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NUMERICAL METHODS

SYLLABUS

UNIT I

Algebraic and Transcendental Equations: Introduction – Errors in Numerical Computation – Iteration Method – Bisection Method – Numerical Computation – Iteration Method – Bisection Method – Regular Falsi Method – Newton – Raphson Method – Horner's Method.

UNIT II

Simultaneous Equations: Introduction – Simultaneous equations – Back Substitution – Gauss Elimination Method – Gauss – Jordan Elimination Method – Calculation of Matrix – Crout's Method – Iterative Methods – Gauss Jacobi Iteration Method – Gauss – Seidel Iteration Method – Relaxation Method.

UNIT III

Eigen Values and Eigen Vectors of a Matrix. Finite Differences: Introduction – Difference Operators – Other Difference Operators – Error Propagation in a difference table – Summation of Series.

UNIT IV

Interpolation: Introduction – Newton's Interpolation Formulae – Central Difference Interpolation Formulae – Lagrange's Interpolation Formulae – Divided Differences – Newton's Divided Difference Formula – Inverse Interpolation.

UNIT V

Numerical Differentiation and Integration: Derivatives using Newton's forward difference formula – Derivatives using Newton's backward difference formula – Derivatives using Newton's central difference formula – Maxima and Minima of the interpolating polynomial – Numerical integration – Difference equations: Basic Definition – Formation of Difference Equations – Linear Difference Equations.

UNIT I

LESSON

1

ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

CONTENTS

- 1.0 Aims and Objectives
- 1.1 Introduction
- 1.2 Errors in Numerical Computation
- 1.3 Iteration Method
- 1.4 Bisection Method
 - 1.4.1 Numerical Computation
- 1.5 Regula-Falsi Methods
- 1.6 Newton-Raphson Method
- 1.7 Horner's Method
- 1.8 Let us Sum up
- 1.9 Keywords
- 1.10 Questions for Discussion
- 1.11 Suggested Readings

1.0 AIMS AND OBJECTIVES

After studying this lesson, you will be able to:

- Identify errors in numerical computation
- Describe iteration method
- Discuss bisection method
- Explain Regula Falsi method
- Discuss Newton Raphson method
- Describe Horner's method

1.1 INTRODUCTION

In Engineering Mathematics we often encounter problems of obtaining solutions of equations of the form $f(x) = 0$. In other words we have to find a number x_0 such that $f(x_0) = 0$. If $f(x)$ is a polynomial then the equation $f(x)$ is called an algebraic equation.

Equations which involve transcendental functions like $\sin x$, $\cos x$, $\tan x$, $\log x$, e^x etc. are called transcendental equations.

$$x^2 + 5x + 6 = 0; \quad 2x^3 - x + 4 = 0; \quad x^5 - x^3 + 3x + 3 = 0$$

are some examples of algebraic equations.

$2e^x + 1 = 0$; $2x + \cos x - 1 = 0$; $\log_{10} x - 2x = 12$; $a + b \sin x + c \cos x + d \log x = 0$; $x^2 + \log_e x - 12 = 0$ are some examples of transcendental equations.

If $f(x)=0$ is a quadratic equation $ax^2 + bx + c = 0$ we have simple formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find

its roots. However if $f(x)$ is a polynomial of higher degree or an expression involving transcendental functions we have no simple formula to find the roots.

1.2 ERRORS IN NUMERICAL COMPUTATION

A computer has a finite word length and so only a fixed number of digits are stored and used during computation. This would mean that even in storing an exact decimal number in its converted form in the computer memory, an error is introduced. This error is machine dependent and is called machine epsilon. After the computation is over, the result in the machine form (with base β) is again converted to decimal form understandable to the users and some more errors may be introduced at this stage.

We now discuss the effect of the errors on the results. The quantity,

$$\text{True value} - \text{Approximate Value}$$

is called the error. In order to determine the accuracy in an approximate solution to a problem, either we find the bound of the

$$\text{Relative Error} = \frac{|\text{Error}|}{|\text{True value}|}$$

or of the

$$\text{Absolute Error} = |\text{Error}|$$

Neglecting a blunder or mistake, the errors may be classified into the following types:

- The inherent error is that quantity which is already present in the statement of the problem before its solution.

The inherent error arises either due to the simplified assumptions in the mathematical formulation of the problem or due to the errors in the physical measurement of the parameters of the problem.

- The round-off error is the quantity R which must be added to the finite representation of a computed number in order to make it the true representation of that number.

Thus, if x is the computed number

$$x = \cdot d_1 d_2 \cdots d_t d_{t+1} \cdots \times \beta^e$$

then the relative error for t-digit mantissa standard form representation of x becomes

$$\frac{|x - fl(x)|}{|x|} \leq \begin{cases} \beta^{1-t} & \text{for chopping} \\ \frac{1}{2}\beta^{1-t} & \text{for rounding.} \end{cases}$$

Thus, the bound on the relative error of a floating-point number is reduced by half when rounding is used than chopping. It is for this reason that on most computers rounding is used. We write

$$fl(x) = x(1 + \delta)$$

Where $\delta = \delta(x)$, some number depending on x, is called the relative round-off error in $fl(x)$. The number δ is called the machine epsilon and is denoted by EPS.

$$|\delta(x)| = EPS = \begin{cases} \beta^{1-t} & \text{for chopping} \\ \frac{1}{2}\beta^{1-t} & \text{for rounding.} \end{cases}$$

- The truncation error is the quality T which must be added to the true representation of the quality in order that the result is exactly equal to the quality we are seeking to generate.

This error is a result of the approximation formulas used which is generally based on truncated series. The Taylor series with a remainder is an invaluable tool in the study of the truncation error.

Example 1:

Obtain a second degree polynomial approximation to

$$f(x) = (1+x)^{1/2}, x \in [0, 0.1]$$

Using the Taylor series expansion about $x = 0$. Use the expansion to approximate $f(0.05)$ and find a bound of truncation error.

We have

$$\begin{aligned} f(x) &= (1+x)^{1/2}, & f(0) &= 1 \\ f'(x) &= \frac{1}{2}(1+x)^{-1/2}, & f'(0) &= \frac{1}{2} \\ f''(x) &= \frac{1}{4}(1+x)^{-3/2}, & f''(0) &= -\frac{1}{4} \\ f'''(x) &= \frac{3}{8}(1+x)^{-5/2} \end{aligned}$$

Thus, the Taylor series expansion with remainder term may be written as

$$(1+x)^{1/2} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{1}{16} \frac{x^3}{[(1+\xi)^{1/2}]^5}, 0 < \xi < 0.1.$$

The truncation term is given by

$$T = (1-x)^{1/2} - \left(1 + \frac{x}{2} - \frac{x^2}{8}\right)$$

$$= \frac{1}{16} \frac{x^3}{\left[(1+\xi)^{1/2}\right]^5}$$

We have $f(0.05) \approx 1 + \frac{0.05}{2} - \frac{(0.05)^2}{8} = 0.10246875 \times 10^1$.

The bound of the truncation error, for $x \in [0, 0.1]$ is

$$\begin{aligned} |T| &\leq \max_{0 \leq x \leq 0.1} \frac{(0.1)^3}{16 \left[(1+x)^{1/2}\right]^5} \\ &\leq \frac{(0.1)^3}{16} = 0.6.25 \times 10^{-4}. \end{aligned}$$

1.3 ITERATION METHOD

These methods are based on the idea of successive approximations i.e. starting with one or more initial approximations to the root, we obtain a sequence of approximations or iterates (x_k), which in the limit converges to the root. The methods give only one root at a time. For example, to solve the quadratic equation we may choose any one of the following iteration methods:

$$(a) x_{k+1} = -\frac{a_2 + a_0 x_k^2}{a_1}, k = 0, 1, 2, \dots$$

$$(b) x_{k+1} = -\frac{a_2}{a_0 x_k + a_1}, k = 0, 1, 2, \dots$$

$$(c) x_{k+1} = -\frac{a_2 + a_1 x_k}{a_0 x_k}, k = 0, 1, 2, \dots$$

The convergence of the sequence $\{x_k\}$ to the number ξ , the root of the equation depends on the rearrangement and the choice of the starting approximation x_0 .

A sequence of iterates $\{x_k\}$ is said to converge to the root ξ , if

$$\lim_{k \rightarrow \infty} |x_k - \xi| = 0 \text{ or } \lim_{k \rightarrow \infty} x_k = \xi.$$

If $x_k, x_{k-1}, \dots, x_{k-m+1}$ are m approximations to the root, then a multipoint iteration method is defined as

$$x_{k+1} = \phi(x_k, x_{k-1}, \dots, x_{k-m+1}).$$

The function ϕ is called the multipoint iteration function.

For $m = 1$, we get the point iteration method

$$x_{k+1} = \phi(x_k).$$

Thus, given one or more initial approximations to the root, we require a suitable iteration function ϕ for a given function $f(x)$, such that the sequence of iterates obtained from converges to the root ξ . In practice, except in rare cases, it is not possible to find ξ which satisfies the given equation exactly. We, therefore, attempts to find an approximate root ξ^* such that either

$$|f(\xi^*)| < \epsilon$$

$$|x_{k+1} - x_k| < \epsilon$$

Where x_k and x_{k-1} are two consecutive iterates and ϵ is the prescribed error tolerance.

Initial Approximation

Initial approximations to the root are often known from the physical considerations of the problem. Otherwise, graphical methods are generally used to obtain initial approximations to the root. Since the value of x , at which the graph of $y=f(x)$ intersects the x -axis, gives the root of $f(x) = 0$, any value in the neighbourhood of this point may be taken as an initial approximation to the root (see Figure 1.1 and 1.2). If the equation $f(x) = 0$ can be conveniently written in the form $f_1(x) = f_2(x)$, then the point of intersection of the graphs of $y=f_1(x)$ and $y=f_2(x)$ gives the root of $f(x) = 0$ and therefore any value in the neighbourhood of this point can be taken as initial approximation to the root (see Figure 1.3)

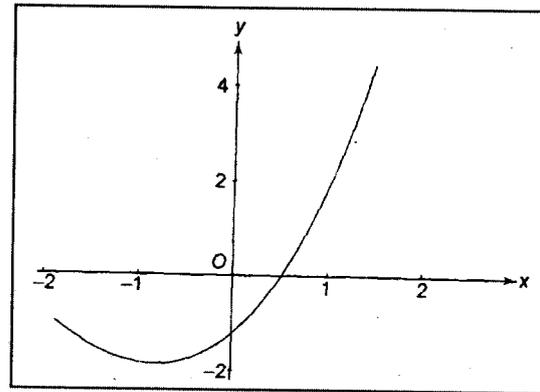


Figure 1.1: Graph of $y = x^2 + 2x - 1$.

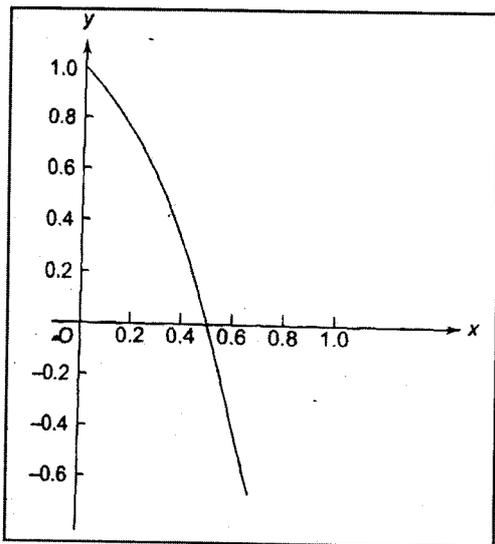


Figure 1.2: Graph of $y = \cos x - xe^x$.

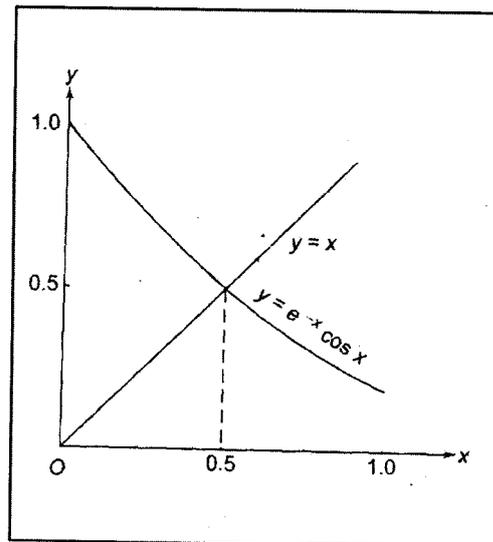


Figure 1.3: $y = x$ and $y = e^{-x} \cos x$.

Another commonly used method to obtain the initial approximation to the root is based upon the **Intermediate Value Theorem**.

Example 2:

The equation

$$8x^3 - 12x^2 - 2x + 3 = 0$$

has three real roots. Find the intervals each of unit length containing each one of these roots. We prepare a table of the value of the function $f(x)$ for various value of x .

Values of $f(x)$

x	-2	-1	0	1	2	3
$f(x)$	-105	-15	3	-3	15	105

From the table, we find that the equation $f(x) = 0$ has root in the intervals $(-1, 0)$, $(0, 1)$ and $(1, 2)$. The exact roots are -0.5 , 0.5 and 1.5 .

Example 3:

Obtain an interval which contains a root of the equation

$$f(x) = \cos x - x e^x = 0.$$

We prepare a table of the value of the function $f(x)$ for various values of x .

Values of $f(x)$

x	0	0.5	1	1.5	2
$f(x)$	1	0.0532	-2.1780	-6.6518	-15.1942

From the table we find that the equation $f(x) = 0$ has at least one root in the interval $(0.5, 1)$. The exact root correct to ten decimal places is 0.5177573637 .

1.4 BISECTION METHOD

This method is based on the repeated application of the intermediate value theorem. If we know that a root of $f(x) = 0$ lies in the interval $I_0 = (a_0, b_0)$, we bisect I_0 at the point $m_1 = (a_0 + b_0)/2$. Denoted by I_1 the interval (a_0, m_1) if $f(a_0)f(m_1) < 0$ or the interval (m_1, b_0) if $f(m_1)f(b_0) < 0$. Therefore the interval I_1 also contains the root. E bisects the interval I_1 and gets a subinterval I_1 at whose end points $f(x)$ takes the value of opposite signs and therefore contains the root. Continuing this procedure, we obtain a sequence of nested sets of sub-intervals $I_0 \supset I_1 \supset I_2 \dots$ such that each subinterval contains the roots. After repeating the bisection process q times, we either find the root or find the interval I_q of length $(b_0 - a_0)/2^q$ which contains the root. This root has error not greater than one-half of the length of the interval of which it is the midpoint. Thus, we have

$$m_{k+1} = a_k + \frac{1}{2}(b_k - a_k), \quad k = 0, 1, 2, \dots$$

$$(a_{k+1}, b_{k+1}) = \begin{cases} (a_k, m_{k+1}), & \text{if } f(a_k)f(m_{k+1}) < 0 \\ (m_{k+1}, b_k), & \text{if } f(m_{k+1})f(b_k) < 0 \end{cases}$$

We notice that this method uses only the end points of the interval $[a_k, b_k]$ for which $F(a_k)F(b_k) < 0$ and not the values of $f(x)$ at these end points, to obtain the next approximation to the root. The

method is simple to use and the sequence of approximations always converges to the root for any $f(x)$ which is continuous in the interval that contains the root. If the permissible error is ϵ , then the approximation number of iterations required may be determined from the relation

$$\frac{b_0 - a_0}{2^n} \leq \epsilon \quad \text{or} \quad n \geq \frac{\log(b_0 - a_0) - \log \epsilon}{\log 2}$$

Since n is an integer, we take n as the next nearest integer.

The minimum number of iterations required for converging to a root in the interval $(0, 1)$ for a given ϵ are listed in Table 1.1

Table 1.1: Number of Iterations

ϵ	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}
n	7	10	14	17	20	24*

Thus the bisection method requires a large number of iterations to achieve a reasonable degree of accuracy for the root. It requires one function evaluation for each iteration.

1.4.1 Numerical Computation

Example 4:

Perform five iterations of the bisection method to obtain the smallest positive root of the equation

$$f(x) = x^3 - 5x + 1 = 0.$$

Since $f(0) > 0$ and $f(1) < 0$, the smallest positive root line in the interval $(0, 1)$. Taking $a_0 = 0$, $b_0 = 1$, we get

$$m_1 = \frac{1}{2}(a_0 + b_0) = \frac{1}{2}(0 + 1) = 0.5$$

$$f(m_1) = -1.375 \text{ and } f(a_0)f(m_1) < 0.$$

Thus, the root lies in the interval $(0, 0.5)$. Taking $a_1 = 0$, $b_1 = 0.5$. We get

$$m_2 = \frac{1}{2}(a_1 + b_1) = \frac{1}{2}(0 + 0.5) = 0.25$$

$$f(m_2) = f(0.25) = -0.234375 \text{ and } f(a_1)f(m_2) < 0.$$

Thus, the root lies in the interval $(0, 0.25)$. The sequence of intervals is given in table below.

Sequence of Intervals for the Bisection Method

k	a_{k-1}	b_{k-1}	m_k	$f(m_k) f(a_{k-1})$
1	0	1	0.5	< 0
2	0	0.5	0.25	< 0
3	0	0.25	0.1875	> 0
4	0.125	0.25	0.1875	> 0
5	0.1875	0.25	0.21875	< 0

Hence, the root lies in (0.1875, 0.21875). The approximate root is taken as the midpoint of this interval, that is 0.203125.

Example 5:

Perform five iterations of the bisection method to obtain a root of the equation

$$f(x) = \cos x - xe^x = 0$$

Since $f(0) = 1 > 0$ and $f(1) = -2.1780 < 0$, the root lies in the interval (0, 1). Taking the initial approximations as $a_0 = 0, b_0 = 1$, we get

$$m_1 = \frac{1}{2}(a_0 + b_0) = \frac{1}{2}(0 + 1) = 0.5$$

$$f(m_1) = f(0.5) = 0.0532 \text{ and } f(a_0)f(m_1) > 0.$$

Therefore, the root lies in the interval (0.5, 1.0).

Taking

$$a_1 = 0.5, b_1 = 1.0, \text{ we get}$$

$$m_2 = \frac{1}{2}(a_1 + b_1) = \frac{1}{2}(0.5 + 1.0) = 0.75$$

$$f(m_2) = 0.8561 \text{ and } f(a_1)f(m_2) < 0.$$

Therefore, the root in the interval (0.5, 0.75). The sequence of intervals is given in table below.

Sequence of Intervals for the Bisection Method

k	a_{k-1}	b_{k-1}	m_k	$f(m_k)f(a_{k-1})$
1	0	1	0.5	> 0
2	0.5	1	0.75	< 0
3	0.5	0.75	0.625	< 0
4	0.5	0.625	0.5625	< 0
5	0.5	0.5625	0.53125	< 0

Hence, the root lies in the interval (0.5, 0.53125). The approximate root is taken as the midpoint of this interval, that is, 0.515625.

1.5 REGULA-FALSI METHODS

If x_{k-1} and x_k are two approximations to the root, then we determine a_0 and a_1 by using the conditions

$$f_{k-1} = a_0 x_{k-1} + a_1$$

$$f_k = a_0 x_k + a_1$$

Where

$$f_{k-1} = f(x_{k-1}) \text{ and } f_k = f(x_k).$$

On solving, we obtain

$$a_0 = (f_k - f_{k-1}) / (x_k - x_{k-1})$$

$$a_1 = (x_k f_{k-1} - x_{k-1} f_k) / (x_k - x_{k-1}).$$

The next approximation x_{k+1} to the root is given by

$$x_{k+1} = \frac{x_{k-1} f_k - x_k f_{k-1}}{f_k - f_{k-1}}$$

which may also be written as

$$x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f_k - f_{k-1}} f_k, k = 1, 2, \dots$$

This is called the secant or chord method.

Geometrically, in this method we replace the function $f(x)$ by a straight line or a chord passing through the points (x_k, f_k) and (x_{k-1}, f_{k-1}) and take the point of intersection of the straight line with the x -axis as the next approximation to the root (Figure 1.4). If the approximations are such that $f_k f_{k-1} < 0$, then the method is known as Regula-Falsi method. The method is shown graphically in Figure 1.5. Since $(x_{k-1}, f_{k-1} < 0)$ are known before the start of the iteration, the secant and the Regula-Falsi methods require one function evaluation per iteration.

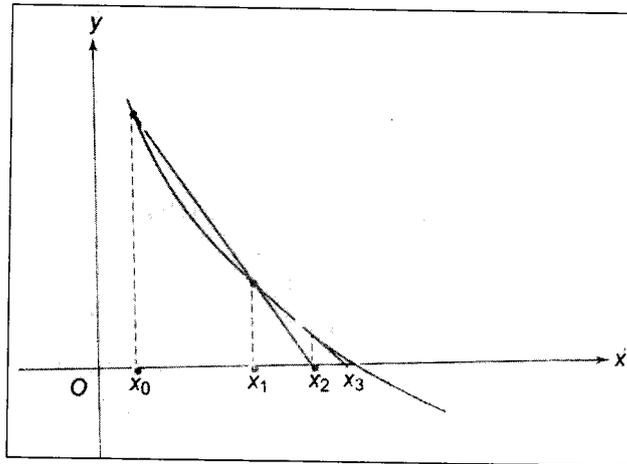


Figure 1.4: Secant Method

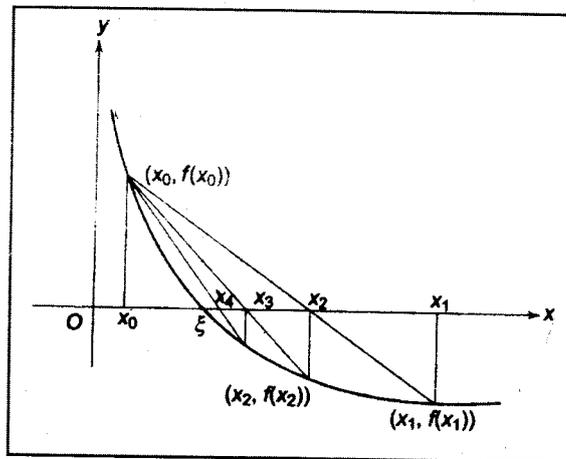


Figure 1.5: The Regula-Falsi Method

Example 6:

A real root of the equation

$$f(x) = x^3 - 5x + 1 = 0$$

Lies on the interval (0, 1) Perform four iterations of the secant method and the Regula-Falsi method to obtain this root.

$$x_0 = 0, x_1 = 1, f_0 = f(x_0) = 1, f_1 = f(x_1) = -3$$

Secant method

$$x_2 = x_1 - \left[\frac{x_1 - x_0}{f_1 - f_0} \right] f_1 = 0.25, f_2 = f(x_2) = -0.234375.$$

$$x_3 = x_2 - \left[\frac{x_2 - x_1}{f_2 - f_1} \right] f_2 = 0.186441, f_3 = f(x_3) = -0.074276.$$

$$x_4 = x_3 - \left[\frac{x_3 - x_2}{f_3 - f_2} \right] f_3 = 0.201736, f_4 = f(x_4) = -0.000470.$$

$$x_5 = x_4 - \left[\frac{x_4 - x_3}{f_4 - f_3} \right] f_4 = 0.201640.$$

Regula-Falsi method

$$x_2 = x_1 - \left| \frac{x_1 - x_0}{f_1 - f_0} \right| f_1 = 0.25, f_2 = f(x_2) = -0.234375.$$

Since $f(x_0)f(x_2) < 0, \xi \in (x_0, x_2)$. Therefore,

$$x_3 = x_2 - \left| \frac{x_2 - x_0}{f_2 - f_0} \right| f_2 = 0.202532, f_3 = f(x_3) = -0.004352.$$

Since $f(x_0)f(x_3) < 0, \xi \in (x_0, x_3)$. Therefore,

$$x_4 = x_3 - \left| \frac{x_3 - x_0}{f_3 - f_0} \right| f_3 = 0.201654, f_4 = f(x_4) = -0.000070.$$

Since, $f(x_0)f(x_4) < 0, \xi \in (x_0, x_4)$. Therefore,

$$x_5 = x_4 - \left[\frac{x_4 - x_0}{f_4 - f_0} \right] f_4 = 0.201640.$$

1.6 NEWTON-RAPHSON METHOD

We determine a_0 and a_1 using the conditions

$$f_k = a_0 x_k + a_1$$

$$f'_k = a_0$$

Where a prime denotes differentiates with respect to x.

On substituting a_0 and a_1 and representing the approximate value of x by x_{k+1} , we obtain

$$x_{k+1} = x_k - \frac{f_k}{f'_k}, k = 0, 1, \dots$$

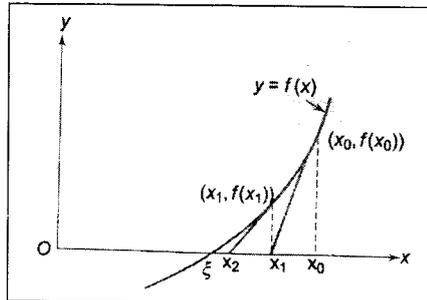


Figure 1.6: The Newton-Raphson Method

This method is called the Newton-Raphson method. The method may also be obtained directly by taking the limit $x_{k-1} \rightarrow x_k$. In the limit when $x_{k-1} \rightarrow x_k$, the chord passing through the points (x_k, f_k) and (x_{k-1}, f_{k-1}) becomes the tangent at the point (x_k, f_k) . Thus, in this case the problem of finding the root of the equation is equivalent to finding the point of intersection of the tangent to the curve $y=f(x)$ at the point (x_k, f_k) with the x-axis. The method is shown graphically in Figure 1.6. The Newton-Raphson method requires two evaluations f_k, f'_k for each iteration.

Alternative

Let x_k be an approximation to the root of the equation $f(x) = 0$. Let Δx be an increment in x such that $x_k + \Delta x$ is an exact root. Therefore,

$$f(x_k + \Delta x) \equiv 0.$$

Expanding in Taylor series about the point x_k , we get

$$f(x_k) + \Delta x f'(x_k) + \frac{1}{2!}(\Delta x)^2 f''(x_k) + \dots = 0.$$

Neglecting the second and higher powers of Δx , we obtain

$$f(x_k) + \Delta x f'(x_k) \approx 0$$

Or

$$\Delta x \approx -\frac{f(x_k)}{f'(x_k)}$$

Hence, we obtain the iteration method

$$x_{k+1} = x_k + \Delta x = x_k - \frac{f(x_k)}{f'(x_k)}, k = 0, 1, \dots$$

1.7 HORNER'S METHOD

Horner's method is the most convenient way of finding approximation values of the irrational roots of the equation $f(x) = 0$, where $f(x)$ is any polynomial. The root is calculated in decimal form and the figures of the decimal are obtained in succession. We describe below the steps to be followed

Step I. Consider the equation $f(x) = 0$. Suppose this has a single root α in the interval $(a, a + 1)$ where a is a positive integer. Then a can be located by using the condition that $f(a)$ and $f(a + 1)$ are of opposite signs.

Step II. Suppose the exact value of the root is $a.a_1 a_2 \dots$. Diminish the roots of $f(x) = 0$ by a . Then we get the transformed equation $f_1(x) = 0$ having $0.1 a_1 a_2 \dots$ as a root.

Step III. Multiply the roots of $f_1(x)$ by 10 and we obtain the transformed equation $f_2(x) = 0$ having $a_1.a_2 \dots$ as a root.

Step IV. By inspection we locate the root by finding two consecutive integer b and $b + 1$ such that $f_2(b)$ and $f_2(b + 1)$ are of opposite signs. Then $b = a_1$ is the first decimal in the root making $a.a_1$ as the first approximation of the root.

Repeat this process (Step I to IV) as many times as needed to get the roots of $f(x) = 0$ to any desired number of decimal places.

Check Your Progress

Perform four iterations of the Newton-Raphson method to find the smallest positive root of the equation

$$f(x) = x^3 - 5x + 1 = 0.$$

1.8 LET US SUM UP

A computer has a finite word length and so only a fixed number of digits are stored and used during computation. This would mean that even in storing an exact decimal number in its converted form in the computer memory, an error is introduced. Iteration methods are based on the idea of successive approximations i.e. starting with one or more initial approximations to the root. Bisection method is based on the repeated application of the intermediate value theorem. The problem of finding the root of the equation is equivalent to finding the point of intersection of the tangent to the curve $y = f(x)$. Horner's method (or synthetic division) is a technique for evaluating polynomials.

1.9 KEYWORDS

Iteration Method: These methods are based on the idea of successive approximations i.e. starting with one or more initial approximations to the root.

Bisection Method: This method is based on the repeated application of the intermediate value theorem.

Regula Falsi Method: It requires one function evaluation per iteration.

Newton Raphson Method: It requires two evaluations for each iteration.

Horner's Method: The most convenient way of finding approximation values.

1.10 QUESTIONS FOR DISCUSSION

1. Obtain an interval which contains a root of the equation

$$f(x) = \cos x - x e^x = 0.$$

2. Perform five iterations of the bisection method to obtain a root of the equation

$$f(x) = \cos x - x e^x = 0$$

3. Use the Regula-Falsi methods to determine the root of the equation

$$\cos x - x e^x = 0$$

Check Your Progress: Model Answers

The smallest positive root lies in the interval (0, 1). Take the initial approximation as $x_0 = 0.5$. We have

$$f(x) = x^3 - 5x + 1, f'(x) = 3x^2 - 5.$$

Using the Newton-Raphson method.

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

we get

$$x_{k+1} = x_k - \frac{x_k^3 - 5x_k + 1}{3x_k^2 - 5} = \frac{2x_k^3 - 1}{3x_k^2 - 5}, k = 0, 1, \dots$$

Starting with $x_0 = 0.5$, we obtain.

$$x_1 = 0.176471, x_2 = 0.201568.$$

$$x_3 = 0.201640, x_4 = 0.201640.$$

The exact value correct to six decimal places is 0.201640.

1.11 SUGGESTED READINGS

S.S. Sastri., *Introductory Methods of Numerical Analysis*, Second Edition, Prentice Hall.

S. Mohan Naidu., *A Text Book of Mathematical Methods for Scientific Engineers*, Students Helpline Series, Hyd - 2005.

Jain, M.K., Iyengar, S.R.K., Jain, R.K., *Numerical Methods for Scientific and Engineering Computation*, 3rd Edition, New Age International (P) Ltd., 1993.

UNIT II

LESSON

2

SIMULTANEOUS EQUATIONS

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2.0 AIMS AND OBJECTIVES

After studying this lesson, you will be able to:

- Discuss simultaneous equations
- Explain back substitution
- Describe Gauss elimination method
- Identify Gauss Jordan elimination method
- Understand calculation of matrix
- Explain Crout's method

2.1 INTRODUCTION

Simultaneous linear algebraic equations occur in several engineering and statistical problems. In this lesson we deal with several numerical methods for solving such system of equations.

2.2 SIMULTANEOUS EQUATIONS

Matrix Operations

MathCAD is designed to be a tool for quick and easy manipulation of matrix forms of data. We've seen the matrix before as a 2-D array. That is, many pieces of information are stored under a single name. Different pieces of information are then retrieved by pointing to different parts of the matrix by row and column indexes. Here we will learn some basic matrix operations: Adding and Subtracting, Transpose, Multiplication.

Adding Matrices

Add two matrices together is just the addition of each of their respective elements. If A and B are both matrices of the same dimensions (size), then

$$C := A + B$$

produces C, where the i^{th} row and j^{th} column are just the addition of the elements (numbers) in the i^{th} row and j^{th} column of A and B

$$\text{Given: } A = \begin{bmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

$$\text{so that the addition is: } C := A + B = \begin{bmatrix} 3 & 7 & 11 \\ 15 & 19 & 23 \end{bmatrix}$$

The Mathcad commands to perform these matrix assignments and the addition are:

$$A := \text{Ctrl-M (choose } 2 \times 3) \ 1 \ 3 \ 5 \ 7 \ 9 \ 11$$

$$B := \text{Ctrl-M (choose } 2 \times 3) \ 2 \ 4 \ 6 \ 8 \ 10 \ 12$$

$$C := A + B \quad C =$$

Rule: A, B, and C must all have the same dimensions.

Transpose

Transposing a matrix means swapping rows and columns of a matrix. No matrix dimension restrictions.

Some examples:

$$\text{1-D } A = [5 \ 2 \ 9], \quad A^T = \begin{bmatrix} 5 \\ 2 \\ 9 \end{bmatrix} \quad 1 \times 3 \text{ becomes } \implies 3 \times 1$$

$$\text{2-D } B = \begin{bmatrix} 8.1 & -4.5 & -7.6 \\ 3.2 & 3.1 & 3.9 \end{bmatrix}, \quad B^T = \begin{bmatrix} 8.1 & 3.2 \\ -4.5 & 3.1 \\ -7.6 & 3.9 \end{bmatrix} \quad 2 \times 3 \text{ becomes } \implies 3 \times 2$$

In general

$$B(i, j) = B^T(j, i)$$

In Mathcad, **The transpose is can be keystroked by Ctrl - 1 (the number one)** or you can view the matrix pallet (view → toolbars → matrix) and click the M^T symbol

$$B = \begin{bmatrix} 5 & 3 & 6 & 2 \\ 9 & 8 & 4 & 7 \end{bmatrix} \quad B \text{ Ctrl-1} = \text{ans} = \begin{bmatrix} 5 & 9 \\ 3 & 8 \\ 6 & 4 \\ 2 & 7 \end{bmatrix}$$

Multiplication

Multiplication of matrices is not as simple as addition or subtraction. It is not an element by element multiplication as you might suspect it would be. Rather, matrix multiplication is the result of the dot products of rows in one matrix with columns of another. Consider:

$$C := A * B$$

matrix multiplication gives the i^{th} row and k^{th} column spot in C as the scalar results of the dot product of the i^{th} row in A with the k^{th} column in B. In equation form this looks like:

$$C_{i,k} = \sum_{j=1}^{\text{\# of columns in A}} A_{i,j} * B_{j,k}$$

Let's break this down in a step-by-step example:

Step 1: Dot Product (a 1-row matrix times a 1-column matrix)

The Dot product is the **scalar** result of multiplying one row by one column

$$\begin{bmatrix} 2 & 5 & 3 \end{bmatrix}_{1 \times 3} * \begin{bmatrix} 6 \\ 8 \\ 7 \end{bmatrix}_{3 \times 1} = 2*6 + 5*8 + 3*7 = 73_{1 \times 1} \quad \text{DOT PRODUCT OF ROW AND COLUMN}$$

Rule:

1. # of elements in the row and column must be the same
2. must be a row times a column, not a column times a row.

Step 2: General matrix multiplication is taking a series of dot products.

Each row in pre-matrix by each column in post-matrix.

$$\begin{bmatrix} 1 & 4 & 2 \\ 9 & 3 & 7 \end{bmatrix}_{2 \times 3} * \begin{bmatrix} 5 & 6 \\ 8 & 12 \\ 10 & 11 \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 1*5+4*8+2*10 & 1*6+4*12+2*11 \\ 9*5+3*8+7*10 & 9*6+3*12+7*11 \end{bmatrix} = \begin{bmatrix} 57 & 76 \\ 139 & 167 \end{bmatrix}_{2 \times 2}$$

$C(i,k)$ is the result of the dot product of row i in A with column k in B

Matrix Multiplication Rules:

1. The # of columns in the pre-matrix must equal # of rows in post-matrix inner matrix dimensions must agree.

2. The result of the multiplication will have the outer dimensions # rows in pre-matrix by # columns in post-matrix.

For this example, apply rules

$$C := A * B$$

A is $nra \times nca$ (# rows in a by # columns in a)

B is $nrb \times ncb$

Rule 1 says:

$$nca = nrb \text{ or else we can't multiply (can't take dot products with different number of terms in row and column)}$$

Rule 2 says:

C will be of size $nra \times ncb$

result C has outer dimensions

$$\overbrace{nra \times nca * nrb \times ncb}^{\text{outer dimensions}}$$

inner dimensions must agree

How to perform matrix multiplication in Mathcad???

$$A := \begin{pmatrix} 4 & 5 \\ 2 & 1 \end{pmatrix} \quad B := \begin{pmatrix} 9 & 1 \\ 6 & 12 \end{pmatrix} \quad C := A \cdot B \quad C = \begin{pmatrix} 66 & 64 \\ 24 & 14 \end{pmatrix}$$

Note: If inner matrix dimensions don't match, Mathcad can't perform the operation since it violates the rules of matrix multiplication, and you'll get an error that says:

“the number of rows and or columns in these arrays do not match”

Example1: Let's try to multiply a 2×3 by another 2×3 (rules say we can't do this)

$$A := \begin{pmatrix} 3 & 4 & 1 \\ 0 & 4 & 9 \end{pmatrix} \quad B := \begin{pmatrix} 2 & 9 & 5 \\ 9 & 4 & 5 \end{pmatrix} \quad C := A \cdot B$$

Mathcad will tell you:

“the number of rows and or columns in these arrays do not match”

Since the # of columns in A was not equal to # of rows in B, we can't multiply $A * B$

IMPORTANT: Another example: Say we create a 1-D vector x with the following:

$$x := (3 \ 8 \ 9 \ 5)$$

Now say we want to square each number in x. It would seem natural to do this:

$$X^2 =$$

But Mathcad tells us:

“This Matrix must be square. It should have the same number of rows as columns”

Note that $x^2 =$ is the same as saying $x*x =$

Mathcad by default will always interpret any multiplication as a standard dot product type matrix multiplication, thus we can't take a dot product of two row vectors, since rules of matrix multiplication are violated in this case.

The exception to this default assumption in Mathcad is if the vector is a column instead of a row. In that case, Mathcad will assume you want to square each element in the vector rather than apply standard matrix multiplication.

If we just want to square the numbers in x , we can do this:

$$y := \begin{pmatrix} 3 \\ 8 \\ 9 \\ 5 \end{pmatrix} \quad y^2 = \begin{pmatrix} 9 \\ 64 \\ 81 \\ 25 \end{pmatrix}$$

Or we can first convert a row into a column vector using transpose, and then square

Try this out

$$a := (2 \ 5 \ 4) \quad (a^T)^2 = \begin{pmatrix} 4 \\ 25 \\ 16 \end{pmatrix}$$

Solving Simultaneous Linear Equations using Matrix Methods

Now we'll use matrices to represent sets of algebraic equations. The solution to these sets of equations can be solved using matrix methods. The simultaneous solution of multiple equations finds its way in to many common engineering problems. In fact, modern structural engineering analysis techniques are ALL ABOUT solving systems of equations simultaneously. You'll see the following material in CES 4141 (structures II) for sure.

- Matrices - an organized way of presenting a set of coupled equations.
- We seek a single unique solution that satisfies all the equations at the same time.

Consider the three *coupled linear* equations below:

$$3X_1 + 5X_2 + 2X_3 = 8$$

$$2X_1 + 3X_2 - 1X_3 = 1$$

$$1X_1 - 2X_2 - 3X_3 = -1$$

- *Coupled* because each equation has one or more terms in common with the others, X_1 , X_2 , X_3 , so that a change in one of these variables will affect more than one equation.
- *Linear* because each equation contains only first order terms of X_1 , X_2 , X_3 . There are no terms
- like X_1^2 , or $\sqrt{X_2}$, or $\log(X_3)$, or $1/(X_1X_2)$, etc.

- Using the rules of matrix multiplication, we can represent the above equations in matrix form:

$$\begin{array}{ccc} \begin{bmatrix} 3 & 5 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & -3 \end{bmatrix} & \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} & = & \begin{bmatrix} 8 \\ 1 \\ -1 \end{bmatrix} \\ \swarrow & \downarrow & & \swarrow \\ \text{Coefficient matrix A} & \text{unknown vector X} & & \text{solution vector B} \end{array}$$

Try multiplying the matrices A and X together, make sure you can get the original equations above.

There are several ways to solve for unknown vector. Each method involves some manipulations to the coefficient matrix using algebraic rules, creating a new and equivalent problem in a more easily solvable form. These manipulations involve the addition of multiples of one row to another.

Adding one row to another result in an equivalent equation, since both sides are equal.

For example, starting with the two equations:

$$\begin{aligned} X_1 + 5X_2 &= 3 \\ -2X_1 - 3X_2 &= 5 \end{aligned}$$

their addition gives:

$$-1X_1 + 2X_2 = 8$$

This addition does not add any new information, but it does present a new form of the old information.

2.3 BACK SUBSTITUTION

Consider a system of simultaneous linear equations given by $AX = B$ where A is an $n \times n$ coefficient matrix.

Suppose the matrix A is upper triangular.

$$\text{Let } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

Then the given system takes the form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & a_{3n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{pmatrix}$$

$$\begin{aligned}
 \text{(i.e.)} \quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
 a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
 \cdots \quad \cdots \quad \cdots \quad \cdots & \\
 a_{n-1n-1}x_{n-1} + a_{n-1n}x_n &= b_{n-1} \\
 a_{nn}x_n &= b_n.
 \end{aligned}$$

From the last equation we get $x_n = \frac{b_n}{a_{nn}}$.

Substituting the value of x_n in the previous equation we get

$$x_{n-1} = \frac{1}{a_{n-1n-1}} \left[b_{n-1} - a_{n-1n} \left(\frac{b_n}{a_{nn}} \right) \right].$$

Proceeding like we can find all x_i 's. This procedure is known as back substitution

Similarly considering lower triangular matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix}$$

the given system takes the form

$$\begin{aligned}
 a_{11}x_1 &= b_1 \\
 a_{21}x_1 + a_{22}x_2 &= b_2 \\
 \cdots \quad \cdots \quad \cdots \quad \cdots & \\
 a_{n1}x_1 + \cdots + a_{nn}x_n &= b_n
 \end{aligned}$$

From the first equation we get $x_1 = \frac{b_1}{a_{11}}$. Substituting the value x_1 in the next equation we get

$$x_2 = \frac{1}{a_{22}} \left[a_2 - a_{21} \left(\frac{b_1}{a_{11}} \right) \right].$$

Proceeding like this we can find all x_i 's. This procedure is known as forward substitution.

2.4 GAUSS ELIMINATION METHOD

Here, the unknowns are eliminated by combining equations such that the n equations in n unknowns are reduced to an equivalent upper triangular system which is then solved by back substitution method. Consider the 3×3 system

$$\begin{aligned}
 a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\
 a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\
 a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3.
 \end{aligned}$$

In the first stage of elimination, multiply the first row in by a_{21}/a_{11} and a_{31}/a_{11} respectively and subtract from the second and third rows. We get

$$a_{22}^{(2)} x_2 + a_{23}^{(2)} x_3 = b_2^{(2)}$$

$$a_{32}^{(2)} x_2 + a_{33}^{(2)} x_3 = b_3^{(2)}$$

Where

$$a_{22}^{(2)} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}, \quad a_{23}^{(2)} = a_{23} - \frac{a_{21}}{a_{11}} a_{13},$$

$$a_{32}^{(2)} = a_{32} - \frac{a_{31}}{a_{11}} a_{12}, \quad a_{33}^{(2)} = a_{33} - \frac{a_{31}}{a_{11}} a_{13}$$

$$b_2^{(2)} = b_2 - \frac{a_{21}}{a_{11}} b_1, \quad b_3^{(2)} = b_3 - \frac{a_{31}}{a_{11}} b_1$$

In the second stage of elimination, multiply the first row by $(a_{32}^{(2)}/a_{22}^{(2)})$ and subtract from the second row. We get

$$a_{33}^{(3)} x_3 = b_3^{(3)}$$

Where

$$a_{33}^{(3)} = a_{33}^{(2)} - \frac{a_{32}^{(2)}}{a_{22}^{(2)}} a_{23}^{(2)}, \quad b_3^{(3)} = b_3^{(2)} - \frac{a_{32}^{(2)}}{a_{22}^{(2)}} b_2^{(2)}.$$

Collecting the first equation from each stage, we obtain the system

$$a_{11}^{(1)} x_1 + a_{12}^{(1)} x_2 + a_{13}^{(1)} x_3 = b_1^{(1)}$$

$$a_{22}^{(2)} x_2 + a_{23}^{(2)} x_3 = b_2^{(2)}$$

$$a_{33}^{(3)} x_3 = b_3^{(3)}$$

Where

$$a_{ij}^{(1)} = a_{ij}, \quad b_i^{(1)} = b_i, \quad ij = 1, 2, 3.$$

The system is an upper triangular system and can be solved using the back substitution method. Therefore, the Gauss elimination method gives

$$[a|b] \xrightarrow[\text{Elimination}]{\text{Gauss}} [U|c]$$

Where $[A|b]$ is the augmented matrix. The elements $a_{11}^{(1)}$, $a_{22}^{(2)}$ and $a_{33}^{(3)}$ which have been assumed to be non-zero are called pivot elements. The elimination procedure described above to determine the elimination method. We may also make the pivot as 1 before elimination, at each step. At the end of the elimination procedure, we produce 1 at each of the positions of the diagonal elements.

We now solve the system in n unknowns by performing the Gauss elimination on the augmented matrix $[A|b]$. Denote

$$b_i^{(k)} = a_{i,n+1}^{(k)}, \quad i, k = 1(1)n.$$

The elements $a_{ij}^{(k)}$ with $i, j \geq k$ are given

$$a_{ij}^{(k+1)} = a_{ij}^{(k)} - \frac{a_{ik}^{(k)}}{a_{kk}^{(k)}} a_{kj}^{(k)}$$

$$i = k+1, k+2, \dots, n \quad j = k+1, \dots, n, n+1$$

Where

$$a_{ij}^{(1)} = a_{ij}$$

The elimination is performed in $(n-1)$ steps, $k = 1, 2, \dots, n-1$. In the elimination process, if any one of the pivot elements $a_{11}^{(1)}, a_{22}^{(2)}, \dots, a_{nn}^{(n)}$ vanishes or becomes very small compared to other elements in that column, then we attempt to rearrange the remaining rows so as to obtain a non-vanishing pivot or to avoid the multiplication by a large number. This strategy is called pivoting. The pivoting is of the following two types.

Partial Pivoting

In the first stage of elimination, the first column is searched for the largest element in magnitude and brought as the first pivot by interchanging the first equation with the equation having the largest element in magnitude. In the second elimination stage, the second column is searched for the largest element in magnitude among the $n-1$ elements leaving the first element, and this element is brought as the second pivot by an interchange of the second equation with equation having the largest element in magnitude. This procedure is continued until we arrive at the equations. We are thus led to the following algorithm to find the pivot.

Choose j , the smallest integer for which

$$|a_{jk}^{(k)}| = \max |a_{ik}^{(k)}|, \quad k \leq i \leq n$$

And interchange rows k and j .

Complete Pivoting

We search the matrix A for the largest element in magnitude and bring it as the first pivot. This requires not only an interchange of equations but also an interchange of the position of the variables. This leads us to the following algorithm to find the pivot.

Choose l and m as the smallest integers for which

$$|a_{lm}^{(k)}| = \max |a_{ij}^{(k)}|, \quad k \leq i, j \leq n$$

And the interchange rows k and l and columns k and m .

If the matrix A is diagonally dominant or real, symmetric and positive definite, then no pivoting is necessary.

Example 2:

Solve the equations

$$\begin{aligned}x_1 + x_2 + x_3 &= 6 \\3x_1 + 3x_2 + 4x_3 &= 20 \\2x_1 + x_2 + 3x_3 &= 13\end{aligned}$$

using the Gauss elimination method.

In the first step we eliminate x_1 from the last two equations and obtain

$$\begin{aligned}x_1 + x_2 + x_3 &= 6 \\x_3 &= 2 \\-x_2 + x_3 &= 1\end{aligned}$$

Here, the pivot in the second equation is zero and so we cannot proceed as usual. We interchange the equations 2 and 3 before the second step. We obtain the upper triangular system

$$\begin{aligned}x_1 + x_2 + x_3 &= 6 \\-x_2 + x_3 &= 1 \\x_3 &= 2\end{aligned}$$

which has the solution

$$x_1 = 3, x_2 = 1 \text{ and } x_3 = 2.$$

2.5 GAUSS-JORDAN ELIMINATION METHOD

Here, the coefficient matrix is reduced to a diagonal matrix rather than a triangular matrix. At all steps of the Gauss elimination method, the elimination is done not only in the equations below but also the equations above the pivots, producing the solution without using the back substitution method. On the completion of the Gauss-Jordan method the equation become

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

The solution is given by

$$x_i = d_i, i = 1, 2, \dots, n,$$

Hence, the Gauss-Jordan method gives

$$[A|b] \xrightarrow[\text{Jordan}]{\text{Gauss}} [I|d].$$

Generally, this method is not used for the solution of a system of equations as it is more expensive from the computation view-point than the Gauss-elimination method. However, it gives a simple

method to find the inverse of a given matrix A. We start with the augmented matrix of A with the identity matrix I of the same order. When the Gauss-Jordan procedure is completed we obtain

$$[A|I] \xrightarrow[\text{Jordan}]{\text{Gauss}} [I|A^{-1}].$$

Example 3:

Find the inverse of the coefficient matrix of the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix}$$

By the Gauss-Jordan method with partial pivoting and hence solve the system. Using the augmented matrix $[A|I]$ we obtain

$$\begin{aligned} & \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 4 & 3 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right] \\ & \sim \left[\begin{array}{ccc|ccc} 1 & 3/4 & -1/4 & 0 & 1/4 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 3/4 & -1/4 & 0 & 1/4 & 0 \\ 0 & 1/4 & 5/4 & 1 & -1/4 & 0 \\ 0 & 11/4 & 15/4 & 0 & -3/4 & 1 \end{array} \right] \\ & \sim \left[\begin{array}{ccc|ccc} 1 & 3/4 & -1/4 & 0 & 1/4 & 0 \\ 0 & 11/4 & 15/4 & 1 & -3/4 & 1 \\ 0 & 1/4 & 5/4 & 1 & -1/4 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 3/4 & -1/4 & 0 & 1/4 & 0 \\ 0 & 1 & 15/11 & 0 & -3/11 & 4/11 \\ 0 & 1/4 & 15/4 & 1 & -1/4 & 0 \end{array} \right] \\ & \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -14/11 & 0 & 5/11 & -3 \\ 0 & 1 & 15/11 & 0 & -3/11 & 4/11 \\ 0 & 0 & 10/11 & 1 & -2/11 & -1/11 \end{array} \right] \\ & \sim \left[\begin{array}{ccc|ccc} 1 & 0 & -14/11 & 0 & 5/11 & -3/11 \\ 0 & 1 & 15/11 & 0 & -3/11 & 4/11 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array} \right] \\ & \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/5 & 1/5 & -2/5 \\ 0 & 1 & 0 & -3/5 & 0 & 1/2 \\ 0 & 0 & 1 & 11/10 & -1/5 & -1/10 \end{array} \right] \end{aligned}$$

Therefore, the solution of the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7/5 & 1/5 & -2/5 \\ -3/5 & 0 & 1/2 \\ 11/10 & -1/5 & -1/10 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ -1/2 \end{bmatrix}$$

2.6 CALCULATION OF A MATRIX

Let A be an $n \times n$ nonsingular matrix

$$\text{Let } X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

be the inverse of A .

$\therefore AX=I$ where I is the unit matrix of order n .

$\therefore AX=I$ gives

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

This equation is equivalent to the following n system of simultaneous equations

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}$$

.....

.....

and

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

Each of the system of the above n systems of equations can be solved by Gauss elimination method or Gauss Jordan method.

2.7 CROUT'S METHOD

This method is also known as the decomposition method or the factorization method. In this method the coefficient matrix A of the system of equations is decomposed or factorized into the product of a lower triangular matrix L and an upper triangular matrix U . We write the matrix A as

$$A = LU$$

Where

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix}, \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{bmatrix}$$

Using the matrix multiplication rule to multiply the matrices L and U and comparing the elements of the resulting matrix with those of A we obtain

$$l_{i1}u_{1j} + l_{i2}u_{2j} + \cdots + l_{in}u_{nj} = a_{ij}, j = 1(1)n$$

Where

$$l_{ij} = 0, j > i \text{ and } u_{ij} = 0, i > j.$$

The system of equations involves $n^2 + n$ unknowns. Thus, there are n parameters family of solutions. To produce a unique solution it is convenient to choose either $u_{ii} = 1$ or $l_{ii} = 1, i = 1(1)n$. When we choose

- (a) $l_{ii} = 1$, the method is called the Doolittle's method.
- (b) $u_{ii} = 1$, the method is called the Crout's Methods.

When we take $u_{ii} = 1, i = 1(1)n$, the solution of the equation may be written as

$$l_{ij} = a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}, i \geq j$$

$$u_{ij} = (a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj}) / l_{ii}, i < j$$

$$u_{ii} = 1.$$

We note that the first column of the matrix L is identical with the first column of the matrix A . That is

$$l_{i1} = a_{i1}, \quad i = 1(1)n.$$

We also note that

$$u_{1j} = a_{1j} / l_{11}, j = 2(1)n.$$

The first column of L and the first row of U have been determined. We can now proceed to determine the second column of L , and the second row of U

$$l_{i2} = a_{i2} - l_{i1}u_{12}, i = 2(1)n$$

$$u_{2j} = (a_{2j} - l_{21}u_{1j})/l_{22}, j = 3(1)n.$$

Next, we find the third column of L followed by the third row of U. Thus, for the relevant indices i and j , the elements are computed in the order

$$l_{i1}, u_{1j}; l_{i2}, u_{2j}; l_{i3}, u_{3j}; \dots; l_{i, n-1}, u_{n-1, j}; l_{nn}.$$

Having determined the matrices L and U, the system of equations becomes

$$L U x = b$$

We write equation as the following two systems of the equations

$$U x = z$$

$$L z = b$$

The unknowns z_1, z_2, \dots, z_n in the equations are determined by forward substitution and the unknowns x_1, x_2, \dots, x_n in equation are obtained by back substitution. Alternatively we find L^{-1} and U^{-1} to get

$$z = L^{-1}b \quad \text{and} \quad x = U^{-1}z.$$

The inverse of A can also be determined from

$$A^{-1} = U^{-1}L^{-1}.$$

This method fails if any of the diagonal elements l_{ii} or u_{ii} is zero. The LU decomposition is guaranteed when the matrix A is positive definite. However, it is only a sufficient condition.

Check Your Progress

Solve the system of equations

$$\begin{bmatrix} 2 & 1 & 1 & -2 \\ 4 & 0 & 2 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -10 \\ 8 \\ 7 \\ -5 \end{bmatrix}$$

using the Gauss elimination method with partial pivoting

2.8 LET US SUM UP

The unknowns are eliminated by combining equations such that the n equations in n unknowns are reduced to an equivalent upper triangular system which is then solved by back substitution method. In the first stage of elimination, the first column is searched for the largest element in magnitude and brought as the first pivot by interchanging the first equation with the equation having the largest element in magnitude. The coefficient matrix is reduced to a diagonal matrix rather than a triangular matrix. The elimination procedure described above to determine the elimination method.

2.9 KEYWORDS

Gauss Elimination Method: The unknowns are eliminated by combining equations.

Gauss-Jordan Elimination Method: The coefficient matrix is reduced to a diagonal matrix rather than a triangular matrix.

2.10 QUESTIONS FOR DISCUSSION

1. Consider the equations

$$x_1 + x_2 + x_3 = 1$$

$$4x_1 + 3x_2 - x_3 = 6$$

$$3x_1 + 5x_2 + 3x_3 = 4$$

Use the decomposition method to solve the system.

2. Find the inverse of the matrix

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}$$

using LU decomposition method. Take $u_{11} = u_{22} = u_{33} = 1$.

3. Solve the system of equations $Ax = b$, where

$$A = \begin{bmatrix} 2 & 1 & 1 & -2 \\ 4 & 0 & 2 & 1 \\ 3 & 2 & 2 & 0 \\ 1 & 3 & 2 & -1 \end{bmatrix}, b = \begin{bmatrix} -10 \\ 8 \\ 7 \\ -5 \end{bmatrix}$$

using the LU decomposition method. Take all the diagonal elements of L as 1. Also find A^{-1} .

4. Solve the following matrix operation:

$$(a) [9 \ 12 \ 5] * \begin{bmatrix} 2 \\ 8 \\ 3 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$(b) \begin{bmatrix} 9 & 7 \\ 3 & 6 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix} = \begin{bmatrix} & \\ & \\ & \end{bmatrix}$$

$$(c) \begin{bmatrix} 4 \\ 9 \\ 7 \end{bmatrix} * [2 \ 1 \ 8] = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$(d) \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix} * \begin{bmatrix} 8 & 7 \\ 9 & 6 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$(e) [4 \ 9 \ 8]^T * [2 \ 7 \ 1] = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$(f) \begin{bmatrix} 1 & 4 & 9 \\ 8 & 6 & 4 \end{bmatrix} * \begin{bmatrix} 3 & 0 \\ 9 & 3 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$(g) \left(\begin{bmatrix} 4 & 7 \\ 8 & 9 \end{bmatrix} + \begin{bmatrix} 5 & 3 \\ 0 & 1 \end{bmatrix} \right)^T * \begin{bmatrix} 10 & 2 & 7 \\ 6 & 3 & 5 \end{bmatrix} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

Check Your Progress: Model Answers

From the augmented matrix $[A|b]$, we obtain

$$[A|b] = \left[\begin{array}{cccc|c} 2 & 1 & 1 & -2 & -10 \\ 4 & 0 & 2 & 1 & 8 \\ 3 & 2 & 2 & 0 & 7 \\ 1 & 3 & 2 & -1 & -5 \end{array} \right] R_2 \sim R_1$$

$$\approx \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 2 & 1 & 1 & -2 & -10 \\ 3 & 2 & 2 & 0 & 7 \\ 1 & 3 & 2 & -1 & -5 \end{array} \right] \begin{array}{l} R_2 - \frac{1}{2}R_1 \\ R_3 - \frac{3}{4}R_1 \\ R_4 - \frac{1}{4}R_1 \end{array}$$

$$\approx \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 1 & 0 & -5/2 & -14 \\ 0 & 2 & 1/2 & -3/4 & 1 \\ 0 & 3 & 3/2 & -5/4 & -7 \end{array} \right] R_4 \sim R_2$$

$$\approx \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 3 & 3/2 & -5/4 & -7 \\ 0 & 2 & 1/2 & -3/4 & 1 \\ 0 & 1 & 0 & -5/2 & -14 \end{array} \right] \begin{array}{l} R_3 - \frac{2}{3}R_2 \\ R_4 - \frac{1}{3}R_2 \end{array}$$

$$\approx \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 3 & 3/2 & -5/4 & -7 \\ 0 & 0 & -1/2 & 1/12 & 17/3 \\ 0 & 0 & -1/2 & -25/12 & -35/3 \end{array} \right] R_4 - R_3$$

Contd...

$$\approx \left[\begin{array}{cccc|c} 4 & 0 & 2 & 1 & 8 \\ 0 & 3 & 3/2 & -5/4 & -7 \\ 0 & 0 & -1/2 & 1/12 & 17/3 \\ 0 & 0 & 0 & -13/6 & -52/3 \end{array} \right]$$

Using back substitution, we get the solution as

$$x_4 = 8, x_3 = -10, x_2 = 6, x_1 = 5.$$

2.11 SUGGESTED READINGS

S.S. Sastri., *Introductory Methods of Numerical Analysis*, Second Edition, Prentice Hall.

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Jain, M.K., Iyengar, S.R.K., Jain, R.K., *Numerical Methods for Scientific and Engineering Computation*, 3rd Edition, New Age International (P) Ltd., 1993.

LESSON

3

ITERATIVE METHODS

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3.0 AIMS AND OBJECTIVES

After studying this lesson, you will be able to:

- Explain Gauss elimination method
- Describe Gauss Seidal methods and their properties
- Discuss relaxation method

3.1 INTRODUCTION

In computational mathematics, an iterative method attempts to solve a problem (for example an equation or system of equations) by finding successive approximations to the solution starting from an initial guess. This approach is in contrast to direct methods, which attempt to solve the problem by a finite sequence of operations, and, in the absence of rounding errors, would deliver an exact solution (like solving a linear system of equations $Ax = b$ by Gaussian elimination). Iterative methods are usually the only choice for nonlinear equations. However, iterative methods are often useful even for linear problems involving a large number of variables (sometimes of the order of millions), where

direct methods would be prohibitively expensive and in some cases impossible even with the best available computing power.

Probably the first iterative method for solving a linear system appeared in a letter of Gauss to by one of his students. He proposed solving a 4-by-4 system of equations by repeatedly solving the component in which the residual was the largest.

The theory of stationary iterative methods was solidly established with the work of D.M. Young starting in the 1950s. The Conjugate Gradient method was also invented in the 1950s, with independent developments by Cornelius Lanczos, Magnus Hestenes and Eduard Stiefel, but its nature and applicability were misunderstood at the time. Only in the 1970s was it realized that conjugacy based methods work very well for partial differential equations, especially the elliptic type.

3.2 ITERATIVE METHODS

In the case of a system of linear equations, the two main classes of iterative methods are the stationary iterative methods, and the more general Krylov subspace methods.

3.2.1 Stationary Iterative Methods

Stationary iterative methods solve a linear system with an operator approximating the original one; and based on a measurement of the error (the residual), form a correction equation for which this process is repeated. While these methods are simple to derive, implement, and analyse, convergence is only guaranteed for a limited class of matrices. Examples of stationary iterative methods are the Jacobi method and the Gauss-Seidel method.

3.2.2 Krylov Subspace Methods

Krylov subspace methods form an orthogonal basis of the sequence of successive matrix powers times the initial residual (the Krylov sequence). The approximations to the solution are then formed by minimizing the residual over the subspace formed. The prototypical method in this class is the Conjugate Gradient Method (CG). Other methods are the Generalized Minimal Residual Method (GMRES) and the Biconjugate Gradient Method (BiCG).

Convergence

Since these methods form a basis, it is evident that the method converges in N iterations, where N is the system size. However, in the presence of rounding errors this statement does not hold; moreover, in practice N can be very large, and the iterative process reaches sufficient accuracy already far earlier. The analysis of these methods is hard, depending on a complicated function of the spectrum of the operator.

Preconditioners

The approximating operator that appears in stationary iterative methods can also be incorporated in Krylov subspace methods such as GMRES (alternatively, preconditioned Krylov methods can be considered as accelerations of stationary iterative methods), where they become transformations of the original operator to a presumably better conditioned one. The construction of preconditioners is a large research area.

3.3 GAUSS JACOBI ITERATION METHOD

All the previous methods seen in solving the system of simultaneous algebraic linear equations are direct methods. Now we will see some indirect methods or iterative methods.

This iterative method is not always successful to all systems of equations. If this method is to succeed, each equation of the system must possess one large coefficient and the large coefficient must be attached to a different unknown in that equation. This condition will be satisfied if the large coefficients are along the leading diagonal of the coefficient matrix. When this condition is satisfied, the system will be solvable by the iterative method.

The system,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

will be solvable by this method if

$$\begin{aligned} |a_{11}| &> |a_{12}| + |a_{13}| \\ |a_{22}| &> |a_{21}| + |a_{23}| \\ |a_{33}| &> |a_{31}| + |a_{32}| \end{aligned}$$

In other words, the solution will exist (iteration will converge) if the absolute values of the leading diagonal elements of the coefficient matrix A of the system $AX=B$ are greater than the sum of the absolute values of the other coefficients of that row. The condition is sufficient but not necessary.

Let us explain this method in the case of three equations in three unknowns.

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

.....(1)

Let us assume

$$\begin{aligned} |a_1| &> |b_1| + |c_1| \\ |b_2| &> |a_2| + |c_2| \\ |c_3| &> |a_3| + |b_3| \end{aligned}$$

Then, iterative method can be used for the system (1). Solve for x, y, z (whose coefficients are the larger values) in terms of the other variables. That is,

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z)$$

$$y = \frac{1}{b_2}(d_2 - a_2x - c_2z) \quad \dots(2)$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$$

If $x^{(0)}, y^{(0)}, z^{(0)}$ are the initial values of x, y, z respectively, then

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)})$$

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2x^{(0)} - c_2z^{(0)}) \quad \dots(3)$$

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3x^{(0)} - b_3y^{(0)})$$

Again using these values in (2), we get

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1y^{(1)} - c_1z^{(1)})$$

$$y^{(2)} = \frac{1}{b_2}(d_2 - a_2x^{(1)} - b_3y^{(1)}) - b_2y^{(1)} \quad \dots(4)$$

$$z^{(2)} = \frac{1}{c_3}(d_3 - a_3x^{(1)} - b_3y^{(1)})$$

Proceeding in the same way, if r th iterates are, the iteration scheme reduces to

$$x^{(2)} = \frac{1}{a_1}(d_1 - b_1y^{(1)} - c_1z^{(1)})$$

$$y^{(2)} = \frac{1}{b_2}(d_2 - a_2x^{(1)} - b_2y^{(1)}) \quad \dots(5)$$

$$z^{(2)} = \frac{1}{c_3}(d_3 - a_3x^{(1)} - b_3y^{(1)})$$

The procedure is continued till the convergence is assured (correct to required decimals).

Note:

1. To get the $(r+1)$ the iterates, we use the value of the r th iterates in the scheme (5)
2. In the absence of the initial values of x, y, z we take, usually, $(0,0,0)$ as the initial estimate.

3.4 GAUSS-SEIDEL ITERATION METHOD

This is only a refinement of Gauss-Jacobi method. As before,

$$x = \frac{1}{a_1}(d_1 - b_1y - c_1z)$$

$$y = \frac{1}{b_2}(d_2 - a_2x - c_2z) \quad \dots(6)$$

$$z = \frac{1}{c_3}(d_3 - a_3x - b_3y)$$

We start with the initial values for y and z and get from the first equation. That is,

$$x^{(1)} = \frac{1}{a_1}(d_1 - b_1y^{(0)} - c_1z^{(0)})$$

While using the second equation, we use for z and for x instead of as in the Jacobi's method, we get

$$y^{(1)} = \frac{1}{b_2}(d_2 - a_2x^{(1)} - c_2z^{(0)})$$

Now having known $x^{(1)}$ and $y^{(1)}$ use $x^{(1)}$ for x and $y^{(1)}$ for y in the third equation, we get

$$z^{(1)} = \frac{1}{c_3}(d_3 - a_3x^{(1)} - b_3y^{(1)})$$

In finding the values of the unknowns, we use the latest available values on the right hand side. If $x^{(r)}$, $y^{(r)}$, $z^{(r)}$ are the r^{th} iterates, then the iteration scheme will be

$$\begin{aligned} x^{(r+1)} &= \frac{1}{a_1}(d_1 - b_1y^{(r)} - c_1z^{(r)}) \\ y^{(r+1)} &= \frac{1}{b_2}(d_2 - a_2x^{(r+1)} - c_2z^{(r)}) \\ z^{(r+1)} &= \frac{1}{c_3}(d_3 - a_3x^{(r+1)} - b_3y^{(r+1)}) \end{aligned}$$

This process of iteration is continued until the convergence is assured. As the current values of the unknowns at each stage of iteration are used in getting the values of unknowns, the convergence in Gauss-Seidel method is very fast when compared to Gauss-Jacobi method. The rate of convergence in Gauss-Seidel method is roughly two times than that of Gauss-Jacobi method. As we saw the sufficient conditions already, the sufficient condition for the convergence of this method is also the same as we stated earlier. That is, the method of iteration will converge if in each equation of the given system, the absolute value of the largest coefficient is greater than the sum of the absolute values of all the remaining coefficients.

Remarks: The largest coefficients must be the coefficients for different unknowns.

Note:

1. For all system of equations, this method will not work (since convergence is not assured). It converges only for special systems of equations.
2. Iteration method is self correcting method. That is, any error made in computation, is corrected in the subsequent iterations.
3. The iteration is stopped when the values of x, y, z start repeating with the required degree of accuracy.

Examples

1. Solve the following system by Gauss-Jacobi and Gauss-Siedel methods

$$10x - 5y - 2z = 3$$

$$4x - 10y + 3z = -3$$

$$x + 6y + 10z = -3$$

Solution:

Here, we see that the diagonal elements are dominant. Hence, the iteration process can be applied.

That is, the coefficient matrix $\begin{pmatrix} 10 & -5 & -2 \\ 4 & -10 & 3 \\ 1 & 6 & 10 \end{pmatrix}$ is diagonally dominant, since

$$|10| > |5| + |-2|, \quad |-10| > |4| + |3| \quad \text{and} \quad |10| > |1| + |6|$$

Gauss Jacobi Method

Solving for x, y, z we have

$$x = \frac{1}{10}(3 + 5y + 2z) \quad \dots(1)$$

$$y = \frac{1}{10}(3 + 4x + 3z) \quad \dots(2)$$

$$z = \frac{1}{10}(-3 - x - 6y) \quad \dots(3)$$

First iteration: Let the initial values be (0,0,0).

Using these initial values in (1), (2), (3) we get

$$x^{(1)} = \frac{1}{10}[3 + 5(0) + 2(0)] = 0.3$$

$$y^{(1)} = \frac{1}{10}[3 + 4(0) + 3(0)] = 0.3$$

$$z^{(1)} = \frac{1}{10}[-3 - (0) - 6(0)] = -0.3$$

Second iteration: Using these values in (1), (2), (3), we get

$$x^{(2)} = \frac{1}{10}[3 + 5(0.3) + 2(-0.3)] = 0.39$$

$$y^{(2)} = \frac{1}{10}[3 + 4(0.3) + 3(-0.3)] = 0.33$$

$$z^{(2)} = \frac{1}{10}[-3 - (0.3) - 6(0.3)] = -0.51$$

Third iteration: Using the values of $x^{(2)}, y^{(2)}, z^{(2)}$ in (1), (2), (3) we, get

$$x^{(3)} = \frac{1}{10} [3 + 5(0.33) + 2(-0.51)] = 0.363$$

$$y^{(3)} = \frac{1}{10} [3 + 4(0.39) + 3(-0.51)] = 0.303$$

$$z^{(3)} = \frac{1}{10} [-3 - (0.39) - 6(0.33)] = -0.537$$

Fourth iteration:

$$x^{(4)} = \frac{1}{10} [3 + 5(0.303) + 2(-0.537)] = 0.3441$$

$$y^{(4)} = \frac{1}{10} [3 + 4(0.363) + 3(-0.537)] = 0.2841$$

$$z^{(4)} = \frac{1}{10} [-3 - 0.363 - 6(0.3030)] = -0.5181$$

Fifth iteration:

$$x^{(5)} = \frac{1}{10} [3 + 5(0.2841) + 2(-0.5181)] = 0.33843$$

$$y^{(5)} = \frac{1}{10} [3 + 4(0.3441) + 3(-0.5181)] = 0.2822$$

$$z^{(5)} = \frac{1}{10} [-3 - 0.3441 - 6(0.2841)] = -0.50487$$

Sixth iteration:

$$x^{(6)} = \frac{1}{10} [3 + 5(0.2822) + 2(-0.50487)] = 0.340126$$

$$y^{(6)} = \frac{1}{10} [3 + 4(0.33843) + 3(-0.50487)] = 0.283911$$

$$z^{(6)} = \frac{1}{10} [-3 - (0.33843) - 6(0.2822)] = -0.503163$$

Seventh iteration:

$$x^{(7)} = \frac{1}{10} [3 + 5(0.283911) + 2(-0.503163)] = 0.3413229$$

$$y^{(7)} = \frac{1}{10} [3 + 4(0.340126) + 3(-0.503163)] = 0.2851015$$

$$z^{(7)} = \frac{1}{10} [-3 - (0.340126) - 6(0.283911)] = -0.5043592$$

Eighth iteration:

$$x^{(8)} = \frac{1}{10} [3 + 5(0.2851015) + 2(-0.5043592)] = 0.34167891$$

$$y^{(8)} = \frac{1}{10} [3 + 4(0.3413229) + 3(-0.5043592)] = 0.2852214$$

$$z^{(8)} = \frac{1}{10} [-3 - (0.3413229) - 6(0.2851015)] = -0.50519319$$

Ninth iteration:

$$x^{(9)} = \frac{1}{10} [3 + 5(0.2852214) + 2(-0.50519319)] = 0.341572062$$

$$y^{(9)} = \frac{1}{10} [3 + 4(0.34167891) + 3(-0.50519311)] = 0.285113607$$

$$z^{(9)} = \frac{1}{10} [-3 - (0.34167891) - 6(0.285221)] = -0.505300731$$

Hence, correct to 3 decimal places, the values are

$$x = 0.342, y = 0.285, z = -0.505$$

Gauss Seidel Method

Initial values; $y = 0, z = 0$

First iteration:

$$x^{(1)} = \frac{1}{10} [3 + 5(0) + 2(0)] = 0.3$$

$$y^{(1)} = \frac{1}{10} [3 + 4(0.3) + 3(-0)] = 0.42$$

$$z^{(1)} = \frac{1}{10} [-3 - (0.3) - 6(0.42)] = -0.582$$

Second iteration:

$$x^{(2)} = \frac{1}{10} [3 + 5(0.42) + 2(-0.582)] = 0.3936$$

$$y^{(2)} = \frac{1}{10} [3 + 4(0.3936) + 3(-0.582)] = 0.28284$$

$$z^{(2)} = \frac{1}{10} [-3 - (0.3936) - 6(0.28284)] = -0.509064$$

Third iteration:

$$x^{(3)} = \frac{1}{10} [3 + 5(0.28284) + 2(-0.509064)] = 0.3396072$$

$$y^{(3)} = \frac{1}{10} [3 + 4(0.3396072) + 3(-0.509064)] = 0.28312368$$

$$z^{(3)} = \frac{1}{10} [-3 - (0.3396072) - 6(0.2812368)] = -0.503834928$$

Fourth iteration:

$$x^{(4)} = \frac{1}{10} [3 + 5(0.28312368) + 2(-0.503834928)] = 0.34079485$$

$$y^{(4)} = \frac{1}{10} [3 + 4(0.34079485) + 3(-0.50383492)] = 0.28516464$$

$$z^{(4)} = \frac{1}{10} [-3 - (0.34079485) - 6(0.28516746)] = -0.50517996$$

Fifth iteration:

$$x^{(5)} = \frac{1}{10} [3 + 5(0.28516746) + 2(-0.50517996)] = 0.34155477$$

$$y^{(5)} = \frac{1}{10} [3 + 4(0.34155477) + 3(-0.50517966)] = 0.28506792$$

$$z^{(5)} = \frac{1}{10} [-3 - (0.34155477) - 6(0.28506792)] = -0.505196229$$

Sixth iteration:

$$x^{(6)} = \frac{1}{10} [3 + 5(0.28506792) + 2(-0.505196229)] = 0.341494714$$

$$y^{(6)} = \frac{1}{10} [3 + 4(0.341494714) + 3(-0.505196229)] = 0.285039017$$

$$z^{(6)} = \frac{1}{10} [-3 - (0.341494714) - 6(0.285039017)] = -0.5051728$$

Seventh iteration:

$$x^{(7)} = \frac{1}{10} [3 + 5(0.285039017) + 2(-0.5051728)] = 0.3414849$$

$$y^{(7)} = \frac{1}{10} [3 + 4(0.3414849) + 3(-0.5051728)] = 0.28504212$$

$$z^{(7)} = \frac{1}{10} [-3 - (0.3414849) - 6(0.28504212)] = -0.5051737$$

The values at each iteration by both methods are tabulated below:

Iteration	Gauss Jacobi method			Gauss - Seidel method		
	x	y	z	x	y	z
1	0.3	0.3	-0.3	0.3	0.42	-0.582
2	0.39	0.33	-0.51	0.3936	0.28284	-0.509064
3	0.363	0.303	-0.537	0.3396072	0.28312364	-0.503834928
4	0.3441	0.2841	-0.5181	0.34079485	0.28516746	-0.50517966
5	0.33843	0.2822	-0.50487	0.34515547	0.28506792	-0.505196229
6	0.340126	0.283911	-0.503163	0.3414947	0.2850390	-0.5051728
7	0.3413229	0.2851015	-0.5043592	0.3414849	0.28504212	-0.5051737
8	0.34167891	0.2852214	-0.50519319			
9	0.341572062	0.285113607	-0.505300731			

The values correct to three decimal places are

$$x = 0.342, y = 0.285, z = -0.505$$

Note: After getting the values of the unknowns, substitute these values in the given equations, and check the correctness of the results.

2. Solve the following system of equations by Gauss Jacobi and Gauss Seidel method correct to three decimal places:

$$x + y + 54z = 110$$

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

Solution:

As the coefficient matrix is not diagonally dominant as it is we rewrite the equation as noted below, so that the coefficient matrix becomes diagonally dominant.

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

Solving for x, y, z we get

$$x = \frac{1}{27}(85 - 6y + z) \quad \dots(1)$$

$$y = \frac{1}{15}[72 - 6x - 2z] \quad \dots(2)$$

$$z = \frac{1}{54}[110 - x - y] \quad \dots(3)$$

Starting with the initial value $x = 0, y = 0, z = 0$ and using (1), (2), (3) and repeating the process we get the values of x, y, z as the tabulated by both methods. (Gauss-Jacobi and Gauss-Seidel)

Iteration	Gauss Jacobi method			Gauss-Seidel method		
	x	y	z	x	y	z
1	3.14815	4.8	2.03704	3.14815	3.54074	1.91317
2	2.15693	3.26913	1.88985	2.43218	3.57204	1.92585
3	2.49167	3.68525	1.93655	2.42569	3.57294	1.92595
4	2.40093	3.54513	1.92265	2.42549	3.57301	1.92595
5	2.43155	3.58327	1.92692	2.42548	3.57301	1.92595
6	2.42323	3.57046	1.92565	2.42548	3.57301	1.92595
7	2.42603	3.57395	1.92604			
8	2.42527	3.57278	1.92593			

Hence $x = 2.425, y = 3.573$ and $z = 1.926$ (Correct to 3 decimal places)

3.5 RELAXATION METHOD

This method is a generalization of the Gauss-Seidel Method. This method is often used when the coefficient matrix of the system is symmetric and has ‘property A’. We define an auxiliary vector \hat{x} as

$$\hat{x}^{(k+1)} = -D^{-1}Lx^{(k+1)} - D^{-1}Ux^{(k)} + D^{-1}b.$$

The final solution is now written as

$$x^{(k+1)} = x^{(k)} + w \left(\hat{x}^{(k+1)} - x^{(k)} \right)$$

$$x^{(k+1)} = (1-w)x^{(k)} + w\hat{x}^{(k+1)}$$

By substituting and simplifying we obtain

$$\begin{aligned} x^{(k+1)} &= (D+wL)^{-1} [(1-w)D - wU] x^{(k)} + w(D+wL)^{-1} b \\ &= Hx^{(k)} + c, \quad k = 0, 1, 2, \dots \end{aligned}$$

Where $H = (D+wL)^{-1} [(1-w)D - wU]$

And $c = w(D+wL)^{-1} b.$

Alternatively

$$\begin{aligned} x^{(k+1)} &= x^{(k)} - (D+wL)^{-1} [(D+wL) - (1-w)D + wU] x^{(k)} \\ &\quad + w(D+wL)^{-1} b \\ &= x^{(k)} + w(D+wL)^{-1} r^{(k)} \end{aligned}$$

Where $r^{(k)} = b - Ax^{(k)}$ is the residual

We may write

$$v^{(k)} = w(D+wL)^{-1} r^{(k)}$$

Or $(D+wL)v^{(k)} = wr^{(k)}$

This equation describes the relaxation method in its error format. For computational purpose, it is convenient to use this equation.

When $w=1$, equation reduces to the Gauss-Seidel method. The quantity w is called the relaxation parameter and $x^{(k+1)}$ is a weighted mean of $\hat{x}^{(k+1)}$ and $x^{(k)}$. From the equation, we find that the weights are non-negative for $0 \leq w \leq 1$. If $w > 1$, then the method is called an over relaxation method and if $w < 1$, then it is called an under relaxation method.

Check Your Progress

- If the system of equations are $8x - y + z = 18$, $2x + 5y - 2z = 3$, $x + y - 3z = -6$. The first iteration values by Gauss seidal method is
 - 2.25, -.3, 2.65
 - 2.5, -.3, 2.65
 - 2.65, .4, 2.5
 - 2.5, -.3, 2.0
- The values of x and z of first iteration by Gauss seidel method are $4x + 2y + z = 14$, $x + 5y - z = 10$, $x + y + 8z = 20$.
 - 3, 1.3, 1.9
 - 3.5, 1.3, 1.9
 - 4, 1.5, 2
 - 3.5, 2, 2
- If the equations are $4x + 2y + z = 14$, $x + 5y - z = 10$, $x + y + 8z = 20$ and the first iteration values are 3.5, 1.3 and 1.9 then the second iteration of x is
 - 3.12
 - 2.375
 - 2.99
 - 3.5
- If the system of equations are $5x - y + z = 10$, $2x + 4y = 12$ and $x + y + 5z = 1$, then the first iteration values by Gauss seidal method are
 - 2, 1, 1
 - 3, 2, 1
 - 2, 2, -1
 - 2, 2, 0

3.6 LET US SUM UP

Iterative methods provide an alternative to the direct methods for solving linear equations. These methods are particularly suitable for solving ill-conditioned systems. We considered the following three iterative methods:

- Jacobi method
- Gauss-Seidel method

We have shown that a sufficient condition for convergence is that, for each row, the absolute value of the diagonal element should be greater than the sum of absolute values of the other elements in the equation.

3.7 KEYWORDS

System of Equations:

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

Jacobi Method of Iteration:

$$x^{(r+1)} = \frac{1}{a_1} (d_1 - b_1y^{(r)} - c_1z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2} (d_2 - a_2x^{(r)} - c_2z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3} (d_3 - a_3x^{(r)} - b_3y^{(r)})$$

$$\text{Gauss-Jacobi Method: } x^{(r+1)} = \frac{1}{a_1} (d_1 - b_1 y^{(r)} - c_1 z^{(r)})$$

$$y^{(r+1)} = \frac{1}{b_2} (d_2 - a_2 x^{(r+1)} - c_2 z^{(r)})$$

$$z^{(r+1)} = \frac{1}{c_3} (d_3 - a_3 x^{(r+1)} - b_3 y^{(r+1)})$$

3.8 QUESTIONS FOR DISCUSSION

Solve the following system of equations using (i) Jacobin's and (ii) Gauss-seidal's methods:

1. $8x - y + z - 18 = 0$
 $2x + 5y - 2z - 3 = 0$
 $x + y - 3z + 6 = 0$
2. $9x_1 + 2x_2 + 4x_3 = 20$
 $x_1 + 10x_2 + 4x_3 = 6$
 $2x_1 - 4x_2 + 10x_3 = 15$
3. $1.02x_1 - 0.05x_2 - 0.10x_3 = 0.795$
 $-0.11x_1 + 1.03x_2 - 0.05x_3 = 0.849$
 $-0.11x_1 - 0.12x_2 + 1.04x_3 = 1.398$
4. $8x + y + z = 8$
 $2x + 4y + z = 4$
 $x + 3y + 5z = 5$
5. $3x + 4y + 15z = 54.8$
 $x + 12y + 3z = 39.66$
 $10x + y - 2z = 7.74$

Check Your Progress: Model Answers
1. (a) 2. (b) 3. (b) 4. (c)

3.9 SUGGESTED READINGS

S.S. Sastri., *Introductory Methods of Numerical Analysis*, Second Edition, Prentice Hall.

S. Mohan Naidu., *A Text Book of Mathematical Methods for Scientific Engineers*, Students Helpline Series, Hyd -2005.

Jain, M.K., Iyengar, S.R.K., Jain, R.K., *Numerical Methods for Scientific and Engineering Computation*, 3rd Edition, New Age International (P) Ltd., 1993.

UNIT III

LESSON

4

FINITE DIFFERENCES

CONTENTS

- 4.0 Aims and Objectives
- 4.1 Introduction
- 4.2 Eigen Values and Eigen Vectors of Matrix
- 4.3 Finite Differences
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- 4.5 Other Difference Operators
- 4.6 Error Propagation in Difference Table
- 4.7 Summation of Series
- 4.8 Let us Sum up
- 4.9 Keyword
- 4.10 Questions for Discussion
- 4.11 Suggested Readings

4.0 AIMS AND OBJECTIVES

After studying this lesson, you will be able to:

- Understand Eigen values and Eigen vectors of a matrix
- Explain finite differences
- Discuss difference operators
- Describe other difference operator

4.1 INTRODUCTION

In this lesson we introduce the idea of finite differences and associated concepts, which have important applications in numerical analysis.

For example Interpolation formulae are based in finite differences. Through finite differences we study the relation that exists between the values change by finite jumps.

In this lesson we study the variations when the independent variables change by equal intervals.

4.2 EIGEN VALUES AND EIGEN VECTORS OF MATRIX

The eigenvalue problem is a problem of considerable theoretical interest and wide-ranging application. For example, this problem is crucial in solving systems of differential equations, analyzing population growth models, and calculating powers of matrices (in order to define the exponential matrix). Other areas such as physics, sociology, biology, economics and statistics have focused considerable attention on "eigenvalues" and "eigenvectors" their applications and their computations. Before we give the formal definition, let us introduce these concepts on an example.

Example 1:

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}$$

Consider the three column matrices

$$C_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$$

We have

$$AC_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad AC_2 = \begin{pmatrix} 4 \\ -8 \\ -4 \end{pmatrix}, \quad AC_3 = \begin{pmatrix} 6 \\ 9 \\ -6 \end{pmatrix}$$

In other words, we have

$$AC_1 = 0C_1, \quad AC_2 = -4C_2, \quad \text{and} \quad AC_3 = 3C_3.$$

Next consider the matrix P for which the columns are C_1 , C_2 , and C_3 , i.e.,

$$P = \begin{pmatrix} 1 & -1 & 2 \\ 6 & 2 & 3 \\ -13 & 1 & -2 \end{pmatrix}$$

We have $\det(P) = 84$. So this matrix is invertible. Easy calculations give

$$P^{-1} = \frac{1}{84} \begin{pmatrix} -7 & 0 & -7 \\ -27 & 24 & 9 \\ 32 & 12 & 8 \end{pmatrix}$$

Next we evaluate the matrix $P^{-1}AP$. We leave the details to the reader to check that we have

$$\frac{1}{84} \begin{pmatrix} -7 & 0 & -7 \\ -27 & 24 & 9 \\ 32 & 12 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 6 & 2 & 3 \\ -13 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

In other words, we have

$$P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Using the matrix multiplication, we obtain

$$A = P \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 3 \end{pmatrix} P^{-1}$$

which implies that A is similar to a diagonal matrix. In particular, we have

$$A^n = P \begin{pmatrix} 0 & 0 & 0 \\ 0 & (-4)^n & 0 \\ 0 & 0 & 3^n \end{pmatrix} P^{-1}$$

for $n = 1, 2, \dots$

Note that it is almost impossible to find A^{75} directly from the original form of A .

This example is so rich of conclusions that many questions impose themselves in a natural way. For example, given a square matrix A , how do we find column matrices which have similar behaviors as the above ones? In other words, how do we find these column matrices which will help find the invertible matrix P such that $P^{-1}AP$ is a diagonal matrix?

From now on, we will call column matrices **vectors**. So the above column matrices C_1 , C_2 , and C_3 are now vectors. We have the following definition.

Definition

Let A be a square matrix. A non-zero vector C is called an **eigenvector** of A if and only if there exists a number (real or complex) λ such that

$$AC = \lambda C.$$

If such a number λ exists, it is called an **eigenvalue** of A . The vector C is called eigenvector associated to the eigenvalue λ .

Remark

The eigenvector C must be non-zero since we have

$$A0 = 0 = \lambda 0$$

for any number λ .

Example 2:

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}.$$

We have seen that

$$AC_1 = 0C_1, AC_2 = -4C_2, \text{ and } AC_3 = 3C_3$$

where

$$C_1 = \begin{pmatrix} 1 \\ 6 \\ -13 \end{pmatrix}, C_2 = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, C_3 = \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}.$$

So C_1 is an eigenvector of A associated to the eigenvalue 0. C_2 is an eigenvector of A associated to the eigenvalue -4 while C_3 is an eigenvector of A associated to the eigenvalue 3.

It may be interesting to know whether we found all the eigenvalues of A in the above example. In the next page, we will discuss this question as well as how to find the eigenvalues of a square matrix.

4.3 FINITE DIFFERENCES

Let $y=f(x)$ be a discrete function. If $x_0, x_0 + h, x_0 + 2h, \dots, x_0 + nh$ are the successive values of x , where two consecutive values differ by a quantity h , then the corresponding values of y are $y_0, y_1, y_2, \dots, y_n$. The values of the independent variable x is usually called the argument and the corresponding functional value is known as the entry. The arguments and entries can be shown in a tabular form as follows:

Argument	x_0	x_1	x_2	...	$x_n = x_0 + nh$
	$x = x_0 + h = x_0 + 2h$				
Entry	y_0	y_1	y_2	$y_n = f(x_0 + nh)$	
	$y = f(x) = f(x_0) = f(x_0 + h) = f(x_0 + 2h)$				

To determine the value of or etc., for some intermediate arguments, the following three types of differences are found useful:

- (a) Forward differences
 - (b) Backward differences and
 - (c) Central differences
- (a) **Forward differences:** If we subtract from each value of y (except y_0) the preceding value of y , we get $y_1 - y_0, y_2 - y_1, \dots, y_n - y_{n-1}$ respectively, known as the first differences of y . These results which may be denoted $\Delta y_0, \Delta y_1, \dots, \Delta y_n$.

i.e.,
$$\Delta y_0 = y_1 - y_0, \Delta y_1 = y_2 - y_1, \dots, \Delta y_{n-1} = y_n - y_{n-1}$$

Where Δ is a symbol representing an operation of forward difference, are called first forward differences. Thus, the first forward differences are given by

$$\Delta y_i = y_{i+1} - y_i; i = 0, 1, 2, \dots$$

Now, the second forward differences are defined as the differences of the first differences, that is,

$$\begin{aligned} \Delta^2 y_0 &= \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0 \\ &= (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0 \end{aligned}$$

$$\Delta^2 y_1 = \Delta(\Delta y_1) = \Delta y_2 - \Delta y_1 = y_3 - 2y_2 + y_1$$

... ..

$$\Delta^2 y_n = \Delta y_{n+1} - \Delta y_n = y_{n+2} - 2y_{n+1} + y_n$$

Here, Δ^2 is called second forward difference operator.

Similarly, the third forward differences are:

$$\begin{aligned} \Delta^3 y_0 &= \Delta(\Delta^2 y_0) = \Delta^2 y_1 - \Delta^2 y_0 = \Delta(\Delta y_1) - \Delta(\Delta y_0) \\ &= \Delta(y_2 - y_1) - \Delta(y_1 - y_0) = \Delta y_2 - 2\Delta y_1 + \Delta y_0 \\ &= (y_3 - y_2) - 2(y_2 - y_1) + y_1 - y_0 \\ &= y_3 - 3y_2 + 3y_1 - y_0 \end{aligned}$$

$$\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1 = y_4 - 3y_3 + 3y_2 - y_1$$

... ..

$$\Delta^3 y_n = \Delta^2 y_{n+1} - \Delta^2 y_n = y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n$$

In general, the nth differences are defined as

$$\Delta^n y_k = \Delta^{n-1} y_{k+1} - \Delta^{n-1} y_k$$

in function notation, the forward differences are as written below:

$$\Delta f(x) = f(x+h) - f(x)$$

$$\Delta^2 f(x) = f(x+2h) - 2f(x+h) + f(x)$$

$$\Delta^3 f(x) = f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$$

and so on, where h is the interval of differencing.

The forward differences are usually arranged in a tabular form in the following manner:

x	y = f(x)	1st	2nd	3rd	4th	5th
argument	entry	diff.	differences	diff.	diff.	differences
x_0	$y_0 = f(x_0)$					
		Δy_0				
$x_1 = x_0 + h$	$y_1 = f(x_1)$		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_2 = x_0 + 2h$	$y_2 = f(x_2)$		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_3 = x_0 + 3h$	$y_3 = f(x_3)$		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_4 = x_0 + 4h$	$y_4 = f(x_4)$		$\Delta^2 y_3$			
		Δy_4				
$x_5 = x_0 + 5h$	$y_5 = f(x_5)$					

The first term in the table y_0 is called the leading term and the differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \dots$ are called leading differences. It can be seen that the differences $\Delta^k y_i$ with a subscript 'i' lie along the diagonal sloping downwards; that is, forward with respect to the direction of x . The above difference table is known as Forward difference table or Diagonal difference table.

Properties of Δ

The operator Δ satisfies the following properties:

- (i) $\Delta[f(x) \pm g(x)] = \Delta f(x) \pm \Delta g(x)$, i.e., Δ is linear
- (ii) $\Delta[\alpha f(x)] = \alpha \Delta f(x)$, α being a constant
- (iii) $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x) = \Delta^n \Delta^m f(x)$, where m and n are positive integers.
- (iv) $\Delta[f(x).g(x)] \neq f(x).\Delta g(x)$

Observation 1: We can express any higher order forward difference of y_0 in terms of the entries $y_0, y_1, y_2, \dots, y_n$

$$\begin{aligned} \text{From } \Delta y_0 &= y_1 - y_0 \\ \Delta^2 y_0 &= y_2 - 2y_1 + y_0 \\ \Delta^3 y_0 &= y_3 - 3y_2 + 3y_1 - y_0 \end{aligned}$$

and so on, we can see that the coefficients of the entries on the RHS are binomial coefficients. Therefore, in general

$$\Delta^n y_0 = y_n - {}^n C_1 y_{n-1} + {}^n C_2 y_{n-2} - \dots - (-1)^n y_0$$

Observation 2: We can express any value of y in terms of leading entry y_0 .

We know that $y_1 = y_0 + \Delta y_0$

$$y_1 = y_0 + \Delta y_0 = (1 + \Delta)y_0$$

Now, $y_2 = y_1 + \Delta y_1 = (1 + \Delta)y_1 = (1 + \Delta)^2 y_0$

Similarly, $y_3 = (1 + \Delta)^3 y_0$ and so on. In general,

$$y_n = (1 + \Delta)^n y_0 = y_0 - {}^n C_1 \Delta y_0 + {}^n C_2 \Delta^2 y_0 + \dots + \Delta^n y_0$$

- (b) **Backward differences:** The differences $y_1 - y_0, y_2 - y_1, \dots, y_n - y_{n-1}$ when denoted by $\nabla y_1, \nabla y_2, \dots, \nabla y_n$, respectively, are called the first backward differences, where is the backward difference operator called nabla operator.

$$\therefore \nabla y_1 = y_1 - y_0, \nabla y_2 = y_2 - y_1, \dots, \nabla y_n = y_n - y_{n-1}$$

Now the second backward differences are defined as the differences of the first backward differences, i.e.,

$$\begin{aligned} \nabla^2 y_2 &= \nabla(\nabla y_2) = \nabla(y_2 - y_1) = \nabla y_2 - \nabla y_1 \\ &= (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0 \end{aligned}$$

$$\nabla^2 y_3 = \nabla y_3 - \nabla y_2 = y_3 - 2y_2 + y_1 \text{ and so on.}$$

In general,

$$\nabla^n y_k = \nabla^{n-1} y_k - \nabla^{n-1} y_{k-1}$$

In function notation, these are written as

$$\nabla f(x) = f(x) - f(x - h)$$

$$\nabla f(x + h) = f(x + h) - f(x)$$

$$\nabla^2 f(x + 2h) = f(x + 2h) - 2f(x + h) + f(x)$$

$$\nabla^3 f(x + 3h) = f(x + 3h) - 3f(x + 2h) + 3f(x + h) - f(x)$$

and so on, where h is the interval of differencing.

These backward differences are arranged in a tabular form in the following manner. In this table, the difference $\nabla^k y_i$ with a fixed subscript 'i' lies along the diagonal sloping upwards; that is, backwards with respect to the direction of increasing argument x.

x	y = f(x)	1st	2nd	3rd	4th	5th
argument	entry	diff.	differences	diff.	diff.	Differences

$$x_0 \qquad y_0$$

$$\nabla y_1$$

$$x_1 = x_0 + h \qquad y_1$$

$$\nabla^2 y_2$$

$$\nabla y_2$$

$$\nabla^3 y_3$$

$$x_2 = x_0 + 2h \qquad y_2$$

$$\nabla^2 y_3$$

$$\nabla^4 y_4$$

$$\nabla y_3$$

$$\nabla^3 y_4$$

$$\nabla^5 y_5$$

$$x_3 = x_0 + 3h \qquad y_3$$

$$\nabla^2 y_4$$

$$\nabla^4 y_5$$

$$\nabla y_4$$

$$\nabla^3 y_5$$

$$x_4 = x_0 + 4h \qquad y_4$$

$$\nabla^2 y_5$$

$$\nabla y_5$$

$$x_5 = x_0 + 5h \qquad y_5$$

Properties of ∇

(a) $\nabla[f(x) \pm g(x)] = \nabla f(x) \pm \nabla g(x)$, i.e., is linear

(b) $\nabla[\alpha f(x)] = \alpha \nabla f(x)$, being a constant

(c) $\nabla^m \nabla^n f(x) = \nabla^{m+n} f(x)$, where m and n are positive integers.

(d) $\nabla[f(x).g(x)] \neq [\nabla f(x)].g(x)$

Observation we can express any value of y in terms of y_n and the backward differences $\nabla y_n, \nabla^2 y_n,$ etc.

By def., $y_n - y_{n-1} = \nabla y_n$

or $y_{n-1} = y_n - \nabla y_n = (1 - \nabla)y_n$

Now, $y_{n-2} = y_{n-1} - \nabla y_{n-1} = (1 - \nabla)y_{n-1} = (1 - \nabla)^2 y_n$

Similarly, $y_{n-3} = (1 - \nabla)^3 y_n$ and so on.

In general, $y_{n-k} = (1 - \nabla)^k y_n$

$y_{n-k} = y_n -^k C_1 \nabla y_n +^k C_2 \nabla^2 y_n - \dots + (-1)^k \nabla^k y_n$

(c) **Central differences:** Sometimes, it is more convenient to employ another system of differences known as central differences. In this system the symbol δ is used instead of Δ and is known as central difference operator. The subscript of for any difference is the average of the subscripts of the two members of the difference.

$y_1 - y_0 = \delta y_{1/2}, y_2 - y_1 = \delta y_{3/2}, y_3 - y_2 = \delta y_{5/2}, \dots$

For higher order differences, we have

$\delta y_{3/2}, -\delta y_{1/2} = \delta^2 y_1, \delta y_{1/2}, -\delta^2 y_{1/2} = \delta^2 y_{1/2} = \delta^2 y_2, \dots, \delta^2 y_2 - \delta^2 y_1 = \delta^2 y_{3/2},$

and so on.

The central differences are tabulated below:

x	y = f(x)	1st	2nd	3rd	4th	5th
argument	entry	diff.	differences	diff.	diff.	differences
x_0	y_0					
		$\delta y_{1/2}$				
$x_1 = x_0 + h$	y_1		$\delta^2 y_1$			
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$		
$x_2 = x_0 + 2h$	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$
$x_3 = x_0 + 3h$	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		
$x_4 = x_0 + 4h$	y_4		$\delta^2 y_4$			
		$\delta y_{9/2}$				
$x_5 = x_0 + 5h$	y_5					

We can see from the table that central differences on the same horizontal line have the same suffix. Also, all odd differences have a fractional suffix, and the even differences have integer suffix.

Note:

1. From all the three tables, we can see that only the notation changes, not the differences. For examples,

$$y_1 - y_0 = \Delta y_0 = \Delta y_1 = \delta y_{\frac{1}{2}}$$

2. If we write $y = f(x)$ as $y = f_x$ or $y = y_x$ then the entries corresponding to $x, x+h, x+2h, \dots$ are $y_x, y_{x+h}, y_{x+2h}, \dots$ respectively, and

$$\Delta y_x = y_{x+h} - y_x, \quad \Delta^2 y_x = \Delta y_{x+h} - \Delta y_x \text{ so on.}$$

similarly, $\nabla y_x = y_x - y_{x-h}$,

$$\delta y_x = y_{x+\frac{1}{2}h} - y_{x-\frac{1}{2}h} \text{ and so on.}$$

4.4 DIFFERENCE OPERATORS

In this section we introduce three difference operators namely forward, backward and central difference operations. Consider the function $y = f(x)$. Suppose we are given table of values of the function at the points

$$x_0, x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \dots, x_n = x_0 + nh.$$

Let $f(x_0) = y_0, f(x_1) = y_1 \dots, f(x_n) = y_n$.

We define

$$\Delta[f(x)] = f(x+h) - f(x).$$

Thus $\Delta y_0 = f(x_0 + h) - f(x_0) = f(x_1) - f(x_0) = y_1 - y_0$.

Similarly $\Delta y_1 = y_2 - y_1$

... ..

$$\Delta y_{n-1} = y_n - y_{n-1}$$

Δ is called the forward difference operator and $\Delta y_0, \Delta y_1, \dots, \Delta y_{n-1}$ are called the first forward difference of the functions $y = f(x)$.

The second order differences of the function are defined by

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

... ..

$$\Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1}$$

In the similar manner higher order difference can be defined. In general the n th order differences are defined by the equations

$$\Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i.$$

These differences of the function $y = f(x)$ can be systematically represented in the form of the table called forward difference table. We can construct the difference table for any number of arguments and a sample difference table is given for six consecutive arguments.

Forward Differences Table

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
$x_1 = x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_2 = x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_3 = x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_4 = x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_5 = x_0 + 5h$	y_5					

Note: In this table y_0 is known as the first entry and $\Delta y_0, \Delta^2 y_0, \dots, \Delta^5 y_0$ are called leading differences.

Remark. Since each higher order difference is defined in the terms of the previous lower differences by continuous substitution each higher order difference can be expressed in terms of the values of the function.

Thus, $\Delta^2 y_0 = \Delta y_1 - \Delta y_0$

$$= (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$

$$= (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0) = y_3 - 3y_2 + 3y_1 - y_0.$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$$

$$= (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0)$$

$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0.$$

We observe that the coefficients occurring in the RHS are simply the binomial coefficients in $(1 - x)^n$. Hence in the general we have

$$\Delta^n y_0 = y_n - n_1 y_{n-1} + n_2 y_{n-2} + \dots + (-1)^n y_0$$

Properties of the Operator Δ

1. Δ is linear. i.e. $\Delta[af(x) + bg(x)] = a\Delta[f(x)] + b\Delta[g(x)]$ where a, b are constants.

Proof.
$$\begin{aligned}\Delta[af(x) + bg(x)] &= [af(x+h) + bg(x+h)] - [af(x) + bg(x)] \\ &= a[f(x+h) - f(x)] + b[g(x+h) - g(x)] \\ &= a\Delta[f(x)] + b\Delta[g(x)]\end{aligned}$$

2. $\Delta^m \Delta^n [f(x)] = \Delta^{m+n} [f(x)]$

Proof.
$$\begin{aligned}\Delta^m \Delta^n [f(x)] &= (\Delta \Delta \dots m \text{ times}) (\Delta \Delta \dots n \text{ times}) f(x) \\ &= [\Delta \Delta \dots (m+n) \text{ times}] f(x) \\ &= \Delta^{m+n} [f(x)].\end{aligned}$$

3. $\Delta[f(x)g(x)] = f(x+h)\Delta[g(x)] + g(x)\Delta[f(x)]$

Proof.
$$\begin{aligned}\Delta[f(x)g(x)] &= f(x+h)g(x+h) - f(x)g(x) \\ &= f(x+h)g(x+h) - f(x+h)g(x) + [f(x+h)g(x) - f(x)g(x)] \\ &= f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)] \\ &= f(x+h)\Delta[g(x)] + g(x)\Delta[f(x)]\end{aligned}$$

4. $\Delta \left[\frac{f(x)}{g(x)} \right] = \frac{g(x)\Delta[f(x)] - f(x)\Delta[g(x)]}{g(x+h)g(x)}$

Proof.
$$\begin{aligned}\Delta \left[\frac{f(x)}{g(x)} \right] &= \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} \\ &= \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \\ &= \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x+h)g(x)} \\ &= \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x+h)g(x)} \\ &= \frac{g(x)\Delta[f(x)] - f(x)\Delta[g(x)]}{g(x+h)g(x)}\end{aligned}$$

Backward Differences

Consider the function $y = f(x)$. Suppose we are given a table of values of the function at the points.

$$x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, \dots, x_n = x_0 + nh$$

Let $f(x_0) = y_0, f(x_1) = y_1, \dots, f(x_n) = y_n$.

We define

$$\nabla[f(x)] = f(x) - f(x-h)$$

$$\begin{aligned} \text{Thus, } \nabla y_1 &= y_1 - y_0 \\ \nabla y_2 &= y_2 - y_1 \\ &\dots \dots \dots \\ \nabla y_n &= y_n - y_{n-1}. \end{aligned}$$

∇ is called the **backward difference operator** and $\nabla y_1, \nabla y_2, \dots, \nabla y_n$ are called the **first order backward differences** of the function $y = f(x)$.

The **second order difference** of the function are defined by

$$\begin{aligned} \nabla^2 y_2 &= \nabla y_2 - \nabla y_1 \\ \nabla^2 y_3 &= \nabla y_3 - \nabla y_2 \\ &\dots \\ \nabla^2 y_n &= \nabla y_n - \nabla y_{n-1}. \end{aligned}$$

In a similar manner higher order difference can be defined. In general the n th order difference are defined by

$$\nabla^n y_i = \nabla^{n-1} y_i - \nabla^{n-1} y_{i-1}.$$

These difference of the function $y = f(x)$ can be systematically represented in the form of a table called **backward difference table**.

Backward Difference Table (for 6 arguments)

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_1				
$x_1 = x_0 + h$	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
$x_2 = x_0 + 2h$	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
$x_3 = x_0 + 3h$	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
$x_4 = x_0 + 4h$	y_4		$\nabla^2 y_5$			
		∇y_5				
$x_5 = x_0 + 5h$	y_5					

Remark 1. The relation between the two difference operations is given by $\nabla[f(x+h)] = \Delta f(x)$.

For, $\nabla[f(x+h)] = f(x+h) - f(x) = \Delta f(x)$

$$\begin{aligned} \text{Similarly} \quad \nabla^2[f(x+2h)] &= \nabla[f(x+2h) - f(x+h)] \\ &= \nabla f(x+2h) - \nabla f(x+h) \\ &= \Delta f(x+h) - \Delta f(x) \\ &= \Delta[f(x+h) - f(x)] \\ &= \Delta^2 f(x) \end{aligned}$$

In general $\nabla[f(x+nh)] = \Delta^n f(x)$.

Hence from the forward difference table of the function $f(x)$ we can obtain backward difference of all orders.

Central Difference Operator

Sometimes it is convenient to employ another system of differences known as central differences. We define **central difference operator** δ as

$$\delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right).$$

Thus if $f(x_i) = y_i$ then we have

$$\begin{aligned} \delta y_{\frac{1}{2}} &= y_1 - y_0 \\ \delta y_{\frac{3}{2}} &= y_2 - y_1 \\ &\vdots \\ \delta y_{\frac{n-1}{2}} &= y_n - y_{n-1} \end{aligned}$$

Here the subject of δy is the average of the subscripts of the two members of the difference. The higher order differences can be defined similar to forward and backward differences.

$$\begin{aligned} \delta^2 y_1 &= \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}} \\ \delta^2 y_2 &= \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}} \\ \delta^2 y_{\frac{3}{2}} &= \delta^2 y_2 - \delta^2 y_1 \text{ etc.} \end{aligned}$$

These differences of the function $y = f(x)$ can be systematically represented in the form of a table called central difference table.

Central Difference Table

x	$y = f(x)$	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{\frac{1}{2}}$			
$x_1 = x_0 + h$	y_1		$\delta^2 y_1$		
		$\delta y_{\frac{3}{2}}$		$\delta^3 y_{\frac{3}{2}}$	
$x_2 = x_0 + 2h$	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{\frac{5}{2}}$		$\delta^3 y_{\frac{5}{2}}$	
$x_3 = x_0 + 3h$	y_3		$\delta^2 y_3$		
		$\delta y_{\frac{7}{2}}$			
$x_4 = x_0 + 4h$	y_4				

Example 3: Form the forward difference table for the following data.

$x:$	0	1	2	3	4
$y:$	8	11	9	15	6

Solution:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	8				
		3			
1	11		-5		
		-2		13	
2	9		8		-36
		6		-23	
3	15		-15		
		-9			
4	6				

Example 4. Find the first and second order differences for $f(x) = ab^{cx}$

Solution:

$$\begin{aligned}
 \Delta f(x) &= f(x+h) - f(x) \\
 &= ab^{c(x+h)} - ab^{cx} \\
 &= ab^{cx}b^{ch} - ab^{cx} \\
 &= ab^{cx}(b^{ch} - 1) \\
 &= (b^{ch} - 1)ab^{cx} \\
 \Delta^2 f(x) &= (b^{ch} - 1)\Delta(ab^{cx}) \\
 &= (b^{ch} - 1)[(b^{ch} - 1)ab^{cx}] \\
 &= (b^{ch} - 1)^2 ab^{cx}.
 \end{aligned}$$

Example 5: Find the second difference of the polynomial.

$$f(x) = x^4 - 12x^3 + 42x^2 - 30x + 9 \text{ with } h = 2.$$

Solution: First we shall express the given polynomial $f(x)$ in terms of factorial polynomial by synthetic division with $h = 2$.

$$\begin{array}{r|rrrrrr}
 0 & 1 & -12 & 42 & -30 & 9 & \\
 & & 0 & 0 & 0 & 0 & \\
 \hline
 2 & 1 & -12 & 42 & -30 & 9 & \\
 & & 2 & -20 & 44 & & \\
 \hline
 4 & 1 & -10 & 22 & 44 & & \\
 & & 4 & -24 & & & \\
 \hline
 6 & 1 & -6 & -2 & & & \\
 & & 6 & & & & \\
 \hline
 & 1 & 0 & & & &
 \end{array}$$

$$\therefore f(x) = x^{(4)} - 2x^{(2)} + 14x^{(1)} + 9.$$

$$\therefore \Delta f(x) = 8x^{(3)} - 8x^{(1)} + 28 \quad (\because h = 2)$$

$$\begin{aligned}
 \Delta^2 f(x) &= 48x^{(2)} - 16 \\
 &= 48x(x-2) - 16 \\
 &= 48x^2 - 96x - 16.
 \end{aligned}$$

Example 6: Find the function whose first difference is $x^3 + 3x^2 + 5x + 12$.

Solution: Given $\Delta y = x^3 + 3x^2 + 5x + 12$.

We express this in terms of factorial polynomial.

$$\begin{array}{r|rrrr}
 & 1 & 3 & 5 & 12 \\
 & & 0 & 0 & 0 \\
 1 & 1 & 3 & 5 & 12 \\
 & & 1 & 4 & \\
 2 & 1 & 4 & 9 & \\
 & & 2 & & \\
 & 1 & 6 & &
 \end{array}$$

$$\therefore \Delta y = x^{(3)} + 6x^{(2)} + 9x^{(1)} + 12$$

$$\therefore y = \Delta^{-1} [x^{(3)} + 6x^{(2)} + 9x^{(1)} + 12]$$

$$= \frac{x^{(4)}}{4} + 2x^{(3)} + \frac{9x^{(2)}}{2} + 12x^{(1)} + c$$

$$= \frac{1}{4} [x(x-1)(x-2)(x-3)] + 2[x(x-1)(x-2)] + \frac{9}{2}x(x-1) + 12x + c$$

Example 7: If $y = \frac{1}{(3x+1)(3x+4)(3x+7)}$ show that

$$\Delta^2 y = \frac{108}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}$$

Solution:

$$\begin{aligned}
 y &= \frac{1}{(3x+1)(3x+4)(3x+7)} \\
 &= \frac{1}{27 \left(x + \frac{1}{3}\right) \left(x + \frac{4}{3}\right) \left(x + \frac{7}{3}\right)} \\
 &= \frac{1}{27 \left(x - \frac{2}{3} + 1\right) \left(x - \frac{2}{3} + 2\right) \left(x - \frac{2}{3} + 3\right)} \\
 &= \frac{1}{27} \left(x - \frac{2}{3}\right)^{(-3)}
 \end{aligned}$$

$$\begin{aligned}
\therefore \Delta y &= \frac{1}{27}(-3)\left(x-\frac{2}{3}\right)^{(-4)} \\
\Delta^2 y &= \frac{1}{27}3 \times 4\left(x-\frac{2}{3}\right)^{(-5)} \\
&= \frac{12}{27\left[\left(x-\frac{2}{3}\right)+1\right]\left[\left(x-\frac{2}{3}\right)+2\right]\left[\left(x-\frac{2}{3}\right)+3\right]\left[\left(x-\frac{2}{3}\right)+4\right]\left[\left(x-\frac{2}{3}\right)+5\right]} \\
&= \frac{12 \times 3^5}{27(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)} \\
&= \frac{108}{(3x+1)(3x+4)(3x+7)(3x+10)(3x+13)}
\end{aligned}$$

4.5 OTHER DIFFERENCE OPERATORS

In this section we introduce the shift operator E and averaging operator μ .

Definition. The shift operator E is defined by

$$\boxed{Ef(x) = f(x+h)}$$

Hence $E^2f(x) = Ef(x+h) = f(x+2h)$

In general for any positive integer n

$$\boxed{E^n f(x) = f(x+nh)}$$

In particular we have

$$\begin{aligned}
Ey_0 &= y_1, \\
E^2y_0 &= y_2, \\
&\dots \\
E^ny_0 &= y_n,
\end{aligned}$$

This inverse operator E^{-1} is define as

$$\boxed{E^{-1}f(x) = f(x-h)}$$

For any real number n we have

$$\boxed{E^n f(x) = f(x-nh)}$$

Note. $E^m E^n f(x) = E^{m+n} f(x)$.

Definition. The averaging operator μ is defined by

$$\mu f(x) = \frac{f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)}{2}$$

There are several relations connecting the operators Δ , ∇ , δ , E , μ and the differentiation operator D .

4.6 ERROR PROPAGATION IN DIFFERENCE TABLE

Formation of difference table for a set of given data involves entry of data followed by successive subtractions of these data to find the difference of various orders. Hence if there is an error in the value of $f(x)$ for a given x when higher order differences are calculated the errors spreads out and magnified. The following table shows how an error in ϵ in y_3 is propagated,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	y_0				
		Δy_0			
1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1 + \epsilon$
		Δy_3		$\Delta^3 y_2 + \epsilon$	
4	y_4		$\Delta^2 y_3 + \epsilon$		$\Delta^4 y_2 - 4\epsilon$
		$\Delta y_4 + \epsilon$		$\Delta^3 y_3 - 3\epsilon$	
5	$y_5 + \epsilon$		$\Delta^2 y_4 - 2\epsilon$		$\Delta^4 y_3 + 6\epsilon$
		$\Delta y_5 - \epsilon$		$\Delta^3 y_4 + 3\epsilon$	
6	y_6		$\Delta^2 y_5 + \epsilon$		$\Delta^4 y_4 - 4\epsilon$
		Δy_6		$\Delta^3 y_5 - \epsilon$	
7	y_7		$\Delta^2 y_6$		$\Delta^4 y_5 + \epsilon$
		Δy_7		$\Delta^3 y_6$	
8	y_8		$\Delta^2 y_7$		
		Δy_8			
9	y_9				

From the above difference table the following facts are immediate.

1. The error increases with the order of differences.
2. The error spreads out fanwise and the error propagation is confined to a triangular region with vertex at the point where the error is committed.
3. The errors in r^{th} column are given by the binomial coefficients of the expansion $(1 - \epsilon)^r$. Thus in the fourth difference Column $\nabla^4 y$ the errors are $\epsilon, -4\epsilon, 6\epsilon, -4\epsilon, \epsilon$.
4. The algebraic sum of the errors in any column of the difference table is zero.
5. The maximum error, in each column, appears opposite to us.

The above facts enable us to detect errors in the difference table.

Example 8: The following is a table of values of a polynomial of degree 5. It is known that there is an error in y_3 . Correct the error.

x	0	1	2	3	4	5	6
y	1	4	43	264	1093	3256	7999

Solution: Given $y_3 = 264$ is not correct. Let ϵ be the error in 264. Let the actual value of $y_3 = 264 + \epsilon$.

The difference table is formed as follows.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	1					
		3				
1	4		36			
		39		$146 + \epsilon$		
2	43		$182 + \epsilon$		$280 - 4\epsilon$	
		$221 + \epsilon$		$426 - 3\epsilon$		$20 + 10\epsilon$
3	$264 + \epsilon$		$608 - 2\epsilon$		$300 + 6\epsilon$	
		$829 - \epsilon$		$726 + 3\epsilon$		$220 - 10\epsilon$
4	1093		$1344 + \epsilon$		$520 - 4\epsilon$	
		2163		$1246 - \epsilon$		
5	3256		2580			
		4743				
6	7999					

Since $y = f(x)$ is a polynomial of degree 5, by fundamental theorem of finite differences $\Delta^5 y$ is constant for all x .

\therefore In particular $\Delta^5 y_0 = \Delta^5 y_1 = \text{constant}$

$$\therefore 20 + 10\epsilon = 220 - 10\epsilon$$

$$\therefore 20\epsilon = 200. \text{ Hence } \epsilon = 10.$$

$$\therefore y_3 = 264 + \epsilon = 264 + 10 = 274.$$

Example 9: In the following table one value of y is incorrect and y is a cubic polynomial in x . Construct a difference table for y and use it to locate and correct the wrong value.

x	0	1	2	3	4	5	6	7
y	25	21	18	18	27	45	76	123

Solution: We form the difference table as follows.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	Error in $\Delta^3 y$
0	25				
		-4			
1	21		1		
		-3		2	
2	18		3		$3 + (-1)$
		0		6	
3	18		9		$3 + (+3)$
		9		0	
4	27		9		$3 + (-3)$
		18		4	
5	45		13		$3 + (1)$
		31		3	
6	76		16		
		47			
7	123				

Since y is a polynomial of degree 3 the third difference $\Delta^3 y$ must be constant.

The sum, of the entries in $\Delta^3 y$ column is 15. Hence each entry in this column must be $\frac{15}{5} = 3$.

There is one entry 3. Hence there are errors in the first four entries in $\Delta^3 y$ column.

Writing these 4 incorrect entries as in error propagation table, we have

$$2 = 3 + (-1); \quad 6 + 3 - 3(-1); \quad 0 = 3 + 3(-1); \quad 4 = 3 - (-1)$$

We find the error $\epsilon = -1$ and it is against the entry 18 corresponding to $x = 3$.

\therefore Correct value of $y(3) = 18 - (-1) = 19$.

4.7 SUMMATION OF SERIES

The concept of finite differences can be applied to find the sum to n terms of a given series.

$$\text{Let } s_n = v_1 + v_2 + \dots + v_n = \sum_{i=1}^n v_i$$

$$\text{Let } v_i = \Delta u_i \text{ so that } u_i = \Delta^{-1} v_i$$

$$\therefore v_i = \Delta u_i = u_{i+1} - u_i \quad (\text{Taking } h = 1)$$

$$\begin{aligned} \text{Thus} \quad v_1 &= u_2 - u_1 \\ v_2 &= u_3 - u_2 \\ \dots &\quad \dots \\ v_n &= u_{n+1} - u_n. \end{aligned}$$

$$\text{Hence } \boxed{s_n = v_1 + v_2 + \dots + v_n = u_{n+1} - u_1 = \Delta^{-1} v_{n+1} - \Delta^{-1} v_1.}$$

Montmorts Theorem

$$u_0 + u_1 x + u_2 x^2 + \dots = \frac{u_0}{1-x} + \frac{x \Delta u_0}{(1-x)^2} + \frac{x^2 \Delta^2 u_0}{(1-x)^3} + \dots$$

Proof.

$$\begin{aligned} u_0 + u_1 x + u_2 x^2 + \dots &= u_0 + x u_1 + x^2 u_2 + \dots \\ &= u_0 + x E u_0 + x^2 E^2 u_0 + \dots \\ &= [1 + x E + x^2 E^2 + \dots] u_0 \\ &= (1 - x E)^{-1} u_0 \\ &= \frac{1}{(1 - x E)} u_0 \\ &= \frac{1}{1 - x(1 + \Delta)} u_0 \\ &= \frac{1}{1 - x - x \Delta} u_0 \\ &= \frac{1}{(1-x) \left[1 - \frac{x \Delta}{1-x} \right]} u_0 \\ &= \frac{1}{1-x} \left[1 - \frac{x \Delta}{1-x} \right]^{-1} u_0 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{1-x} \left[1 + \frac{x\Delta}{1-x} + \frac{x^2\Delta^2}{(1-x)^2} + \dots \right] u_0 \\
 &= \frac{u_0}{1-x} + \frac{x\Delta u_0}{(1-x)^2} + \frac{x^2\Delta^2 u_0}{(1-x)^3} + \dots
 \end{aligned}$$

Example 10: Sum the series to n terms of

$$1.2.3 + 2.3.4 + 3.4.5 + \dots$$

Solution: The n^{th} term of the series is given by

$$\begin{aligned}
 v_n &= n(n+1)(n+2) = (n+2)(n+1)n \\
 &= (n+2)^{(3)} \text{ (with } h=1)
 \end{aligned}$$

\therefore

$$\begin{aligned}
 s_n &= \sum_{i=1}^n v_i = \Delta^{-1}v_{n+1} - \Delta^{-1}v_1 \\
 &= \Delta^{-1}(n+3)^3 - \Delta^{-1}v_1 \\
 &= \frac{(n+3)^{(4)}}{4} - 0 \\
 &= \frac{1}{4}(n+3)(n+2)(n+1)n \\
 &= \frac{1}{4}n(n+1)(n+2)(n+3)
 \end{aligned}$$

Example 11: Sum of n terms of the series $1.3.5 + 2.4.6 + \dots$

Solution: The n^{th} term of the series is

$$\begin{aligned}
 v_n &= n(n+2)(n+4) \\
 &= n^3 + 6n^2 + 8n
 \end{aligned}$$

We express $n^3 + 6n^2 + 8n$ as factorial polynomial with $h = 1$.

$$\begin{array}{r|l}
 0 & 1 \quad 6 \quad 8 \quad 0 \\
 & \quad 0 \quad 0 \\
 \hline
 1 & 1 \quad 6 \quad 8 \quad 0 \\
 & \quad 1 \quad 7 \\
 \hline
 2 & 1 \quad 7 \quad 15 \\
 & \quad 2 \\
 \hline
 & 1 \quad 9
 \end{array}$$

\therefore

$$v_n = n^{(3)} + 9n^{(2)} + 15n^{(1)}$$

Now

$$\begin{aligned}
 s_n &= \sum_{i=1}^n v_i = \Delta^{-1}v_{n+1} - \Delta^{-1}v_1 \\
 &= \Delta^{-1} \left[(n+1)^{(3)} + 9(n+1)^{(2)} + 15(n+1)^{(1)} \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{(n+1)^4}{4} + \frac{9(n+1)^3}{3} + \frac{15(n+1)^2}{2} \\
&= \frac{(n+1)n(n-1)(n-2)}{4} + 3(n+1)n(n-1) + \frac{15}{2}(n+1)n \\
&= \frac{n(n+1)}{4} [n^2 - 3n + 2 + 12n - 12 + 30] \\
&= \frac{n(n+1)(n^2 + 9n + 20)}{4} \\
&= \frac{n(n+1)(n+4)(n+5)}{4}
\end{aligned}$$

Check Your Progress

1. $\Delta \nabla = \dots$
 - (a) $\Delta \nabla$
 - (b) $\Delta + \nabla$
 - (c) $\Delta - \nabla$
 - (d) $E \Delta \nabla$
2. Which of the following result is true?
 - (a) $\Delta x^{(n)} = nx^{(n-1)}$
 - (b) $\Delta x^{(n)} = nx^{(n-1)}$
 - (c) $\Delta^n e^x = e^x$
 - (d) $\Delta \cos x = \sin x$
3. The relation between E (shifting operator) and D (operator of differentiation) is
 - (a) $E = e^{hD}$
 - (b) $E = e^{-hD}$
 - (c) $D = \log E$
 - (d) $D = e^{hE}$
4. If the interval of differencing being unity, then $\Delta^3 [(1-x)(1-3x)(1-5x)] =$
 - (a) -6
 - (b) -90
 - (c) -15
 - (d) none
5. If Δ denotes the forward difference operator, $f(x_k) = f_k$ and $g(x_k) = g_k$ then $\Delta(f_k g_k)$ is equal to
 - (a) $f_k \Delta g_k + g_k \Delta f_k$
 - (b) $f_{k+1} \Delta g_k + g_{k+1} \Delta f_k$
 - (c) $f_k \Delta g_{k+1} + g_k \Delta f_{k+1}$
 - (d) $f_k \Delta g_{k+1} + g_{k+1} \Delta f_k$

4.8 LET US SUM UP

The eigenvalue problem is a problem of considerable theoretical interest and wide-ranging application. For example, this problem is crucial in solving systems of differential equations, analyzing population growth models, and calculating powers of matrices (in order to define the exponential matrix). The values of the independent variable x are usually called the argument and the corresponding functional value is known as the entry. In engineering and science one often has a number of data points, as obtained by sampling or experimentation, and tries to construct a function which closely fits those data points. This is called curve fitting or regression analysis. Interpolation is a specific case of curve fitting, in which the function must go exactly through the data points.

4.9 KEYWORD

Stirling's formula:

$$y_p = y_0 + p \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \times \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \frac{p^2(p^2-1)}{4!} \Delta^4 y_{-2} + \dots$$

4.10 QUESTIONS FOR DISCUSSION

1. Show that

$$(i) \delta = \nabla(1 - \nabla)^{-1/2} \quad (ii) \mu = \left[1 + \frac{\delta^2}{4} \right]^{1/2}$$

2. Construct the difference table for the sequence values

$$f(x) = (0, 0, 0, \varepsilon, 0, 0, 0)$$

where ε is an error. Also show that (i) the error spreads and increases in magnitude as the order of differences is increased, (ii) the errors in each column have binomial coefficient.

3. For the matrix

$$(i) A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \quad (ii) A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 5 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

(a) Find all the eigen values and the corresponding eigenvectors.

(b) Verify that A is a diagonal matrix, where S is the matrix of eigenvectors

Check Your Progress: Model Answers

1. (a)

2. (b)

3. (a)

4. (b)

5. (d)

4.11 SUGGESTED READINGS

S.S. Sastri., *Introductory Methods of Numerical Analysis*, Second Edition, Prentice Hall.

S. Mohan Naidu., *A Text Book of Mathematical Methods for Scientific Engineers*, Students Helpline Series, Hyd - 2005.

Jain, M.K., Iyengar, S.R.K., Jain, R.K., *Numerical Methods for Scientific and Engineering Computation*, 3rd Edition, New Age International (P) Ltd., 1993.

UNIT IV

LESSON

5

INTERPOLATION

CONTENTS

- 5.0 Aims and Objectives
- 5.1 Introduction
- 5.2 Newton's Interpolation Formulae
- 5.3 Central Difference Interpolation Formulae
- 5.4 Lagrange's Interpolation
- 5.5 Let us Sum up
- 5.6 Keywords
- 5.7 Questions for Discussion
- 5.8 Suggested Readings

5.0 AIMS AND OBJECTIVES

After studying this lesson, you will be able to:

- Explain Newton's Interpolation Formulae
- Discuss Central Difference Interpolation Formulae
- Understand Lagrange's Interpolation Formula

5.1 INTRODUCTION

In our daily life we are sometimes confronted with the problem of finding the value of a function for some value in the given interval with the help of a given set of observations. For example if we are to find out the population of India in 1954 when we know that the census in India is done in 1941, 1951, 1961, 1971, and so on i.e. the figures of population are available for 1941, 1951, 1961, 1971 etc., then the process of finding this figure is known as interpolation.

5.2 NEWTON'S INTERPOLATION FORMULAE

We now derive two important interpolation formulae by means of the forward and backward differences of a function. These formulae are often employed in engineering and scientific problems.

Let the function $y=f(x)$ take the values y_0, y_1, y_2, \dots corresponding to the values $x_0, x_0+h, x_0+2h, \dots$ of x . Suppose it is required to evaluate $f(x) = x = x_0 + ph$, where p is any real number.

For any real number p , we have defined E such that

$$E^p f(x) = f(x + ph)$$

$$\begin{aligned} \therefore y_p &= f(x_0 + ph) = E^p f(x_0) = (1 + \Delta)^p y_0 \\ &= \left\{ 1 + p\Delta + \frac{p(p-1)}{2!} \Delta^2 + \frac{p(p-1)(p-2)}{3!} \Delta^3 + \dots \right\} y_0 \end{aligned}$$

[using Binomial theorem]

$$\text{i.e. } y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots \quad (1)$$

It is called Newton's forward interpolation formula as (1) contains y_0 and the forward difference of y_0 .

Obs. This formula is used for interpolating the values of y near the end of a set of tabulated values and also for extrapolating values of y a little ahead (to the right) of y_0 .

5.3 CENTRAL DIFFERENCE INTERPOLATION FORMULAE

Gauss's forward interpolation formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p+2)}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} + \dots \quad \dots(1)$$

Gauss's backward interpolation formula is

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!} \Delta^4 y_{-2} + \dots \quad \dots(2)$$

Taking the mean of (1) and (2), we obtain

$$y_p = y_0 + p \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \times \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \frac{p^2(p^2-1)}{4!} \Delta^4 y_{-2} + \dots \quad \dots(3)$$

which is called Stirling's formula.

Cor. In the central differences notation, (3) takes the form

$$y_p = y_0 + p\mu \delta y_0 + \frac{p^2}{2!} \delta^2 y_0 + \frac{p(p^2+1^2)}{3!} \mu \delta^3 y_0 + \frac{p^2(p^2-1^2)}{4!} \delta^4 y_0 + \dots$$

$$\text{for } \frac{1}{2} (\Delta y_0 + \Delta y_{-1}) = \frac{1}{2} (\delta y_{1/2} + \delta y_{-1/2}) = \mu \delta y_0$$

$$\frac{1}{2} (\Delta^3 y_{-1} + \Delta^3 y_{-2}) = \frac{1}{2} (\delta^3 y_{1/2} + \delta^3 y_{-1/2}) = \mu \delta^3 y_0 \text{ etc.}$$

Obs. This formula involves means of the odd differences just above and below the central line and even differences on this line as shown below:

$$\dots y_0 \dots \begin{pmatrix} \Delta y_{-1} \\ \Delta y_0 \end{pmatrix} \dots \Delta^2 y_{-1} \dots \begin{pmatrix} \Delta^3 y_{-2} \\ \Delta^3 y_{-1} \end{pmatrix} \dots \Delta^4 y_{-2} \dots \begin{pmatrix} \Delta^5 y_{-3} \\ \Delta^5 y_{-2} \end{pmatrix} \dots \Delta^6 y_{-3} \dots \text{Central line}$$

5.4 LAGRANGE'S INTERPOLATION

Let $f(x_0), f(x_1), \dots, f(x_n)$ be the values of the uncton $y=f(x)$ corresponding to the arguments x_0, x_1, \dots, x_n , not necessarily equally spaced.

Let $f(x)$ be a polynomial in x of degree n . Then we can represent $f(x)$ as

$$f(x) = a_0(x-x_1)(x-x_2)\dots(x-x_n) + a_1(x-x_0)(x-x_2)\dots(x-x_n) + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1}) \dots(1)$$

where a_0, a_1, \dots, a_n are constants.

Now we have to determine the $(n + 1)$ constant a_0, a_1, \dots, a_n

Putting $x = x_0$ in (1), we get

$$f(x_0) = a_0(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)$$

i.e.,
$$a_0 = \frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} \dots(2)$$

Putting $x = x_1$ in (1), we get,

$$f(x_1) = a_1(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)$$

i.e.
$$a_1 = \frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} \dots(3)$$

Similarly
$$a_2 = \frac{f(x_2)}{(x_2-x_0)(x_2-x_1)\dots(x_2-x_n)} \dots(4)$$

.....

$$a_n = \frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} \dots(5)$$

Substituting (2), (3), (4), (5) in (1), we get,

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_n)} f(x_1) + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)(x_n-x_{n-1})} f(x_n)$$

If we denote $f(x_0), f(x_1), \dots, f(x_n)$ by y_0, y_1, \dots, y_n we get,

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} y_1 + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} y_n$$

which is Lagrange's interpolation formula.

Notes:

1. When the arguments x_0, x_1, \dots, x_n are not equally spaced then we can use this formula to find y for any x .
2. The above formula can also be written as

$$f(x) = \sum_{r=0}^n \frac{\phi(x)f(x_r)}{(x-x_r)\phi'(x_r)} \text{ where} \quad \dots(6)$$

$$\phi(x) = \prod_{r=0}^n (x-x_r) \text{ and } f'(x_r) = \left[\frac{d}{dx} \phi(x) \right]_{x=x_r}$$

Proof: Given $\phi(x) = \prod_{r=0}^n (x-x_r)$
 $= (x-x_0)(x-x_1)(x-x_2) \dots (x-x_n)$

Since $\phi(x)$ is the product of $(n+1)$ factors, the usual process of differentiation produces $\phi'(x)$ as the sum of $(n+1)$ terms, in each of which one factors has been differentiated as given below:

$$\begin{aligned} \phi'(x) &= (x-x_1)(x-x_2)\dots(x-x_n) + (x-x_0)(x-x_2)\dots(x-x_n) \\ &+ \dots + \dots + \dots \\ &+ (x-x_0)(x-x_1)\dots(x-x_{n-1}) \end{aligned}$$

Now, $\phi'(x_0) = (x_0-x_1)(x_0-x_2)\dots(x_0-x_n)$
 $\phi'(x_1) = (x_1-x_0)(x_1-x_2)\dots(x_1-x_n)$

\dots
 \dots
 $\phi'(x_n) = (x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})$

$$\begin{aligned} \therefore & \frac{\phi(x)f(x_0)}{(x-x_0)\phi'(x_0)} + \frac{\phi(x)f(x_1)}{(x-x_1)\phi'(x_1)} + \dots + \frac{\phi(x)f(x_n)}{(x-x_n)\phi'(x_n)} \\ &= \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(x-x_0)(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} f(x_0) \\ &+ \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(x-x_1)(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} f(x_1) \\ &+ \dots \\ &+ \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(x-x_n)(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} f(x_n) \end{aligned}$$

$$= \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} f(x_0) + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} f(x_1) + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} f(x_n) = f(x)$$

Check Your Progress

1. For the factorial function $x^{(r)} = x(x-h)(x-2h)\dots(x-rh+h)$. The value of $\Delta^2 x^{(r)}$ and $\Delta^{r+1} x^{(r)}$ are:

(a) $r(r-1)h^2 x^{(h-2)}$;	(b) $\lfloor r, 0$;
(c) $r(r-1)h^2 x^{(r-2)}$,	(d) $rx^{(r-1)}, 0$
2. If the 5th and higher order differences of a function vanish, then the function represents a polynomial of degree

(a) 4	(b) 2
(c) 5	(d) 3
3. Which of the following is not correct

(a) $\Delta = E - 1$	(b) $\nabla = 1 - E^{-1}$
(c) $\delta = E^{1/2} - E^{-1/2}$	(d) none
4. Newton's forward interpolation formula is used for interpolating the values of the function.

(a) near of beginning of a set of tabulated values	(b) near the end of a set of tabulated values
(c) near the middle of a set of tabulated values	(d) none
5. Lagrange interpolation formula is used

(a) near the beginning of the tabulated values	(b) near the end of the tabulated values
(c) near the middle of the tabulated values	(d) all of these

5.5 LET US SUM UP

In the mathematical subfield of numerical analysis, interpolation is a method of constructing new data points within the range of a discrete set of known data points.

A different problem which is closely related to interpolation is the approximation of a complicated function by a simple function. Suppose we know the function but it is too complex to evaluate efficiently. Then we could pick a few known data points from the complicated function, creating a lookup table, and try to interpolate those data points to construct a simpler function. Of course, when

using the simple function to calculate new data points we usually do not receive the same result as when using the original function, but depending on the problem domain and the interpolation method used the gain in simplicity might offset the error.

5.6 KEYWORDS

Newton's forward interpolation:

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots$$

Newton's backward interpolation:

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!} \nabla^2 y_n + \frac{p(p+1)(p+2)}{3!} \nabla^3 y_n + \dots$$

Stirling's formula:

$$y_p = y_0 + p \left(\frac{\Delta y_0 + \Delta y_{-1}}{2} \right) + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2-1)}{3!} \times \left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right) + \frac{p^2(p^2-1)}{4!} \Delta^4 y_{-2} + \dots$$

Gauss's forward interpolation formula:

$$y_p = y_0 + p\Delta y_0 + \frac{p(p+2)}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!} \Delta^4 y_{-2} + \dots$$

Gauss's backward interpolation formula :

$$y_p = y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!} \Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!} \Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!} \Delta^4 y_{-2} + \dots$$

Lagrange's interpolation:

$$f(x) = \frac{(x-x_1)(x-x_2)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_2)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)\dots(x_1-x_n)} y_1$$

$$+ \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} y_n$$

5.7 QUESTIONS FOR DISCUSSION

1. Prove that $1 + \mu^2 \delta^2 = \left(1 + \frac{1}{2} \delta^2 \right)^2$
2. Prove that $f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$
3. The following data give I, the indicated HP and V, the speed in knots developed by a ship.

V	8	10	12	14	16
I	1000	1900	3250	5400	8950

Find I when V = 9, using Newton's forward interpolation formula.

4. The amount A of substance remaining in a reacting system after an interval of time t in a certain chemical experiment is tabulated below.

$t(\text{min})$	2	5	8	11
$A(\text{gm})$	94.8	87.9	81.3	75.1

obtain the value of A where $t = 9$ using Newton's backward interpolation formula.

5. Find a polynomial which take the following values

x	1	3	5	7	9	11
y_x	3	14	19	21	23	28

and hence compute y_x at $x = 2, 12$

6. The following are data from the steam table:

temp $i(t)$	140	150	160	170	180
Pressure $\text{kgf/cm}^2(p)$	3.685	4.854	6.302	8.076	10.225

using Newton's formula, find the pressure of the steam for temperatures 142° and 175° .

7. Give the following table, find $y(35)$ by using Stirling's formula and Bessel's formula

$x :$	20	30	40	50
$y :$	512	439	346	243

8. Use Lagrange and the divided difference formula to calculate $f(3)$ from the following table :

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	14

Check Your Progress: Model Answers

1. (c)
2. (a)
3. (d)
4. (a)
5. (d)

5.8 SUGGESTED READINGS

S.S. Sastri., *Introductory Methods of Numerical Analysis*, Second Edition, Prentice Hall.

S. Mohan Naidu., *A Text Book of Mathematical Methods for Scientific Engineers*, Students Helpline Series, Hyd - 2005.

Jain, M.K., Iyengar, S.R.K., Jain, R.K., *Numerical Methods for Scientific and Engineering Computation*, 3rd Edition, New Age International (P) Ltd., 1993.

LESSON

6

DIVIDED DIFFERENCES

CONTENTS

- 6.0 Aims and Objectives
- 6.1 Introduction
- 6.2 Newton's Divided Difference Formula
- 6.3 Inverse Interpolation
- 6.4 Let us Sum up
- 6.5 Keywords
- 6.6 Question for Discussion
- 6.7 Suggested Readings

6.0 AIMS AND OBJECTIVES

After studying this lesson, you will be able to:

- Explain Newton's Divided Difference Formula
- Discuss Inverse Interpolation

6.1 INTRODUCTION

A polynomial $P(x)$ is called the interpolating polynomial if the values of $P(x)$ and/or its certain derivatives coincide with those of $f(x)$ and/or its same order derivatives at one or more tabular points.

6.2 NEWTON'S DIVIDED DIFFERENCE FORMULA

The linear Newton divided difference interpolation is easy to generalize. We define the higher order divided differences as

$$\begin{aligned} f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{1}{(x_2 - x_0)} \left[\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0} \right] \\ &= \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)} - \frac{f(x_1)}{(x_2 - x_0)} \left[\frac{1}{x_2 - x_1} + \frac{1}{x_1 - x_0} \right] + \frac{f(x_0)}{(x_2 - x_0)(x_1 - x_0)} \end{aligned}$$

$$= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}$$

$$f[x_0, x_1, x_2, \dots, x_{k-1}, x_k] = \frac{f[x_1, x_2, \dots, x_k] - f[x_0, x_1, \dots, x_{k-1}]}{x_k - x_0}$$

$k = 3, 4, \dots, n.$

In terms of function values, the nth divided difference can be written as

$$f[x_0, x_1, x_2, \dots, x_n] = \sum_{i=0}^n \frac{f(x_i)}{\prod_{\substack{j=0 \\ i \neq j}}^n (x_i - x_j)}$$

The divided differences may be calculated with the help of Table 6.1

Table 6.1: Divided Difference (d.d) Table

		first d.d.	second d.d.	third d.d.
x_0	$f[x_0]$			
x_1	$f[x_1]$	$f[x_0, x_1]$		
x_2	$f[x_2]$	$f[x_1, x_2]$	$f[x_0, x_1, x_2]$	
x_3	$f[x_3]$	$f[x_2, x_3]$	$f[x_1, x_2, x_3]$	$f[x_0, x_1, x_2, x_3]$

Note that

$$f[x_0, x_1] = f[x_1, x_0]$$

$$f[x_0, x_1, x_2] = f[x_2, x_1, x_0] \text{ etc.}$$

The interpolating polynomial $P_n(x)$, interpolating at the $n+1$ distinct points x_0, x_1, \dots, x_n can also be written as

$$P_n(x) = a_0 + (x - x_0)a_1 + (x - x_0)(x - x_1)a_2 + \dots + (x - x_0) \dots (x - x_{n-1})a_n.$$

Substituting successively $x = x_0, x = x_1, \dots, x = x_n$, we obtain

$$P_n(x_0) = f[x_0] = a_0.$$

$$P_n(x_1) = f[x_1] = a_0 + (x_1 - x_0)a_1 = f[x_0] + (x_1 - x_0)a_1$$

Or $a_1 = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = f[x_0, x_1],$

$$P_n(x_2) = f[x_2] = a_0 + (x_2 - x_0)a_1 + (x_2 - x_0)(x_2 - x_1)a_2$$

Or $a_2 = \frac{1}{(x_2 - x_0)(x_2 - x_1)} \left[f[x_2] - f[x_0] - (x_2 - x_0) \left\{ \frac{f[x_1] - f[x_0]}{x_1 - x_0} \right\} \right]$

$$= \frac{f[x_0]}{(x_0 - x_1)(x_0 - x_2)} + \frac{f[x_1]}{(x_1 - x_0)(x_1 - x_2)} + \frac{f[x_2]}{(x_2 - x_0)(x_2 - x_1)}$$

$$= f[x_0, x_1, x_2].$$

Using induction, we can prove that

$$a_n = f[x_0, x_1, \dots, x_n].$$

The divided difference interpolating polynomial becomes

$$P_n(x) = f[x_0] + (x - x_0)f[x_0, x_1] + \dots + (x - x_0) \dots (x - x_{n-1})f[x_0, x_1, \dots, x_n].$$

Note that, since the interpolating polynomial is unique, Lagrange and divided difference polynomials are two different forms of the same polynomial.

6.3 INVERSE INTERPOLATION

So far given a table of values of x and y , using one of the interpolation formulae we find the value of y corresponding to some value of x for some value of y which is not in the table. On the other hand the process of estimating the value of x for some value of y which is not in the table is called inverse interpolation. There are two methods for inverse interpolation. There are two methods for inverse interpolation, one when the values are unequally spaced and another when the values are equally spaced.

Lagrange's Method

This method is used when the values of x are unequally spaced. Lagrange's interpolation formula can be simple viewed as a relation between two variables and any one of the variable can be taken as an independent variable. Therefore interchanging the variables x and y in Lagrange's Formula we get

$$x = \frac{(y - y_1)(y - y_2) \dots (y - y_n)}{(y_0 - y_1)(y_0 - y_2) \dots (y_0 - y_n)} x_0 + \frac{(y - y_0)(y - y_2) \dots (y - y_n)}{(y_1 - y_0)(y_1 - y_2) \dots (y_1 - y_n)} x_1$$

$$+ \dots + \frac{(y - y_0)(y - y_1) \dots (y - y_{n-1})}{(y_n - y_0)(y_n - y_1) \dots (y_n - y_{n-1})} x_n.$$

which is used for inverse interpolation

Iterative Method

Newton's forward difference formula is

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \dots + \frac{p(p-1) \dots (p-n+1)}{n!} \Delta^n y_0$$

$$\therefore p = \frac{1}{\Delta y_0} \left[y_p - y_0 - \frac{p(p-1)}{2!} \Delta^2 y_0 - \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 - \dots \right].$$

Neglecting the second and higher order differences we obtain the first approximation to p given by

$$p^1 = \frac{1}{\Delta y_0}(y_p - y_0).$$

To find the second approximation to p we retain the term with second difference and replace p by p¹

$$\therefore p^2 = \frac{1}{\Delta y_0} \left[y_p - y_0 - \frac{p_1(p_1 - 1)}{2!} \Delta^2 y_0 \right]$$

To find the third approximation we retain the terms upto third order difference and replace p by p².

$$\therefore p^3 = \frac{1}{\Delta y_0} \left[y_p - y_0 - \frac{p_2(p_2 - 1)}{2!} \Delta^2 y_0 - \frac{p_2(p_2 - 1)(p_2 - 2)}{3!} \Delta^3 y_0 \right].$$

Continue this process till the successive values of p are approximately equal.

Example

Problem 1: Find the value of x correct to one decimal place for which y = 7 given

x	1	3	4
y	4	12	19

Solution: We have to find x when y = 7.

We need to use Lagrange's inverse interpolation formula.

Here

$$x_0 = 1; \quad x_1 = 3; \quad x_2 = 4$$

$$y_0 = 4; \quad y_1 = 12; \quad y_2 = 19$$

$$x = \frac{(y - y_1)(y - y_2)}{(y_0 - y_1)(y_0 - y_2)} \times x_0 + \frac{(y - y_0)(y - y_2)}{(y_1 - y_0)(y_1 - y_2)} \times x_1 + \frac{(y - y_0)(y - y_1)}{(y_2 - y_0)(y_2 - y_1)} \times x_2$$

$$= \frac{(7 - 12)(7 - 19)}{(4 - 12)(4 - 19)} \times 1 + \frac{(7 - 4)(7 - 19)}{(12 - 4)(12 - 19)} \times 3 + \frac{(7 - 4)(7 - 12)}{(19 - 4)(19 - 12)} \times 4$$

$$= 0.5 + 1.9286 - 0.5714$$

$$= 1.8572.$$

Problem 2: Tabulate $y = x^3$ for x = 2, 3, 4, 5 and calculate the cube root of 10 correct of three decimal places.

Solution: For x = 2, 3, 4, 5 the values of y are 8, 27, 64, 125 respectively.

Here h = 1. We form the forward difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
2	8			
		19		
3	27		18	
		37		6
4	64		24	
		61		
5	125			

The first approximation is given by

$$\begin{aligned} p^1 &= \frac{1}{\Delta y_0} (y_p - y_0) \\ &= \frac{1}{19} (10 - 8) \\ &= 0.1. \end{aligned}$$

The second approximation is

$$\begin{aligned} p^2 &= \frac{1}{\Delta y_0} \left[y_p - y_0 - \frac{p_1(p_1 - 1)}{2!} \Delta^2 y_0 \right] \\ &= \frac{1}{19} \left[10 - 8 - \frac{(0.1)(0.1 - 1)}{2!} (18) \right] \\ &= 0.15. \end{aligned}$$

The third approximation is

$$\begin{aligned} p^3 &= \frac{1}{\Delta y_0} \left[y_p - y_0 - \frac{p_2(p_2 - 1)}{2!} \Delta^2 y_0 - \frac{(p_2 - 1)(p_2 - 2)}{3!} \Delta^3 y_0 \right] \\ &= \frac{1}{19} \left[10 - 8 - \frac{0.15(0.15 - 1)}{2!} \times 18 - \frac{0.15(0.15 - 1)(0.15 - 2)}{3!} \times 6 \right] \\ &= 0.1532. \end{aligned}$$

The fourth approximation is

$$\begin{aligned} p^4 &= \frac{1}{\Delta y_0} \left[y_p - y_0 - \frac{p_3(p_3 - 1)}{2!} \Delta^2 y_0 - \frac{p_3(p_3 - 1)(p_3 - 2)}{3!} \Delta^3 y_0 \right] \\ &= \frac{1}{19} \left[10 - 8 - \frac{0.1532(0.1532 - 1)}{2!} \times 18 - \frac{0.1532(0.1532 - 1)(0.1532 - 2)}{3!} \times 6 \right] \\ &= 0.1541. \end{aligned}$$

The fifth approximation is

$$\begin{aligned} p^5 &= \frac{1}{\Delta y_0} \left[y_p - y_0 - \frac{p_4(p_4 - 1)}{2!} \Delta^2 y_0 - \frac{p_4(p_4 - 1)(p_4 - 2)}{3!} \Delta^3 y_0 \right] \\ &= \frac{1}{19} \left[10 - 8 - \frac{0.1541(0.1541 - 1)}{2!} \times 18 - \frac{0.1541(0.1541 - 1)(0.1541 - 2)}{3!} \times 6 \right] \\ &= 0.1542. \end{aligned}$$

Hence $p_4 \approx p_5 = 0.154$ (Correct to 3 places of decimals)

We have to find $\sqrt[3]{10}$. Since 10 lies between the values of y corresponding to $x = 2$ and $x = 3$, the required value of $\sqrt[3]{10}$ is $2 + hp_5 = 2 + 0.154 = 2.154$.

Check Your Progress

Using Newton's divided difference formula evaluate $f(8)$ given that

x	4	5	7	10	11	13
f(x)	48	100	294	900	1210	2028

6.4 LET US SUM UP

Lagrange's interpolation formula has the disadvantage that if another interpolation value were inserted then we have to recompute the interpolation coefficients. Hence it is desirable to have an interpolation formula which has the property that a polynomial of higher degree may be derived from it by simply adding new terms. Newton's divided difference formula is one such formula and this formula employs the concept of divided differences, which had introduced.

6.5 KEYWORDS

Newton's divided difference formula: This formula employs the concept of divided differences.

Inverse Interpolation: The process of estimating the value of x for some value of y which is not in the table.

6.6 QUESTION FOR DISCUSSION

- Find out the equation of the cubic curve which passes through points (4,43) (7,327) and (12, 1053). Hence find $f(10)$.

Check Your Progress: Model Answer

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \dots$$

At $x = 8$ we have

$$\begin{aligned} f(8) &= 48 + (8 - 4)[4, 5] + (8 - 4)(8 - 5)[4, 5, 7] \\ &\quad + (8 - 4)(8 - 5)(8 - 7)[4, 5, 7, 10] \\ &\quad + (8 - 4)(8 - 5)(8 - 7)(8 - 10)[4, 5, 7, 10] \\ &\quad + (8 - 4)(8 - 5)(8 - 7)(8 - 10)(8 - 11)[4, 5, 7, 10, 11] \\ &\quad + (8 - 4)(8 - 5)(8 - 7)(8 - 10)(8 - 11)(8 - 13)[4, 5, 7, 10, 11, 13] \end{aligned}$$

Contd...

The divided difference table is given below

x	f(x)	First div. diff	Second div. diff	Third div. diff	Fourth div. diff	Fifth div. diff
4	48					
		52				
5	100		15			
		97		1		
7	294		21		0	
		202		1		0
10	900		27		0	
		310		1		
11	1210		33			
		409				
13	2028					

Substituting the values of all divided differences from the divided difference table we have

$$\begin{aligned}
 f(8) &= 48 + (4 \times 52) + (4 \times 3 \times 15) + (4 \times 3 \times 1 \times 1) \\
 &= 48 + 208 + 180 + 12 \\
 &= 448
 \end{aligned}$$

6.7 SUGGESTED READINGS

S.S. Sastri., *Introductory Methods of Numerical Analysis*, Second Edition, Prentice Hall.

S. Mohan Naidu., *A Text Book of Mathematical Methods for Scientific Engineers*, Students Helpline Series, Hyd - 2005.

Jain, M.K., Iyengar, S.R.K., Jain, R.K., *Numerical Methods for Scientific and Engineering Computation*, 3rd Edition, New Age International (P) Ltd., 1993.

UNIT V

LESSON

7

NUMERICAL DIFFERENTIATION AND INTEGRATION

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7.0 AIMS AND OBJECTIVES

After studying this lesson, you will be able to:

- Describe derivatives using Newton's forward difference formula
- Discuss derivatives using Newton's backward difference formula
- Explain Newton's Central Difference formula
- Understand Maxima and Minima of the interpolating polynomial
- Identify Numerical Integration

7.1 INTRODUCTION

The Numerical Differentiation methods are very sensitive to round-off errors in addition to truncation errors introduced by the methods themselves. The accuracy attainable by these methods would depend on the given function and the order of the polynomial used. If the polynomial fitted is exact then the

error would be theoretically zero. In practice, however, rounding errors will introduce errors in the calculated values. The error introduced in obtaining derivatives is in general much worse than that introduced in determining integrals. The reason for this is that any errors in approximating a function are amplified while taking the derivative whereas they are smoothed out in integration. Thus, Numerical Differentiation should be avoided as far as possible if an alternative method is available.

Numerical Differentiation is carried out below for continuous functions and also for tabulated functions or discrete functions.

Here, we discuss the numerical process of approximating the derivatives $f'(x)$ of a function $f(x)$ when the function itself is available. We discuss the following methods.

1. Forward Difference
2. Backward Difference
3. Central Difference

The methods 1 and 2 are also known as Two-Point Formulae and use First Order Forward/Backward Difference Formula. The method 3 is also known as Three -Point Formula and uses Second Order Central Difference Quotient.

The Numerical Differentiation is very sensitive to round off errors. If the distance 'h' between any two successive points where the function values are known is small, the round-off error has an overriding influence on the total error. Therefore, while reducing the step size, we should exercise proper judgement in choosing the size.

Whenever a function cannot be integrated in finite terms or the evaluation of its integral is too cumbersome, Integration may be conveniently performed by Numerical Methods. Like Numerical Differentiation, we need to seek the help of Numerical Integration Techniques in the following situations:

1. Function does not possess closed form solution.
2. Closed form solutions exist but these solutions are complex and difficult to use for calculations.
3. Data for variables are available in the form of table, but no mathematical relationship between them is known, as is often the case with experimental data.

We know that a definite integral of the form,

$$I = \int_a^b f(x) dx$$

can be treated as the area under the curve $y = f(x)$, enclosed between the limits $x = a$ and $x = b$. The problem of Integration is then simply reduced to the problem of finding the shaded areas for the respective limits.

7.2 NUMERICAL DIFFERENTIATION

Consider a set of values (x_i, y_i) of a function. The process of computing the derivative or derivatives of that function at some values of x from the given set of values is called Numerical differentiation. This may be done by first approximating the function by a suitable interpolation formula and then differentiating it as many times as desired.

If the values of x are equi-spaced and the derivative is required near the beginning of the table, we employ Gregory - Newton forward interpolation formula. If it is required near the end of the table, we use Gregory - Newton backward interpolation formula. For the values near the middle of the table, the derivative is calculated by means of central difference interpolation formulae.

If the values of x are not equi-spaced, we use, Newton's divided difference interpolation formula or Lagrange's interpolation formula to get the derivative value.

7.2.1 Derivative Using Newton's Forward Difference Formula

Newton's forward interpolation formula's:

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \dots \quad \dots(1)$$

where
$$p = \frac{x - x_0}{h}$$

Differentiating both sides of Eqn (1) with respect to p , we have

$$\frac{dy}{dp} = \Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \dots \quad \dots(2)$$

Now
$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{dy}{dp} \cdot \frac{1}{h}$$

$$\therefore \frac{dy}{dx} = \frac{1}{h} \left[\Delta y_0 + \frac{2p-1}{2!}\Delta^2 y_0 + \frac{3p^2-6p+2}{3!}\Delta^3 y_0 + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 y_0 + \dots \right] \quad \dots(3)$$

At $x = x_0$, $p = 0$. Hence, putting $p = 0$ in Eqn (3), we get

$$\left. \frac{dy}{dx} \right|_{x=x_0} = \frac{1}{h} \left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \dots \right] \quad \dots (4)$$

Differentiating Eqn (3) again w.r.t. x , we get

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{d}{dp} \left(\frac{dy}{dx} \right) \frac{dp}{dx} = \frac{1}{h} \times \frac{d}{dp} \left(\frac{dy}{dx} \right) \\ &= \frac{1}{h^2} \left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 + \dots \right] \end{aligned} \quad \dots (5)$$

Putting $p = 0$ in Eqn (5), we get

$$\left. \frac{d^2 y}{dx^2} \right|_{x=x_0} = \frac{1}{h^2} \left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \dots \right] \quad \dots (6)$$

Similarly,

$$\left. \frac{d^3 y}{dx^3} \right|_{x=x_0} = \frac{1}{h^3} \left[\Delta^3 y_0 - \Delta^3 y_0 + \frac{3}{2}\Delta^4 y_0 + \dots \right] \quad \dots (7)$$

and so on.

Aliter: We know that $1 + \Delta = E = e^{hD}$. where

$$hD = \log(1 + \Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \dots$$

$$\therefore D = \frac{1}{h} \left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \dots \right]$$

$$\begin{aligned} D^2 &= \frac{1}{h^2} \left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \dots \right]^2 \\ &= \frac{1}{h^2} \left[\Delta^2 - \Delta^3 - \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \dots \right] \end{aligned}$$

$$\text{and } D^3 = \frac{1}{h^3} \left[\Delta^3 - \frac{3}{2}\Delta^4 + \frac{7}{4}\Delta^5 - \dots \right]$$

Now applying these identities to y_0 ,

$$Dy_0 = \left. \frac{dy}{dx} \right|_{x=x_0} = \frac{1}{h} \left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \dots \right]$$

$$D^2 y_0 = \left. \frac{d^2 y}{dx^2} \right|_{x=x_0} = \frac{1}{h^2} \left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{2}\Delta^4 y_0 + \frac{5}{6}\Delta^5 y_0 + \dots \right]$$

and

$$D^3 y_0 = \left. \frac{d^3 y}{dx^3} \right|_{x=x_0} = \frac{1}{h^3} \left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \frac{7}{4}\Delta^5 y_0 - \dots \right]$$

which are same as Eqns (4) - (7), respectively

7.2.2 Derivative Using Newton's Backward Difference Formula

Newton's backward interpolation formula is

$$y = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \dots \quad \dots(8)$$

$$\text{where } \frac{x - x_n}{h} = p \quad \dots(9)$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{dy}{dp} \cdot \frac{1}{h} \left[\because \frac{dp}{dx} = \frac{1}{h} \right] \\ &= \frac{1}{h} \left[\nabla y_n + \frac{2p+1}{2!}\nabla^2 y_n + \frac{3p^2+6p+2}{3!}\nabla^3 y_n + \dots \right] \end{aligned} \quad \dots(10)$$

At $x = x_n$, $p = 0$ Hence, putting $p = 0$ in Eqn (10)

$$\left. \frac{dy}{dx} \right|_{x=x_n} = \frac{1}{h} \left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \dots \right] \quad \dots(11)$$

Again, differentiating Eqn (10) w.r.t. x , we get

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dp}\left(\frac{dy}{dx}\right) \cdot \frac{dp}{dx} = \frac{1}{h} \cdot \frac{d}{dp}\left(\frac{dy}{dx}\right) \\ &= \frac{1}{h^2} \left[\nabla^2 y_n + \frac{6p+6}{3!} \nabla^3 y_n + \frac{6p^2+18p+11}{12} \nabla^4 y_n + \dots \right] \quad \dots(12)\end{aligned}$$

Putting $p = 0$, we get

$$\left. \frac{d^2x}{dx^2} \right|_{x=x_n} = \frac{1}{h^2} \left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \dots \right] \quad \dots(13)$$

$$\left. \frac{d^3x}{dx^3} \right|_{x=x_n} = \frac{1}{h^3} \left[\nabla^3 y_n + \frac{3}{2} \nabla^4 y_n + \dots \right] \quad \dots(14)$$

and so on.

Aliter: We know that $1 - \nabla = E^{-1} = e^{-hD}$

$$-hD = \log(1 - \nabla) = - \left[\nabla + \frac{1}{2} \nabla^2 + \frac{1}{3} \nabla^3 + \frac{1}{4} \nabla^4 + \dots \right]$$

$$\therefore D = \frac{1}{h} \left[\nabla + \frac{1}{2} \nabla^2 + \frac{1}{3} \nabla^3 + \frac{1}{4} \nabla^4 + \dots \right]$$

$$\begin{aligned}D^2 &= \frac{1}{h^2} \left[\nabla + \frac{1}{2} \nabla^2 + \frac{1}{3} \nabla^3 + \frac{1}{4} \nabla^4 + \dots \right]^2 \\ &= \frac{1}{h^2} \left[\nabla^2 + \nabla^3 + \frac{11}{12} \nabla^4 + \frac{5}{6} \nabla^5 + \dots \right]\end{aligned}$$

Similarly,

$$D^3 = \frac{1}{h^3} \left[\nabla^3 + \frac{3}{2} \nabla^4 + \frac{7}{4} \nabla^5 + \dots \right]$$

Applying these identities to y_n , we get

$$Dy_n = \left. \frac{dy}{dx} \right|_{x=x_n} = \frac{1}{h} \left[\nabla y_n + \frac{1}{2} \nabla^2 y_n + \frac{1}{3} \nabla^3 y_n + \frac{1}{4} \nabla^4 y_n + \dots \right]$$

$$D^2 y_n = \left. \frac{d^2y}{dx^2} \right|_{x=x_n} = \frac{1}{h^2} \left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12} \nabla^4 y_n + \frac{5}{6} \nabla^5 y_n + \dots \right] \text{ and}$$

$$D^3 y_n = \left. \frac{d^3y}{dx^3} \right|_{x=x_n} = \frac{1}{h^3} \left[\nabla^3 y_n + \frac{3}{2} \nabla^4 y_n + \frac{7}{4} \nabla^5 y_n + \dots \right]$$

which are same as Eqn (11) - (14), respectively.

7.2.3 Derivatives Using Stirling's Formula

Stirling's formula is

$$y = y_0 + \frac{p}{1!} \left[\frac{\Delta y_0 + \Delta y_{-1}}{2} \right] + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2 - 1^2)}{3!} \left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right] + \frac{p^2(p^2 - 1^2)}{4!} \Delta^4 y_{-2} + \dots \quad \dots (15)$$

$$\text{where } p = \frac{x - x_0}{h} \quad \dots (16)$$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{dy}{dp} \cdot \frac{1}{h} \quad \left[\because \frac{dp}{dx} = 1 \right]$$

$$\frac{dy}{dx} = \frac{1}{h} \left[\left\{ \frac{\Delta y_0 + \Delta y_{-1}}{2} \right\} + p \Delta^2 y_{-1} + \frac{3(p^2 - 1)}{6} \left\{ \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right\} + \frac{(2p^3 - p)}{12} \Delta^4 y_{-2} + \dots \right] \quad \dots (17)$$

At $x = x_0$, $p = 0$. Hence, putting $p = 0$, we get

$$\left. \frac{dy}{dx} \right|_{x=x_0} = \frac{1}{h} \left[\left\{ \frac{\Delta y_0 + \Delta y_{-1}}{2} \right\} - \frac{1}{6} \left\{ \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right\} + \frac{1}{30} \left\{ \frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2} \right\} + \dots \right] \quad \dots (18)$$

Differentiating Eqn (17) w.r.t. x , we get

$$\frac{d^2 y}{dx^2} = \frac{1}{h} \left[\Delta^2 y_{-1} + p \left\{ \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right\} + \frac{6p^2 - 1}{12} \Delta^4 y_{-2} + \dots \right]$$

$$\therefore \left. \frac{d^2 y}{dx^2} \right|_{x=x_0} = \frac{1}{h^2} \left[\Delta^2 y_{-1} - \frac{1}{12} \Delta^4 y_{-2} + \frac{1}{90} \Delta^6 y_{-3} - \dots \right] \quad \dots (19)$$

$$\left. \frac{d^3 y}{dx^3} \right|_{x=x_0} = \frac{1}{h^3} \left[\frac{1}{2} \left\{ \Delta^3 y_{-1} + \Delta^3 y_{-2} \right\} + \dots \right] \quad \dots (20)$$

and so on.

In the same manner, we can use any other interpolation formula for computing the derivatives.

Note: Numerical differentiation should be performed only if it is clear from the tabulated values that differences of some order are constant. Otherwise, the method will involve errors of considerable magnitude and they go on increasing significantly as the derivatives of higher orders are computed. This is due to the fact that the original $f(x)$ function and the approximating function $\phi(x)$ may not differ much at the data points but $f'(x) - \phi'(x)$ may be large.

7.2.4 Maxima and Minima of Tabulated Function

Differentiating Newton's forward interpolation formula (Eq (1)) with respect to x , we get

$$\frac{dy}{dx} = \frac{1}{h} \left[\Delta y_0 + \frac{2p-1}{2} \Delta^2 y_0 + \frac{3p^2 - 6p + 2}{3!} \Delta^3 y_0 + \dots \right] \quad \dots (21)$$

We know that the maximum and minimum values of a function $y = f(x)$ can be found by equating $\frac{dy}{dx}$ to zero and solving for x .

∴ From Eqn (21) $\frac{dy}{dx} = 0$

$$\Rightarrow \Delta y_0 + \frac{2p-1}{2} \Delta^2 y_0 + \frac{3p^2-6p+2}{6} \Delta^3 y_0 + \dots = 0$$

Hence, by keeping only up to the third difference, we have

$$\Delta y_0 + \frac{2p-1}{2} \Delta^2 y_0 + \frac{3p^2-6p+2}{6} \Delta^3 y_0 = 0$$

Solving this p, by substituting $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ (which we get from the difference table), we get x as $x_0 + ph$, at which y is a maximum or minimum.

Examples

1. Find the first, second and third derivatives of f(x) at x = 1.5 if

x	1.5	2.0	2.5	3.0	3.5	4.0
f(x)	3.375	7.000	3.625	24.000	38.875	59.000

Solution:

We have find the derivative at the point x = 1.5 which is at the beginning of the given data. Therefore, we use here the derivatives of Newton's forward interpolation formula. The forward difference table is as follows:

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5	3.375				
		3.625			
2.0	7.000		3.000		
		6.625		0.750	
2.5	3.625		3.750		0
		10.375		0.750	
3.0	24.000		4.500		0
		14.875		0.750	
3.5	38.875		5.250		
		20.125			
4.0	59.000				

Here, $x_0 = 1.5$, $y_0 = 3.375$, $\Delta y_0 = 3.625$, $\Delta^2 y_0 = 3$, $\Delta^3 y_0 = 0.75$, and $h = 0.5$

Now, from Eqn above, we are

$$\left. \frac{dy}{dx} \right]_{x=x_0} = f'(x_0) = \frac{1}{h} \left[\Delta y_0 - \frac{1}{2} \Delta^2 y_0 - \frac{1}{3} \Delta^3 y_0 - \frac{1}{4} \Delta^4 y_0 + \dots \right]$$

$$\therefore f'(1.5) = \frac{1}{0.5} \left[3.6250 - \frac{1}{2}(3) + \frac{1}{3}(0.75) \right] = 4.75$$

From Eqn. above, we have

$$\left. \frac{d^2 y}{dx^2} \right]_{x=x_0} = f''(x_0) = \frac{1}{h^2} \left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12} \Delta^4 y_0 - \dots \right]$$

$$\therefore f'(1.5) = \frac{1}{(0.5)^2} [3 - 0.75] = 9$$

Again from Eqn above, we have

$$\left. \frac{d^3 y}{dx^3} \right]_{x=x_0} = f'''(x_0) = \frac{1}{h^3} \left[\Delta^3 y_0 - \frac{3}{2} \Delta^4 y_0 + \dots \right]$$

$$\therefore f'''(1.5) = \frac{1}{(0.5)^3} (0.75) = 6$$

2. The population of a certain town (as obtained from census data) is shown in the following table.

Year	1951	1961	1971	1981	1991
Population	19.96	36.65	58.81	77.21	94.61 (in thousands)

Find the rate of growth of the population in the year 1981.

Solution:

Here, we have to find the derivative at 1981 which is near the end of the table. Hence, we use the derivative of Newton's backward difference formula. The table of differences is as follows:

x (year)	y (population)	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1951	19.96				
		16.69			
1961	36.65		5.47		
		22.16		-9.23	
1971	58.81		-3.76		11.99
		18.40		2.76	
1981	77.21		-1		
		17.40			
1991	94.61				

Hence, $h = 10$, $x_n = 1991$, $\nabla y_n = 17.4$, $\nabla^2 y_n = -1$, $\nabla^3 y_n = 2.76$ and $\nabla^4 y_n = 11.99$

we know from above Eqn that

$$\left. \frac{dy}{dx} \right]_{x=x_n} = \frac{1}{h} \left[\nabla y_n + \frac{2p+1}{2} \nabla^2 y_n + \frac{3p^2+6p+2}{6} \nabla^3 y_n + \frac{2p^3+9p^2+11p+3}{12} \nabla^4 y_n + \dots \right] \quad \dots(1)$$

Now, we have to find out the rate of growth of the population in the year 1981, i.e., $\left. \frac{dy}{dx} \right|_{x=1981}$

i.e., $x_n + ph = 1981 \therefore p = \frac{1981 - 1991}{10} = -1$

\therefore Putting $p = -1$, $h = 10$ and the values ∇y , $\nabla^2 y_n$, $\nabla^3 y_n$ of and $\nabla^4 y_n$ in Eqn (1), we get

$$y'(1981) = \frac{1}{10} \left[17.4 + \frac{2(-1)+1}{2}(-1) + \frac{3(-1)^2+6(-1)+2}{6}(2.76) + \frac{2(-1)^3+9(-1)^2+11(-1)+3}{12}(11.99) \right]$$

$$= \frac{1}{10} [17.4 + 0.5 - 0.46 - 0.9991666] = 1.6440833$$

\therefore The rate of growth of the population in the year 1981 is 1.6440833.

3. Obtain the value of $f'(90)$ using Stirling's formula to the following data.

x	60	75	90	105	120
f(x)	28.2	38.2	43.2	40.9	37.7

Also find the maximum value of the function from the data.

Solution:

Since $x = 90$ is in the middle of the table, we use central difference formula and in particular, Stirling's formula:

The central difference table is as given below.

x	y = f(x)	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
60	28.2				
		10			
75	38.2		-5		
		5		-2.3	
90	43.2		-7.3		8.7
		-2.3		6.4	
105	40.9		-0.9		
		-3.2			
120	37.7				

Here, $x_0 = 90, y_0 = 43.2, \Delta y_0 = -2.3, \Delta y_{-1} = 5, \Delta^3 y_{-1} = -2.3, \Delta^3 y_{-2} = 6.4$ and

Now, from Eqn

$$\left. \frac{dy}{dx} \right|_{x=x_0} = \frac{1}{h} \left[\left\{ \frac{\Delta y_0 + \Delta y_{-1}}{2} \right\} - \frac{1}{6} \left\{ \frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right\} + \dots \right]$$

$$\therefore f'(90) = \frac{1}{15} \left[\left\{ \frac{-2.3 + 5}{2} \right\} - \frac{1}{6} \left\{ \frac{-2.3 + 6.4}{2} \right\} \right]$$

$$= \frac{1}{15} [1.35 - 0.3416666] = 0.0672222.$$

To find the maximum value of the tabular function:

By Stirling's formula,

$$y = y(x_0 + ph) = y_0 \frac{p}{2} (\Delta y_0 + \Delta y_{-1}) + \frac{p^2}{2!} \Delta^2 y_{-1} + \frac{p(p^2 - 1^2)}{3!} \left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} \right] + \frac{p(p^2 - 1^2)}{4!} \Delta^4 y_{-2} + \dots$$

Substituting the values from the table, we get, after simplification,

$$y = 43.2 + 1.35 p - 3.65 p^2 + 0.3417 (p^3 - p)$$

$$\text{or } y = 0.3417 p^3 - 3.65 p^2 + 1.0083 p + 43.2$$

If y is maximum, $\frac{dy}{dx} = 0$,

$$\text{i.e., } 1.0251 p^2 - 7.3 p + 1.0083 = 0$$

$$p = \frac{7.3 \pm \sqrt{(1.3)^2 - 4(1.0251)(1.0083)}}{2(1.0251)} = 6.9803 \text{ or } 0.1409$$

$$p = 6.9803 \text{ is out of range } \therefore p = 0.1409$$

Hence, $x = x_0 + ph = 90 + 15(0.1409) = 92.1135$

and maximum of y

$$= 0.3417 (0.1409)^3 - 3.65 (0.1409)^2 + 1.0083 (0.1409) + 43.2$$

$$= 43.27$$

4. Using Bessel's formula, find the derivative of $f(x)$ at $x = 3.5$ from the following table.

x	3.47	3.48	3.49	3.50	3.51	3.52	3.53
$f(x)$	0.193	0.195	0.198	0.201	0.203	0.206	0.208

Solution:

Bessel's formula is

$$\begin{aligned} y_p &= y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \left[\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} \right] + \frac{\left(p - \frac{1}{2}\right)p(p-1)}{3!} \Delta^3 y_{-1} \\ &+ \frac{(p+1)p(p-1)(p-2)}{4!} \left[\frac{\Delta^2 y_{-2} + \Delta^4 y_{-1}}{2} \right] + \frac{\left(p - \frac{1}{2}\right)(p+1)p(p-1)(p-2)}{5!} \Delta^5 y_{-2} \\ &+ \frac{(p+2)(p+1)p(p-1)(p-2)(p-3)}{6!} \left[\frac{\Delta^6 y_{-3} + \Delta^6 y_{-2}}{2} \right] + \dots \end{aligned} \quad \dots(i)$$

The central difference table is as follows:

x	y = f(x)	Δy	Δ ² y	Δ ³ y	Δ ⁴ y	Δ ⁵ y	Δ ⁶ y
3.47	0.193						
		0.002					
3.48	0.195		0.001				
		0.003		-0.000			
3.49	0.198		0.000		0.000		
		0.003		-0.001		0.003	
3.50	0.201		-0.001		0.003		-0.010
		0.002		0.002		-0.007	
3.51	0.203		0.001		-0.004		
		0.003		-0.002			
3.52	0.206		-0.001				
		0.002					
3.53	0.208						

Where $p = \frac{x - x_0}{h}$. Differentiating Eqn (i) with respect to x, we get

$$\frac{dy}{dx} = \frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{1}{h} \cdot \frac{dy}{dp} \quad \left[\because \frac{dp}{dx} = \frac{1}{h} \right]$$

Now,

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{x=x_0} &= \frac{1}{h} \left. \left[\frac{dy}{dp} \right]_{p=0} \right. \\ &= \frac{1}{h} \left[\Delta y_0 - \frac{1}{4} (\Delta^2 y_{-1} + \Delta^2 y_0) + \frac{1}{12} \Delta^3 y_{-1} \right. \\ &\quad \left. + \frac{1}{24} (\Delta^4 y_{-2} + \Delta^4 y_{-1}) - \frac{1}{120} \Delta^5 y_{-2} - \frac{1}{240} (\Delta^6 y_{-3} + \Delta^6 y_{-2}) \right] \end{aligned}$$

Substituting values from the table in above, we get

$$\begin{aligned} \left. \frac{dy}{dx} \right|_{x=3.5} &= f'(3.5) = \frac{1}{0.01} \left[0.02 - \frac{1}{4} (-0.001 + 0.001) + \frac{1}{12} (0.002) \right. \\ &\quad \left. + \frac{1}{24} (-0.004 + 0.003) + \frac{1}{120} (-0.00) - \frac{1}{240} (-0.010 + 0) \right] \\ &= [0.02 - 0 + 0.01666 - 0.04166 + 0.00583 + 0.04166] \\ &= 0.22249 \end{aligned}$$

5. Given the following data, find the maximum value of y

x	-1	1	2	3
y	-21	15	12	3

Solution:

Since the arguments are not equi-spaced, we will form the divided difference table as follows:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
-1	-21			
		18		
1	15		-7	
		-3		1
2	12		-3	
		-9		
3	3			

Using Newton's divided difference formula, we get

$$\begin{aligned} y &= y_0 + (x - x_0)\Delta y_0 + (x - x_0)(x - x_1)\Delta^2 + (x - x_0)(x - x_1)(x - x_2)\Delta^3 y_0 \\ &= -21 + (x + 1)(18) + (x + 1)(x - 1)(-7) + (x + 1)(x - 1)(x - 2)(1) \\ &= x^3 - 9x^2 + 17x + 6 \end{aligned}$$

Now for maximum $\frac{dy}{dx} = 0 \Rightarrow 3x^2 - 18x + 17 = 0$

$$x = \frac{18 \pm \sqrt{(-18)^2 - 4(3)(17)}}{2(3)} = 4.8257 \text{ or } 1.1743$$

$x = 4.8257$ is out of range

$\therefore 1.1743$ is the value giving maximum of y .

Max of y (at $x = 1.1743$)

$$= (1.1743)^3 - 9(1.1743)^2 + 17(1.1743) + 6 = 15.171612$$

6. Use Stirling's formula to find y_{35} given that $y_{10} = 600, y_{20} = 512, y_{30} = 439, y_{40} = 346, y_{50} = 243$.

Solution:

Take $x_0 = 30, h = 10 \therefore p = \frac{x - 30}{10}$

Now the central difference table is

x	p	y_p	Δy_p	$\Delta^2 y_p$	$\Delta^3 y_p$	$\Delta^4 y_p$
10	-2	600				
			-88			
20	-1	512		15		
			-73		-35	
30	0	439		-20		145
			-93		10	
40	1	346		-10		
			-103			
50	2	243				

Thus, at $x = 35$, $p = \frac{35-30}{10} = 0.5$,

$$y_0 = 439, \Delta y_0 = -93, \Delta y_{-1} = -73, \Delta^2 y_{-1} = -20, \Delta^3 y_{-1} = -10, \Delta^3 y_{-2} = -35 \text{ and } \Delta^4 y_{-2} = 145$$

Substituting the above values in Stirling's formula, we get

$$\begin{aligned} y_{0.5} &= 435 + (0.5) \left[\frac{(-93) + (-73)}{2} \right] + \frac{(0.5)^2}{2!} (-20) \\ &\quad + \frac{(0.5) [(0.5)^2 - 1]}{3!} \left[\frac{10 + (-35)}{2} \right] + \frac{(0.5)^2 [(0.5)^2 - 1]}{4!} (145) \\ &= 435 - 41.5 - 2.5 + 0.78125 - 1.1328125 \\ &= 390.64844 \end{aligned}$$

$$\therefore y_{35} \approx 390.648$$

7.3 NUMERICAL INTEGRATION

We know that $\int_a^b f(x) dx$ represents the area between $y = f(x)$, x -axis and the ordinates $x = a$ and $x = b$.

This integration is possible only if the $f(x)$ is explicitly given and if it is integrable. The problem of numerical integration can be stated as follows: Given a set of $(n+1)$ paired values $(x_i, y_i) = 0, 1, 2, \dots, n$

of the function $y = f(x)$, where $f(x)$ is not known explicitly, it is required to compute $\int_{x_0}^{x_n} y dx$.

As we did in the case of interpolation or numerical differentiation, we replace $f(x)$, by an interpolating polynomial $P_n(x)$ and obtain $\int_{x_0}^{x_n} P_n(x) dx$ which is approximately taken as the value for $\int_{x_0}^{x_n} f(x) dx$.

A general quadrature formula for equidistant ordinates (or) (Newton's cote's formula)

For equally spaced intervals, we have Newton's forward difference formula as

$$y(x) = y(x_0 + uh) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \dots \quad \dots (1)$$

Now, instead of $f(x)$, we will replace it by this interpolating formula of Newton.

Here, $u = \frac{x-x_0}{h}$ where h is interval of differencing.

Since $x_n = x_0 + nh$, and $u = \frac{x-x_0}{h}$ we have $\frac{x-x_0}{h} = n = u$.

$$\int_{x_0}^{x_n} f(x) dx = \int_{x_0}^{x_0+nh} f(x) dx$$

$\approx \int_{x_0}^{x_0+nh} P_n(x) dx$ where $P_n(x)$ is interpolating polynomial of degree n .

$$\approx \int_0^n \left(y_0 + u\Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \dots \right) h du$$

Since $dx = h du$, and when $x = x_0$, $u = 0$ and where $x = x_0 + nh$, $u = n$

$$= h \int_0^n \left(y_0 + u\Delta y_0 + \frac{u^2 - u}{2!} \Delta^2 y_0 + \frac{u^3 - 3u^2 + 2u}{3!} \Delta^3 y_0 + \dots \right) du$$

$$= h \left[y_0 \left(u \frac{u^2}{2} \Delta y_0 + \frac{\left(\frac{u^3}{3} - \frac{u^2}{2} \right)}{2} \Delta^2 y_0 + \frac{1}{6} \left(\frac{u^4}{4} - u^3 + u^2 \right) \Delta^3 y_0 + \dots \right) \right]_0^n$$

$$\int_{x_0}^{x_n} f(x) dx \approx h \left[ny_0 + \frac{n^2}{2} \Delta y_0 + \frac{1}{2} \left(\frac{n^3}{3} - \frac{n^2}{2} \right) \Delta^2 y_0 + \frac{1}{6} \left(\frac{n^4}{4} - n^3 + n^2 \right) \Delta^3 y_0 + \dots \right] \quad \dots(2)$$

The equation (2), called Newton-Cote's quadrature formula is a general quadrature formula. Giving various values for n , we get a number of special formula.

7.4 TRAPEZOIDAL RULE

By putting $n = 1$, in the quadrature formula (i.e., there are only two paired values and interpolating polynomial is linear.)

$$\int_{x_0}^{x_0+h} f(x) dx \approx h \left[1 \cdot y_0 + \frac{1}{2} \Delta y_0 \right] \text{ since other differences do not exist if } n = 1$$

$$\approx h \left[y_0 + \frac{1}{2} (y_1 - y_0) \right]$$

$$= \frac{h}{2} (y_0 + y_1)$$

$$\int_{x_0}^{x_n} f(x) dx \approx \int_{x_0}^{x_0+nh} f(x) dx$$

$$= \int_{x_0}^{x_0+h} f(x) dx + \int_{x_0+h}^{x_0+2h} f(x) dx + \dots + \int_{x_0+(n-1)h}^{x_0+nh} f(x) dx$$

$$= \frac{h}{2} (y_0 + y_1) + \frac{h}{2} (y_1 + y_2) + \dots + \frac{h}{2} (y_{n-1} + y_n)$$

$$\begin{aligned}
&= \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \dots + \frac{h}{2}(y_{n-1} + y_n) \\
&= \frac{h}{2} (\text{Sum of the first and the last ordinates}) + 2 (\text{Sum of the remaining ordinates})]
\end{aligned}$$

This is known as Trapezoidal rule.

7.4.1 Truncation Error in Trapezoidal Rule

In the neighbourhood of $x = x_0$, we can expand $y = f(x)$ by Taylor series in powers of $x - x_0$. That is

$$y(x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \dots \quad \dots (1)$$

where $y_0' = [y'(x)]_{x=x_0}$

$$\begin{aligned}
\int_{x_0}^{x_1} y dx &= \int_{x_0}^{x_1} \left[y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \dots \right] dx \\
&= \left[y_0 x + \frac{(x - x_0)^2}{2!} y_0' + \frac{(x - x_0)^3}{3!} y_0'' + \dots \right]_{x_0}^{x_1} \\
&= y_0 (x_1 - x_0) + \frac{(x_1 - x_0)^2}{2!} y_0' + \frac{(x_1 - x_0)^3}{3!} y_0'' + \dots \\
&= h y_0 + \frac{h^2}{2!} y_0' + \frac{h^3}{3!} y_0'' + \dots \quad \dots (2)
\end{aligned}$$

if h is the equal interval length.

$$\text{Also } \int_{x_0}^{x_1} y dx \approx \frac{h}{2} (y_0 + y_1) = \text{area of the first trapezium} = A_0 \quad \dots (3)$$

Putting $x = x_1$ in (1),

$$y(x_1) = y_1 = y_0 + \frac{(x_1 - x_0)}{1!} y_0' + \frac{(x_1 - x_0)^2}{2!} y_0'' + \dots$$

$$\text{i.e., } y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \dots \quad \dots (4)$$

$$A_0 \approx \frac{h}{2} \left[y_0 + y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \dots \right] \quad \text{using (4) in (3)}$$

$$\approx h y_0 + \frac{h^2}{2} y_0' + \frac{h^3}{2 \times 2!} y_0'' + \dots$$

Subtracting A_0 value from (2),

$$\int_{x_0}^{x_1} y \, dx - A_0 = h^3 y_0'' \left(\frac{1}{3!} - \frac{1}{2 \times 2!} \right) + \dots = -\frac{1}{12} h^3 y_0'' + \dots$$

Therefore the error in the first's interval (x_0, x_1) is $-\frac{1}{12} h^3 y_0''$ (neglecting other terms).

Similarly the error in the i^{th} interval = $-\frac{1}{12} h^3 y''_{i-1}$

Therefore, the total cumulative error (approx.),

$$E = -\frac{1}{12} h^3 (y_0'' + y_1'' + y_2'' + \dots + y''_{n-1})$$

$|E| < \frac{nh^3}{12} \cdot M$ where M is the maximum value of

$|y_0''|, |y_1''|, |y_2''|, \dots$

$< \frac{(b-a)h^2}{12} \cdot M$ if the interval is (a, b) and $h = b-a/n$

Hence, the error in the Trapezoidal rule is of order h^2

7.5 SIMPSON'S ONE-THIRD RULE

Setting $n = 2$ in Newton-cote's quadrature formula, we have

$$\begin{aligned} \int_{x_0}^{x_2} f(x) \, dx &\approx h \left[2y_0 + \frac{4}{2} \Delta y_0 + \frac{1}{2} \left(\frac{8}{3} - \frac{4}{2} \right) \Delta^2 y_0 \right] \text{ (since other terms vanish)} \\ &\approx h \left[2y_0 + 2(y_1 - y_0) + \frac{1}{3} (E - 1)^2 y_0 \right] \\ &= h \left[2y_0 + 2(y_1 - y_0) + \frac{1}{3} (E - 1)^2 y_0 \right] \\ &= h \left[\frac{1}{3} y_2 + \frac{4}{3} y_1 + \frac{1}{3} y_0 \right] \\ &= \frac{h}{3} (y_2 + 4y_1 + y_0) \end{aligned}$$

Similarly, $\int_{x_2}^{x_4} f(x) \, dx = \frac{h}{3} (y_2 + 4y_3 + y_4)$

$$\int_{x_i}^{x_{i+2}} f(x) \, dx = \frac{h}{3} (y_i + 4y_{i+1} + y_{i+2})$$

In n is even integer, last integral will be

$$\int_{x_{n-2}}^{x_n} f(x) dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

Adding all these integrals, if n is an even positive integer, that is, the number of ordinates y_0, y_1, \dots, y_n is odd, we have

$$\begin{aligned} \int_{x_0}^{x_n} f(x) dx &= \int_{x_0}^{x_2} f(x) dx + \int_{x_2}^{x_4} f(x) dx + \dots + \int_{x_{n-2}}^{x_n} f(x) dx \\ &= \frac{h}{3} [(y_0 + 4y_1 + 4y_2) + (y_2 + 4y_3 + 4y_4) + \dots + (y_{n-2} + 4y_{n-1} + y_n)] \\ &= \frac{h}{3} [(y_0 + y_n) + 2(y_2 + y_4 + \dots) + 4(y_1 + y_3 + \dots)] \\ &= \frac{h}{3} [\text{Sum of the first and last ordinates} + 2(\text{sum of remaining odd ordinates}) + 4(\text{sum of even ordinates})] \end{aligned}$$

Note: Though has suffix even, it is the third ordinate (odd).

7.6 SIMPSON'S THREE-EIGHTHS RULE

Putting $n = 3$ in Newton-cotes formula

$$\begin{aligned} \int_{x_0}^{x_3} f(x) dx &= h \left[3y_0 + \frac{9}{2} \Delta y_0 + \frac{1}{2} \left(\frac{9}{2} \right) \Delta^2 y_0 + \frac{1}{6} \left(\frac{81}{4} - 27 + 9 \right) \Delta^3 y_0 \right] \\ &= h \left[3y_0 + \frac{9}{2} (y_1 - y_0) + \frac{9}{4} (E - 1)^2 y_0 + \frac{3}{8} (E - 1)^3 y_0 \right] \\ &= h \left[3y_0 + \frac{9}{2} y_1 - \frac{9}{4} y_0 + \frac{9}{4} (y_2 - 2y_1 + y_0) + \frac{8}{3} (y_3 - 3y_2 + 3y_1 - y_0) \right] \\ &= \frac{3h}{8} [y_3 + 3y_2 + 3y_1 + y_0] \end{aligned} \tag{1}$$

If n is a multiple of 3,

$$\begin{aligned} \int_{x_0}^{x_0+nh} f(x) dx &= \int_{x_0}^{x_0+3h} f(x) dx + \int_{x_0+3h}^{x_0+6h} f(x) dx + \dots + \int_{x_0+(n-3)h}^{x_0+nh} f(x) dx \\ &= \frac{3h}{8} [(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) + \dots + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n)] \\ &= \frac{3h}{8} [(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \dots + y_n)] \end{aligned} \tag{2}$$

Equation (2) is called Simpson's three - eights rule which is applicable only when n is a multiple of 3.

Examples

1. Evaluate $\int_{-3}^3 x^4 dx$ by using (1) Trapezoidal rule (2) Simpson's rule. Verify your results by actual integration.

Solution:

Here $y(x) = x^4$. Interval length $(b-a) = 6$. So, we divide 6 equal intervals with $h = \frac{6}{6} = 1$.

We form below the table

x :	-3	-2	-1	0	1	2	3
y :	81	16	1	0	1	16	81

- (i) By Trapezoidal rule, $\int_{-3}^3 y dx \approx \frac{h}{2} [(\text{sum of the first and last ordinates} + 2(\text{sum of the remaining ordinates})]$

$$\approx \frac{1}{2} [(81+81) + 2(16+1+0+1+16)] \approx 115$$

- (ii) By Simpson's one-third rule (since number of ordinates is odd)

$$\int_{-3}^3 y dx \approx \frac{1}{3} [(81+81) + 2(1+1) + 4(16+0+16)] \approx 98$$

- (iii) Since $n = 6$, (multiple of three), we can also use Simpson's three-eighths rule. By this rule,

$$\int_{-3}^3 y dx \approx \frac{3}{8} [(81+81) + 3(16+1+1+16) + 2(0)] \approx 99.$$

- (iv) By actual integration

$$\int_{-3}^3 x^4 dx = 2 \times \left(\frac{x^5}{5} \right)_0^3 = \frac{2 \times 243}{5} = 97.2$$

From the results obtained by various methods, we see that Simpson's rule gives better result than Trapezoidal rule.

2. By dividing the range into ten equal parts, evaluate $\int_0^{\pi} \sin x dx$ by Trapezoidal and Sampson's rule.

Verify your answer with integration.

Solution:

$$\text{Range} = \pi - 0 = \pi \quad \text{Hence } h = \frac{\pi}{10}$$

we tabulated below the values of y at different x's.

x	:	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$
y = sin x	:	0.0	0.3090	0.5878	0.8090	0.9511	1.0
x	:	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$	π	
y = sin x	:	0.9511	0.8090	0.5878	0.3090		

(Note that the values are symmetrical about $x = \frac{\pi}{2}$)

(i) By Trapezoidal rule,

$$I = \frac{\pi}{20} [(0+0) + 2(0.3090 + 0.5878 + 0.8090 + 0.9511 + 1.0 + 0.9511 + 0.8090 + 0.5878 + 0.3090)]$$

$$= 1.9843 \text{ nearly.}$$

(ii) By Simpson's one-third rule (since there are 11 ordinates)

$$I = \frac{1}{3} \left(\frac{\pi}{10} \right) [(0+0) + 2(0.5878 + 0.9511 + 0.9511 + 0.5878) + 4(0.3090 + 0.8090 + 1 + 0.8090 + 0.3090)]$$

$$= 2.00091$$

Note: We cannot use Simpson's three-eighths rule or Weddle's rule here.

(iii) By actual integration, $I = (-\cos x)_0^\pi = 2$

Hence, Simpson's rule is more accurate than the Trapezoidal rule.

3. Evaluate $\int_0^1 e^x dx$ by Simpson's one-third rule correct to five decimal places, by proper choice of h.

Solution:

Here, interval length = $b-a = 1$

$$y = e^x; y^{(iv)} = e^x$$

$$\text{Error} = |E| < \frac{(b-a)}{180} h^4 \cdot M, \text{ where } M = \text{Max}(e^x) \text{ in the range.}$$

$$< \frac{1}{180} h^4 \cdot e$$

We require $(E) < 10^{-6}$

$$\frac{h^4 e}{180} < 10^{-6}; \quad h < \left(\frac{180 \times 10^{-6}}{e} \right)^{1/4} = 0.148$$

Hence we take $h = 0.1$ to have the accuracy required.

$$\begin{aligned}\therefore \int_0^1 e^x dx &= \frac{0.1}{3} [(1+e) + 2(e^{0.2} + e^{0.4} + e^{0.6} + e^{0.8}) + 4(e^{0.1} + e^{0.3} + e^{0.5} + e^{0.7} + e^{0.9})] \\ &= 1.718283\end{aligned}$$

By actual integration, $\int_0^1 e^x dx = (e^x)_0^1 = e - 1 = 1.71828183$

Correct to five decimal places, the answer is 1.71828.

4. The accelerations of a vertical at nine timing instances from $t = 0$ to 40 with an interval of 5 are 40.0, 45.25, 48.50, 51.25, 54.35, 59.48, 61.5, 64.3, 68.7. Find the velocity at $t = 40$ using trapezoidal rule.

Solution:

Trapezoidal rule can be given by

$$\int_{x_0}^{x_n} y dx = \frac{h}{2} [y_0 + 2[y_1 + y_2 + \dots + y_{n-1}] + y_n]$$

According to the given problem

$$\int_0^{40} y dx = \frac{h}{2} [y_0 + 2[y_1 + y_2 + \dots + y_{n-1}] + y_n]$$

Where $h = 5$

$$y_0 = 40.0, y_1 = 45.25, y_2 = 48.5, y_3 = 51.25, y_4 = 54.35, y_5 = 59.48,$$

$$y_6 = 61.5, y_7 = 64.3, y_8 = 68.7$$

$$\int_0^{40} y dx = \frac{5}{2} [40.0 + 2[45.25 + 48.5 + 51.25 + 54.35 + 59.48 + 61.5 + 64.3] + 68.7]$$

$$= 2194.9 \text{ m/sec.}$$

Therefore at $t = 40$, the velocity will be 2194.9 m/sec.

5. The accelerations of a vehicle at nine timing instances from $t = 0$ to 40 with an interval of 5 are 40.0, 45.25, 48.5, 51.25, 54.35, 59.48, 61.5, 64.3, 68.7. Find the velocity at $t = 40$ using Simpson's rule.

Solution:

Simpson's $\frac{3}{8}$ rule $\because h = 5$ (odd)

$$\int_{x_0}^{x_n} y dx = \frac{3h}{8} [(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots) + 2(y_3 + y_6 + y_9 + \dots + y_{n-3})]$$

Here $h=5, y_0=40.0, y_1=45.25, y_2=48.5, y_3=51.25, y_4=54.35, y_5=59.48, y_6=61.5, y_7=64.3, y_8=68.7$

$$\int_0^{40} y \, dx = \frac{3 \times 5}{8} \left[(40.0 + 68.7) + 3(45.25 + 48.5 + 54.35 + 59.48 + 64.3) + 2(51.25 + 61.5) \right] = 2155.14$$

At $t = 40$, the velocity is 2155.14 m/sec.

6. Evaluate $\int_0^{1/2} \left(\frac{x}{\sin x} \right) dx$, taking the step size as $\frac{1}{16}$ using Simpson's rule.

Solution:

Let $y = \int_0^{1/2} \frac{x}{\sin x} \, dx$ and given that the step size $\frac{1}{16} \Rightarrow h = \frac{1}{16} = 0.0625$

x	0	0.625	0.125	0.1875	0.25	
y	0	57.2957	57.2958	57.2958	57.2959	57.2959
		0.3125	0.375	0.4375	0.5	
		57.2960	27.2962	57.2963	57.2965	

By taking Simpson's 1/3rd rule

$$\int_{x_0}^{x_n} y \, dx = \frac{1}{3} h (y_0 + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) + 2(y_2 + y_4 + y_6 + \dots + y_{n-2}) + y_n)$$

$$\begin{aligned} \int_{x=0}^{1/2} \frac{x}{\sin x} \, dx &= \frac{1}{3} \cdot \frac{1}{16} \left[0 + 4(57.2957 + 57.2958 + 57.2960 + 57.2963) + \right. \\ &\quad \left. 2(57.2958 + 57.2959 + 57.2962 + 57.2965) \right] \\ &= \frac{1}{48} [0 + 916.7352 + 343.7758 + 57.2965] = \frac{1317.8075}{48} = 27.4543 \end{aligned}$$

7. Compute $\int_0^{\pi/2} \frac{1}{(\sin x)^{1/4}} \, dx$ with an error $< 10^{-5}$ using Simpson's rule.

Solution:

Let $y = \int_0^{\pi/2} \frac{1}{(\sin x)^{1/4}} \, dx$ and given that error $< 10^{-5}$

By Simpson's rule the error expression is $\Rightarrow \frac{h^2}{90} \leq \text{error}$

$$\frac{h^2}{90} \leq 10^{-5} \Rightarrow h^2 \leq 90 \times 10^{-5} \Rightarrow h \leq 0.24595$$

Let step length $h = \frac{\pi}{4}$

x	0	$\frac{\pi}{14}$	$\frac{2\pi}{14}$	$\frac{3\pi}{14}$	$\frac{4\pi}{14}$	$\frac{5\pi}{14}$	$\frac{6\pi}{14}$	$\frac{7\pi}{14}$
y	3.69	1.46	1.24	1.13	1.06	1.03	1.01	1.00

By Simpson's 1/3rd rule is

$$\int_{x_0}^{x_n} dx = \frac{1}{3}h[y_0 + 4[y_1 + y_3 + y_5 + \dots] + 2[y_2 + y_4 + y_6 + \dots] + y_n]$$

$$\int_0^{\pi/2} \frac{1}{(\sin x)^{1/4}} dx = \frac{1}{3} \times 0.224285 [3.68 + 4[1.46 + 1.13 + 1.03] + 2[1.24 + 1.06 + 1.01] + 1.00]$$

$$= 0.07476 \times 25.79 = 1.92810$$

Hence $\int_0^{\pi/2} (\sin x)^{-1/4} dx = 1.92810$

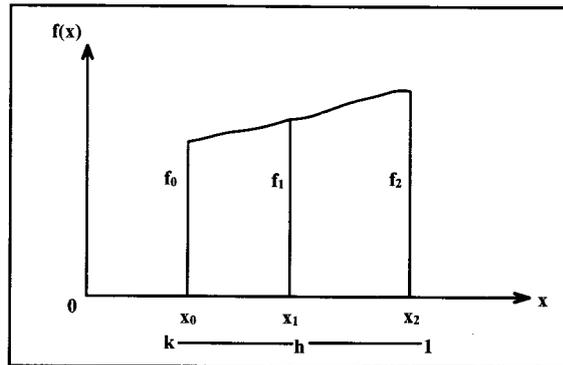
7.7 ROMBERG'S METHOD

Romberg integration is basically Richardson's extrapolation procedure. Romberg's name is attached to the method because he was the first one to describe the algorithm in recursive form. Consider the integral

$$I = \int_a^b f(x) dx \tag{1}$$

To desire Romberg method, we shall return to Trapezoidal rule.

First find the numerically integrated value dividing only one step in the interval $x_0 - x_2$



Using Trapezoidal Rule, we get

$$I_n = \frac{h}{2}[f_0 + f_2] \tag{2}$$

Hence True value of I is

$$I_{True} = I_T + ch^2 \tag{3}$$

Dividing the interval into two steps and applying trapezoidal rule.

$$I_{\frac{h}{2}} = \frac{h}{4}[f_0 + 2f_1 + f_2] = \dots\dots(4)$$

$$I_{\text{True}} = I_{\frac{h}{2}} + ch^2 \dots(5) \left(\frac{ch^2}{4} \dots\text{error} \right) \dots\dots(5)$$

Dividing (3) by (5) we get

$$\frac{I_{\text{True}} - I_n}{I_{\text{True}} - I_{\frac{h}{2}}} = \frac{ch^2}{\frac{ch^2}{4}} = 4$$

$$I_{\text{True}} = \frac{4I_{\frac{h}{2}} - I_n}{3} \dots\dots(6)$$

Substituting for $I_{\frac{h}{2}}$ and for I_n , we get

$$I_{\text{True}} = \frac{h[f_0 + 2f_1 + f_2] - \frac{h}{2}[f_0 + f_2]}{3} \quad [\text{using (2)+(4)}]$$

$$= \frac{h}{3}[2f_0 + 4f_1 + 2f_2 - f_0 - f_2]$$

$$I_{\text{True}} = \frac{h}{3}[f_0 + 4f_1 + f_2] \dots\dots(7)$$

If we assume $T_i^{(0)}$ as the numerically evaluated integral value for i trapezoids and $T_{2i}^{(0)}$ as the numerically evaluated integral value for 2i trapezoids, combining them, we get

$$T_{2i}^{(1)} = \frac{4T_{2i}^{(0)} - T_i^{(0)}}{3} \dots\dots(8)$$

It is same as the one obtained from Simpson's rule.

In a similar manner, let us assume $T_i^{(1)}$ as the numerically integrated value for i interval in Simpson's and $T_{2i}^{(1)}$ evaluated integral value for 2i steps.

$$I_{\text{True}} = T_i^{(1)} + ch^4 \dots\dots(9)$$

$$I_{\text{True}} = T_{2i}^{(1)} + \frac{ch^4}{6} \dots\dots(10) \left(ch^4 \cdot \frac{ch^4}{16} \text{ error in simpson's rule} \right)$$

From (9) & (10) we get

$$\frac{I_{\text{True}} - T_i^{(0)}}{I_{\text{True}} - T_{2i}^{(0)}} + \frac{ch^4}{ch^4} = 16$$

(Dividing (9) by (10))

$$I_{\text{True}} = \frac{16T_{2i}^{(1)} - T_i^{(1)}}{15} = \frac{16T_{2i}^{(1)} - T_i^{(1)}}{4^2 - 1} \dots\dots(11)$$

In general, We get the extrapolation procedure for trapezoidal rule

$$T_{2i}^{(j)} = \frac{4^j T_{2i}^{(j-1)} - T_i^{(j-1)}}{4^j - 1} \dots\dots(12)$$

The extrapolation procedure, for Simpson’s rule becomes

$$T_{2i}^{(j)} = \frac{4^j T_{2i}^{(j-1)} - T_i^{(j-1)}}{4^{j+1} - 1}; \dots\dots(13)$$

The table for Romberg integration is shown in Table 7.1:

Table 7.1: Romberg Table

No. of Trapezoidal h	Trapezoidal Approx	First Order	2 nd order Simpson	3 rd order	4 th order	5 th order
1	T ₁ ⁰					
2	T ₂ ⁰	T ₂ ¹				
4	T ₄ ⁰	T ₄ ¹	T ₄ ²			
8	T ₈ ⁰	T ₈ ¹	T ₈ ²	T ₈ ³		
16	T ₁₆ ⁰	T ₁₆ ¹	T ₁₆ ²	T ₁₆ ³	T ₁₆ ⁴	
32	T ₃₂ ⁰	T ₃₂ ¹	T ₃₂ ²	T ₃₂ ³	T ₃₂ ⁴	T ₃₂ ⁵

Example:

Compute $I_p = \int_0^1 \frac{x^p}{x^3 + 12} dx$

for p=1 using trapezoidal and Simpson’s rule with the number of points 3,5, and 9. Improve the results using Romberg integration.

Solution:

Given that $x_0 = 0, x_N = 1, h = \frac{1-0}{N}, x_i = x_0 + ih; f(x) = \frac{x}{x^3 + 12}$

Using the composite trapezoidal rule.

$x_0 = 0, x_N = 1, h = \frac{1-0}{N}, x_i = x_0 + ih; f(x) = \frac{x}{x^3 + 12}$ for N = 8, we get

$$x_0 = 0, x_1 = \frac{1}{8}, x_2 = \frac{2}{8}, x_3 = \frac{3}{8}, x_4 = \frac{4}{8}, x_5 = \frac{5}{8}, x_6 = \frac{6}{8}, x_7 = \frac{7}{8}, x_8 = 1$$

and the function value are

$$f(x_0) = f(0) = f_0 = 0$$

$$f(x_1) = f\left(\frac{1}{8}\right) = f_1 = 0.01041$$

$$f(x_2) = f\left(\frac{2}{8}\right) = f_2 = 0.02081$$

$$f(x_3) = f\left(\frac{3}{8}\right) = f_3 = 0.03111$$

$$f(x_4) = f\left(\frac{4}{8}\right) = f_4 = 0.04124$$

$$f(x_5) = f\left(\frac{5}{8}\right) = f_5 = 0.05104$$

$$f(x_6) = f\left(\frac{6}{8}\right) = f_6 = 0.06038$$

$$f(x_7) = f\left(\frac{7}{8}\right) = f_7 = 0.06906$$

$$f(x_8) = f(1) = f_8 = 0.07692$$

Using the composite trapezoidal rule, $N = 2, h = 1/2$

$$\begin{aligned} T_2^0 &= \frac{h}{1} [f_0 + 2f_4 + f_8] = \frac{1}{4} [0 + 2(0.4124 + 0.07692)] \\ &= 0.03985 \end{aligned}$$

$$\begin{aligned} N=4, h=\frac{1}{4}; \quad T_4^0 &= \frac{h}{2} [f_0 + 2(f_2 + f_4 + f_6) + f_8] \\ &= \frac{1}{8} [0 + 2(0.02081 + 0.04124 + 0.6038) + 0.07692] \end{aligned}$$

$$T_4^0 = 0.0402225$$

$$N=8, h=\frac{1}{8}; T_8^0 = \frac{h}{2} \left[f_0 + 2 \sum_{i=1}^7 f_i + f_8 \right]$$

$$T_8^0 = \frac{1}{16} [0 + 2(0.01041 + 0.02081 + 0.03111 + 0.04124 + 0.05104 + 0.06038 + 0.06906) + 0.07692]$$

$$T_8^0 = 0.04031375$$

Using Romberg integration we get

$$T_4^{(1)} = \frac{4T_4^0 - T_2^0}{3} = \frac{4(0.402225) - (0.03985)}{3} = 0.0403467$$

$$T_8^{(1)} = \frac{4T_8^0 - T_4^0}{3} = \frac{4(0.4031375) - (0.0402225)}{3} = 0.0403442 \text{ III}^{\text{ly}}$$

$$T_8^{(2)} = \frac{4^2 T_8^{(1)} - T_4^{(1)}}{4^2 - 1} = \frac{16(0.403442) - (0.0403467)}{15} = 0.40344$$

Table 7.2: Trapezoidal Rule with Romberg Integration

h	Fourth order method	Sixth order method	Eight order method
1/2	0.03985		
		0.0403467	
1/4	0.0402225		0.40344
		0.0403442	
1/8	0.04031375		

Similarly, using the composite Simpson's Rule.

$$I = \int_0^1 \frac{x}{x^3 + 12} dx = \frac{h}{3} \left[f_0 + 4 \sum_{i=1}^N f_{2i-1} + 2 \sum_{i=1}^{N-1} f_{2i} + f_{2N} \right]$$

where $x_0 = 0, x_{2N} = 1, h = \frac{b-a}{2N} = \frac{1-0}{2N}$

when $N=1, h = \frac{1}{2}, T_1^0 = \frac{1}{6} [f_0 + 4f_4 + f_8] = \frac{1}{6} [0 + 4(0.04124) + 0.07692]$

$$T_1^0 = 0.040313$$

when $N=2, h = \frac{1}{4}, T_1^0 = \frac{1}{12} [f_0 + [4f_2 + f_6] + 2f_4 + f_8]$

$$T_1^0 = \frac{1}{12} [0 + 4(0.02081 + 0.06038) + 2(0.04124) + 0.07692]$$

$$T_2^0 = 0.040313$$

$$N=4; h = \frac{1}{8}; T_4^0 = \frac{h}{3} (f_0 + 4(f_1 + f_3 + f_5 + f_7) + 2(f_2 + f_4 + f_6) + f_8)$$

$$T_4^0 = \frac{1}{24} [0 + 4(0.01401 + 0.03111 + 0.05104 + 0.06906) + 2(0.02081 + 0.04124 + 0.06038) + 0.07692]$$

$$T_4^0 = 0.040344166$$

Using Romberg Integration

$$T_2^{(1)} = \frac{4^2 T_2^0 - T_1^0}{4^2 - 1} = \frac{16(0.0403467) - (0.040313)}{15} = 0.4034895$$

$$T_4^{(1)} = \frac{4^2 T_4^0 - T_2^0}{4^2 - 1} = \frac{16(0.040344166) - (0.0403467)}{15} = 0.40343997 \text{ III}^{\text{ly}}$$

$$T_4^{(2)} = \frac{4^3 T_4^{(1)} - T_2^0}{4^3 - 1} = \frac{64(0.40343997) - (0.04034895)}{63} = 0.4034392$$

Table 7.3: Simpson's Rule with Romberg Integration

h	Fourth order method	Sixth order method	Eight order method
1/2	0.040313		
		0.04034895	
1/4	0.0403467		0.4034392
		0.040343997	
1/8	0.040344166		

Check Your Progress

- If $h = \frac{1}{2} \int_1^2 (x^3 + 1) dx$ by trapezoidal rule is

(a) 4.525 (b) 5.523 (c) 6.523 (d) 4.938
- x 1 1.5 2 2.5 3

f(x) 2 2.4 2.7 2.8 3

$\int_1^3 f(x) dx =$ using Simpson's 1/3 rule.

(a) 4.03 (b) 3.82 (c) 5.03 (d) 4.52
- If $h = \frac{1}{2} \int_0^1 \frac{dx}{x^2 + 1}$ Simon's 1/3 rd rule

(a) .217 (b) .231 (c) .252 (d) .242
- If $f(0) = 0, f(.25) = .5, f(.5) = .7, f(.75) = 8$ and $f(1) = .9$ by Simpson's $\frac{1}{3}$ rd rule $\int_0^1 f(x) dx =$

(a) .600 (b) .625 (c) .331 (d) 65
- If $y(0) = 3, f(.5) = 3.5, f(1) = 4, f(1.5) = 4.75, f(2) = 4.25, \int_0^2 y dx =$ by sampan's $\frac{1}{3}$ rd =

(a) 24.25 (b) 24.125 (c) 24.000 (d) 23.925
- $f(0), f(2) = 10, f(4) = 18, f(6) = 25, f(8) = 29$ and $h = 2$ then by sampan's $\frac{1}{3} rd \int_0^8 f(x) dx =$

(a) 135 (b) 134.25 (c) 136 (d) 136.66
- If $y_0 = 1, y_1 = .66, y_2 = .5, y_3 = A, y_4 = .33$ by sampan's $\frac{1}{3} rd \int_0^2 f(x) dx$

(a) 1.00 (b) 1.20 (c) 1.31 (d) 1.11

7.8 LET US SUM UP

In this lesson, we discussed the integration of definite integrals using numerical integration techniques. The following Newton-Cotes methods were considered in detail:

- Trapezoidal rule
- Simpson's 1/3 rule
- Simpson's 3/8 rule

We also presented a method known as Romberg integration to improve the accuracy of the results of the trapezoidal method.

We finally discussed another approach known as Gauss integration which is based on the concept that the accuracy can be improved by choosing the sampling points wisely, rather than equally.

7.9 KEYWORDS

Newton's forward interpolation formula's:

$$y = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!} \Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!} \Delta^3 y_0 + \dots$$

Trapezoidal rule:

$$= \frac{h}{2} [(\text{Sum of the first and the last ordinates}) + 2(\text{Sum of the remaining ordinates})]$$

Simpson's 1/3 rule:

$$= \frac{h}{3} [(y_0 + y_n) + 2(y_2 + y_4 + \dots) + 4(y_1 + y_3 + \dots)]$$

Simpson's 3/8 rule:

$$= \frac{3h}{8} [(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_6 + y_9 + \dots + y_n)]$$

7.10 QUESTIONS FOR DISCUSSION

1. Evaluate $\int_0^1 e^{-x^2} dx$ by dividing the range of integration into 4 equal using (i) Trapezoidal rule, (ii) Simpson's rule

2. Using Trapezoidal rule evaluate $\int_{0.6}^2 y dx$ from the following table.

x:	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0
y:	1.23	1.58	2.03	4.32	6.25	8.36	10.23	12.45

3. Find the value of $\log 2^{1/3}$ from $\int_0^1 \frac{x^2}{1+x^3} dx$ using Simpson's $\frac{1}{3}$ rule with $h = 0.25$
4. Evaluate $\int_{0.2}^{1.4} (\sin x - \ln x + e^x) dx$ by Simpson's $\frac{1}{3}$ rule.
5. Use Romberg integration to find $I = \int_0^{\frac{\pi}{4}} \sec x dx$
6. Derive Numerical Integration method for the solution of a differential equations.

Check Your Progress: Model Answers

1. (d)
2. (c)
3. (a)
4. (b)
5. (b)
6. (d)
7. (d)

7.11 SUGGESTED READINGS

S.S. Sastri, *Introductory Methods of Numerical Analysis*, Second Edition, Prentice Hall.

S. Mohan Naidu., *A Text Book of Mathematical Methods for Scientific Engineers*, Students Helpline Series, Hyd -2005.

Jain, M.K., Iyengar, S.R.K., Jain, R.K., *Numerical Methods for Scientific and Engineering Computation*, 3rd Edition, New Age International (P) Ltd., 1993.

LESSON

8

DIFFERENCE EQUATIONS

CONTENTS

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- 8.1 Introduction
- 8.2 Basic Definitions
- 8.3 Formation of Difference Equations
- 8.4 Linear Difference Equations
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- 8.7 Questions for Discussions
- 8.8 Suggested Readings

8.0 AIMS AND OBJECTIVES

After studying this lesson, you will be able to:

- Explain basic definitions of difference equations
- Understand the formation of Difference Equations
- Discuss Linear Difference Equation

8.1 INTRODUCTION

In this lesson we develop the theory of difference equation and discuss methods of solving them. Any situation in which there exists a sequential relation at discrete values of the independent variable leads to difference equations. Difference equation may be thought of as a discrete counterpart of differential equation and there is a striking similarity between the methods of solving difference equations and differential equations.

8.2 BASIC DEFINITIONS

1. An equation involving the difference of an unknown function $y = y(x)$ at one or more general values of the argument n is called a difference equation.

The following are some examples of difference equations.

$$\Delta y_n + 2y_n = n \quad \dots(1)$$

$$\Delta^2 y_n + 5\Delta y_n + 3y_n = 0 \quad \dots(2)$$

$$\Delta^2 u_x - 4\Delta u_x + 4u_x = 3^x \quad \dots(3)$$

Throughout this lesson we assume that the consecutive values of the independent variable differ by unity. With this assumption a difference equation can be written in an alternate form as illustrated below.

Consider $\Delta y_n + 2y_n = n$

Since $\Delta y_n = (E - 1)y_n = y_{n+1} - y_n$, the above equation can be written as

$$y_{n+1} - y_n + 2y_n = n$$

(i.e.) $y_{n+1} + y_n = n.$

Consider (3) given by $\Delta^2 u_x - 4\Delta u_x + 4u_x = 3^x$

$$\Delta^2 u_x = (E - 1)^2 u_x = (E^2 - 2E + 1)u_x = u_{x+2} - 2u_{x+1} + u_x$$

and $\Delta u_x = (E - 1)u_x = u_{x+1} - u_x.$

Hence the above equation can be rewritten as

$$(u_{x+2} - 2u_{x+1} + u_x) - 4(u_{x+1} - u_x) + 4u_x = 3^x$$

i.e. $u_{x+2} - 6u_{x+1} + 9u_x = 3^x.$

1. Difference equation written in the above form is also called recurrence relation.
2. The order of a difference equation is the difference between the largest and smallest subscribes occurring in it, when the equation is expressed in a form free of Δ . The degree of a difference equation, expressed in a form free of Δ , is the highest power of y .

Examples

(a) The difference equation $\Delta y_n + 2y_n = n$, when expressed in a form free of Δ is

$$y_{n+1} + y_n = n$$

The order and degree of the difference equation are both 1.

(b) The difference equation $\Delta^2 u_x - 4\Delta u_x + 4u_x = 3^x$, when expressed in a form free of Δ is

$$u_{x+2} - 6u_{x+1} + 9u_x = 3^x.$$

Its order is 2 and degree is 1.

(c) The difference equation $4y_{n+3}^2 - 2y_n y_{y+1} + y_n^2 y_{n+1}^4 = 0$ is of order 3 and degree 4.

3. A solution of a difference equation is an expression for y_n which satisfies the given difference equation. A solution in which the number of arbitrary constants is equal to the order of the difference equation is called the general solution. Any solution which is obtained from the general solution by giving particular values to the arbitrary constants is called a particular solution.

Solved Problem

Problem 1: Write the difference equation $\Delta^3 y_x + \Delta^2 y_x + \Delta y_x + y_x = 0$ in the subscript notation.

Solution: The given difference equation can be written as

$$(E-1)^3 y_x + (E-1)^2 y_x + (E-1)y_x + y_x = 0.$$

$$\text{i.e. } (E^3 - 3E^2 + 3E - 1)y_x + (E^2 - 2E + 1)y_x + (E-1)y_x + y_x = 0$$

$$\text{i.e. } (E^3 - 2E^2 + 2E)y_x = 0$$

$$\therefore y_{x+3} - 2y_{x+2} + 2y_{x+1} = 0.$$

Problem 2: Find the order of the difference equation

$$\Delta^3 y_n - 3\Delta^2 y_n + 2\Delta y_n + y_n = \cos \pi x$$

Solution: The given difference equation can be written as

$$(E-1)^3 y_n - 3(E-1)^2 y_n + 2(E-1)y_n + y_n = \cos \pi x$$

$$\text{i.e. } (E^3 - 3E^2 + 3E - 1)y_n - 3(E^2 - 2E + 1)y_n + 2(E-1)y_n + y_n = \cos \pi x$$

$$\text{i.e. } (E^3 - 6E^2 + 11E - 5)y_n = \cos \pi x$$

$$\therefore y_{n+3} - 6y_{n+2} + 11y_{n+1} - 5y_n = \cos \pi x$$

\therefore The order of the given difference equation is 3.

Problem 3: Show that $y_n = 1 - \frac{2}{n}$ is a solution of the difference equation $(n+1)y_{n+1} + ny_n = 2n - 3$.

Solution.

$$\begin{aligned} (n+1)y_{n+1} + ny_n &= (n+1)\left(1 - \frac{2}{n+1}\right) + n\left(1 - \frac{2}{n}\right) \\ &= (n+1) - 2 + n - 2 \\ &= 2n - 3. \end{aligned}$$

$\therefore y_n = 1 - \frac{2}{n}$ is a solution of the given difference equation.

8.3 FORMATION OF DIFFERENCE EQUATIONS

Let $x_n, x_{n+1}, \dots, x_{n+k}$ be a set of $k+1$ equispaced tabular points with spacing h and $u_n, u_{n+1}, \dots, u_{n+k}$ be the corresponding values of a function $u(x)$ at these points, that is, $x_{n+i} = x_n + ih, u_{n+i} = u(x_{n+i}), i = 0, 1, \dots, k$, for some integer k . A relationship between u_n and the difference $\Delta u_n, \Delta^2 u_n, \dots, \Delta^k u_n$ is called a difference equation and hence it can be regarded as a relation among $u_n, u_{n+1}, \dots, u_{n+k}$. The order of a difference equation is the number is the number of intervals separating the largest and the smallest argument of the dependent variable.

A difference equation of order k , in its most general form can be written as

$$F(u_n, u_{n+1}, \dots, u_{n+k}) = 0 \quad \dots(1)$$

If the function F is linear in $u_n, u_{n+1}, \dots, u_{n+k}$, then the difference equation is called linear. A linear difference equation of order k can be written as

$$a_0 u_{n+k} + a_1 u_{n+k-1} + \dots + a_k u_n = g_n, a_0 \neq 0. \quad \dots(2)$$

8.4 LINEAR DIFFERENCE EQUATIONS

If the coefficients a_0, a_1, \dots, a_k in above equation are constant, then the equation defines (See equation (1) and (2) in the previous heading) a linear difference equation of order k with constant coefficients.

For example, the difference equation

$$\Delta^2 u_n + 3\Delta u_n + 5u_n = 0$$

$$\text{or } (u_{n+2} - 2u_{n+1} + u_n) + 3(u_{n+1} - u_n) + 5u_n = 0$$

$$\text{or } u_{n+2} + u_{n+1} + 3u_n = 0$$

is a linear difference equation of order 2, with constant coefficients.

If $g_n = 0$, then the difference equation is said to be homogeneous, otherwise, it is said to be inhomogeneous. We shall consider only the linear difference equations with constant coefficient. The solution of the difference equation consists of a solution $u_n^{(H)}$ called the complementary solution of the homogeneous part

$$a_0 u_{n+k} + a_1 u_{n+k-1} + \dots + a_k u_n = 0$$

and a particular solution $u_n^{(P)}$ of the inhomogeneous part. The general solution is written as

$$u_n = u_n^{(H)} + u_n^{(P)}.$$

For solving, we assume a solution of the form $u_n = A\xi^n$, where $A \neq 0$ is a constant. We get

$$A(a_0 \xi^k + a_1 \xi^{k-1} + \dots + a_k) \xi^n = 0$$

$$\text{or } a_0 \xi^k + a_1 \xi^{k-1} + \dots + a_k = 0$$

Which is a polynomial of degree k . The equation is called the characteristic equation of the difference equation. Let $\xi_1, \xi_2, \dots, \xi_k$ be the roots of last above equation. We have the following cases:

Real and Distinct Roots

If $\xi_1, \xi_2, \dots, \xi_k$ are all real and distinct, then we have

$$u_n^{(H)} = c_1 \xi_1^n + c_2 \xi_2^n + \dots + c_k \xi_k^n$$

Where c_1, c_2, \dots, c_k are arbitrary constants

Real and Repeated Roots

Let $\xi_1 (= \xi_2)$ be a double root and all other roots $\xi_3, \xi_4, \dots, \xi_k$ be distinct. It can be verified that ξ_1^n and $n\xi_1^n$ are two linearly independent solutions. Then, we have

$$u_n^{(H)} = (c_1 + nc_2)\xi_1^n + c_3\xi_3^n + \cdots + c_k\xi_k^n.$$

Similarly, when ξ_1 is a root of multiplicity p and the other roots $\xi_{p+1}, \xi_{p+2}, \dots, \xi_k$ are distinct, we have

$$u_n^{(H)} = (c_1 + nc_2 + \cdots + n^{p-1}c_p)\xi_1^n + c_{p+1}\xi_{p+1}^n + \cdots + c_k\xi_k^n.$$

Complex Roots

For a polynomial with real coefficients, the complex roots occur as conjugate pairs. Let $\xi_1 = \alpha + i\beta = re^{i\theta}$ and $\xi_2 = \alpha - i\beta = re^{-i\theta}$, where $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}(\beta/\alpha)$ be the complex roots and the other roots $\xi_3, \xi_4, \dots, \xi_k$ be real and distinct. Then, we have

$$u_n^{(H)} = [c_1 \cos(n\theta) + c_2 \sin(n\theta)]|\xi_1|^n + c_3\xi_3^n + \cdots + c_k\xi_k^n.$$

The constants c_1, c_2, \dots, c_n can be determined from the given conditions.

The particular integral $u_n^{(p)}$ depends on the form of g_n . When $g_n = g$ is a constant, we assume the solution as $u_n^{(p)} = q$, a constant. By Substituting, we get

$$(a_0 + a_1 + \cdots + a_k)q = g$$

or
$$u_n^{(p)} = q = \frac{g}{a_0 + a_1 + \cdots + a_k}$$

When g_n is the form $g_n = B + nC$, then we write the particular integral as $u_n^{(p)} = q + nr$. Substituting and comparing the coefficient of n and the constant term on both sides, we determine q and r .

Consider the homogeneous difference equation with constant coefficients.

If $\xi_1, \xi_2, \dots, \xi_k$ are all real and distinct, then

$$u_n = c_1\xi_1^n + c_2\xi_2^n + \cdots + c_k\xi_k^n.$$

Suppose now, we require that $u_n \rightarrow 0$ as $n \rightarrow \infty$. Then a necessary and sufficient condition is $|\xi_i| < 1, i = 1(1)k$. If u_n is to be bounded, then some of ξ_i may satisfy $|\xi_i| < 1$ and others may satisfy $|\xi_i| = 1$

Assume that $|\xi_1| = 1 = |\xi_2|$ and $|\xi_i| < 1, i = 3(1)k$. Then

$$|u_n| \rightarrow |c_1| + |c_2| \text{ as } n \rightarrow \infty$$

If $\xi_1 = \xi_2$ is a double root, then

$$u_n = (c_1 + nc_2)\xi_1^n + c_3\xi_3^n + \cdots + c_k\xi_k^n.$$

Now, let $|\xi_i| < 1, i = 3(1)k$. If $|\xi_1| < 1$, then $u_n \rightarrow 0$ as $n \rightarrow \infty$ and the solution is unbounded.

However, if $|\xi_1| = |\xi_2| < 1$, then $u_n \rightarrow 0$ as $n \rightarrow \infty$.

If ξ_1 and ξ_2 form a complex pair and $|\xi_i| < 1, i = 1, 2$, then $u_n \rightarrow 0$ as $n \rightarrow \infty$ if $|\xi_i| < 1$.

Hence, $u_n \rightarrow 0$ as $n \rightarrow \infty$, if the roots satisfy $|\xi_i| < 1, i = 1, 2$. If the roots are of magnitude 1 and simple, then u_n is bounded. In other cases, u_n is unbounded.

In many applications, the coefficients involve some parameters and we need to determine the ranges for these parameters so that the roots satisfy $|\xi_i| < 1$ for all i . It is not always possible to find the roots for all values of the parameters to check whether $|\xi_i| < 1$. In such cases, the values of parameters for which $|\xi_i| < 1$ can be determined by using the conformal mapping. Consider the mapping

$$\xi = \frac{1+z}{1-z}$$

which maps the interior of the unit circle $|\xi| = 1$ onto the left half plane $\operatorname{Re}(z) < 0$, and the unit circle $|\xi| = 1$ onto the imaginary axis.

Check Your Progress

Solve the difference equation $u_{n+2} - 2u_{n+1} + 6u_n = 4$.

8.5 LET US SUM UP

Difference equation may be thought of as a discrete counterpart of differential equation and there is a striking similarity between the methods of solving difference equations and differential equations. An equation involving the difference of an unknown function $y = y(x)$ at one or more general values of the argument n is called a difference equation. Any solution which is obtained from the general solution by giving particular values to the arbitrary constants is called a particular solution.

8.6 KEYWORDS

Difference Equation: An equation involving the difference of an unknown function $y = y(x)$ at one or more general values of the argument n is called a difference equation.

Particular Solutions: Any solution which is obtained from the general solution by giving particular values to the arbitrary constants is called a particular solution.

8.7 QUESTIONS FOR DISCUSSIONS

1. Solve $y_{n+2} - 3y_{n+1} + 2y_n = 0$
2. Solve $y_{n+4} - 4y_{n+3} + 6y_{n+2} - 4y_{n+1} + 4y_n = 0$.
3. Solve $u_{x+2} + 2u_{x+1} + u_x = 0$.
4. Solve $y_{n+1} = \sqrt{y_n}$.

Check Your Progress: Model Answers

The given difference equation can be written as

$$(E^2 - 2E + 6)u_n = 4.$$

The auxiliary equation is $E^2 - 2E + 6 = 0$

The roots are $\frac{2 \pm \sqrt{-20}}{2} = 1 \pm i\sqrt{5} = \alpha \pm i\beta$ say.

\therefore C.F. = $r^n(A \cos n\theta + B \sin n\theta)$ where

$$r = \sqrt{\alpha^2 + \beta^2} = \sqrt{1+5} = \sqrt{6}; \text{ and } \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \tan^{-1}(\sqrt{5}).$$

$$\therefore \text{C.F.} = 6^{\frac{n}{2}} [A \cos n\theta + \beta \sin n\theta] \text{ where } \theta = \tan^{-1}(\sqrt{5}).$$

$$\begin{aligned} \text{P.I.} &= \left(\frac{1}{E^2 - 2E + 6} \right) (4) \\ &= \left(\frac{1}{E^2 - 2E + 6} \right) 4 \times 1^n \\ &= \frac{4}{1^2 - 2 \times 1 + 6} \\ &= \frac{4}{5}. \end{aligned}$$

\therefore The complete solution is $u_n = \text{C.F.} + \text{P.I.} = 6^{\frac{n}{2}} (A \cos n\theta + B \sin n\theta) + \frac{4}{5}$.

8.8 SUGGESTED READINGS

S.S. Sastri, *Introductory Methods of Numerical Analysis*, Second Edition, Prentice Hall.

S. Mohan Naidu., *A Text Book of Mathematical Methods for Scientific Engineers*, Students Helpline Series, Hyd - 2005.

Jain, M.K., Iyengar, S.R.K., Jain, R.K., *Numerical Methods for Scientific and Engineering Computation*, 3rd Edition, New Age International (P) Ltd., 1993.