 and P_2 of $[a, b]$ such that

$$\begin{aligned}
U(P_1, f, \alpha_1) - L(P_1, f, \alpha_1) < \epsilon \\
\text{and } U(P_2, f, \alpha_2) - L(P_2, f, \alpha_2) < \epsilon
\end{aligned}$$

Let P^* be the common refinement of P_1 and P_2 of $[a, b]$. $P^* = P_1 \cup P_2$

$$\begin{aligned}
&U(P^*, f, \alpha_1) - L(P^*, f, \alpha_1) < \epsilon........(3C) \\
&U(P^*, f, \alpha_2) - L(P^*, f, \alpha_2) < \epsilon........(4C) \text{ (by Theorem 4.10)}
\end{aligned}$$

Now,

$$\begin{aligned}
U(P^*, f, \alpha) - L(P^*, f, \alpha) &= U(P^*, f, \alpha_1) + U(P^*, f, \alpha_2) \\
&\quad - [L(P^*, f, \alpha_1) + L(P^*, f, \alpha_2)] \text{ (by (1C) and (2C))} \\
&= [U(P^*, f, \alpha_1) - L(P^*, f, \alpha_1)] \\
&\quad + [U(P^*, f, \alpha_2) - L(P^*, f, \alpha_2)] \\
&< \epsilon + \epsilon \text{ (by (3C) and (4C))} \\
U(P^*, f, \alpha) - L(P^*, f, \alpha) &< 2\epsilon.
\end{aligned}$$

Since ϵ arbitrary, we get $f \in \mathcal{R}(\alpha)$ (i.e.) $f \in \mathcal{R}(\alpha_1 + \alpha_2)$.
To Prove:

$$\int_a^b d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

$$\begin{aligned}
(1C) \Rightarrow U(P, f, \alpha) &= U(P, f, \alpha_1) + U(P, f, \alpha_2) \\
&\geq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \\
\Rightarrow \inf U(P, f, \alpha) &\geq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \\
\int_a^b f d\alpha &\geq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 ......(5C) \\
(2C) \Rightarrow L(P, f, \alpha) &= L(P, f, \alpha_1) + L(P, f, \alpha_2) \\
&\leq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \\
\sup U(P, f, \alpha) &\leq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \\
\int_a^b f d\alpha &\leq \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 ......(6C)
\end{aligned}$$

from (5C) and (6C) we get,

$$\begin{aligned}
\int_a^b f d\alpha &= \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \\
(i.e.) \int_a^b d(\alpha_1 + \alpha_2) &= \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2.
\end{aligned}$$

To Prove: Given $f \in \mathcal{R}(\alpha)$ and $c > 0$
To Prove: $f \in \mathcal{R}(\alpha)$, for any partition P,

$$\begin{aligned}
U(P, f, c\alpha) &= \sum_{i=1}^{n} M_i \Delta(c\alpha_i) \\
&= \sum_{i=1}^{n} M_i (c\alpha(x_i) - c\alpha(x_{i-1})) \\
&= \sum_{i=1}^{n} M_i c[\alpha(x_i) - \alpha(x_{i-1})] \\
&= \sum_{i=1}^{n} c M_i \Delta\alpha_i \\
&= cU(P, f, \alpha) ....(7C) \\
\text{Similarly } L(P, f, c\alpha) &= cL(P, f, \alpha) \\
U(P, f, c\alpha) - L(P, f, c\alpha) &= cU(P, f, \alpha) - cL(P, f, \alpha) \\
&= c[U(P, f, \alpha) - L(P, f, \alpha)] ......(8C)
\end{aligned}$$

Since $f \in \mathcal{R}(\alpha)$, given $\epsilon > 0$, there exists partition P of $[a, b]$ such that

$$U(P, f, \alpha) - L(P, f, \alpha) < \frac{\epsilon}{c} ....(9C)$$

sub (9C)in (8C) we get

$$U(P, f, c\alpha) - L(P, f, c\alpha) < c \cdot \frac{\epsilon}{c} = \epsilon$$

$\therefore f \in \mathcal{R}(c\alpha)$. To Prove:

$$\int_a^b f d(c\alpha) = c \int_a^b f d\alpha$$

$$\begin{aligned}(7C) \Rightarrow U(P, f, c\alpha) &= cU(P, f, \alpha)\\ \Rightarrow \inf U(P, f, c\alpha) &= \inf cU(P, f, \alpha)\\ &= c \inf U(P, f, \alpha)\\ \Rightarrow \int_a^b f d(c\alpha) &= c \int_a^b f d\alpha\end{aligned}$$

Theorem 4.17 *If* $f, g \in \mathcal{R}(\alpha)$ *on* $[a, b]$*,then*
(a) $f \cdot g \in \mathcal{R}(\alpha)$
(b) $|f| \in \mathcal{R}(\alpha)$ *and*

$$\left|\int_a^b f d\alpha\right| \leq \int_a^b |f| d\alpha.$$

Proof: **(a)** Let $\phi(t) = t^2$, clearly ϕ is continuous

$$\begin{aligned} h(x) &= \phi(f(x)) \text{ (by Theorem 4.14)}\\ &= f(x)^2\\ &= f^2(x)\\ \therefore f^2 &\in \mathcal{R}(\alpha) ....(1) \ (\because f \in \mathcal{R}(\alpha))\\ \text{Now,} f, g &\in \mathcal{R}(\alpha)\\ \Rightarrow f + g, f - g &\in \mathcal{R}(\alpha) \text{ (by Theorem 4.16)}\\ \Rightarrow (f + g)^2, (f - g)^2 &\in \mathcal{R}(\alpha)\\ \Rightarrow (f + g)^2 - (f - g)^2 &\in \mathcal{R}(\alpha)\\ \Rightarrow 4fg &\in \mathcal{R}(\alpha)\\ \Rightarrow fg &\in \mathcal{R}(\alpha) \text{ (by Theorem 4.16)}\end{aligned}$$

(b) $|f| \in \mathcal{R}(\alpha)$ and $|\int_a^b f d\alpha| \leq \int_a^b |f| d\alpha$.
To Prove: $|f| \in \mathcal{R}(\alpha)$. Let $\phi(t) = |t|$; $h(x) = \phi(f(x)) = |f(x)|$. $\therefore$ By Theorem 4.14, $|f| \in \mathcal{R}(\alpha)$
To prove:

$$\left|\int_a^b f d\alpha\right| \leq \int_a^b |f| d\alpha.$$

Choose $c = \pm 1$ so that $c\int_a^b f d\alpha \geq 0$

$$\begin{aligned}\therefore |\int_a^b f d\alpha| &= c\int_a^b f d\alpha \\ &= \int_a^b cf d\alpha \text{ (by Theorem 4.16(a))} \\ &\leq \int_a^b |f| d\alpha \ (\because cf \leq |f|) \text{ by Theorem 4.16(b)}\end{aligned}$$

Hence the proof.

Definition 4.18 ***Unit Step Function:***

$$I(x) = \begin{cases} 0 & if \quad x \leq 0 \\ 1 & if \quad x > o \end{cases}$$

Theorem 4.19 *If $a < s < b$, f is bounded on $[a, b]$, f is continuous at s and $\alpha(x) = I(x - s)$, then*

$$\int_a^b f d\alpha = f(s).$$

Proof: Consider partitions $P = \{x_0, x_1, x_2, x_b\}$ of $[a, b]$ where $x_0 x_1 = s, s < x_2 < b, x_2 = b$. Now,

$$\begin{aligned}U(P, f, \alpha) &= \sum_{i=1}^{3} M_i \Delta\alpha_i \\ &= M_i\Delta\alpha_1 + M_2\Delta\alpha_2 + M_3\Delta\alpha_3 \\ &= M_1[\alpha(x_1) - \alpha(x_0)] + M_2[\alpha(x_2) - \alpha(x_1)] + M_3[\alpha(x_3) - \alpha(x_2)] \\ &= M_1[I(x_1 - s) - I(x_0 - s)] + M_2[I(x_2 - s) - I(x_1 - s)] \\ &\quad + M_3[I(x_3 - s) - I(x_2 - s)] \\ &= M_1[I(s - s) - I(a - s)] + M_2[I(x_2 - s) - I(s - s)] \\ &\quad + M_3[I(b - s) - I(x_2 - s)] \\ &= M_1[I(0) - I(a - s)] + M_2[I(x_2 - s) - I(0)] \\ &\quad + M_3[I(b - s) - I(x_2 - s)] \\ &= M_1[0 - 0] + M_2[1 - 0] + M_3[1 - 1] \text{ (by definition of } i) \\ &= M_2\end{aligned}$$

In a similar fashion we can get $L(P, f, \alpha) = m_2$.

$$\begin{aligned}\int_a^b f d\alpha &= \inf U(P, f, \alpha) = \sup L(P, f, \alpha) \\ &= \inf M_2 = \sup m_2 \\ &= f(s) \ (\because x_2 \to s, f(x_2) \to f(x) \text{ as } f \text{ is continuous at } s)\end{aligned}$$

Theorem 4.20 *Suppose $c_n \geq 0$ for $1, 2, 3...$, $\sum c_n$ converges, $\{s_n\}$ is a sequence of distinct point in (a, b) and $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$. Let f be continuous on $[a, b]$, then*

$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n).$$

Proof: We have $|I(x - s_n)| \leq 1$. $\therefore |c_n I(x - s_n)| \leq c_n$. Since

$$\sum_{n=1}^{\infty} c_n$$

is convergent, by comparison test,

$$\sum_{n=1}^{\infty} c_n I(x - s_n)$$

also converges. Now,

$$\begin{aligned} \alpha(a) &= \sum_{n=1}^{\infty} c_n I(a - s_n) \\ &= 0.......(1) \ (\because I(a - s_n) = 0) \\ \text{and } \alpha(b) &= \sum_{n=1}^{\infty} c_n I(b - s_n) \\ &= \sum_{n=1}^{\infty} c_n.....(2) \ (\because I(b - s_n) = 0) \end{aligned}$$

Claim: α is monotonically increasing. Let $x < y$ and let $x < s_k < y$

$$\begin{aligned} \alpha(x) &= \sum_{n=1}^{\infty} c_n I(x - s_n) \\ &= c_1 + c_2 + ... + c_{k-1} \\ \alpha(y) &= \sum_{n=1}^{\infty} c_n I(y - s_n) \\ &= c_1 + c_2 + ... + c_{k-1} + c_k \\ \therefore \alpha(x) &\leq \alpha(y) \end{aligned}$$

Hence the claim. Since

$$\sum_{n=1}^{\infty} c_n$$

is convergent, given $\epsilon > 0$, there exists $N >$ such that

$$\sum_{n=N+1}^{\infty} c_n < \epsilon......(3)$$

Let

$$\alpha_1(x) = \sum_{n=1}^{N} c_n I(x - s_n)$$
$$\alpha_2(x) = \sum_{n=N+1}^{\infty} c_n I(x - s_n)$$

Clearly $\alpha(x) = \alpha_1(x) + \alpha_2(x)$. Let $\alpha_{1i} = I(x - s_i), i = 1, 2, ..., N$.

$$\begin{aligned}
\therefore \alpha_1(x) &= \sum_{n=1}^{N} c_n \alpha_{1n}(x) \\
&= (c_1\alpha_{11} + c_2\alpha_{12} + ... + c_N\alpha_{1N})x \\
\text{(or) } \alpha_1 &= c_1\alpha_{11} + c_2\alpha_{12} + ... + c_N\alpha_{1N}
\end{aligned}$$

Now,

$$\begin{aligned}
\int_a^b f d\alpha_1 &= \int_a^b f d(c_1\alpha_{11} + c_2\alpha_{12} + ... + c_N\alpha_{1N}) \\
&= c_1 \int_a^b f d\alpha_{11} + c_2 \int_a^b f d\alpha_{12} + ...c_N \int_a^b f d\alpha_{1N} \text{ (by Theorem 4.16(e))} \\
&= c_1 f(s_1) + c_2 f(s_2) + ... + c_N f(s_N) \text{ (by Theorem 4.19)} \\
&= \sum_{n=1}^{N} c_n f(s_n) ....(4)
\end{aligned}$$

Now,

$$\begin{aligned}
\alpha_2(a) &= \sum_{n=N+1}^{\infty} c_n I(a - s_n) \\
&= 0........(5) \\
\alpha_2(b) &= \sum_{n=N+1}^{\infty} c_n I(b - s_n) \\
&= \sum_{n=N+1}^{\infty} c_n \\
&< \epsilon \text{ (by (3))......(6)}
\end{aligned}$$

Let $M = |f(x)|, x \in [a, b]$. By Theorem 4.16(d),

$$\left|\int_a^b f d\alpha_2\right| \leq [\alpha_2(b) - \alpha_2(a)]$$
$$\leq M\epsilon \text{ (by (5)and(6))},$$
$$(i.e.) \left|\int_a^b f d\alpha_2\right| \leq M\epsilon$$
$$\Rightarrow \left|\int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 - \int_a^b f d\alpha_1\right| \leq M\epsilon$$

$$\Rightarrow \left|\int_a^b f d(\alpha_1 + \alpha_2) - \int_a^b f d\alpha_1\right| \leq M\epsilon \text{ (by theorem 4.16(d))}$$
$$\Rightarrow \left|\int_a^b f d\alpha - \sum_{n=1}^{N} c_n f(s_n)\right| \leq M\epsilon \text{ (by (4))}$$

Taking limits as $N \to \infty$,

$$\left|\int_a^b f d\alpha - \sum_{n=1}^{\infty} c_n f(s_n)\right| \leq M\epsilon$$
$$\therefore \left|\int_a^b f d\alpha\epsilon\right| = \sum_{n=1}^{\infty} c_n f(s_n)$$

Theorem 4.21 *Assume α increases monotonically and $\alpha' \in \mathcal{R}$ on $[a, b]$, Let f be a bounded real function on $[a, b]$, then $f \in \mathcal{R}(\alpha)$ iff $f\alpha' \in \mathcal{R}$. In that case $\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx$.*

Proof: Let $\epsilon > 0$ be given. Since $\alpha' \in R$, there exists a partition $P = \{x_1, x_2, ..., x_n\}$ of $[a, b]$ such that $U(P, \alpha') - L(P, \alpha') < \epsilon$........ (1)

By mean value theorem , there exists $t :\in [x_{i-1}, x_i]$ such that $\alpha(x_i) - \alpha(x_{i-1}) = \alpha'(t_i)(x_i - x_{i-1})$ (i.e.) $\Delta\alpha_i = \alpha'(t_i)\Delta x_i$..... (2)

By Theorem 4.10(b), $\forall s_i, t_i \in [x_{i-1}, x_i]$

$$\sum_{i=1}^{n} |\alpha'(s_i) - \alpha'(t_i)|\Delta x_i < \epsilon......(3)$$

Now,

$$\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right|$$
$$= \left|\sum_{i=1}^{n} f(s_i)\alpha'(t_i)\Delta x_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right|$$
$$= \left|\sum_{i=1}^{n} f(s_i)[\alpha'(t_i) - \alpha'(s_i)]\Delta x_i\right|$$
$$\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right|$$
$$\leq \sum_{i=1}^{n} |f(s_i)||\alpha'(t_i) - \alpha'(s_i)|\Delta x_i$$
$$\leq \sum_{i=1}^{n} M|\alpha'(t_i) - \alpha'(s_i)|\Delta x_i \quad \text{where } M = \sup|f(x)|$$
$$= M\sum_{i=1}^{n} |\alpha'(t_i) - \alpha'(s_i)|\Delta x_i$$
$$\leq M\epsilon \text{ (by (3))}$$
$$(i.e.)\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(s_i)\alpha'(s_i)\Delta x_i\right| \leq M\epsilon$$
$$\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} f(\alpha')(s_i)\Delta x_i\right| \leq M\epsilon.....(4)$$

Since inequality (4) is true for any s_i in $[x_{i-1}, x_i]$, we can replace $(f\alpha')(s_i)$ by M_i' and m_i', where $m_i' = \inf(f\alpha')s_i,\ M_i' = \sup(f\alpha')(s_i), s_i \in [x_{i-1}, x_i]$

$$\left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} M_i'\Delta x_i\right| \leq M\epsilon.......(5)$$
$$\text{and } \left|\sum_{i=1}^{n} f(s_i)\Delta\alpha_i - \sum_{i=1}^{n} m_i'\Delta x_i\right| \leq M\epsilon......(6)$$

Again by replacing $f(s_i)$ by M_i in (5) and by m_i in (6)
we get

$$\left|\sum_{i=1}^{n} M_i'\Delta\alpha_i - \sum_{i=1}^{n} M_i'\Delta x_i\right| \leq M\epsilon \text{ and}$$
$$\left|\sum_{i=1}^{n} m_i'\Delta\alpha_i - \sum_{i=1}^{n} m_i'\Delta x_i\right| \leq M\epsilon$$
$$\Rightarrow |U(P, f, \alpha) - U(P, f, \alpha')| \leq M\epsilon......(7) \text{ and}$$
$$|L(P, f, \alpha) - L(P, f, \alpha')| \leq M\epsilon.......(8)$$

Since ϵ is arbitrary, (7) and (8)

$$\begin{aligned}
&\Rightarrow U(P,f,\alpha) = U(P,f,\alpha') \text{ and}\\
&L(P,f,\alpha) = L(P,f,\alpha')\\
&\Rightarrow \inf U(P,f,\alpha) = \inf U(P,f,\alpha') \text{ and}\\
&\sup L(P,f,\alpha) = \sup L(P,f,\alpha')\\
&\Rightarrow \int_a^{\bar{b}} f d\alpha = \int_a^{\bar{b}} (f\alpha')d\alpha ....(9) \text{ and}\\
&\int_{\underline{a}}^{b} f d\alpha = \int_{\underline{a}}^{b} (f\alpha')d\alpha ....(10)\\
&\therefore f \in \mathcal{R}(\alpha) \Leftrightarrow \int_{\underline{a}}^{b} f d\alpha = \int_a^{\bar{b}} f d\alpha\\
&\Leftrightarrow \int_{\underline{a}}^{b} (f\alpha')d\alpha = \int_a^{\bar{b}} (f\alpha')d\alpha \text{ (by (9) and (10))}\\
&\Leftrightarrow f(\alpha') \in \mathcal{R}.\\
&\text{Now, } \int_a^b f d\alpha = \int_a^{\bar{b}} f d\alpha\\
&= \int_a^{\bar{b}} (f\alpha')dx \ (by(9))\\
&= \int_a^b (f\alpha')dx\\
&= \int_a^b f(x)\alpha'(x)dx\\
&\therefore \int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)dx
\end{aligned}$$

Remark 4.22 *The above theorem gives the relation of $\mathcal{R}$ integral and $\mathcal{R}(\alpha)$ integral.*

Theorem 4.23 ***Change of Variable:*** *Suppose ϕ is a strictly increasing function that maps an interval $[A,B]$ onto $[a,b]$. Suppose α is monotonically increasing on $[a,b]$ and $f \in \mathcal{R}(\alpha)$ on $[a,b]$. Define β and g on $[A,B]$ by $\beta(y) = \alpha(\phi(y)), g(y) = f(\phi(y))$, then $g \in \mathcal{R}(\beta)$ and $\int_A^B gd(\beta) = \int_a^b f d\alpha$.*
Proof: $g(y) = (f\cdot\phi)x = f(\phi(y)) = f(x)$

$$[A,B] \xrightarrow{\phi} [a,b] \xrightarrow{f} \mathcal{R}$$

$$[A,B] \xrightarrow{\phi} [a,b] \xrightarrow{\alpha} \mathcal{R}$$

$$\begin{aligned}
\beta(y) &= (\alpha\cdot\phi)y\\
&= \alpha(\phi(y))\\
&= \alpha(x)
\end{aligned}$$

Let $P = \{x_0, x_1, x_2, ..., x_n\}$ be any partition of $[a, b]$. Since ϕ is onto for each i, there exists $y_i \in [A, B]$ such that $\phi(y_i) = x_i,\ i = 0, 1, 2, ..., n$. $\therefore$ $\{y_0, y_1, y_2, ..., y_n\}$ is a partition of $[A, B]$ every partition of $[A, B]$ can be obtained in this way (since ϕ is monotonically increasing)

$$\begin{aligned}
\text{For } y &\in [y_{i-1}, y_i] \\
g(y) &= (f \cdot \phi)y \\
g(y) &= f(\phi(y)) \\
&= f(x) \text{ where } x = \phi(y),\ x \in [x_{i-1}, x_i] \\
\Rightarrow \sup g(y) &= \sup f(x) \\
\Rightarrow M_{i'} &= M_i ....(1) \\
\text{Similarly } \inf g(y) &= \inf f(x) \\
m_{i'} &= m_i ....(2) \\
\text{Now } \Delta\beta_i &= \beta(y_i) - \beta(y_{i-1}) \\
&= (\alpha \circ \phi)y_i - (\alpha \circ \phi)y_{i-1} \\
&= \alpha(\phi(y_i)) - \alpha(\phi(y_{i-1})) \\
&= \alpha(x_i) - \alpha(x_{i-1}) \\
&= \Delta\alpha_i ......(3) \\
\therefore U(Q, g, \beta) &= \sum_{i=1}^{n} M_i' \Delta\beta_i \\
&= \sum_{i=1}^{n} M_i \Delta\alpha_i \text{ (by (1) and (3))} \\
&= U(P, f, \alpha) ......(4) \\
\text{Similarly } L(Q, g, \beta) &= L(P, f, \alpha) ......(5)
\end{aligned}$$

Since $f \in \mathcal{R}(\alpha)$, given $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$\begin{aligned}
U(P, f, \alpha) - L(P, f, \alpha) &< \epsilon \\
\Rightarrow U(Q, g, \beta) - L(Q, g, \beta) &< \epsilon \text{ (by (4) and (5))} \\
\therefore g &\in \mathcal{R}(\beta)
\end{aligned}$$

$$\begin{aligned}
\text{Also } \int_A^B g d\beta &= \inf U(Q, g, \beta) \\
&= \inf U(P, f, \alpha) \text{ (by (4))} \\
&= \int_a^b f d\alpha.
\end{aligned}$$

Note 4.24 *Let* $\alpha(x) = x$ *and* $\phi' \in \mathcal{R}$ *on* $[A, B]$.

$$\begin{aligned}
\therefore \beta(y) &= (\alpha \circ \phi)y, \\
&= \alpha(\phi(y)) \\
&= \phi(y) \ \forall y \in [A, B] \\
\therefore \beta &= \phi \\
\int_A^B g d\beta &= \int_a^b f d\alpha \ \textit{(by previous theorem)} \\
\int_a^b f(x)dx &= \int_A^B g d\beta \\
&= \int_A^B g d\phi \\
&= \int_A^B g(y)\phi'(y)dy \ \textit{(by theorem 4.21)}
\end{aligned}$$

Integrations and Differentiations:

Theorem 4.25 *Let* $f \in R$ *on* $[a, b]$*, for* $a \leq x \leq b$*, put* $F(x) = \int_a^x f(t)dt$*, then* F *is continuous on* $[a, b]$*, further more if* f *is continuous at some point* x_0 *of* $[a, b]$*, then* F *is differentiable at* x_0 *and* $F'(x_0) = f(x_0)$.

Proof: Given $F(x) = \int_a^x f(t)dt$. To Prove: $F(x)$ is continuous on $[a, b]$. Let $a \leq x \leq y \leq b$. Now,

$$\begin{aligned}
F(y) - F(x) &= \int_a^y f(t)dt - \int_a^x f(t)dt \\
&= \int_a^x f(t)dt + \int_x^y f(t)dt - \int_a^x f(t)dt \\
&= \int_x^y f(t)dt \\
\Rightarrow |F(y) - F(x)| &= |\int_x^y f(t)dt| \\
&\leq \int_x^y |f(t)|dt \\
&\leq \int_x^y M dt \text{ where } M = \sup |f(t)|, \ t \in [a, b] \\
&= M(y - x) \\
(i.e.) \ |F(y) - F(x)| &\leq M|y - x| \ (\because (y - x) = 0)
\end{aligned}$$

Given $\epsilon > 0$, there exists $\delta = \frac{\epsilon}{M}$ such that $|y - x| < \delta \Rightarrow |F(y) - F(x)| < \epsilon$ (i.e.) F is continuous on $[a, b]$. (infact F is uniformly continuous on $[a, b]$). Suppose f is continuous at $x_0 \in [a, b]$. To Prove: $F'(x_0) = f(x_0)$. Given $\epsilon > 0$, there exists $\delta > 0$ such that $|t - x_0| < \delta \Rightarrow |f(t) - f(x_0)| < \epsilon$ for $t \in [a, b]$...... (1)

Let $x_0 - \delta < s \le x_0 \le t \le x_0 + \delta$. Now,

$$
\begin{aligned}
F(t) - F(s) &= \int_a^t f(t)dt - \int_a^s f(t)dt \\
&= \int_a^s f(t)dt + \int_s^t f(t)dt - \int_a^s f(t)dt \\
F(t) - F(s) &= \int_s^t f(t)dt \\
\Rightarrow \frac{F(t) - F(s)}{t-s} &= \frac{1}{t-s}\int_s^t f(t)dt \\
\Rightarrow \frac{F(t) - F(s)}{t-s} - f(x_0) &= \frac{1}{t-s}\int_s^t f(t)dt - f(x_0) \\
\frac{F(t) - F(s)}{t-s} - f(x_0) &= \frac{1}{t-s}\{\int_s^t f(t)dt - (t-s)f(x_0)\} \\
&= \frac{1}{t-s}\{\int_s^t f(t)dt - \int_s^t f(x_0)dt\} \\
&= \frac{1}{t-s}\int_s^t (f(t) - f(x_0))dt \\
\left|\frac{F(t) - F(s)}{t-s} - f(x_0)\right| &= \left|\frac{1}{t-s}\int_s^t (f(t) - f(x_0))dt\right| \\
&\le \frac{1}{t-s}\int_s^t |f(t) - f(x_0)|dt \\
&< \frac{\in}{t-s}\int_s^t dt \text{ (by (1))} \\
\left|\frac{F(t) - F(s)}{t-s} - f(x_0)\right| &< \epsilon
\end{aligned}
$$

It follows that $F'(x_0) = f(x_0)$.

Theorem 4.26 ***The Fundamental Theorem of Calculus:*** *If $f \in R$ on $[a, b]$ and if there is a differentiable function F such that $F' = f$, then $\int_a^b f(x)dx = F(b) - F(a)$.*

Proof: Since $f \in R$ on $[a, b]$, given $\in 0$, there exists a partition $P = \{x_0, x_1, x_2, ..., x_n\}$ of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$...... (1)

Since F is differentiable we can apply the mean value theorem to it on $[x_{i-1}, x_i]$. There exists $t_i \in [x_{i-1}, x_i]$ such that

$$
\begin{aligned}
F(x_i) - F(x_{i-1}) &= (x_{i-1} - x_i)F'(t_i) \\
&= \Delta x_i f(t_i) \ (\because F' = f)
\end{aligned}
$$

Summing over i, we get,

$$
\begin{aligned}
\sum_{i=1}^{n}[F(x_i) - F(x_{i-1})] &= \sum_{i=1}^{n} \Delta x_i f(t_i) \\
F(b) - F(a) &= \sum_{i=1}^{n} f(t_i)\Delta x_i \text{.......(2)}
\end{aligned}
$$

By Theorem 4.10(c), (1) implies that

$$\left|\sum_{i=1}^{n} f(t_i)\Delta x_i - \int_a^b f(x)dx\right| < \epsilon.......(3)$$

Using (2) and (3) we get, $|(F(b)-F(a))-\int_a^b f(x)dx| < \epsilon$. Since ϵ is arbitrary, $\int_a^b f(x)dx = F(b) - F(a)$. Hence the proof.

Theorem 4.27 ***Integration by parts:*** *Suppose F and G are differentiable functions on $[a,b], F' = f \in \mathcal{R}, G' = g \in \mathcal{R}$, then*

$$\int_a^b f(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx.$$

Proof: Let $H(x) = F(x)G(x)$. $\therefore H'(x) = F(x)G'(x) + F'(x)G(x) = F(x)g(x) + f(x)G(x)$....... (1)
Given f and $g \in \mathcal{R}$. Since F and G are differentiable, they are continuous. $\therefore$ By Theorem 4.11, F and G are integrable ($\in \mathcal{R}$). $\therefore$ By Theorem 4.16 $F(x)g(x) + f(x)G(x) \in \mathcal{R}$ (i.e.) $H'(x) \in R$. By fundamental theorem of calculus,

$$\int_a^b H'(x)dx = H(b) - H(a)$$

$$(i.e.)\ \int_a^b (F(x)g(x) + f(x)G(x))dx = F(b)G(b) - F(a)G(a)$$

$$\Rightarrow \int_a^b F(x)g(x)dx + \int_a^b f(x)G(x)dx = F(b)G(b) - F(a)G(a)$$

$$\Rightarrow \int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$$

Hence the proof.

Definition 4.28 ***Integration of vector valued functions:*** *Let $f_1, f_2, ..., f_k$ be real functions on $[a,b]$ and let $\bar{f} = (f_1, f_2, ..., f_k)$ be a mapping of $[a,b] \to \mathbb{R}^k$. Suppose α increases monotonically on $[a,b]$, then $\bar{f} \in \mathcal{R}(\alpha) \Leftrightarrow$ for each $f_i \in \mathcal{R}(\alpha)$, and in this case*

$$\int_a^b \bar{f}d\alpha = (\int_a^b f_1 d\alpha, \int_a^b f_2 d\alpha, ..., \int_a^b f_k d\alpha)$$

Theorem 4.29 ***Fundamental Theorem of calculus for vector valued functions:*** *If $\bar{F}, \bar{f}$ map $[a,b]$ into $\mathbb{R}^k$ and if $\bar{f} \in \mathcal{R}$ on $[a,b]$ and if $\bar{F}' = \bar{f}$ then $\int_a^b \bar{f}(t)dt = \bar{F}(b) - \bar{F}(a)$.*
Proof: Let

$$\bar{f} = (f_1, f_2, ..., f_k)$$
$$\bar{F} = (F_1, F_2, ..., F_k)$$
$$\bar{F}' = (F_1', F_2', ..., F_k')$$

Given $\bar{F}' = \bar{f}$. $\therefore (F_1', F_2', ..., F_k') = (f_1, f_2, ..., f_k) \Rightarrow F_i' = f_i \quad \forall i = 1, 2, ..., k$. Since $\bar{f} \in \mathcal{R}$, each $f_i \in \mathcal{R}$. $\therefore$ By fundamental theorem of calculus, for any i.

$$\int_a^b F_i'(t)dt = F_i(b) - F_i(a)$$

$$(i.e.) \int_a^b f_i(t)dt = F_i(b) - F_i(a)........(1)$$

Now,

$$\begin{aligned}
\int_a^b \bar{f}(t)dt &= \left(\int_a^b f_1(t)dt, \int_a^b f_2(t)dt, ..., \int_a^b f_k(t)dt\right) \text{ (by definition)} \\
(1) \Rightarrow &= (F_1(b) - F_1(a), F_2(b) - F_2(a), ..., F_k(b) - F_k(a)) \\
&= (F_1(b), F_2(b), ..., F_k(b)) - (F_1(a), F_2(a), ..., F_k(a)) \\
&= \bar{F}(b) - \bar{F}(a)
\end{aligned}$$

$$\therefore \int_a^b \bar{f}(t)dt = \bar{F}(b) - \bar{F}(a)$$

Note 4.30 ***Schwartz inequality:***

$$\left|\sum_{j=1}^n a_j \bar{b_j}\right|^2 \leq \left(\sum_{j=1}^n |a_j|^2\right)\left(\sum_{j=1}^n |b_j|^2\right) \text{ (or)}$$

$$\left|\sum_{j=1}^n a_j \bar{b_j}\right| \leq \left(\sum_{j=1}^n |a_j|^2\right)^{\frac{1}{2}}\left(\sum_{j=1}^n |b_j|^2\right)^{\frac{1}{2}}$$

Theorem 4.31 *If $\bar{f}$ maps $[a, b]$ into $\mathbb{R}^k$ and if $\bar{f} \in \mathcal{R}(\alpha)$ for some monotonically increasing function $[a, b]$, then $|\bar{f}| \in \mathcal{R}(\alpha)$ and $|\int_a^b \bar{f}(t)d\alpha| \leq \int_a^b |\bar{f}(t)|d\alpha$.*
Proof:

$$\begin{aligned}
\bar{f} &= (f_1, f_2, ..., f_k) \\
|\bar{f}| &= (f_1^2 + f_2^2 + f_3^2 + ... + f_k^2)^{1/2} \\
\text{Since } \bar{f} &\in \mathcal{R}(\alpha) \\
\Rightarrow f_i &\in \mathcal{R}(\alpha) \ \forall i = 1, 2, ..., k \\
\Rightarrow f_i^2 &\in \mathcal{R}(\alpha) \\
\Rightarrow (f_1^2 + f_2^2 + f_3^2 + ... + f_k^2) &\in \mathcal{R}(\alpha) \\
\Rightarrow (f_1^2 + f_2^2 + f_3^2 + ... + f_k^2)^2 &\in \mathcal{R}(\alpha) \text{(by Theorem } 4.17, \phi(t) = t^{1/2}) \\
\Rightarrow |\bar{f}| &\in \mathcal{R}(\alpha)
\end{aligned}$$

To Prove:

$$\left|\int_a^b \bar{f}(t)d\alpha\right| \leq \int_a^b |\bar{f}(t)|d\alpha$$

Let $\bar{y} = \int_a^b \bar{f}(t)d\alpha$. If $\bar{y} = 0$, then the inequality is trivial (for, $\bar{y} = 0 \Rightarrow$ L.H.S=0 and $|\bar{f}| \geq 0 \Rightarrow \int_a^b |\bar{f}(t)|d\alpha \geq 0$ (i.e.) R.H.S ≥ 0)
Let $\bar{y} \neq 0$

$$\therefore \bar{y} = \int_a^b \bar{f}d\alpha = \left(\int_a^b f_1 d\alpha, \int_a^b f_2 d\alpha, ..., \int_a^b f_k d\alpha\right)$$

$$= (y_1, y_2, ..., y_k) \text{ where } y_i = \int_a^b f_i d\alpha$$

$$\text{Now } |\bar{y}|^2 = y_1^2 + y_2^2 + ... + y_k^2$$

$$(i.e.)\ |\bar{y}|^2 = \sum_{i=1}^k y_i^2$$

$$= \sum_{i=1}^k y_i y_i$$

$$= \sum_{i=1}^k y_i (\int_a^b f_i d\alpha)$$

$$= \sum_{i=1}^k \int_a^b (y_i f_i) d\alpha$$

$$= \int_a^b (\sum_{i=1}^k y_i f_i) d\alpha$$

$$\leq \int_a^b \left(\sum_{i=1}^k |y_i|^2\right)^{1/2} \left(\sum_{i=1}^k |f_i|^2\right)^{1/2} d\alpha \text{ (by schwartz inequality)}$$

$$(i.e.)\ |\bar{y}|^2 \leq \int_a^b \left(\sum_{i=1}^k y_i^2\right)^{1/2} \left(\sum_{i=1}^k f_i^2\right)^{1/2} d\alpha$$

$$= \int_a^b |\bar{y}||\bar{f}|d\alpha$$

$$= |\bar{y}| \int_a^b |\bar{f}|d\alpha$$

$$(i.e.)\ |\bar{y}|^2 \leq |\bar{y}| \int_a^b |\bar{f}|d\alpha$$

$$\Rightarrow |\bar{y}| \leq \int_a^b |\bar{f}|d\alpha$$

$$\left|\int_a^b \bar{f}d\alpha\right| \leq \int_a^b |\bar{f}|d\alpha$$

Uniform Convergence:

Definition 4.32 ***Uniform Convergence:*** *We say that $\{f_n\}$ of function $n = 1, 2, ...$ converges uniformly on E to a function f is every $\epsilon > 0$ there is an integer N such that $n \geq N \Rightarrow |f_n(x) - f(x)| < \epsilon$.*

Note 4.33 *If $\{f_n\}$ converges pointwise on E, then there exists a function f such that for every $\epsilon > 0$ and for every x in E there is an integer N depending on ϵ and x such that $|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N$. If $\{f_n\}$ converges uniformly on E, it is possible for each $\epsilon > 0$, to find one integer N which will do for all x in E. We say that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on E if the $\{s_n\}$ of partial sums defined by $s_n(x) = \sum_{i=1}^{n} f_i(x)$ converges uniformly on E.*

Theorem 4.34 ***Cauchy's Criterian for Uniform Convergence:*** *The sequence of functions $\{f_n\}$, defined on E, converges uniformly on E iff for every $\epsilon > 0$ there exists an integer N such that $n, m \geq N, x \in E \Rightarrow |f_n(x) - f_m(x)| < \epsilon$.*

Proof: For the 'only if' part we assume that $\{f_n\} \to f$ uniformly. To Prove: There exists N such that $x \in E \;\; n, m \geq N \Rightarrow |f_n(x) - f_m(x)| < \epsilon$. Let $\epsilon > 0$ such that $|f_n(x) - f(x)| \leq \epsilon/2$...... (1) $\quad \forall n \geq N \;\; \forall x \in E$
Now, for $n, m \geq N$

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f(x) + f(x) - f_m(x)| \\ &\leq |f_n(x) - f(x)| + |f(x) - f_m(x)| \\ &\leq \epsilon/2 + \epsilon/2 \text{ (by (1))} \\ (i.e.)\ |f_n(x) - f_m(x)| &\leq \epsilon \end{aligned}$$

For the $'if'$ part we assume that there exists $N > 0$ such that $n, m \geq N, x \in E \Rightarrow |f_n(x) - f_m(x)| \leq \epsilon$........ (2)
For fixed x, (2) implies that $\{f_n(x)\}$ is a cauchy sequence $\therefore \{f_n(x)\} \to f(x)(|f_n(x) - f(x)| \to 0)$. To Prove: $\{f_n\} \to f$ uniformly. In (2), keeping n fixed and taking limit as $m \to \infty$ we get $|f_n(x) - f(x)| \leq \epsilon \quad \forall n \geq N$ $\forall x \in E$. $\therefore \{f_n\} \to f$ uniformly.

Theorem 4.35 *Suppose*

$$\lim_{n\to\infty} f_n = f(x), \;\; (x \in E).$$

Put $M_n = \sup_{x\in E} |f_n(x) - f(x)|$, then $\{f_n\} \to f$ uniformly on E iff $M_n \to 0$ as $n \to \infty$.

Proof: For the 'only if' part, we assume that $\{f_n\} \to f$. To Prove: $M_n \to 0$ as $n \to \infty$. By hypothesis, given $\epsilon > 0$, there exists $N > 0$ such that $|f_n(x) - f(x)| \leq \epsilon \quad \forall n \geq N \;\; \forall x \in E \Rightarrow \sup x \in E|f_n(x) - f(x)| \leq \epsilon$ $\forall n \geq N \Rightarrow M_n \leq \epsilon \; \forall n \geq N$ (i.e.) $M_n \to 0$ as $n \to \infty$. For the 'if' part, let $M_n \to 0$ as $n \to \infty$. Then there exists $N > 0$ such that $M_n \leq \epsilon$ $\forall n \geq N \Rightarrow \sup_{x\in E} |f_n(x) - f(x)| \leq \epsilon \quad \forall n \geq N \Rightarrow |f_n(x) - f(x)| \leq \epsilon$ $\forall n \geq N, x \in E \Rightarrow \{f_n\} \to f$ uniformly.

Theorem 4.36 ***Weristress M test for uniform convergence:*** *Suppose $\{f_n\}$ is a sequence of function defined on E and suppose that $|f_1(x)| \leq M_n$*

$(x \in E, n = 1, 2...)$ *then* $\sum f_n$ *converges uniformly on* E *its* $\sum M_n$ *converges.*
Proof: Assume that $\sum M_n$ converges. To Prove: $\sum f_n$ converges uniformly. Let $\epsilon > 0$ be given. Let $\{s_n\}$ and $\{t_n\}$ be the sequences of partial sums of $\sum f_n$ and $\sum M_n$ respectively. Since $\sum M_n$ converges, $\{t_n\}$ also converges. Since any convergence sequence is a Cauchy sequence $\{t_n\}$ is also a Cauchy sequence. Then there exists $N > 0$ such that $|t_n - t_m| \le \epsilon \ \ \forall n, m \ge N$. Let $m > n(\ge N)$

$$|t_n - t_m| = \left|\sum_{n+1}^{m} M_k\right| \le \epsilon ....(1)$$

Now, for $x \in E$,

$$\begin{aligned} |s_n(x) - s_m(x)| &= \left|\sum_{n+1}^{m} f_k(x)\right| \\ &\le \sum_{n+1}^{m} |f_k(x)| \\ &\le \sum_{n+1}^{m} M_k \le \epsilon \text{ (by (1))} \\ \therefore |s_n(x) - s_m(x)| &< \epsilon \end{aligned}$$

$\therefore$ By Cauchy's criteria 4.34 the $\{s_n\}$ converges uniformly on E. $\therefore \sum f_n$ converges uniformly.

Theorem 4.37 ***[Uniform Convergence and Continuity]*** *Suppose* $\{f_n\}$ *converges to* f *uniformly on a set* E*, in a metric space. Let* x *be a limit point of* E *and suppose that* $\lim_{t\to x} f_n(t) = A_n(n = 1, 2, 3...)$*, then* $\{A_n\}$ *converges* $\lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$*. In other words* $\lim_{t\to x} \lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} \lim_{t\to x} f_n(t)$*.*
Proof: Let $\epsilon > 0$ be given. Since $\{f_n\}$ converges to f uniformly on E, by Theorem 4.34, there exists an integer $N > 0$ such that $|f_n(t) - f_m(t)| \le \epsilon$ $\forall n, m \ge N, t \in E......$ (1)
Letting $t \to x$ in (1) we get $|A_n - A_m| \le \epsilon \quad \forall n, m \ge N(\because \lim_{t\to x} = A_n)$ (i.e.) $\{A_n\}$ is a Cauchy sequence of real numbers. Since $\mathbb{R}$ is complete, $\{A_n\}$ converges to some $A($ in $\mathbb{R})$ (i.e.) $\{A_n\} \to A$. $\therefore$ there exists $N_1 > 0$ such that $|A_n - A| \le \epsilon/3, \ \forall n \ge N_1......$ (2)
Now,

$$\begin{aligned} |f(t) - A| &= |f(t) - f_n(t)| + (f_n(t) - A_n) + |(A_n - A)| \\ &\le |f(t) - f_n(t)| + |f_n(t) - A_n| + (A_n - A)|.......(3) \end{aligned}$$

Since $\{f_n\} \to f$ uniformly, there exists $N_2 > 0$ such that $|f_n(t) - f(t)| \le \epsilon/3$ $\forall n \ge N_2, t \in E.......$ (4)

Since x is a limit point of E and $\because \lim_{t\to x} f_n(t) = A_n$, there exists a neighbourhood V of x such that $|f_n(t) - A_n| \leq \epsilon/3 \ \ \forall t \in V \cap E$....... (5)
Let $N_3 = max\{N_1, N_2\}$. Now using (2),(4) and (5) in (3) we get

$$|f(t) - A| \leq \epsilon/3 + \epsilon/3 + \epsilon/3 \ \forall n \geq N_3 \ \forall t \in V \cap E.$$
$$(i.e.)\ |f(t) - A| \leq \epsilon$$
$$(i.e.)\ \lim_{t\to x} f(t) = A \text{ (or)}$$
$$\lim_{t\to x}\lim_{n\to\infty} f_n(t) = \lim_{n\to\infty} A_n$$
$$= \lim_{n\to\infty}\lim_{t\to x} f_n(t))$$
$$\therefore \lim_{t\to x} f(t) = \lim_{n\to\infty} A_n$$

Theorem 4.38 *If $\{f_n\}$ is a sequence of continuous functions on E, and if $\{f_n\}$ converges to f uniformly on E then f is continuous on E.*
Proof: Enough To Prove: $\lim_{t\to x} f(t) = f(x)$

$$\lim_{t\to x} f(t) = \lim_{t\to x}\lim_{n\to\infty} f_n(t)) \ (\because f_n \to f \ \text{ uniformly})$$
$$\lim_{t\to x} f(t) = \lim_{n\to\infty}(\lim_{t\to x} f_n(t)) \ \text{(by Theorem 4.37)}$$
$$= \lim_{n\to\infty} f_n(x) \ (\because f_n \text{ is continuous})$$
$$= f(x) \ (\because f_n \to f \text{ uniformly})$$

Remark 4.39 *The converse of the above theorem need not be true. (i.e.) a sequence of continuous function may converse to a continuous function, although the convergence is not uniform.*

Example 4.40 $f_n(x) = n^2x(1-x^2)^n,\ 0 \leq x \leq 1,\ n = 1, 2, 3, ...$ *Clearly, each f_n is continuous. Also f is continuous. But the convergence is not uniform. By Theorem 4.35, for let*

$$M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$$
$$= \sup_{x\in[0,1]} |n^2x(1-x^2)^n - 0|$$
$$= n^2 \sup_{x\in[0,1]} \{x(1-x^2)^n\}$$
$$\nrightarrow 0 \ as \ n \to \infty.$$

By Theorem 4.35, the convergence is not uniform.

Theorem 4.41 ***[Dini's Theorem]*** *Suppose K is compact and*
(a) $\{f_n\}$ is a sequence of continuous functions on K.
(b) $\{f_n\}$ converges pointwise to a continuous functions f on K.
(c) $f_n(x) \geq f_{n+1}(x) \ \ \forall x \in K,\ n = 1, 2, 3...$

then $f_n \to f$ uniformly on K.

Proof: Given K is compact. Let $g_n = f_n - f$. Since each f_n is continuous and f is continuous, g_n is continuous for all n. Since $\{f_n\}$ converges pointwise to $f, \{g_n\}$ converges pointwise to 0. Since $f_n(x) \geq f_{n+1}(x)$ $\forall x \in K,\ n = 1, 2...$ $f_n(x) - f(x) \geq f_{n+1}(x) - f(x)$. (i.e.) $g_n(x) \geq g_{n+1}(x)$ $\forall x,\ n = 1, 2...$ (i.e.) $\{g_n\}$ is also a monotonic decreasing sequence. To prove that $\{f_n\}$ converges to f uniformly. It is enough to prove that $\{g_n\}$ converges to 0 uniformly. Let $\epsilon > 0$ be given. For each n, let $K_n = \{x \in K | g_n(x) \geq \epsilon\}$. Now,

$$\begin{aligned} K_n &= \{x \in K | g_n(x) \geq \in [\epsilon, \infty)\} \\ &= \{x \in K | x \in g_n^{-1}[\epsilon, \infty)\} \\ &= g_n^{-1}[\epsilon, \infty). \end{aligned}$$

Since $[\epsilon, \infty)$ is closed in R and g_n is continuous, $g_n^{-1}[\epsilon, \infty)$ is closed in K. (i.e.) K_n is a closed subspace of the compact space K. $\therefore K_n$ is compact ($\because$ every closed subspace of a compact space is compact). Claim: $K_n \supset K_{n+1},\ n = 1, 2, 3...$ Let $x \in K_{n+1} \Rightarrow g_{n+1}(x) \geq \epsilon$. But $g_n(x) \geq g_{n+1}(x)$ (by (1)). $\therefore g_n(x) \geq g_{n+1}(x) \geq \epsilon \Rightarrow g_n(x) \geq \epsilon \Rightarrow x \in K_n \therefore K_{n+1} \subset K_n$. Fix $x \in K$. Since $\{g_n\}$ converges pointwise to 0. $\{g_n(x)\} \to 0$. Then there exists $N(x) > 0$ such that $|g_n(x) - 0| < \epsilon \ \ \forall n \geq N(x) \Rightarrow g_n(x) < \epsilon \ \ \forall n \geq N(x) \Rightarrow$ $x \notin K_n \ \ \forall n \geq N(x) \Rightarrow x \notin \bigcap_{n=1}^{\infty} K_n$. Since x is arbitrary,$\bigcap_{n=1}^{\infty} K_n = \phi \Rightarrow$ $K_N = \phi$ for some N. $\therefore g_N(x) < \epsilon \ \ \forall x \in K$. But

$$\begin{aligned} 0 \leq g_n(x) \leq g_N(x) < \epsilon\ \forall x \in K,\ \forall n \geq N \\ g_n(x) < \epsilon\ \forall x \in K,\ \forall n \geq N \\ (i.e.)\ |g_n(x) - 0| < \epsilon\ \forall x \in K,\ \forall n \geq N \end{aligned}$$

Hence $\{g_n\} \to 0$ uniformly.

Note 4.42 *Compactness is really needed in the above theorem.*

Example 4.43 *$f_n(x) = \frac{1}{nx+1},\ 0 < x < 1,\ n = 1, 2, 3...$ $\{f_n\} \to f$ pointwise where $f(x) = 0 \forall x \in (0, 1)$ and $(0, 1)$ is not compact. Clearly, each f_n is continuous. Also f is continuous. Now,*

$$\begin{aligned} n + 1 &> n \\ \Rightarrow (n + 1)x &> nx \\ \Rightarrow (n + 1)x + 1 &> nx + 1 \\ \Rightarrow \frac{1}{(n + 1)x + 1} &< \frac{1}{nx + 1} \\ \Rightarrow f_{n+1}(x) &< f_n(x) \end{aligned}$$

$\Rightarrow \{f_n\}$ *is a decreasing sequence. But* $\{f_n\} \to f$ *uniformly. For, if* $\{f_n\} \to f$ *uniformly then, given* $\epsilon > 0$*, there exists* $N > 0$ *such that*

$$\begin{aligned}
|f_n(x) - f(x)| &\leq \epsilon \ \forall n \geq N, \ \forall x \in (0,1) \\
(i.e.) \ \left|\frac{1}{nx+1} - 0\right| &\leq \epsilon \ \forall x \in (0,1) \\
\left|\frac{1}{nx+1}\right| &\leq \epsilon \ \forall x \in (0,1) \\
Put\ x = \frac{1}{n}.\ Then\ \frac{1}{2} &\leq \epsilon \\
&\Rightarrow\Leftarrow
\end{aligned}$$

$\therefore$ *The convergence is not uniform.*

Definition 4.44 *If* X *is a metric space* $\mathscr{C}(x)$ *denotes the set of all complex valued continuous bounded functions with domain* X*.* $\mathscr{C}(X) = \{f/f : X \to c, f$ *is continuous and bounded*$\}$*. If* X *is compact,* $\mathscr{C}(X) = \{f/f : X \to c, f$ *is continuous*$\}$ *(*$\because$ *any continuous function on a compact space is bounded). For any* f *in* $\mathscr{C}(f)$*,* $\sup \|f\| = \sup_{x\in X} |f(x)|$*, since* f *is bounded* $\|f\| < \infty$*.*

Result 4.45 $\mathscr{C}(X)$ *is a metric space. Given* $f, g \in \mathscr{C}(X)$ *define*

$$\begin{aligned}
(i)\ d(f,g) &= \|f-g\| \\
&= \sup_{x\in E} |f(x) - g(x)| \\
&\geq 0 \\
\therefore d(f,g) &\geq 0 \\
(ii)\ d(f,g) &= \sup_{x\in E} |f(x) - g(x)| \\
&= \sup_{x\in E} |g(x) - f(x)| \\
&= \|g-f\| \\
&= d(f,g) \\
(iii)\ d(f,g) = 0 &\Leftrightarrow \|f-g\| = 0 \\
&\Leftrightarrow \sup_{x\in E} |f(x) - g(x)| \\
&\Leftrightarrow |f(x) - g(x)| = 0 \forall x \in E \\
&\Leftrightarrow f(x) = g(x) \\
&\Leftrightarrow f = g
\end{aligned}$$

$$
\begin{aligned}
(iv)\ d(f,g) &= \|f-g\| \\
&= \sup_{x\in E}|f(x)-g(x)| \\
&= \sup_{x\in E}|(f(x)-h(x))+(h(x)-g(x))| \\
&\le \sup_{x\in E}\{|(f(x)-h(x))|+|(h(x)-g(x))|\} \\
&\le \sup_{x\in E}|(f(x)-h(x))|+\sup_{x\in E}|(f(x)-g(x))| \\
&= \|f-h\|+\|h-g\| \\
&= d(f,h)+d(h,g) \\
(i.e.)\ d(f,g) &\le d(f,h)+d(h,g)
\end{aligned}
$$

$\therefore$ *$(\mathscr{C}(X), d)$ is a metric space.*

Result 4.46 ***(Analogue of Theorem 4.35)*** *A sequence $\{f_n\} \to f$ with respect to the metric space $\mathscr{C}(X)$ iff $\{f_n\} \to f$ uniformly on X.*
Proof: 'only if' part:
Assume that $\{f_n\} \to f$ in $\mathscr{C}(X)$. $\|f_n - f\| \to 0$ as $n \to \infty$ (i.e.) $\sup_{x\in E}|f_n(x) - f(x)| \to 0$ as $n \to \infty$ (i.e.) $M_n \to 0$ as $n \to \infty$ (Theorem 4.35). $\{f_n\} \to f$ uniformly (by Theorem 4.35)
'if' part:
Suppose $\{f_n\} \to f$ uniformly. Then $M_n \to 0$ as $n \to \infty$ (Theorem 4.35) (i.e.) $\sup x \in E|f_n(x) - f(x)| \to 0$ as $n \to \infty$ (i.e.)$\|f_n - f\| \to 0$ as $n \to \infty$. $\therefore \{f_n\} \to f$ in $\mathscr{C}(X)$

Note 4.47 *(i) Closed subsets of $\mathscr{C}(X)$ are called uniformly closed subsets. (ii) If $A \subset \mathscr{C}(X)$ then the closure of A is called the uniform closure of A.*

Theorem 4.48 *$\mathscr{C}(X)$ is a complete metric space.*
Proof: Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{C}(X)$. Let $\epsilon > 0$ be given. Then there exists $N > 0$ such that $\|f_n - f_m\| < \epsilon \quad \forall n, m \ge N$...... (1)
(i.e.) $\sup_{x\in E}|f_n(x) - f_m(x)| \le \epsilon \quad \forall n, m \ge N$. $\Rightarrow |f_n(x) - f_m(x)| \le \epsilon$ $\forall n, m \ge N, x \in X$. By Theorem 4.34, guarantees that $\{f_n\}$ converges uniformly, say f. (i.e.) $\lim_{n\to\infty} f_n(x) = f(x), x \in X$. Claim: $f \in \mathscr{C}(X)$. Since each f_n is continuous and $\{f_n\} \to f$ uniformly (Theorem 4.38). Theorem 4.38 demands that f is also continuous. Again, since $\{f_n\} \to f$ uniformly, there exists $N_1 > 0$ such that $|f_n(x) - f(x)| < 1\ \forall n \ge N_1, x \in X$. In particular, $|f_{N_1}(x) - f(x)| < 1$....... (2) $\forall x \in X$
Since $f_{N_1}(x) \in \mathscr{C}(X), |f_{N_1}(x)| \le K$......... (3) $\forall x \in X$
Now,

$$
\begin{aligned}
|f(x)| &= |(f(x) - f_{N_1}(x)) + f_{N_1}(x)| \\
|f(x)| &\le |f(x) - f_{N_1}(x)| + |f_{N_1}(x)| \\
&< 1 + K \text{ (by (2) and (3)) } \forall x \in X \\
(i.e.)\ |f(x)| &< 1 + K\ \forall x \in K.
\end{aligned}
$$

$\therefore f$ is bounded. Hence $f \in \mathscr{C}(X)$. It remains to prove that $\{f_n\} \to f$ in $\mathscr{C}(X)$. For, $\{f_n\} \to f$ uniformly $\Rightarrow M_n \to 0 \Rightarrow \sup_{x\in X} |f_n(x) - f(x)| \to 0$ as $n \to \infty$ (by Theorem 4.35) $\Rightarrow \|f_n - f\| \to 0$ as $n \to \infty$. So $\{f_n\} \to f$ in the metric space $\mathscr{C}(X)$. $\therefore \mathscr{C}(X)$ is a complete metric space.

Uniform Convergence and Integration

Theorem 4.49 *Let α be monotonically increasing on $[a,b]$. Suppose $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ for $n = 1,2,3....$ and suppose $f_n \to f$ uniformly on $[a,b]$ then $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ and $\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f d\alpha$.*
Proof: Let $\epsilon_n = \sup_{a\le x\le b} |f(x) - f_n(x)|$....... (1) (Theorem 4.35)

$$
\begin{aligned}
\therefore |f - f_n| &\le \epsilon_n \ \forall n = 1,2,3... \\
-\epsilon &\le f - f_n \le \epsilon_n \\
\Rightarrow f_n - \epsilon_n &\le f \le f_n + \epsilon_n \\
\Rightarrow \int_a^b (f_n - \epsilon_n) d\alpha \le \underline{\int_a^b} f d\alpha &\le \overline{\int_a^b} f d\alpha \le \int_a^b (f_n + \epsilon_n) d\alpha........(2) \\
\Rightarrow \int_a^b f_n d\alpha - \int_a^b \epsilon_n d\alpha \le \underline{\int_a^b} f d\alpha &\le \overline{\int_a^b} f d\alpha \le \int_a^b f_n d\alpha + \int_a^b \epsilon_n d\alpha \\
\Rightarrow \overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha &\le \left(\int_a^b f_n d\alpha + \int_a^b \epsilon_n d\alpha\right) - \left(\int_a^b f_n d\alpha - \int_a^b \epsilon_n d\alpha\right) \\
&= 2\int_a^b \epsilon_n d\alpha \\
&= 2\epsilon_n \int_a^b d\alpha \\
&= 2\epsilon_n[\alpha(b) - \alpha(a)] \\
(i.e.)\ \overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha &\le 2\epsilon_n(\alpha(b) - \alpha(a)) \\
&\to 0 \ (\because \epsilon_n \to 0 \ \text{ as } \ f_n \to f \ \text{ uniformly by theorem } 4.35) \\
\therefore \overline{\int_a^b} f d\alpha &= \underline{\int_a^b} f d\alpha
\end{aligned}
$$

Hence $f \in \mathcal{R}(\alpha)$. **II part:** To prove:

$$\int_a^b f d\alpha = \lim_{n\to\infty} \int_a^b f_n d\alpha$$

Now, (2)⇒

$$\begin{aligned}
\int_a^b (f_n - \epsilon_n) d\alpha \leq \int_a^b f d\alpha &\leq \int_a^b (f_n + \epsilon_n) d\alpha \\
\int_a^b f_n d\alpha - \int_a^b \epsilon_n d\alpha \leq \int_a^b f d\alpha &\leq \int_a^b f_n d\alpha + \int_a^b \epsilon_n d\alpha \\
\Rightarrow \int_a^b f_n d\alpha - \epsilon_n \int_a^b d\alpha \leq \int_a^b f d\alpha &\leq \int_a^b f_n d\alpha + \epsilon_n \int_a^b d\alpha \\
\Rightarrow -\epsilon_n \int_a^b d\alpha \leq \int_a^b f d\alpha - \int_a^b f_n d\alpha &\leq \epsilon_n \int_a^b d\alpha \\
\Rightarrow \left| \int_a^b f d\alpha - \int_a^b f_n d\alpha \right| &\leq \epsilon_n \int_a^b d\alpha \\
&= \epsilon_n (\alpha(b) - \alpha(a)) \\
&\to 0 \text{ as } n \to \infty \ (\because \epsilon_n \to 0) \\
\lim_{n\to\infty} \int_a^b f_n d\alpha &= \int_a^b f d\alpha.
\end{aligned}$$

Corollary 4.50 *If $f_n \in \mathcal{R}(\alpha)$ on $[a,b]$ and if $f(x) = \sum_{n=1}^{\infty} f_n(x) (a \leq x \leq b)$, the series converges uniformly on $[a,b]$, then $\int_a^b f d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n d\alpha$.(the series may be integrated term by term)*

Proof: Given $\sum f_n = f$ (uniformly). Let $s_n = \sum_{k=1}^{n} f_k$. By hypothesis $\{s_n\} \to f$ uniformly. By Theorem 4.49,

$$\begin{aligned}
\int_a^b f d\alpha &= \lim_{n\to\infty} \int_a^b s_n d\alpha \\
&= \lim_{n\to\infty} \int_a^b \left(\sum_{k=1}^{n} f_k \right) d\alpha \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\int_a^b f_k d\alpha \right) \\
&= \sum_{k=1}^{\infty} \int_a^b f_k d\alpha
\end{aligned}$$

5. UNIT V

Uniform Convergence and Differentiation

Theorem 5.1 *Suppose $\{f_n\}$ is a sequence of functions, differentiable on $[a, b]$ such that $\{f_n(x_0)\}$ converges for some point x_0 in $[a, b]$. If $\{f'_n\}$ converges uniformly on $[a, b]$, then $\{f_n\}$ converges uniformly on $[a, b]$ to a function f and $f'(x) = \lim_{n\to\infty} f'_n(x), a \le x \le b$.*

Proof: Since $\{f_n(x_0)\}$ is convergent, it is a Cauchy sequence. Also $\{f'_n\}$ converges uniformly. Therefore, there exists an integer $N > 0$ such that

$$|f_n(x_0) - f_m(x_0)| \le \epsilon/2 ....(1)\ \forall n, m \ge N$$
$$|f'_n(x) - f'_m(x)| \le \frac{\epsilon}{2(b-a)} ......(2)\ \forall n, m \ge N,\ \forall x \in [a, b]$$

By applying mean value theorem to $f_n - f_m$ in $[t, x]$,

$$\begin{aligned}
(f_n - f_m)(x) - (f_n - f_m)(t) &= (x - t)(f'_n - f'_m)(y) \\
&\qquad \text{where } y \in (a, b),\ \text{for } t, x \in [a, b] \\
f_n(x) - f_m(x) - f_n(t) + f_m(t) &= (x - t)(f'_n(y) - f'_m(y)) \\
|f_n(x) - f_m(x) - f_n(t) + f_m(t)| &= |(x - t)(f'_n(y) - f'_m(y))| \\
&= |(x - t)||f'_n(y) - f'_m(y)| \\
&\le \frac{|x - t|\epsilon}{2(b - a)} ......(3)\ (by (2)) \\
&\le \frac{(b - a)\epsilon}{2(b - a)}\ (\because |x - t| \le b - a) \\
&= \epsilon/2 \\
|f_n(x) - f_m(x) - f_n(t) + f_m(t)| &\le \epsilon/2 ......(4)\ \forall x, t \in [a, b],\ \forall n, m \ge N.
\end{aligned}$$

Now,

$$\begin{aligned}
|f_n(x) - f_m(x)| &= |(f_n(x) - f_m(x)) - (f_n(x_0) - f_n(x_0)) + (f_m(x_0) - f_m(x_0))| \\
&\le |f_n(x) - f_m(x) - f_n(x_0) + f_m(x_0)| + |(f_n(x_0) - f_m(x_0))| \\
&\le \epsilon/2 + \epsilon/2 \text{ (by (4) and (1))} \\
|f_n(x) - f_m(x)| &\le \epsilon \quad \forall n, m \ge N,\ \forall x \in [a, b]
\end{aligned}$$

Cauchy's criteria guarantees that $\{f_n\}$ converges uniformly, say f. (i.e.) $\lim_{n\to\infty} f_n = f$. To Prove: $f'(x) = \lim_{n\to\infty} f'_n(x)$. Fix $x \in [a, b]$, define

$\phi_n(t) = \frac{f_n(t)-f_n(x)}{t-x}$ and $\phi(t) = \frac{f(t)-f(x)}{t-x}$. Now,

$$\begin{aligned}
\lim_{t\to x}\phi_n(t) &= \lim_{t\to x}\frac{f_n(t)-f_n(x)}{t-x} \\
&= f_n'(x)......(5) \\
\lim_{t\to x}\phi(t) &= \lim_{t\to x}\frac{f(t)-f(x)}{t-x} \\
&= f'(x)......(6) \\
\text{Also, } |\phi_n(t)-\phi_m(t)| &= \left|\frac{f_n(t)-f_n(x)}{t-x} - \frac{f_m(t)-f_m(x)}{t-x}\right| \\
&\le \frac{1}{|t-x|}|f_n(t)-f_n(x)-f_m(t)+f_m(x)| \\
&\le \frac{1}{|t-x|}\cdot\frac{|t-x|\epsilon}{2(b-a)} \text{ (by (3))} \\
&= \frac{\epsilon}{2(b-a)} \\
|\phi_n(t)-\phi_m(t)| &\le \frac{\epsilon}{2(b-a)}
\end{aligned}$$

Cauchy's criteria for uniform convergence demands that $\{\phi_n\}$ converges uniformly. Now,

$$\begin{aligned}
\lim_{n\to\infty}\phi_n(t) &= \lim_{n\to\infty}\frac{f_n(t)-f_n(x)}{t-x} \\
&= \frac{f(t)-f(x)}{t-x} \\
&= \phi(t) \\
(i.e.)\phi(t) &= \lim_{n\to\infty}\phi_n(t)......(7) \\
\text{Finally, } f'(x) &= \lim_{t\to x}\phi(t) \text{ (by (6))} \\
&= \lim_{t\to x}(\lim_{n\to\infty}\phi_n(t)) \text{ (by (7))} \\
&= \lim_{n\to\infty}\lim_{t\to x}\phi_n(t) \text{ } (\because \{\phi_n\}\to\phi \text{ uniformly and by Theorem 4.37)} \\
&= \lim_{n\to\infty}f_n'(x) \text{ (by (5))}
\end{aligned}$$

Therefore $f'(x) = \lim_{n\to\infty} f_n'(x)$.

Theorem 5.2 *There exists a real continuous function on the real line which is no where differentiable.*
Proof: Let $\phi(x) = |x|, -1 \le x \le 1$ and $\phi(x+2) = \phi(x) \quad \forall x \in R$. Define $f(x) = \sum_{n=0}^{\infty}(3/4)^n\phi(4^n x), x \in R$. We observe that,

$$\begin{aligned}
|\phi(s)-\phi(t)| &\le |s-t|......(1) \ \forall s,t \in R \\
|(3/4)^n\phi(4^n x)| &\le (3/4)^n,
\end{aligned}$$

$\sum_{n=0}^{\infty}(3/4)^n$ is a geometric series with common ratio $\frac{3}{4} < 1$ and hence it converges to $\frac{1}{1-3/4} = 4$. Now, Weierstrass M test for uniform convergence demands that $\sum(3/4)^n\phi(4^n x)$ converges uniformly to f. Clearly $f(x)$ is continuous. Fix a real number x and a positive integer m define $\delta_m = \pm\frac{1}{2}(4-m)$ where the sign is chosen such that no integer lies between $4^m(x)$ and $4^m(x+\delta_m)$. This is possible since $|4^m\delta_m| = 1/2$. Let $\gamma_n = \frac{\phi(4^m(x+\delta_m))-\phi(4^m x)}{\delta_m}$. Now,

$$4^n\delta_m = \pm\frac{1}{2}4^{n-m} = \begin{cases}\text{an integer} & n \geq m\\ \text{not an integer} & 0 \leq n \leq m\end{cases}$$

when $n > m$,

$$\begin{aligned}\gamma_n &= \frac{\phi(4^n(x+\delta_m))-\phi(4^n x)}{\delta_m}\\ \gamma_n &= \frac{\phi(4^m x+4^n\delta_m)-\phi(4^n x)}{\delta_m}\\ \gamma_n &= \frac{\phi(4^n x)-\phi(4^n x)}{\delta_m}\ (\because 4^n\delta_m\ is\ even)\\ &= 0\\ (i.e.)\gamma_n &= 0\ \forall n \geq m.......(2)\end{aligned}$$

when $n < m$,

$$\begin{aligned}|\gamma_n| &= \left|\frac{\phi(4^n(x+\delta_m))-\phi(4^n x)}{\delta_m}\right|\\ &\leq \frac{|4^n(x+\delta_m)-4^n x|}{|\delta_m|}\\ |\gamma_n| &\leq \left|\frac{4^n\delta_m}{\delta_m}\right|\\ (or)|\gamma_n| &\leq 4^n, \forall n < m.......(3)\end{aligned}$$

when $n = m$

$$\begin{aligned}|\gamma_n| &= \phi|\gamma_m|\\ &= \left|\frac{\phi(4^m(x+\delta_m))-\phi(4^m x)}{\delta_m}\right|\\ &= \left|\frac{4^m\delta_m}{\delta_m}\right|\\ |\gamma_n| &= 4^m\ n = m........(4)\end{aligned}$$

Now,

$$\begin{aligned}
\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| &= \left|\frac{\sum_{n=0}^{\infty}(3/4)^n\phi(4^n(x+\delta_m)) - \sum_{n=0}^{\infty}(3/4)^n\phi(4^n x)}{\delta_m}\right| \\
&= \left|\sum_{n=0}^{\infty}(3/4)^n\frac{\{\phi(4^m(x+\delta_m)) - \phi(4^m x)\}}{\delta_m}\right| \\
&= \left|\sum_{n=0}^{\infty}(3/4)^n\gamma_n\right| \\
&= \left|\sum_{n=0}^{m}(3/4)^n\gamma_n\right| \text{ (by (2))} \\
&\geq |(3/4)^m\gamma_m| - \left|\sum_{n=0}^{m-1}(3/4)^n\gamma_n\right| \\
&\geq (3/4)^m|\gamma_m| - \sum_{n=0}^{m-1}(3/4)^n|\gamma_n| \\
&\geq (3/4)^m 4^m - \sum_{n=0}^{m-1}(3/4)^n 4^n \text{ (by (4) and (3))} \\
&= 3^m - \sum_{n=0}^{m-1}3^n \\
&= 3^m - \frac{3^m-1}{3-1} \\
&= \frac{3^m+1}{2} \\
\left|\frac{f(x+\delta_m)-f(x)}{\delta_m}\right| &\geq \frac{3^m+1}{2}
\end{aligned}$$

As $m \to \infty, \delta_m \to 0$ and $\frac{3^m+1}{2} \to \infty$. It follows that $f'(x)$ does not exists.

Equicontinuous family of functions:

Definition 5.3 ***Pointwise bounded:*** *Let f_n be a sequence of functions defined on E. We say $\{f_n\}$ is pointwise bounded if $\{f_n(x)\}$ is bounded for every $x \in E$. (i.e.) there exists a finite valued function ϕ defined on E such that $|f_n(x)| \leq \phi(x), \ \forall x \in E, n = 1, 2, 3, ...$*

Definition 5.4 ***Uniform boundedness:*** *$\{f_n\}$ is said to be uniformly bounded on E if there exists a number M such that $|f_n(x)| \leq M, \ \forall x \in E, n = 1, 2, 3, ...$*

Example 5.5 *Even if $\{f_n\}$ is a uniformly bounded sequence of continuous function on a compact set E, there need not exists a subsequence which* *converges pointwise on E.*

Solution:

$$f_n(x) = \sin nx, 0 \le x \le 2\pi, n = 1, 2, 3....$$
$$|f_n(x)| = |\sin nx| \le 1$$

$\therefore$ f_n is uniformly bounded. To Prove: $[0, 2\pi]$ is compact. Claim: This does not have any convergent subsequence. Suppose it has any convergent subsequence $\{\sin n_k x\}$,

$$\lim_{k\to\infty} \sin\ n_k x = A$$
$$\lim_{k\to\infty} (\sin\ n_k x - \sin n_{k+1} x) = 0$$
$$\lim_{n\to\infty} (\sin\ n_k x - \sin\ n_{k+1} x)^2 = 0$$
$$\int_0^{2\pi} \lim_{k\to\infty} (\sin\ n_k x - \sin\ n_{k+1} x)^2 dx = \int_0^{2\pi} 0 dx$$
$$\int_0^{2\pi} \lim_{k\to\infty} (\sin n_k x - \sin n_{k+1} x)^2 dx = 0.......(1)$$

But,

$$\int_0^{2\pi} \lim_{k\to\infty} (\sin n_k x - \sin n_{k+1} x)^2 dx$$
$$= \lim_{k\to\infty} \int_0^{2\pi} (\sin n_k x - \sin n_{k+1} x)^2 dx$$
$$= \lim_{k\to\infty} \int_0^{2\pi} (\sin^2 n_k x + \sin^2 n_{k+1} x - 2 \sin n_k x \sin n_{k+1} x) dx$$
$$= \lim_{k\to\infty} \left[\int_0^{2\pi} \sin^2 n_k x dx + \int_0^{2\pi} \sin^2 n_{k+1} x dx \right]$$
$$- \lim_{k\to\infty} \left[2 \int_0^{2\pi} \sin n_k x \sin n_{k+1} x dx \right]$$
$$= \lim_{k\to\infty} \left[\int_0^{2\pi} \frac{1 - \cos 2n_k x}{2} dx + \int_0^{2\pi} \frac{1 - \cos 2n_{k+1} x}{2} dx \right]$$
$$+ \lim_{k\to\infty} \left[\int_0^{2\pi} (\cos(n_k + n_{k+1})x - \cos(n_k - n_{k+1})x) dx \right]$$
$$= \lim_{k\to\infty} \left[\left[\frac{x}{2} - \frac{\sin 2n_k x}{4n_k} \right]_0^{2\pi} + \left[\frac{x}{2} - \frac{\sin 2n_{k+1} x}{4n_{k+1}} \right]_0^{2\pi} \right]$$
$$+ \lim_{k\to\infty} \left[\left[\frac{\sin(n_k + n_{k+1})x}{(n_k + n_{k+1})} - \frac{\sin(n_k - n_{k+1})x}{(n_k - n_{k+1})} \right]_0^{2\pi} \right]$$
$$= \lim_{k\to\infty} \left[\left[\frac{2\pi}{2} - 0 \right] + \left[\frac{2\pi}{2} - 0 \right] - [0] + [0 - 0] \right]$$
$$= \lim_{k\to\infty} 2\pi$$
$$= 2\pi.......(2)$$
$$\Rightarrow\Leftarrow to (1)$$

$\therefore$ There does not exists a subsequence which converges pointwise on E.

Example 5.6 *A uniformly bounded convergent sequence of a function, even if defined on a compact set, need not contain a uniformly convergent subsequence,*

$$f_n(x) = \frac{x^2}{x^2 + (1-nx)^2},\ \ 0 \le x \le 1, n = 1, 2, 3....$$

Solution: Clearly $[0, 1]$ is compact.

$$\begin{aligned}
|f_n(x)| &= \left|\frac{x^2}{x^2 + (1-nx)^2}\right| \le 1 \\
\lim_{n\to\infty} f_n(x) &= \lim_{n\to\infty} \frac{x^2}{x^2 + (1-nx)^2}, 0 \le x \le 1 \\
&= 0.........(1) \\
\text{But, } f_n\left(\frac{1}{n}\right) &= \frac{\frac{1}{n^2}}{\frac{1}{n^2} + (1 - n\frac{1}{n})^2} \\
&= \frac{\frac{1}{n^2}}{\frac{1}{n^2} + 0} \\
&= 1......(2)
\end{aligned}$$

Therefore f_{n_k} has no subsequence of $\{f_n\}$ which converges uniformly, if there is a subsequence $\{f_{n_k}\}$ converging uniformly. Then,

$$\begin{aligned}
|f_{n_k}(x) - 0| < \epsilon,\ \forall n_k \ge N. \\
&\Rightarrow \left|f_{n_k}\left(\frac{1}{n_k}\right) - 0\right| < \epsilon \ when\ x = \frac{1}{n_k} \\
&\Rightarrow |1 - 0| < \epsilon \\
&\Rightarrow 1 < \epsilon \\
&\Rightarrow\Leftarrow .
\end{aligned}$$

Definition 5.7 ***Equicontinuity:*** *A family $\mathscr{F}$ of complex functions f defined on a set E in a metric space X is said to be equicontinuous on E if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $d(x, y) < \delta$, ‘$x, y \in E$, $f \in \mathscr{F}$.*

Note 5.8 *(i) Every member of an equicontinuous family is uniformly continuous.*
(ii) Example 5.6 is not equicontinuous.

Proof: Let $x = \frac{1}{n}$ and $y = \frac{1}{n+1}$.

$$\begin{aligned}
d(x,y) &= \left|\frac{1}{n} - \frac{1}{n+1}\right| \\
&= \left|\frac{n+1-n}{n(n+1)}\right| \\
&= \left|\frac{1}{n(n+1)}\right| \\
&< \delta
\end{aligned}$$

$$\begin{aligned}
\text{But } |f_n(x) - f_n(y)| &= \left|\frac{\frac{1}{n^2}}{\frac{1}{n^2}} + (1 - n\frac{1}{n})^2 - \frac{\frac{1}{(n+1)^2}}{\frac{1}{(n+1)^2}} + (1 - n\frac{1}{n+1})^2\right| \\
&= \left|1 - \frac{\frac{1}{(n+1)^2}}{\frac{1}{(n+1)^2}} + (1 - \frac{n}{n+1})^2\right| \\
&= \left|1 - \frac{\frac{1}{(n+1)^2}}{\frac{1}{(n+1)^2} + (\frac{n+1-n}{n+1})^2}\right| \\
&= \left|1 - \frac{\frac{1}{(n+1)^2}}{\frac{1}{(n+1)^2}} + (\frac{1}{n+1})^2\right| \\
&= \left|1 - \frac{\frac{1}{(n+1)^2}}{\frac{2}{(n+1)^2}}\right| \\
&= \left|1 - \frac{1}{2}\right| = \frac{1}{2} \\
|f_n(x) - f_n(y)| < \epsilon &\Rightarrow \frac{1}{2} < \epsilon \\
&\Rightarrow\Leftarrow \ (\because \epsilon \text{ is arbitrarily small})
\end{aligned}$$

$\therefore$ The family is not equicontinuous.

Theorem 5.9 *If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every x in E.*

Proof: Since E is countable, we can arrange the elements of E in a sequence $\{x_i\}$, $i = 1, 2, ..., \infty$. As $\{f_n\}$ is pointwise bounded $\{f_{n_k}(x_1)\}$ is also a bounded sequence. $\therefore$ This sequence has a convergent subsequence. (i.e.) There exists a subsequence $\{f_{1k}\}$ of $\{f_n\}$ such that $\{f_{1k}(x_1)\}$ converges as $k \to \infty$. Let $S_1 : f_{11} \ \ f_{12} \ \ f_{13}$..... Now, $\{f_{1k}(x_1)\}$ is bounded. $\therefore$ There exists a subsequence $\{f_{2k}\}$ of $\{f_{1k}\}$ such that $\{f_{2k}(x_2)\}$ converges. Let $S_2 : f_{21} \ \ f_{22} \ \ f_{23}$..... Similarly we get S_3, $S_3 : f_{31} \ \ f_{32} \ \ f_{33}$..... The sequences S_n's have the following properties.

(a) S_n is a subsequence of S_{n-1}

(b) $\{f_{nk}(x_n)\}$ converges as $k \to \infty$
(c) The functions f_n's appear in the same order in all the subsequences.
Consider the diagonal sequence, $S : f_{11} \quad f_{22} \quad f_{33}.....$, by condition (a) S is a subsequence of S_n for $n = 1, 2, 3...$ except possibly its first $n-1$ terms and (b)$\Rightarrow \{f_{nn}(x_i)\}$ converges as $n \to \infty$ for every x_i in E.

Theorem 5.10 *If K is a compact metric space and $f_n \in \mathscr{C}(K)$, $n = 1, 2...$ and if $\{f_n\}$ converges uniformly on K, then $\{f_n\}$ is equicontinuous on K.*
Proof: Let $\epsilon > 0$ be given. Since $\{f_n\}$ converges uniformly on $K, \{f_n\}$ converges to some f in $\mathscr{C}(K)$. (i.e.) There exists $N > 0$ such that

$$\begin{aligned} \|f_n - f\| &< \epsilon/2 \ \forall n \geq N \\ Now, \|f_n - f_N\| &= \|(f_n - f) + (f - f_N)\| \\ &\leq \|(f_n - f)\| + \|(f - f_N)\| \\ &< \epsilon/2 + \epsilon/2 \\ &< \epsilon \ \forall n \geq N \\ (i.e.)\ \|(f_n - f_N)\| &< \epsilon \ \forall n \geq N \\ (i.e.)\ \sup_{x \in k} |(f_n(x) - f_N(x))| &< \epsilon \ \forall n \geq N \\ \Rightarrow |(f_n(x) - f_N(x))| &< \epsilon........(1) \quad \forall n \geq N \quad \forall x \in K. \end{aligned}$$

Since all continuous functions are uniformly continuous on the compact set K, there exists δ_i >0 such that $d(x, y) < \delta_i \Rightarrow |f_i(x) - f_i(y)| < \epsilon$....... (2) for $x, y \in K$, $i = 1, 2, ..., N$. Let $\delta = min\{\delta_1, \delta_2, ..., \delta_N\}$. Therefore $d(x, y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon$.......... (3) for $x, y \in K$, $n = 1, 2, ..., N$. For $n > N$,

$$\begin{aligned} d(x, y) &< \delta \\ \Rightarrow |f_n(x) - f_n(y)| &= |(f_n(x) - f_N(x)) + (f_N(x) - f_N(y)) + f_N(y) - f_n(y)| \\ &\leq |(f_n(x) - f_N(x))| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| \\ &< \epsilon + \epsilon + \epsilon \text{ (by (1) and(2))} \end{aligned}$$
$$\Rightarrow |(f_n(x) - f_n(y))| < 3\epsilon........(4)$$

Combination (3) and (4) proves the result.

Theorem 5.11 *If K is compact and if $f_n \in \mathscr{C}(K)$ for $n = 1, 2, 3...$ and if $\{f_n\}$ is pointwise bounded and equicontinuous on K, then*
(a) $\{f_n\}$ is uniformly bounded on K
(b) $\{f_n\}$ contains a uniformly convergent subsequence.
Proof:(a) Let ϵ >0 be given. By hypothesis $\{f_n\}$ is equicontinuous. Accordingly, there exists $\delta > 0$ such that $d(x, y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon$......(1) for $x, y \in K$, $n = 1, 2, ...$ Clearly, $K \subset \bigcup_{x \in K} N_\delta(x)$ where $N_\delta(x)$ is an neighbourhood of radius δ with center x. Since K is compact, there

are finitely many points $p_1, p_2, ..., p_r$ in K such that $K \subset \bigcup_{i=1}^{N} N_\delta(p_i)$......(2). Since $\{f_n\}$ is pointwise bounded, $\{f_n(p_i)\}$ is bounded for $i = 1, 2, ..., r$. $\therefore$ There exists $M_i < \infty, i = 1, 2, ..., r$ such that $|f_n(p_i)| < M_i$.
Let $M = \max\{M_1, M_2, ..., M_r\}$. Then $|f_n(p_i)| < M$......(3) $\forall i = 1, 2, ..., r$ and $\forall n$.
Let $x \in K$. Then (2) implies $x \in N_\delta(p_i)$ for some $i, 1 \leq i \leq r$. Therefore,

$$d(x, p_i) < \delta \Rightarrow |f_n(x) - f_n(p_i)| < \epsilon......(4) \text{ (by (1))}$$

Now,

$$\begin{aligned} |f_n(x)| &= |f_n(x) - f_n(p_i) + f_n(p_i)| \\ &\leq |f_n(x) - f_n(p_i)| + |f_n(p_i)| \\ &< \epsilon + M. \text{ (by (3) and (4))} \end{aligned}$$

Hence $\{f_n\}$ is uniformly bounded on K.
(b)Given K is compact and $\{f_n\}$ is pointwise bounded, equicontinuous on K. To Prove: $\{f_n\}$ contains a uniformly convergent subsequence. Since K is compact, there exists a countable dense subset $E \subseteq K$ (i.e.) $\bar{E} \subset K$. Theorem 5.9 shows that $\{f_{n_i}(x)\}$ converges for all $x \in E$. Let $g_i = f_{n_i}$. We shall show that $\{g_i\}$ converges uniformly on K. Let $\epsilon > 0$ be given. Since $\{f_n\}$ is equicontinuous on K, there exists $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon......(1) for \ x, y \in K.$$

Let $V(x, \delta) = \{y \in K | d(x, y) < \delta\} (= N_\delta(x))$. Clearly, $K \subseteq \bigcup_{x \in K} V(x, \delta)$. Since K is compact and E is dense in K, there exists $x_1, x_2, ..., x_m$ in E such that

$$K \subseteq V(x_1, \delta) \cup V(x_2, \delta) \cup ... \cup V(x_m, \delta)......(2)$$

. For $1 \leq s \leq m, \{g_i(x_s)\}$ converges. Then there exists N >0 such that

$$|g_i(x_s) - g_j(x_s)| < \epsilon......(3) \ \forall i, j \geq \ N.$$

Let $x \in K$, then (2) $\Rightarrow x \in V(x_s, \delta)$ for some $1 \leq s \leq m$.

$$d(x, x_s) < \delta \ \Rightarrow |g_i(x) - g_i(x_s)| < \epsilon......(4) \forall i$$

(by (1)$\because g_i = f_n$ for some n)
Now,

$$\begin{aligned} |g_i(x) - g_j(x)| &= |g_i(x) - g_i(x_s) + g_i(x_s) - g_j(x_s) + g_j(x_s) - g_j(x)| \\ &\leq |g_i(x) - g_i(x_s)| + |g_i(x_s) - g_j(x_s)| + |g_j(x_s) - g_j(x)| \\ &< \epsilon + \epsilon + \epsilon \text{ (by (3) and (4))} \ \forall i, j \geq N \end{aligned}$$
$$(i.e.) |g_i(x) - g_i(x)| < 3\epsilon \ \forall i, j \geq N.$$

Since x is arbitrary, the Cauchy's criteria guarantees that $\{g_i\}$ converges uniformly on K.

Theorem 5.12 ***Stone Weierstrass Theorem- the original form of Weierstrass theorem:*** *If f is a continuous complex function on $[a, b]$, then there exists a sequence of polynomials p_n such that*

$$\lim_{n\to\infty} p_n(x) = f(x)$$

uniformly on $[a, b]$. If f is real, p_n may be taken real.
Proof: Without loss of generality, we assume that $[a, b] = [0, 1]$, $f(x) = 0$ outside [0,1], $f(0) = 0$ and $f(1) = 0$.
For, suppose the result is true for this case, let

$$\begin{aligned} g(x) &= f(x) - f(0) - x[f(1) - f(0)] \\ g(1) &= f(1) - f(0) - 1[f(1) - f(0)] \\ &= 0 \\ g(0) &= f(0) - f(0) \\ &= 0 \\ \text{But} \quad f(x) - g(x) &= f(0) + x[f(1) - f(0)]. \end{aligned}$$

Since $g(x)$ is the uniform limit of a sequence of polynomials, $f(x)$ can also be obtained as the uniform limit of a sequence of polynomials.
Let

$$Q_n(x) = c_n(1 - x^2)^n, n = 1, 2, 3...$$

where we choose c_n such that

$$\int_{-1}^{1} Q_n(x)dx = 1......(1)$$

Now

$$\begin{aligned} \int_{-1}^{1}(1 - x^2)^n dx = 2\int_{0}^{1}(1 - x^2)^n dx \\ 2 \geq \int_{-1}^{\frac{1}{\sqrt{n}}}(1 - x^2)^n dx \ (\because [0, \frac{1}{\sqrt{n}}] \subseteq [0, 1]) \\ 2 \geq \int_{-1}^{\frac{1}{\sqrt{n}}}(1 - nx^2)dx \text{ (by binomial theorem)} \\ = 2\left[x - \frac{nx^3}{3}\right]_0^{\frac{1}{\sqrt{n}}} \end{aligned}$$

$$= 2\left[\frac{1}{\sqrt{n}} - \frac{n}{3n^{3/2}}\right]$$
$$= 2\left[\frac{1}{\sqrt{n}} - \frac{1}{3\sqrt{n}}\right]$$
$$= 2\left(\frac{2}{3\sqrt{n}}\right)$$
$$= \frac{4}{3\sqrt{n}}$$
$$> \frac{1}{\sqrt{n}}......(2)\ (\because 4/3 > 1)$$

$$(1) \Rightarrow \int_{-1}^{1} Q_n(x)dx = 1$$
$$\Rightarrow \int_{-1}^{1} C_n(1-x^2)^n dx = 1$$
$$\Rightarrow C_n \int_{-1}^{1} (1-x^2)^n dx = 1$$
$$\Rightarrow \int_{-1}^{1} (1-x^2)^n dx = \frac{1}{C_n}$$
$$\Rightarrow \frac{1}{C_n} = \int_{-1}^{1} (1-x^2)^n dx$$
$$\Rightarrow \frac{1}{C_n} > \frac{1}{\sqrt{n}} \text{ (by (2))}$$
$$\Rightarrow C_n > \sqrt{n}......(3)$$

$$Now, \delta \le |x| \le 1 \Rightarrow \delta^2 \le x^2$$
$$\Rightarrow -\delta^2 \ge -x^2$$
$$\Rightarrow 1-\delta^2 \ge 1-x^2$$
$$\Rightarrow (1-\delta^2)^n \ge (1-x^2)^n$$
$$\Rightarrow C_n(1-\delta^2)^n \ge C_n(1-x^2)^n$$
$$\Rightarrow C_n(1-x^2)^n \le C_n(1-\delta^2)^n$$
$$\Rightarrow C_n(1-x^2)^n \le \sqrt{n}(1-\delta^2)^n \text{ (by (3))}$$
$$\Rightarrow Q_n(x) \le \sqrt{n}(1-\delta^2)^n......(4)$$
$$\to 0 \text{ as } n \to \infty$$

$$\text{Let } p_n(x) = \int_{-1}^{1} f(x+t)Q_n(t)dt$$
$$p_n(x) = \int_{-1}^{-x} f(x+t)Q_n(t)dt + \int_{-x}^{1-x} f(x+t)Q_n(t)dt + \int_{1-x}^{1} f(x+t)Q_n(t)dt$$
$$= 0 + \int_{-x}^{1-x} f(x+t)Q_n(t)dt + 0$$

$$\therefore p_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t)dt$$
$$= \int_0^1 f(T)Q_n(T-x)dT......(5)$$

Obviously $p_n(x)$ is a polynomial in x. Moreover $p_n(x)$ is real when f is real. Claim: $p_n(x) \to f(x)$ uniformly. Since $f(x)$ is continuous on [0,1] it is uniformly continuous also. Let $\epsilon > 0$ be given, then there exists $\delta > 0$ such that

$$|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon/2 \text{ ...... } (6) for\ x,\ y\ \in\ [0,1].$$

Let $M = \sup|f(x)|$ for any $x \in$ [0,1].

$$|p_n(x) - f(x)| = \left|\int_{-1}^{1} f(x+t)Q_n(t)dt - f(x)\right|$$
$$= \left|\int_{-1}^{1} f(x+t)Q_n(t)dt - f(x)\int_{-1}^{1} Q_n(t)dt\right| \left(\because \int_{-1}^{1} Q_n(x)dx = 1\right)$$
$$= \left|\int_{-1}^{1} f(x+t)Q_n(t)dt - \int_{-1}^{1} f(x)Q_n(t)dt\right|$$
$$= \left|\int_{-1}^{1} [f(x+t) - f(x)]Q_n(t)dt\right|$$
$$\leq \int_{-1}^{1} |f(x+t) - f(x)|Q_n(t)dt$$
$$= \int_{-1}^{-\delta} |f(x+t) - f(x)|Q_n(t)dt + \int_{-\delta}^{\delta} |f(x+t) - f(x)|Q_n(t)dt$$
$$+ \int_{\delta}^{1} |f(x+t) - f(x)|Q_n(t)dt$$
$$\leq 2M\int_{-1}^{-\delta} Q_n(t)dt + \epsilon/2\int_{-\delta}^{\delta} Q_n(t)dt + 2M\int_{0}^{1} Q_n(t)dt$$
$$\leq 2M\sqrt{n}(1-\delta^2)^n\int_{-1}^{-\delta} dt + \epsilon/2\int_{-1}^{1} Q_n(t)dt$$
$$+ 2M\sqrt{n}(1-\delta^2)^n\int_{0}^{1} dt \text{ (by (4))}$$
$$\leq 2M\sqrt{n}(1-\delta^2)^n \cdot 1 + \epsilon/2 \cdot 1 + 2M\sqrt{n}(1-\delta^2)^n \cdot 1)$$
$$\left(\because \int_{-1}^{\delta} dt = 1 - \delta < 1, \int_{\delta}^{1} dt = 1 - \delta < 1\right)$$
$$\leq 4M\sqrt{n}(1-\delta^2)^n + \epsilon/2 \to 0 \text{ as } n \to \infty$$

$\therefore p_n(x) \to f(x)$ uniformly.

Some Special Functions

Definition 5.13 ***Power Series:*** *A function of the form*

$$f(x) = \sum_{n=0}^{\infty} C_n x^n \quad (or)\ f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n$$

is called a power series.

Theorem 5.14 *Suppose the series* $\sum_{n=0}^{\infty} C_n x^n$*......(1) converges for* $|x| < R$ *and define* $f(x) = \sum_{n=0}^{\infty} C_n x^n$*...... (2)* $(|x| < R)$*, then (1) converges uniformly on* $[-R+\epsilon, R-\epsilon]$ *no matter which* $\epsilon > 0$ *is choosen. The function* f *is continuous and differentiable in* $(-R, R)$ *and*
$f'(x) = \sum_{n=0}^{\infty} nC_n x^{n-1}$*...... (3)* $(|x| < R)$.
Proof: Let ϵ >0 be given. For $|x| \leq R-\epsilon$; $|C_n x^n| \leq |C_n(R-\epsilon)^n|$....... (4). We know, by Cauchy's root test , any power series $\sum_{n=0}^{\infty} C_n Z_n$ converges in $|x| < R$, where R is the radius of convergence and is given by

$$R = \frac{1}{\lim_{n\to\infty} \sqrt[n]{|C_n|}}$$

$\therefore$ The power series $\sum_0^{\infty} C_n(R-\epsilon)^n$ also converges. $\sum_{n=0}^{\infty} C_n x_n$ converges uniformly (by Weierstrass M test for uniform convergence), for $x \in [-R+\epsilon, R-\epsilon]$. Since $\lim_{n\to\infty} \sup \sqrt[n]{|C_n|} = \lim_{n\to\infty} \sqrt[n]{|C_n|}$, (1),(3) have the same radius of convergence. (i.e.) By applying Theorem 5.1 for series we see that (3) holds for $x \in [-R+\epsilon, R-\epsilon]$. But when $|x| < R$, we can find ϵ >0 such that $|x| \leq R-\epsilon$. Hence (3) holds for $|x| < R$. Since f' exists, f is continuous.

Corollary 5.15 *Under the hypothesis of Theorem 5.14,* f *has derivatives of all orders in* $(-\mathbb{R}, \mathbb{R})$ *which are given by*

$$f^k(x) = \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)C_n x^{n-k}.$$

In particular $f^k(0) = k!C_k$ *for* $k = 0, 1, 2, ...$
Proof: Let $f(x) = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + ... + C_n x^n + ...$

$$\begin{aligned}
f'(x) &= C_1 + 2C_2 x + 3C_3 x^2 + ... + nC_n x^{n-1} \\
f'(0) &= 1!C_1 \\
f''(x) &= 2C_2 + 3 \cdot 2C_3 x + ... + n(n-1)C_n x^{n-2} + ... \\
f''(0) &= 2!c_2 \\
f'''(x) &= 3 \cdot 2 \cdot 1 \cdot C_3 + ... + n(n-1)(n-2)C_n x^{n-3} + ... \\
f'''(0) &= 3!C_3 \\
f^k(x) &= \sum_{n=k}^{\infty} n(n-1)(n-2)\cdots(n-k+1)C_n x^{n-k} \\
\therefore f^k(0) &= C_k k(k-1)(k-2)\cdots 1 = k!C_k.
\end{aligned}$$

Theorem 5.16 ***Abel's theorem:*** *Suppose* $\sum C_n$ *converges. Put* $f(x) = \sum_{n=0}^{\infty} C_n x^n$ $(-1 < x < 1)$, *then*

$$\lim_{x \to 1} f(x) = \sum_{n=0}^{\infty} C_n.$$

Proof: Let $S_n = C_0 + C_1 + C_2 + ... + C_{n-1} + C_n,\ S_{-1} = 0$
Now,

$$\begin{aligned}
\sum_{n=0}^{m} C_n x^n &= \sum_{n=0}^{m} (S_n - S_{n-1}) x^n \ (\because S_n - S_{n-1} = C_n) \\
&= \sum_{n=0}^{m} S_n x^n - \sum_{n=0}^{m} S_{n-1} x^n \\
&= \sum_{n=0}^{m} S_n x^n - \sum_{n=1}^{m} S_{n-1} x^n \ (S_{-1} = 0) \\
&= \sum_{n=0}^{m-1} S_n x^n - \sum_{n=1}^{m} S_{n-1} x^n + S_m x^m \\
&= \sum_{n=0}^{m-1} S_n x^n - \left(\sum_{n=1}^{m} S_{n-1} x^{n-1} \right) x + S_m x^m \\
&= \sum_{n=0}^{m-1} S_n x^n - \left(\sum_{n=0}^{m-1} S_n x^n \right) x + S_m x^m \\
\sum_{n=0}^{m} C_n x^n &= (1 - x) \left(\sum_{n=0}^{m-1} S_n x^n \right) x + S_m x^m
\end{aligned}$$

Taking limits as $m \to \infty$ we get

$$\begin{aligned}
&\sum_{n=0}^{\infty} C_n x^n = (1 - x) \left(\sum_{n=0}^{\infty} S_n x^n \right) x + 0 \ (|x| < 1 \Rightarrow x^m \to 0 as m \to \infty) \\
&(i.e.) f(x) = (1 - x) \sum_{n=0}^{\infty} S_n x^n ......(1)
\end{aligned}$$

Since $\sum C_n$ converges, $\{S_n\}$ also converges, say to s.$\therefore$ for $\epsilon > 0$, there exists $N > 0$ such that

$$|S_n - S| < \epsilon/2 \ ......(2) \ \forall n \geq N$$

Now, since $|x|$ <1,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x} \Rightarrow (1 - x) \left(\sum_{n=0}^{\infty} x^n \right) = 1 \ ......(3)$$

Now,

$$\begin{aligned}
|f(x)-S| &= \left|(1-x)\sum_{n=0}^{\infty} S_n x^n - S\right| \text{ (by (1))}\\
&= \left|(1-x)\sum_{n=0}^{\infty} S_n x^n - S(1-x)\sum_{n=0}^{\infty} x^n\right| \text{ (by (3))}\\
&= \left|(1-x)\left(\sum_{n=0}^{\infty}(S_n x^n - Sx^n)\right)\right|\\
&= \left|(1-x)\left(\sum_{n=0}^{\infty}(S_n - S)x^n\right)\right|\\
&= \left|(1-x)\left(\sum_{n=0}^{N}(S_n - S)x^n + \sum_{n=N+1}^{\infty}(S_n - S)x^n\right)\right|\\
&\leq |(1-x)|\left(\sum_{n=0}^{N}|S_n - S||x|^n + \sum_{n=N+1}^{\infty}|S_n - S||x|^n\right)\\
&= |(1-x)|k + |(1-x)|\sum_{n=N+1}^{\infty}|S_n - S||x|^n \ where \ k = \sum_{n=0}^{N}|S_n - S||x|^n\\
&< |(1-x)|k + |(1-x)|\epsilon/2\sum_{n=N+1}^{\infty}|x|^n \text{ (by (2))}\\
&< |(1-x)|k + |(1-x)|\epsilon/2\sum_{n=0}^{\infty}|x|^n\\
&= |(1-x)|k + |(1-x)|\epsilon/2\frac{1}{1-|x|} \text{......(4)}
\end{aligned}$$

we choose $\delta = \epsilon/2k, \therefore |x-1| < \delta \Rightarrow |x-1| < \epsilon/2k.$
when $x \to 1, 1-|x| = |1-x|$

$$\begin{aligned}
\therefore |f(x)-S| &< \frac{\epsilon}{2k}k + |1-x|\epsilon/2 \cdot \frac{1}{|1-x|}\\
&= \epsilon, |x-1| < \delta\\
(i.e.) \lim_{x\to 1} f(x) = S \text{ (or) } & \lim_{x\to 1} f(x) = \sum_{n=0}^{\infty} C_n
\end{aligned}$$

Corollary 5.17 *If* $\sum a_n, \sum b_n, \sum c_n$ *converge to* A, B, C *and if* $c_n = a_0 b_n + a_1 b_{n-1} + ... + a_n b_0$ *then* $C = AB$.

Proof:

$$\begin{aligned} Let\ f(x) &= \sum_{n=0}^{\infty} a_n x^n \\ g(x) &= \sum_{n=0}^{\infty} b_n x^n \\ h(x) &= \sum_{n=0}^{\infty} c_n x^n, \text{ where } 0 \le x \le 1. \end{aligned}$$

For $x < 1$, all these series converge (by Theorem 5.14). Hence the series can be multiplied. (i.e.) $f(x)g(x) = h(x)$

$$\begin{aligned} &\Rightarrow \lim_{x\to 1}\{f(x)g(x)\} = \lim_{x\to 1} h(x) \\ &\Rightarrow \lim_{x\to 1} f(x) \lim_{x\to 1} g(x) = \lim_{x\to 1} h(x) \\ &\Rightarrow (\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n) = (\sum_{n=0}^{\infty} a_n) \text{ (by Abel's theorem)} \\ &\Rightarrow AB = C. \ (\because \sum a_n = A, \sum b_n = B, \sum c_n = C). \\ \therefore C = AB. \end{aligned}$$

Theorem 5.18 *Given a double sequence* $\{a_{ij}\}$, *i=1,2,3..., j=1,2,3... Suppose that* $\sum_{j=1}^{\infty} |a_{ij}| = b_i$ *(i=1,2,3, ...) and* $\sum b_i$ *converges, then*

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

(Inversion in the order of summation).
Proof: Let $E = \{x_0, x_1, x_2, ...\}$ be a countable set such that $x_n \to x_0$. Define

$$\begin{aligned} f_i(x_0) &= \sum_{j=1}^{\infty} a_{ij} \ (i = 1, 2, 3, ...) \\ f_i(x_n) &= \sum_{j=1}^{n} a_{ij} \ (n, i = 1, 2, 3, ...) \ and \\ g(x) &= \sum_{i=1}^{\infty} f_i(x) \ (x \in E). \end{aligned}$$

Clearly then,

$$
\begin{aligned}
\lim_{n\to\infty} f_i(x_n) &= \lim_{n\to\infty} \sum_{j=1}^{n} a_{ij} \\
&= \sum_{j=1}^{\infty} a_{ij} \\
&= f_i(x_0) \\
\therefore \lim_{x_n\to x_0} f_i(x_n) &= f_i(x_0).
\end{aligned}
$$

$\therefore$ Each f_i is continuous at x_0. ($\because \sum_{j=1}^{\infty} a_{ij}$ converges to $b_i \Rightarrow \sum a_{ij}$ converges, $f_i(x_0)$ exists $\forall i$)
Now,

$$
\begin{aligned}
|f_i(x_n)| &= \left|\sum_{j=1}^{n} a_{ij}\right| \\
&\leq \sum_{j=1}^{n} |a_{ij}| \\
&\leq \sum_{j=1}^{\infty} |a_{ij}| \\
&= b_i \text{ (by hypothesis)} \\
(i.e.)|f_i(x_n)| &\leq b_i \ (\forall n, \text{ hence } \forall x_n \in E) \\
(or)|f_i(x)| &\leq b_i ......(1)\ \forall x \in E.
\end{aligned}
$$

Since $\sum b_i$ converges, (1) and weierstrass test guarantees that $\sum_{i=1}^{\infty} f_i(x)$ converges uniformly ((i.e.) $g(x)$).
Now,

$$
\begin{aligned}
\lim_{x_n\to x_0} g(x_n) &= \lim_{x_n\to x_0} \left(\sum_{i=1}^{\infty} f_i(x_n)\right) \\
&= \sum_{i=1}^{\infty} \left(\lim_{x_n\to x_0} f_i(x)\right) \\
&= \sum_{i=1}^{\infty} f_i(x_0) \text{ (by uniform convergence and continuity theorem)} \\
&= g(x_0)
\end{aligned}
$$

(i.e.) $g(x)$ is continuous at x_0

$$g(x_0) = \lim_{n\to\infty} g(x_n)$$
$$\Rightarrow \sum_{i=1}^{\infty} f_i(x_0) = \lim_{n\to\infty} \sum_{i=1}^{\infty} f_i(x_n)$$
$$\Rightarrow \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \lim_{n\to\infty} \sum_{i=1}^{\infty}\left(\sum_{j=1}^{n} a_{ij}\right)$$
$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \lim_{n\to\infty} \sum_{j=1}^{n}\left(\sum_{i=1}^{\infty} a_{ij}\right)$$
$$\sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$$
$$\therefore \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} a_{ij}\right) = \sum_{j=1}^{\infty}\left(\sum_{i=1}^{\infty} a_{ij}\right)$$

Theorem 5.19 ***Taylor's theorem:*** *Suppose $f(x) = \sum_{n=0}^{\infty} C_n x^n$, the series converging in $|x| < R$. If $-R < a < R$ then f can be expanded in a power series about the point $x = a$ which converges in $|x-a| < R - |a|$ and*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n \quad (|x-a| < R - |a|).$$

Proof:

$$\text{Let } f(x) = \sum_{n=0}^{\infty} C_n x^n$$
$$= \sum_{n=0}^{\infty} C_n((x-a)+a)^n$$
$$= \sum_{n=0}^{\infty} C_n \left[\sum_{m=0}^{n} \binom{n}{m}(x-a)^m a^{n-m}\right]$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} C_n \binom{n}{m}((x-a)^m a^{n-m})$$
$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n} C_n \binom{n}{m}((x-a)^m a^{n-m}) ......(1)$$
$$\left(\because \binom{n}{m} = 0 \ if \ m \geq n\right)$$

Consider the series,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} |C_n \binom{n}{m}((x-a)^m a^{n-m})|.$$

The series,

$$\sum_{n=0}^{\infty}|C_n|\sum_{m=0}^{n}\binom{n}{m}|x-a|^m|a|^{n-m}=\sum_{n=0}^{\infty}|C_n|(|x-a|+|a|)^n,$$

this being the power series converges in $|x-a|+|a|<R$ (by Theorem 5.14).
(i.e.) in $|x-a|<R-|a|$. (i.e.) the series (1) converge absolutely in $|x-a|<R-|a|$. Hence by Theorem 5.18, order of summation in (1) can be changed.

$$\begin{aligned}
f(x)&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}C_n\binom{n}{m}(x-a)^m a^{n-m}\\
&=\sum_{n=0}^{\infty}\sum_{n=m}^{n}C_n\binom{n}{m}(x-a)^m a^{n-m}(\because\binom{n}{m}=0\ if\ n<m)\\
&=\sum_{n=0}^{\infty}\sum_{n=m}^{n}C_n\frac{n(n-1)...(n-m+1)}{m!}(x-a)^m a^{n-m}\\
&=\sum_{n=0}^{\infty}\frac{1}{m!}\left(\sum_{n=m}^{n}C_n n(n-1)...(n-m+1)a^{n-m}\right)(x-a)^m\\
\therefore f(x)&=\sum_{m=0}^{\infty}\frac{f^m(a)}{m!}(x-a)^m \text{ (by Corollary 5.15)}
\end{aligned}$$

Theorem 5.20 *Suppose the series $\sum a_n x^n$ and $\sum b_n x^n$ converge in the segment $S=(-R,R)$. Let E be the set of all x in S at which*

$$\sum_{n=0}^{\infty}a_n x^n=\sum_{n=0}^{\infty}b_n x^n......(1).$$

If E has a limit point in S, then $a_n=b_n, n=0,1,2,....$ hence (1) holds for all $x\in S$.
Proof: Put $C_n=a_n-b_n, \forall n=0,1,2,...$ Define

$$\begin{aligned}
f(x)&=\sum_{n=0}^{\infty}C_n x^n\\
\text{Now, } f(x)&=\sum_{n=0}^{\infty}(a_n-b_n)x^n\\
&=\sum_{n=0}^{\infty}a_n x^n-\sum_{n=0}^{\infty}b_n x^n.
\end{aligned}$$

Therefore $E=\{x\in S|f(x)=0\}......$ (2) $(\because \sum a_n x^n=\sum b_n x^n \forall x\in E)$. Let A be the set of all limit points of E in S and let $B=S-A$. Obviously, B is open in S. Also $S=A\cup B......$ (3)

We first show that A is open. Let $x_0 \in A$ ((i.e.) x_0 is a limit point of E in S). Since $-R < x_0 < R, f(x)$ can be expanded by Taylor's theorem as a power series about x_0, $|x - x_0| < R - |x_0|$.

$$(i.e.)\ f(x) = \sum_{n=0}^{\infty} d_n(x - x_0)^n ......(4), |x - x_0| < R - |x_0|.$$

Claim: All d_n's are zero. Otherwise, let k be the smallest non-negative integer such that $d_k \neq 0$. ((i.e.) $d_1 = d_2 = ... = d_{k-1} = 0$).

$$\begin{aligned} \therefore f(x) &= \sum_{n=k}^{\infty} d_n(x - x_0)^n \\ &= d_k(x - x_0)^k + d_{k+1}(x - x_0)^{k+1} + ... + d_{k+2}(x - x_0)^{k+2} + ... \\ &= (x - x_0)^k(d_k + d_{k+1}(x - x_0) + ... + d_{k+2}(x - x_0)^2 + ...) \\ f(x) &= (x - x_0)^k g(x)......(5) \text{ where } g(x) = d_k + d_{k+1}(x - x_0) + ... \\ &\qquad = \sum_{m=0}^{\infty} d_{m+k}(x - x_0)^m \end{aligned}$$

Since $g(x)$ is continuous and $g(x_0) \neq 0$, there exists $\delta > 0$ such that $g(x) \neq 0$ for all $|x - x_0| < \delta$. It follows from (5) that $f(x) \neq 0,\ \forall\ 0 < |x - x_0| < \delta$. But this contradicts that x_0 is a limit point of E. $\therefore$ All $d'_n s$ are zero. (i.e.) $f(x) = 0,\ \forall |x - x_0| < R - |x_0|$ (by (4)). Hence $(|x - x_0| < R - |x_0|) \subset A$ and A is open. Since S is connected, it cannot be expressed as a disjoint union of open sets. $\therefore$ (3) $\Rightarrow A = \phi$ (or) $B = \phi$ ($\because A \cap B = \phi$). Since E has limit points, by hypothesis in $S, A \neq \phi$. $\therefore B = \phi$. Hence $S = A$ (by (3)). Claim: $A \subset E$. Let $y \in A$ (i.e.) y is a limit point of E(in S) (i.e.) there exists a sequence $\{x_n\}$ in E such that $x_n \to y \ \therefore f(x_n) \to f(y) \ \therefore f(y) = 0$ ($\because x_n \in E \Rightarrow f(x_n) = 0 \ \ \forall n) \Rightarrow y \in E$. $\therefore A \subset E$. So,$A \subset E \subset S = A \Rightarrow E = S(= A)$. Now, by the definition of $E, f(x) = 0 \ \ \forall x \in E$

$$\begin{aligned} &\Rightarrow f(x) = 0\ \forall x \in S\ (\because E = S) \\ \Rightarrow \sum_{0}^{\infty} a_n x_n - \sum_{n=0}^{\infty} b_n x_n &= 0\ \forall x \in S \\ \Rightarrow \sum_{0}^{\infty} a_n x_n &= \sum_{n=0}^{\infty} b_n x_n\ \forall x \in S \end{aligned}$$

(i.e.) (1) holds for $\forall x \in S$. Again, $f(x) = 0 \forall x \in S \Rightarrow C_n = 0 \ \ \forall n$ (by Corollary 5.15) $\Rightarrow a_n = b_n \ \ \forall n$. Hence the proof.

The Exponential and logarithmic functions:

Definition 5.21 $E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. *This series is called the exponential series. The ratio test shows that the series converges for every complex number* z.

Definition 5.22 *We define $E(x) = e^x$ for all real x. E is called the exponential function.*

Note 5.23 $E(1) = \sum_{n=0}^{\infty} \frac{1}{n!}$ $(= e)$.

Result 5.24 *(1) $E(z)E(w) = E(z+w)$.*
Proof:

$$\begin{aligned} E(z)E(w) &= \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right)\left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\left(\frac{z^k}{k!}\right)\left(\frac{w^{n-k}}{(n-k)!}\right)\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{k=0}^{n} \frac{n! z^k w^{n-k}}{k!(n-k)!}\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^k w^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}(z+w)^n \\ &= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!} \\ &= E(z+w). \end{aligned}$$

(2) $E(z) \neq 0$ for any z.
Proof:

$$\begin{aligned} E(z)E(-z) &= E(z-z) \text{ (by result (1))} \\ &= E(0) \\ &= 1 \ (\because E(0) = 1) \\ &\Rightarrow E(z) \neq 0 \\ \text{also } E(-z) &= \frac{1}{E(z)} \end{aligned}$$

(3) $E(x) > 0$ for all real x.
Proof: Case(i): Let $x > 0$.

$$\begin{aligned} E(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \\ &> 0 \ (\because x > 0 \Rightarrow \frac{x^i}{i!} > 0 \ \forall i) \end{aligned}$$

Case(ii): Let $x < 0$. Then $x = -y$ where y is positive.

$$\begin{aligned}\therefore E(x) &= E(-y)\\ &= \frac{1}{E(y)} \text{ (by result (2))}\\ &> 0 \ (\because y > 0 \Rightarrow E(y) > 0 \text{ (by Case (i))}\\ \therefore E(x) &> 0\end{aligned}$$

Case(iii): $x = 0$.

$$\begin{aligned}E(x) &= E(0)\\ &= 1 > 0\\ hence\ E(x) &> 0\ for\ all\ real\ x.\end{aligned}$$

(4) $E(x) \to \infty$ as $x \to \infty$ and $E(x) \to 0$ as $x \to -\infty$.
Proof:

$$\begin{aligned}\textbf{(i)} E(x) &= \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...\\ &> \infty \text{ (as } x \to \infty)\end{aligned}$$

(ii) Let $x = -y$.

$$\begin{aligned}x \to -\infty &\Rightarrow -y \to -\infty\\ &\Rightarrow y \to \infty\\ &\Rightarrow E(y) \to \infty \text{ (by (i))}\\ E(x) = E(-y) &= \frac{1}{E(y)} \to 0\\ (i.e.)\ E(x) &\to 0 \text{ as } x \to -\infty.\end{aligned}$$

(5) $E(x)$ is strictly increasing on the whole real line.
Proof: **(i)** Let $x < y$. Then $x^n < y^n$.

$$\begin{aligned}&\Rightarrow \frac{x^n}{n!} < \frac{y^n}{n!}\\ &\Rightarrow \sum_{n=0}^{\infty} \frac{x^n}{n!} < \sum_{n=0}^{\infty} \frac{y^n}{n!}\\ &\Rightarrow E(x) < E(y).\end{aligned}$$

(ii) Let $x, y < 0$ and $x < y$.

$\therefore x = -x_1, y = -y_1$ where x_1 and y_1 are positive.

$$\begin{aligned} x < y &\Rightarrow -x_1 < -y_1 \\ &\Rightarrow x_1 > y_1 \\ &\Rightarrow E(x_1) > E(y_1) \text{ (by (i))} \\ &\Rightarrow \frac{1}{E(x_1)} < \frac{1}{E(y_1)} \\ &\Rightarrow E(-x_1) < E(-y_1) \text{ (by result (2))} \\ &\Rightarrow E(x) < E(y). \end{aligned}$$

(6) $E'(z) = E(z)$.

Proof:

$$\begin{aligned} E'(z) &= \lim_{h\to 0} \frac{E(z+h) - E(z)}{n} \\ &= \lim_{h\to 0} \frac{E(z)E(h) - E(z)}{h} \text{ (by (1))} \\ &= \lim_{h\to 0} E(z) \left(\frac{E(h)-1}{h}\right) \\ &= E(z) \lim_{h\to 0} \left(\frac{E(h)-1}{h}\right) \\ &= E(z) \lim_{h\to 0} \left(\frac{\sum_0^\infty \frac{h^n}{n!} - 1}{h}\right) \\ &= E(z) \lim_{h\to 0} \left(\frac{1 + \sum_0^\infty \frac{h^n}{n!} - 1}{h}\right) \\ &= E(z) \lim_{h\to 0} \frac{\sum_0^\infty \frac{h^n}{n!}}{h} \\ &= E(z) \lim_{h\to 0} \left(\sum_{n=1}^{\infty} \frac{h^{n-1}}{n!}\right) \\ &= E(z) \lim_{h\to 0} \left(1 + \frac{h}{2!} + \frac{h^3}{3!} + ...\right) \\ &= E(z) \cdot 1 \\ &= E(z). \end{aligned}$$

(7) $E(n) = e^n$ for all n.

Proof: Case(i): $n > 0$. we have $E(z_1 + z_2 + ... + z_n) = E(z_1)E(z_2)\cdots E(z_n)$ (by result 1). Put $z_i = 1 \ \forall i$, we have

$$\begin{aligned} E(1+1+1+...+1) &= E(1)E(1)\cdots E(1) \\ E(n) &= ee\cdots e \ (\because E(1) = e). \\ &= e^n \end{aligned}$$

Case(ii): $n < 0$.
Let $n = -m$ where m is a positive integer.

$$\begin{aligned}E(n) = E(-m) &= \frac{1}{E(m)}\\ &= \frac{1}{e^m} \text{ (by Case(i) as } m \text{ is a positive integer)}\\ &= e^{-m}\\ &= e^n\end{aligned}$$

Case(iii): $p = \frac{n}{m}$, n and m are integers and $m \neq 0$.
Now,

$$\begin{aligned}(E(p))^m &= E(p)E(p)\cdots E(p)\\ &= E(p+p+...+p)\\ &= E(mp)\\ &= E(n) \ (\because p = \frac{n}{m})\end{aligned}$$

$$(E(p))^m = e^n \text{ (by Case (i) and (ii))}$$

$$\begin{aligned}E(p) &= (e^n)^{1/m}\\ &= e^{n/m}\\ &= e^p\end{aligned}$$

(8) $\lim_{x\to\infty} x^n e^{-x} = 0$ *for every* n.
Proof:

$$\begin{aligned}e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}\\ &> \frac{x^{n+1}}{(n+1)!}\\ \Rightarrow e^x &> \frac{x^{n+1}}{(n+1)!}\\ \Rightarrow e^x &> \frac{x^n \cdot x}{(n+1)!}\\ \Rightarrow \frac{(n+1)!}{x} &> \frac{x^n}{e^x}\\ x^n e^{-x} &< \frac{(n+1)!}{x}\\ &\to 0 \text{ as } x \to \infty\\ (i.e.) \lim_{x\to\infty} x^n e^{-x} &= 0.\end{aligned}$$

Theorem 5.25 *Let* e^x *be defined on* R. *Then*

1. e^x *is continuous and differentiable for all* x.

2. $(e^x)' = e^x$.

3. e^x *is strictly increasing function of* x *and* $e^x > 0$.

4. $e^{x+y} = e^x e^y$.

5. $e^x \to \infty$ *as* $x \to \infty$ *and* $e^x \to 0$ *as* $x \to -\infty$.

6. $\lim_{x\to\infty} x^n e^{-x} = 0$ *for every* n. *(i.e.)* $e^x \to \infty$ *faster than any power of* x

Logarithmic function:

Definition 5.26 *Inverse of* E *is* L. $E(L(y)) = y, (y > 0);\ L(E(x)) = x, (x$ *real).*

Result 5.27 *(1)* $L(1) = 0$ *(i.e.)* $\log 1 = 0$.
Proof: $L(E(x)) = x$. Put $x = 0$, we have

$$\begin{aligned} E(x) &= E(0) \\ L(1) &= L(E(0)) \\ &= 0 \end{aligned}$$

(2) $\int_1^x \frac{1}{x} dx = L(x)$
Proof:

$$\begin{aligned} E(L(y)) &= y \\ \text{Differentiate w.r.t } y \text{, we get } E'(L(y))L'(y) &= 1 \\ yL'(y) &= 1 \\ L'(y) &= \frac{1}{y} \\ L(y) &= \int_1^y \frac{1}{y} dy \\ \text{(or) } L(x) &= \int_1^x \frac{1}{x} dx. \end{aligned}$$

(3) $L(uv) - L(u) + L(v)$
Proof: Put $u = E(x);\ v = E(y)$

$$\begin{aligned} L(E(x)E(y)) &= L(uv) \\ &= L(E(x+y)) \\ &= x + y \\ &= L(E(x)) + L(E(y)) \\ &= L(u) + L(v) \end{aligned}$$

(4) $L(\frac{u}{v}) = L(u) - L(v)$
Proof: Put $u = E(x);\ v = E(y)$

$$\begin{aligned} L\left(\frac{u}{v}\right) &= L\left(\frac{E(x)}{E(y)}\right) \\ &= L(E(x)E(-y)) \\ &= x - y \\ &= L(E(x)) - L(E(y)) \\ &= L(u) - L(v) \end{aligned}$$

(5) $\log x \to \infty$ *as* $x \to \infty$ *and* $\log x \to -\infty$ *as* $x \to 0$
Proof: $L(E(y)) = y$. Put $E(y) = x$. $y \to \infty, x \to \infty;\ y \to -\infty, x \to 0$. $\log x = y;\ \log x \to \infty$ as $x \to \infty$ and $\log x \to -\infty$ as $x \to 0$
(6) $L(x^n) = nL(x)$
Proof: Case(i): n is a positive integer.

$$\begin{aligned} L(x^n) &= L(x \cdot x \cdots x) \\ &= L(x) + L(x) + ... + L(x) \text{ (by (3))} \\ &= nL(x) \end{aligned}$$

Case(ii): n is a negative integer. $n = -m$, where m is a positive integer.

$$\begin{aligned} L(x^n) &= L(x^{-m}) \\ &= L(\frac{1}{x^m}) \\ &= L(1) - L(x^m) \text{ (by result (4))} \\ &= 0 - L(x^m) \text{ (by result (1))} \\ &= -mL(x) \text{ (by Case(i))} \\ &= nL(x) \end{aligned}$$

Case(iii): $n = \frac{1}{m}$. Let $x^{1/m} = y$. (i.e.) $y^m = x$.

$$\begin{aligned} L(x) &= L(y^m) \\ &= mL(y) \text{ (by Case (i) and (ii))} \\ \Rightarrow \frac{1}{m}L(x) &= L(y) \\ \Rightarrow L(y) &= \frac{1}{m}L(x) \\ \Rightarrow L(x^{1/m}) &= \frac{1}{m}L(x) \\ \Rightarrow L(x^n) &= nL(x) \end{aligned}$$

Case(iv): $n = p/q$.

$$\begin{aligned} L(x^n) &= L(x^{p/q}) \\ &= L(x^{1/q})^p \\ &= pL(x^{1/q}) \text{ (by Case (i) and (ii))} \\ &= p\frac{1}{q}L(x) \text{ (by Case (iii))} \\ L(x^n) &= nL(x) \end{aligned}$$

(7) $x^n = E(nL(x))$.
Proof: $E(nL(x)) = E(L(x^n))$ (by (6)) $=x^n$
(8) $(x^\alpha)' = \alpha x^{\alpha-1}$.
Proof: $x^\alpha = E(\alpha L(x))$
Differentiate w.r.t x, we get

$$\begin{aligned} (x^\alpha)' &= E'(\alpha L(x)) \cdot \alpha L'(x) \\ &= E(\alpha L(x)) \cdot \alpha\frac{1}{x} \\ &= \alpha x^{\alpha-1} \\ (x^\alpha)' &= \alpha x^{\alpha-1} \end{aligned}$$

(9) $\lim_{x\to\infty} x^{-\alpha}\log x = 0$.
Proof: Let $0 < E < \alpha$.

$$\begin{aligned} x^{-\alpha}\log x &= x^{-\alpha}\int_1^x \frac{1}{t}dt \\ &= x^{-\alpha}\int_1^x t^{-1}dt \\ &< x^{-\alpha}\int_1^x t^{\epsilon-1}dt \ (\because \epsilon - 1 > -1) \\ &= x^{-\alpha}(\frac{t^\epsilon}{\epsilon})_1^x \\ &= x^{-\alpha}(\frac{x^\epsilon}{\epsilon} - \frac{1}{\epsilon}) \\ &< \frac{x\alpha^{\epsilon-\alpha}}{\epsilon} \to 0 as x \to \infty \end{aligned}$$

$$\therefore \lim_{x\to\infty} x^{-\alpha}\log x = 0.$$

The Trignometric functions

Definition 5.28

$$\begin{aligned} C(x) &= \frac{E(ix) + E(-ix)}{2} \\ S(x) &= \frac{E(ix) - E(-ix)}{2i} \end{aligned}$$

Result 5.29 *(1) $C(x)$ and $S(x)$ are real if x is real.*
Proof:

$$
\begin{aligned}
E(ix) &= 1 + \frac{(ix)}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + ... \\
&= 1 + \frac{ix}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + ... \quad ......(1) \\
E(-ix) &= 1 + \frac{(-ix)}{1!} + \frac{(-ix)^2}{2!} + \frac{(-ix)^3}{3!} + \frac{(-ix)^4}{4!} + ... \\
&= 1 - \frac{ix}{1!} - \frac{x^2}{2!} + \frac{ix^3}{3!} + \frac{x^4}{4!} + ... \quad ......(2)
\end{aligned}
$$

(1)+(2)

$$
\begin{aligned}
\Rightarrow E(ix) + E(-ix) &= 2\left\{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right\} \\
\frac{E(ix) + E(-ix)}{2} &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ... \\
C(x) &= \frac{E(ix) + E(-ix)}{2}
\end{aligned}
$$

$\therefore$ C(x) is real if x is real.
(1)-(2)

$$
\begin{aligned}
\Rightarrow E(ix) - E(-ix) &= 2\left\{\frac{ix}{1!} - \frac{x^2}{2!} - \frac{ix^3}{3!} + ...\right\} \\
\Rightarrow \frac{E(ix) - E(-ix)}{2} &= \left\{x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right\} \\
\Rightarrow S(x) &= \frac{E(ix) - E(-ix)}{2} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ...
\end{aligned}
$$

$\therefore S(x)$ is real when x is real.
(2) $E(ix) = C(x) + iS(x)$.
Proof:

$$
\begin{aligned}
C(x) + iS(x) &= \frac{E(ix) + E(-ix)}{2} + i\frac{E(ix) - E(-ix)}{2i} \\
&= \frac{2E(ix)}{2} \\
&= E(ix).
\end{aligned}
$$

(3) $\overline{E(z)} = E(\bar{z})$.
(4) $|E(ix)| = 1$.

Proof:

$$\begin{aligned}
|E(ix)|^2 &= E(ix)\overline{E(ix)} \\
&= E(ix)E(-ix) \\
&= E(ix-ix) \\
&= E(0) \\
|E(ix)|^2 &= 1 \\
|E(ix)| &= 1
\end{aligned}$$

(5) $C(0)=1,\ \ S(0)=0$ *and* $C'(x)=-S(x),\ \ S'(x)=C(x).$

Proof:

$$\begin{aligned}
C(x) &= \frac{E(ix)+E(-ix)}{2} \\
C(0) &= \frac{E(0)+E(0)}{2} \\
&= \frac{1+1}{2} \\
&= 1 \\
S(x) &= \frac{E(ix)-E(-ix)}{2} \\
S(0) &= \frac{E(0)+E(0)}{2i} \\
&= \frac{1-1}{2i} \\
&= 0. \\
C(x) &= \frac{E(ix)+E(-ix)}{2} \\
C'(x) &= \frac{E'(ix)i+E'(-ix)(-i)}{2} \\
&= \frac{i(E(ix)-E(-ix))}{2} \\
&= \frac{i^2}{i}\frac{(E(ix)-E(-ix))}{2} \\
&= \frac{-(E(ix)-E(-ix))}{2i} \\
&= -S(x) \\
S(x) &= \frac{E(ix)-E(-ix)}{2i} \\
S'(x) &= \frac{E'(ix)i+E'(-ix)(-i)}{2i}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{i(E(ix) - E(-ix))}{2i} \\
&= \frac{E(ix) + E(-ix)}{2} \\
S'(x) &= C(x)
\end{aligned}
$$

(6) There exists positive numbers x such that $C(x) = 0$.

Proof: Suppose there is no such real number x. Since $C(0) = 1$, we get $C(x) > 0 \quad \forall x$. (i.e.) $S'(x) > 0, \quad \forall x \Rightarrow S(x)$ is an increasing function. $\therefore 0 < x \Rightarrow S(0) < S(x)$ (or) $S(x) > 0 \quad \forall x > 0$. Let $0 < x < t < y$.

$$
\begin{aligned}
\Rightarrow S(x) &< S(t) \\
\Rightarrow \int_x^y S(x)dt &< \int_x^y S(t)dt \\
\Rightarrow S(x)(y - x) &< (-C(t))_x^y \\
&< C(x) - C(y) \\
&\leq |C(x) - C(y)| \leq |C(x)| - |C(y)| \\
&\leq 1 + 1 \\
S(x)(y - x) &\leq 2 ......(1)
\end{aligned}
$$

Since $S(x) > 0$, inequality (1) does not hold for larger value of y. This contradiction proves the assertion. $\therefore$ There exist positive numbers x such that $C(x) = 0$.

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